Electronic Journal of Differential Equations, Vol. 2024 (2024), No. 64, pp. 1–19.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2024.64

EXISTENCE AND UNIQUENESS OF THE SOLUTION TO INITIAL AND INVERSE PROBLEMS FOR INTEGRO-DIFFERENTIAL HEAT EQUATIONS WITH FRACTIONAL LOAD

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Abstract. This work is devoted to the unique solvability of the direct and inverse problems for a multidimensional heat equation with a fractional load in Holder spaces. In the problem under consideration, the loaded term is in the form of a fractional integral operator for the time variable. We prove the existence and uniqueness of the solution to these problems by the contraction mapping theorem and the theory of integral equations.

1. Introduction

The theory of differential equations with fractional operators has been widely used in various fields of science and engineering [46]. These equations are multidisciplinary and used in diverse fields such as dynamical systems, control theory, elasticity, electric drives, circuits systems, continuum mechanics, heat transfer, quantum mechanics, fluid mechanics, signal analysis, biomathematics, biomedicine, social systems, and bioengineering for modeling of the anomalous diffusion processes [2, 32, 47]; see the references therein.

Fractional diffusion equations are extensions of the basic equations of mathematical physics [8, 9, 35, 42]. The analytical methods used for solving these equations are of minimal use. In general, such equations began to be studied at the end of the previous century and have been intensively developed in recent decades [11, 16, 19, 25, 30, 40, 44, 52]. One of the actively studied fractional diffusion equations in recent years is the time-fractional diffusion equation which describes the mathematical processes of slow and super slow anomalous diffusion [15, 41, 51, 53].

But what about the diffusion equation with the time-fractional diffusion equation when the fractional operator includes a trace or combination of traces of the desired function? Here, we should also note the nonlocal problems [45] studied for the diffusion and wave equations, which have been successfully used in [5, 6, 10, 17, 26, 27, 39, 43] with applications to optimal control problems for dynamic populations. Such equations are also closely related to loaded equations [1, 12].

2020 *Mathematics Subject Classification.* 35K15, 35R30.
Key words and phrases. Heat equation; Cauchy problem; inverse problem; loaded equation; fractional operator.

Submitted August 15, 2024. Published October 23, 2024.

If we dwell, especially on problems for the loaded equation of parabolic type [45], or diffusion equation with the loaded term, then it should be noted the first results in these directions were obtained in the work of Nakhushev, Borisov, and Kreferov. Further, in [36] we find studies on the existence and uniqueness for the problem for a nonlinear loaded parabolic equation. Also in [24] we find studies of the Cauchy problem and related inverse problems for a one-dimensional nonlinear loaded parabolic equation in a special form.

Also in [7, 29] there are studies of the initial and boundary value problems (Cauchy problem, Cauchy-Dirichlet problem) for equations of essential loaded parabolic type loaded at a fixed time variable. A feature of these problems is the presence of a loaded term in an equation with a derivative of any integer order of the desired solution. In [37, 48] there are studies of boundary value problems for a fractionally loaded heat equation in two and three-dimensional domains. There when the order of the derivative in the loaded term is less than the order of the differential part, the load point moves. We should also note that in the recent papers [1, 28, 31, 50] there are interesting mixed-type loaded equations, which include parabolic equations with fractional operators in two and three-dimensional domains. Therefore, loaded fractional-diffusion equations with Riemann-Liouville or Caputo operators and parabolic type equations with fractional loads in three and higher dimensional domains need to be studied. This is for the completeness of the theory of fractional diffusion and integro-differential equations, and for their numerous applications.

In this work, along with the Cauchy problem, we study the inverse problem of determining the coefficient for the fractional time-loaded equation, which has attracted some interest in inverse problems research. Here, we also note the work [21] in two-dimensional problem of determining the diffusion coefficient for the fractional time equation. In [33] there are studies of inverse problems for the perturbed fractional diffusion time equation with a final redefinition. In [33] we find an explicit formula for solving the anomalous diffusion equation in a multidimensional space. Inverse problems for time-fractional diffusion equations with nonclassical conditions were investigated in [14, 34, 49].

The aim of this work is to study the existence and uniqueness of the solution of direct and inverse problems for the heat equation in a multidimensional domain. As far as we know, very few researchers have studied problems for fractional diffusion equations in a multidimensional domain and fractional load equations. In this work, we generalize the study to the heat equation with a fractional load and equations of convolution type [23] in an n-dimensional domain.

2. Cauchy problem for integro-differential heat equations with fractional load

In this section, we prove the existence and uniqueness of a solution to the Cauchy problem for the integro-differential equation of heat dissipation loaded with variable coefficients. Before proceeding to the formulation of the problem, we give some definitions and propositions for Holder spaces [38]. We introduce the following notation:

Let $\mathbb{R}^n$ be the n dimensional Euclidean space, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$;

$\mathbb{R}^n_T$ is $(n+1)$ dimensional Euclidean space, consisting of points (x,t), where $x \in \mathbb{R}^n$ and $t \in (0,T]$, $T > 0$; $\mathbb{R}^{n-1}_T = \{(x',t) : x' \in \mathbb{R}^{n-1},\ 0 < t < T\}$.

Let $f(x)$ be a position function in $\mathbb{R}^n$.

Definition 2.1. If for any two points $x_1, x_2 \in \mathbb{R}^n$, there is a positive constant A such that

$$|f(x_1) - f(x_2)| \le A|x_1 - x_2|^l,$$

then, the function f is said to satisfy the Holder condition with exponent l. The class of functions satisfying this condition is denoted as $H^l(\mathbb{R}^n)$.

Definition 2.2. If for any given pair of values $(x^{(1)}, t_1)$ and $(x^{(2)}, t_2)$ in $\mathbb{R}^n_T$, with $x^{(1)} = x_1^{(1)}, x_2^{(1)}, \dots, x_n^{(1)}$ and $x^{(2)} = x_1^{(2)}, x_2^{(2)}, \dots, x_n^{(2)}$, if holds

$$|f(x^{(1)}, t_1) - f(x^{(2)}, t_2)| \le \sum_{i=1}^n A_i |x_i^{(1)} - x_i^{(2)}|^l + A_{n+1}|t_1 - t_2|^{l/2},$$

where A_i is a positive constant and $l \in (0,1)$, then the function $f(x,t)$ is said to satisfy the Holder condition with exponent $l, l/2$ on $\mathbb{R}^n_T$. The class of such functions is denoted as $H^{l,l/2}(\mathbb{R}^n_T)$.

Cauchy problem. Find a solution $u(x,t)$ in the domain $(x,t) \in \mathbb{R}^n_T$ of the loaded heat equation

$$u_t - a(t)\Delta u = \lambda D_{0t}^{-\alpha} u(x', t) + \int_0^t k(x', \tau) u(x, t-\tau) d\tau, \quad (x,t) \in \mathbb{R}^n_T, \tag{2.1}$$

that satisfies the condition

$$u(x,t)\big|_{t=0} = \varphi(x), \quad x \in \mathbb{R}^n, \tag{2.2}$$

where $D_{0t}^{-\alpha}$ is the Riemann-Liouville fractional integral operator of order α defined by

$$D_{0t}^{-\alpha} u(x', t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(x', \tau) d\tau, \quad \alpha > 0,$$
$$a(t) \in E := \{a(t) \in C^1[0,T] : 0 < a_0 < a(t) \le a_1 < \infty\}, \quad \lambda \in R,$$

$\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator acting on variables $x(x_1, x_2, \dots, x_n)$, $k(x', t)$, and $\varphi(x)$ is a given real-valued function sufficiently smooth.

Theorem 2.3. *If $a(t) \in E$ for all $t \in (0,T]$, and $k(x', t) \in H^{l,l/2}(\overline{R}^{n-1}_T)$,*

$$\varphi(x) \in H^{l+2}(\mathbb{R}^n), \quad \varphi(x) \le \varphi_0,$$

where φ_0 is a positive constant, then there exists a unique solution to the Cauchy problem in the domain $u(x,t) \in H^{l+2,(l+2)/2}(\mathbb{R}^n_T)$, where

$$(\overline{R}^{n-1}_T) = \{(x', t) : x' \in \mathbb{R}^{n-1},\, 0 \le t \le T\}, \quad l \in (0,1).$$

Thus, on the basis of the given functions $k(x', t)$ and $\varphi(x)$, we will consider finding the function $u(x,t)$ from the integro-differential equation (2.1) with the initial condition (2.2), i.e., the Cauchy problem.

Before proceeding to study the Cauchy problem (2.1) and (2.2), we present the well-known solution to the problem for the classical inhomogeneous heat equation:

$$\begin{aligned} v(x,t) &= \int_{\mathbb{R}^n} \phi(\xi) G(x-\xi, \theta(t)) d\xi \\ &\quad + \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} F(\xi, \theta^{-1}(\tau)) G(x-\xi, \theta(t)-\tau) d\xi, \end{aligned} \tag{2.3}$$

which is the solution of the Cauchy problem for the heat equation with a time-varying heat conduction coefficient,

$$\begin{gathered} v_t - a(t)\Delta v = F(x,t), \quad x \in \mathbb{R}^n,\ t > 0, \\ v(x,0) = \phi(x), \quad x \in \mathbb{R}^n. \end{gathered}$$

In (2.3), $\theta^{-1}(t)$ is the inverse functions of $\theta(t) = \int_0^t a(\tau)d\tau$, and $G(x-\xi, \theta(t)-\tau)$ is a fundamental solution of the differential operator with a variable coefficient $\partial/\partial t - a(t)\Delta$, $\xi = (\xi_1, \xi_2, \dots, \xi_n)$; see [20].

Using (2.3), taking into account the properties of the fundamental solution, we have that the Cauchy problem (2.1) and (2.2) is equivalently reduced to the loaded integral equation of Volterra type with the shift [45],

$$\begin{aligned} u(x,t) = &\int_{\mathbb{R}^n} \varphi(\xi) G(x-\xi, \theta(t)) d\xi \\ &+ \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \int_0^{\theta^{-1}(\tau)} k(\xi', \alpha) u(\xi, \theta^{-1}(\tau) - \alpha) \\ &\times G(x-\xi, \theta(t)-\tau) d\xi d\alpha \\ &+ \frac{\lambda}{\Gamma(\alpha)} \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \int_0^{\theta^{-1}(\tau)} (\theta^{-1}(\tau) - \beta)^{\alpha-1} \\ &\times u(\xi', \beta) G(x-\xi, \theta(t)-\tau) d\beta d\xi, \end{aligned} \tag{2.4}$$

where $\xi' = (\xi_1, \xi_2, \dots, \xi_{n-1})$, $|x|^2 = x_1^2 + x_2^2 + \dots + x_n^2$. Thus, the last integral in relation (2.4) is understood as

$$\frac{\lambda}{\Gamma(\alpha)} \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}n-1} \int_0^{\theta^{-1}(\tau)} (\theta^{-1}(\tau) - \beta)^{\alpha-1} u(\xi', \beta) \widetilde{G}(x'-\xi, \theta(t)-\tau) d\beta d\xi',$$

i.e. with a loaded member, where $\widetilde{G}(x'-\xi, \theta(t)-\tau) = \int_{-\infty}^{\infty} G(x-\xi, \theta(t)-\tau) d\xi_n$.

For the solvability of the loaded integral equation (2.4), we use the theory of Volterra integral equations.

Lemma 2.4. *Let $\varphi(x) \in H^{l+2}(\mathbb{R}^n)$, $\varphi(x) \le \varphi_0$ and $k(x', t) \in H^{l,l/2}(\overline{R}_T^{n-1})$. Then there exists a unique solution $u(x,t)$ to the integral equation* (2.4).

Proof. Using the method of successive approximations for (2.4) we define the sequence $\{u_j(x,t)\}_{j=0}^{\infty}$ as follows:

$$u_0(x,t) = \int_{\mathbb{R}^n} \varphi(\xi) G(x-\xi, \theta(t)) d\xi,$$

$$\begin{aligned} u_1(x,t) = &\frac{\lambda}{\Gamma(\alpha)} \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \int_0^{\theta^{-1}(\tau)} (\theta^{-1}(\tau) - \beta)^{\alpha-1} u_0(\xi', \beta) \\ &\times G(x-\xi, \theta(t)-\tau) d\beta d\xi \\ &+ \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(t))} \int_{\mathbb{R}^n} \int_0^{\theta^{-1}(\tau)} k(\xi', \alpha) u_0(\xi, \theta^{-1}(\tau) - \alpha) \\ &\times G(x-\xi, \theta(t)-\tau) d\xi d\alpha, \end{aligned}$$

$$\begin{aligned}
u_2(x,t) = &\frac{\lambda}{\Gamma(\alpha)}\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\int_{\mathbb{R}^n}\int_0^{\theta^{-1}(\tau)}(\theta^{-1}(\tau)-\beta)^{\alpha-1}u_1(\xi',\beta)\\
&\times G(x-\xi,\theta(t)-\tau)d\beta d\xi\\
&+\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(t))}\int_{\mathbb{R}^n}\int_0^{\theta^{-1}(\tau)}k(\xi',\alpha)u_1(\xi,\theta^{-1}(\tau)-\alpha)\\
&\times G(x-\xi,\theta(t)-\tau)d\xi d\alpha\\
&\dots \\
u_j(x,t) = &\frac{\lambda}{\Gamma(\alpha)}\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\int_{\mathbb{R}^n}\int_0^{\theta^{-1}(\tau)}(\theta^{-1}(\tau)-\beta)^{\alpha-1}u_{j-1}(\xi',\beta)\\
&\times G(x-\xi,\theta(t)-\tau)d\beta d\xi\\
&+\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(t))}\int_{\mathbb{R}^n}\int_0^{\theta^{-1}(\tau)}k(\xi',\alpha)u_{j-1}(\xi,\theta^{-1}(\tau)-\alpha)\\
&G(x-\xi,\theta(t)-\tau)d\xi d\alpha,
\end{aligned} \tag{2.5}$$

for $(x,t)\in\mathbb{R}^n_T$, $j=1,2,\dots$.

Using $\varphi_0=|\varphi(x)|^l$ in $\mathbb{R}^n_T$, and

$$\int_{\mathbb{R}^n} G(x-\xi,\theta(t)-\tau)d\xi = 1, \tag{2.6}$$

we estimate the modulus of functions $u_j(x,t)$ defined above,

$$|u_0(x,t)|_T^{l+2,(l+2)/2} \le \int_{\mathbb{R}^n} |\varphi(\xi)|_T^{l+2,(l+2)/2} G(x-\xi,\theta(t))d\xi \le \varphi_0,$$

and

$$\begin{aligned}
&|u_1(x,t)|_T^{l+2,(l+2)/2}\\
&\le \frac{|\lambda|}{\Gamma(\alpha)}\int_0^{\theta(t)}\frac{d\tau}{|a(\theta^{-1}(t))|}\int_{\mathbb{R}^n}\int_0^{\theta^{-1}(\tau)}(\theta^{-1}(\tau)-\beta)^{\alpha-1}|u_0(\xi',\beta)|_T^{l+2,(l+2)/2}\\
&\quad \times G(x-\xi,\theta(t)-\tau)d\beta d\xi + \int_0^{\theta(t)}\frac{d\tau}{|a(\theta^{-1}(t))|}\int_{\mathbb{R}^n}\int_0^{\theta^{-1}(\tau)}|k(\xi',\alpha)|_T^{l+2,(l+2)/2}\\
&\quad \times |u_0(\xi,\theta^{-1}(\tau)-\alpha)|_T^{l+2,(l+2)/2} G d\xi d\alpha\\
&\le \varphi_0\frac{|\lambda|}{\Gamma(\alpha)}\frac{T^\alpha}{\alpha}\frac{a_1}{a_0}\frac{t}{1!} + \varphi_0\frac{a_1k_0T}{a_0}\frac{t}{1!}\\
&= \varphi_0(\frac{|\lambda|T^\alpha}{\Gamma(\alpha+1)} + k_0T)\frac{a_1}{a_0}\frac{t^1}{1!},
\end{aligned}$$

where $k_0 := |k(x',t)|_T^{l,l/2}$.

We also estimate the modulus for $u_2(x,t), \dots$:

$$
\begin{aligned}
&|u_2(x,t)|_T^{l+2,(l+2)/2} \\
&\le \frac{|\lambda|}{\Gamma(\alpha)}\int_0^{\theta(t)}\frac{d\tau}{|a(\theta^{-1}(t))|}\int_{\mathbb{R}^n}\int_0^{\theta^{-1}(\tau)}(\theta^{-1}(\tau)-\beta)^{\alpha-1}|u_1(\xi',\beta)|_T^{l+2,(l+2)/2} \\
&\quad\times G(x-\xi,\theta(t)-\tau)d\beta d\xi \\
&\quad+\int_0^{\theta(t)}\frac{d\tau}{|a(\theta^{-1}(t))|}\int_{\mathbb{R}^n}\int_0^{\theta^{-1}(\tau)}|k(\xi',\alpha)|_T^{l+2,(l+2)/2}| \\
&\quad\times u_1(\xi,\theta^{-1}(\tau)-\alpha)|_T^{l+2,(l+2)/2}Gd\xi d\alpha \\
&\le \varphi_0\Big(\frac{|\lambda|}{\Gamma(\alpha)}\frac{T^\alpha}{\alpha}\frac{a_1}{a_0}\Big)^2\frac{t^2}{2!}+\varphi_0(\frac{a_1k_0T}{a_0})^2\frac{t^2}{2!} \\
&=\varphi_0\Big((\frac{|\lambda|T^\alpha}{\Gamma(\alpha+1)})^2+(k_0T)^2\Big)(\frac{a_1}{a_0})^2\frac{t^2}{2!},
\end{aligned}
\tag{2.7}
$$

$$
|u_j(x,t)|_T^{l+2,(l+2)/2}\le\varphi_0\Big((\frac{|\lambda|}{\Gamma(\alpha)}\frac{T^\alpha}{\alpha})^j+(k_0T)^j\Big)(\frac{a_1}{a_0})^j\frac{t^j}{j!},
$$

$$\dots$$

As a result, we have the functional series

$$
\sum_{j=0}^{\infty}u_j(x,t). \tag{2.8}
$$

Using the above estimates, according to the Weierstrass theorem on the smooth approximation of functional series [4], we can easily see that the obtained functional series converges. Using the definete integral for $u_j(x,t)$, the sequence of functions (2.8) converges uniformly to a function $u(x,t)$ defined in $H^{l+2,(l+2)/2}(\mathbb{R}^n_T)$. Thus, we have shown that there exists a solution of the integral equation (2.4), i.e., there is a solution to the Cauchy problem (2.1)-(2.2) as a mapping $H^{l+2,(l+2)/2}\to(\mathbb{R}^n_T)$.

Next we prove the uniqueness of this solution. Suppose on the contrary that the integral equation (2.4) has two different solutions $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$:

$$
\begin{aligned}
&u^{(1)}(x,t)\\
&=\int_{\mathbb{R}^n}\varphi(\xi)G(x-\xi,\theta(t))d\xi+\frac{\lambda}{\Gamma(\alpha)}\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\\
&\quad\times\int_{\mathbb{R}^n}\int_0^{\theta^{-1}(\tau)}(\theta^{-1}(\tau)-\beta)^{\alpha-1}u^{(1)}(\xi',\beta)G(x-\xi,\theta(t)-\tau)d\beta d\xi\\
&\quad+\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\int_{\mathbb{R}^n}\int_0^{\theta^{-1}(\tau)}k(\xi',\alpha)u^{(1)}(\xi,\theta^{-1}(\tau)-\alpha)\\
&\quad\times G(x-\xi,\theta(t)-\tau)d\alpha d\xi,
\end{aligned}
$$

and

$$
\begin{aligned}
u^{(2)}(x,t)&=\int_{\mathbb{R}^n}\varphi(\xi)G(x-\xi,\theta(t))d\xi+\frac{\lambda}{\Gamma(\alpha)}\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\\
&\quad\times\int_{\mathbb{R}^n}\int_0^{\theta^{-1}(\tau)}(\theta^{-1}(\tau)-\beta)^{\alpha-1}u^{(2)}(\xi',\beta)G(x-\xi,\theta(t)-\tau)d\beta d\xi
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \int_0^{\theta^{-1}(\tau)} k(\xi',\alpha) u^{(2)}(\xi,\theta^{-1}(\tau)-\alpha) \\
&\times G(x-\xi,\theta(t)-\tau) d\alpha d\xi.
\end{aligned}
$$

Let the difference between these two functions be

$$
Z(x,t) = u^{(1)}(x,t) - u^{(2)}(x,t).
$$

Thus $Z(x,t)$ satisfies a homogeneous integral equation, and

$$
\begin{aligned}
Z(x,t) = &\frac{\lambda}{\Gamma(\alpha)} \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \int_0^{\theta^{-1}(\tau)} (\theta^{-1}(\tau)-\beta)^{\alpha-1} Z(\xi',\beta) \\
&\times G(x-\xi,\theta(t)-\tau) d\beta d\xi \\
&+ \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \int_0^{\theta^{-1}(\tau)} k(\xi',\alpha) Z(\xi,\theta^{-1}(\tau)-\alpha) \\
&\times G(x-\xi,\theta(t)-\tau) d\xi d\alpha.
\end{aligned} \tag{2.9}
$$

For $t \in [0,T]$ and $x \in \mathbb{R}^n$ the modular supremum of $Z(x,t)$ is

$$
\overline{Z} = \sup |Z(x,t)|, t \in [0,T].
$$

In this case, it is easy to see that the following integral inequality holds

$$
\overline{Z}(t) \le \frac{a_1}{a_0} (\frac{|\lambda|}{\Gamma(\alpha)} \frac{T^\alpha}{\alpha} + k_0 T) \int_0^{a_1 t} \overline{Z}(t) d\tau.
$$

Therefore, by the Gronwall-Bellman inequality [13], the last integral inequality has a unique solution, i.e. $\bar{Z}(t) \equiv 0$, for all $t \in [0,T]$. From this, we obtain that $Z(x,t) \equiv 0$, i.e. $u^{(1)}(x,t) = u^{(2)}(x,t)$ in $\overline{R}^n_T$. Thus, the integral equation (2.4) has a unique solution, and thus we can conclude that the equivalent problem (2.1) and (2.2) also has a unique solution. Lemma 2.4 is proved. □

3. Inverse problem for the heat equation with fractional load

Problem 3.1. Find functions $u(x,t)$ and $k(x',t)$ in the domains $(x,t) \in \mathbb{R}^n_T$ and $(x',t) \in R^{n-1}_T$ respectively, which satisfy the equation

$$
u_t - a(t)\Delta u = \lambda D^{-\alpha}_{0,t} u(\tilde{x},t) + \int_0^t k(x',\tau) u(x,t-\tau) d\tau, \quad (x,t) \in \mathbb{R}^n_T, \tag{3.1}
$$

and the initial and boundary value conditions

$$
u(x,t)\big|_{t=0} = \varphi(x), \quad x \in \mathbb{R}^n, \tag{3.2}
$$

$$
u\big|_{x_n=0} = f(x',t), \quad (x',t) \in \bar{R}^{n-1}_T, \tag{3.3}
$$

where

$$
a(t) \in E := \{a(t) \in C^1[0,T] : 0 < a_0 < a(t) \le a_1 < \infty\},
$$

Δ is the Laplace operator acting on variables $x(x_1,x_2,\dots,x_n) \in \mathbb{R}^n$, $\widetilde{x} \in \mathbb{R}^{n-2}$, where $x_i = x_j = 0$, at $i \ne j$ $(1 \le i,j \le n)$, $\lambda \in \mathbb{R}$, $D^{-\alpha}_{0t}$ is the Riemann-Liouville fractional integral operator of order α $(\alpha > 0)$, $\varphi(x)$ and $f(x',t)$ are given real-valued functions with $f(x',0) = \varphi(x',0)$.

We use the following notation

$$(\overline{R}_T^{n-2}) = \{(\tilde{x}, t) : \tilde{x} \in \mathbb{R}^{n-2}, 0 \le t \le T\},$$
$$(\overline{R}_T^{n-1}) = \{(x', t) : x' \in \mathbb{R}^{n-1}, 0 \le t \le T\}, \quad l \in (0, 1).$$

Remark 3.2. In loaded equations [28], a loaded operator or loaded term must include the trace of the desired function in the manifolds of dimension less than one from the sought-for solution in $\mathbb{R}^{n+1}$ [3]. Based on this, we assume let $\widetilde{x} \in \mathbb{R}^{n-2}$. (We choose so that there is a difference between the x' functions).

Thus, the problem can be continued in the same way in the cases where the loaded operator $Mu \equiv \lambda D_{0,t}^{-\alpha} u(\tilde{x}, t)$ contains the trace of the desired function, from the domains $\mathbb{R}^{n-3}$, ..., $\mathbb{R}^1$, i.e. respectively when $x_1 = 0$, $x_2 = 0$, ..., $x_n = 0$.

Theorem 3.3. *If* $f(x', t) \in H^{l+4,(l+4)/2}(\bar{R}_T^{n-1})$, $f(x', 0) = \varphi(x', 0)$ *and*

$$\varphi(x) \in H^{l+2}(\mathbb{R}^n), \quad \varphi(x) \le \varphi_0 = \text{const} > 0,$$

then there exists a unique solution to the inverse problem in the domains $\mathbb{R}_T^n$ *and* R_T^{n-1} *respectively.*

This theorem will be proven using the theory of integral equations. First, we show that the inverse problem is equivalent to a system integral equations of Volterra type. We will introduce a new function as $\vartheta(x, t)$. Thus, by replacing $\vartheta(x, t) = u_{x_n x_n}(x, t)$, then the problem (3.1)-(3.2) takes and follows (is taken and followed) the form:

$$\vartheta_t - a(t)\Delta\vartheta = \lambda D_{0,t}^{-\alpha} \vartheta(\tilde{x}, t) + \int_0^t k(x', \tau)\vartheta(x, t - \tau)d\tau, \tag{3.4}$$

$$\vartheta(x, t)\big|_{t=0} = \varphi_{x_n x_n}(x), \tag{3.5}$$

when $x_n = 0$. Taking into account (3.1) and (3.2), we have an additional boundary condition (3.3) for the function $\vartheta(x, t)$ in the form

$$\begin{aligned} &\vartheta(x, t)\big|_{x_n=0} \\ &= \frac{1}{a(t)} f_t(x', t) - \sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2} f(x', t) - \frac{1}{a(t)} \int_0^t k(x', \tau) f(x', t - \tau)d\tau \\ &\quad - \frac{\lambda}{a(t)} D_{0,t}^{-\alpha} f(0, 0, x_3 \dots, x_{n-1}, t). \end{aligned} \tag{3.6}$$

As a result, we obtain the following condition, in agreement with initial and boundary conditions (3.5) and (3.6):

$$\varphi_{x_n x_n}(x', 0) = \frac{1}{a(0)} f_t(x', 0) - \sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2} f(x', 0) \tag{3.7}$$

If matching conditions (3.6) and (3.7) are satisfied, $f(x', t) \in H^{l+4,(l+4)/2}(\bar{R}_T^{n-1})$, $f(x', 0) = \varphi(x', 0)$, and $\varphi(x) \in H^{l+2}(\mathbb{R}^n)$, then it is equivalent to the inverse problem with respect to the function $\vartheta(x, t) = u_{x_n x_n}(x, t)$, where

$$u(x, t) = f(x', t) + x_n \varphi_n(x', 0) + \int_0^{x_n} (x_n - \xi)\vartheta(x', \xi, t)d\xi. \tag{3.8}$$

Taking into account the matching conditions $f(x',0)=\varphi(x',0)$ and (3.7), for $t=0$, we obtain the following conditions

$$\begin{aligned}
u(x,t)|_{t=0} &= f(x',0)+x_n u_{x_n}(x',0,0)+\int_0^{x_n}(x_n-\xi)\varphi_{\xi\xi}(x',\xi)d\xi \\
&= f(x',0)+x_n u_{x_n}(x',0,0)+\int_0^{x_n}(x_n-\xi)d\varphi_\xi \\
&= f(x',0)+x_n u_{x_n}(x',0,0)+(x_n-\xi)\varphi_\xi(x',\xi)|_0^{x_n}-\int_0^{x_n}\varphi_\xi(x',\xi)d\xi \\
&= f(x',0)+x_n u_{x_n}(x',0,0)-x_n\varphi_{x_n}(x',0)+\varphi(x)-\varphi(x',0) \\
&= x_n(u_{x_n}(x',0,0)-\varphi_{x_n}(x',0))+\varphi(x)=\varphi(x).
\end{aligned}$$

It is easy to see from (3.8) that at $x_n=0$, our additional condition arises. Similarly, following the derivation of equation (3.1) from equation (3.4) as follows. Integrating twice from both parts of equation (3.4) from 0 to x_n, we obtain

$$\begin{aligned}
&\int_0^{x_n}(x_n-\xi)\vartheta_t(x',\xi,t)d\xi-a(t)\int_0^{x_n}(x_n-\xi)\Delta\vartheta(x',\xi,t)d\xi \\
&=\int_0^{x_n}(x_n-\xi)\int_0^t k(x',\tau)\vartheta(x',\xi,t-\tau)d\tau d\xi \\
&\quad+\frac{\lambda}{\Gamma(\alpha)}\int_0^{x_n}(x_n-\xi)\int_0^t(t-\tau)^{\alpha-1}u(\tilde{x},\tau)d\tau d\xi.
\end{aligned}$$

Therefore, taking into account the equality

$$\int_0^{x_n}(x_n-\xi)\vartheta(x',\xi,t)d\xi=u(x,t)-f(x',t)-x_n\varphi_n(x',0),$$

we have

$$\begin{aligned}
&\frac{\partial}{\partial t}(u(x,t)-f(x',t)-x_n\varphi_n(x',0))-a(t)\Delta_{x'}(u(x,t) \\
&-f(x',t)-x_n\varphi_n(x',0))-a(t)\int_0^{x_n}(x_n-\xi)\vartheta_{x_nx_n}(x',\xi,t)d\xi \\
&=\int_0^t k(x',\tau)(u(x,t-\tau)-f(x',t-\tau)-x_n\varphi_n(x',0))d\tau \\
&\quad+\frac{\lambda}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}(u(\tilde{x},\tau)-f(\tilde{x},t-\tau)-x_n\varphi_n(x',0))d\tau,
\end{aligned}$$

i.e.

$$\begin{aligned}
&u_t(x,t)-f_t(x',t)-a(t)\Delta_{x'}u(x,t)+a(t)f(x',t)+a(t)x_n\vartheta_{x_n}(x',0,t) \\
&-a(t)\vartheta(x,t)+a(t)\vartheta(x',0,t) \\
&=\int_0^t k(x',\tau)u(x,t-\tau)d\tau-\int_0^t k(x',\tau)f(x',t-\tau)d\tau \\
&\quad-x_n\varphi_n(x',0)\int_0^t k(x',\tau)d\tau+\frac{\lambda}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}u(\tilde{x},\tau)d\tau \\
&\quad-\frac{\lambda}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}(f(\tilde{x},t-\tau)+x_n\varphi_n(x',0))d\tau.
\end{aligned}$$

Thus, the inverse problem of (3.1)-(3.3) of finding the functions $u(x,t)$ and $k(x',t)$ is equivalent to the problem of determining the functions $\vartheta(x,t)$ and $k(x',t)$ from problem (3.4)-(3.6).

In the next step, equations (3.4), (3.6) with differentiated once by the variable t, and $\vartheta_t(x,t)=\omega(x,t)$, as a result, we have (3.4), (3.5) and the auxiliary problems

$$\begin{aligned}\omega_t - a(t)\Delta\omega &= (\ln a(t))'\omega - (\ln a(t))'\int_0^t k(x',\tau)\vartheta(x,t-\tau)d\tau \\ &\quad + \int_0^t k(x',\tau)\omega(x,t-\tau)d\tau - \lambda(\ln a(t))' D_{0,t}^{-\alpha}\vartheta(\tilde{x},t) + \lambda D_{0,t}^{-\alpha}\omega(\tilde{x},t) \\ &\quad + k(x',t)\varphi_{x_nx_n}(x) + \frac{\lambda}{\Gamma(\alpha)} t^{\alpha-1}\varphi_{x_nx_n}(\tilde{x}),\end{aligned} \tag{3.9}$$

and

$$\omega|_{t=0} = a(0)\Delta\varphi_{x_nx_n}(x), \tag{3.10}$$

$$\begin{aligned}\omega|_{x_n=0} &= F_t(x',t) + \frac{a'(t)}{a^2(t)}\int_0^t k(x',\tau)f(x',t-\tau)d\tau \\ &\quad - \frac{1}{a(t)}\int_0^t k(x',\tau)f_t(x',t-\tau)d\tau - \frac{1}{a(t)}k(x',t)\varphi(x',0),\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}F(x',t) &= \frac{1}{a(t)}f_t(x',t) - \sum_{k=1}^{n-1}\frac{\partial^2}{\partial x_k^2}f(x',t) \\ &\quad - \frac{\lambda}{a(t)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(0,0,x_3,\dots,x_{n-1},\tau)d\tau.\end{aligned}$$

From relations (3.9), (3.10), and (3.11) we can find the functions $\vartheta(x,t)$, $k(x',t)$, and $\omega(x,t)$.

When we integrate $\vartheta_t(x,t)=\omega(x,t)$ from 0 to t, we obtain

$$\vartheta(x,t) = \varphi_{x_nx_n}(x) + \int_0^t \omega(x,\tau)d\tau. \tag{3.12}$$

From this equality, if the function $\omega(x,t)$ is known, we can easily determine the function $\vartheta(x,t)$. Thus, from the problem (3.9)-(3.11) arises the problem (3.4)-(3.6), as a result of which (3.1)-(3.3) is passed to the inverse problem. Thus, the inverse problem of (3.1)-(3.3) of finding the functions $u(x,t)$ and $k(x',t)$ is equivalent to the problem of determining the functions $\vartheta(x,t)$, $\omega(x,t)$ and $k(x',t)$ from problem (3.4)-(3.6) and (3.9) and (3.11). Thus, we proved the following lemma.

Lemma 3.4. *Let* $\varphi(x)\in H^{l+6}(\mathbb{R}^n)$, $f(x',t)\in H^{l+4,(l+4)/2}(\bar{R}_T^{n-1})$, $a(t)\in E$ *and satisfies the agreement conditions*

$$f(x',0)=\varphi(x',0), \quad \varphi_{x_nx_n}(x',0) = \frac{1}{a(0)}f_t(x',0) - \sum_{k=1}^{n-1}\frac{\partial^2}{\partial x_k^2}f(x',0),$$

then (3.1), (3.2), (3.3) *inverse problem is equivalent to the problem* (3.4)-(3.6) *and* (3.9)-(3.11), *finding functions* $\vartheta(x,t)$, $k(x',t)$, *and* $\omega(x,t)$

In the next stage, we will transform the lemma on the equivalence of the obtained problem (3.4)-(3.6) and (3.9)-(3.11) into a system of loaded integral equations.

Lemma 3.5. *Equations* (3.4)-(3.6) *and* (3.9)-(3.11) *auxiliary problems, equivalent to finding functions* $\vartheta(x,t)$, $k(x',t)$, $\omega(x,t)$, *from the following system of integral equations:*

$$\begin{aligned}\vartheta(x,t) = \psi_{01}(x,t) &+ \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} d\xi \\ &\times \int_0^{\theta^{-1}(\tau)} k(\xi',\alpha)\vartheta(\xi,\theta^{-1}(\tau)-\alpha)G(x-\xi,\theta(t)-\tau)d\alpha \\ &+ \frac{\lambda}{\Gamma(\alpha)} \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \int_0^{\theta^{-1}(\tau)} (\theta^{-1}(\tau)-\beta)^{\alpha-1}\vartheta(\tilde{\xi},\beta) \\ &\times G(x-\xi,\theta(t)-\tau)d\beta d\xi,\end{aligned} \tag{3.13}$$

$$\begin{aligned}\omega(x,t) = \psi_{02}(x,t) &+ \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \Big[(\ln a(\theta^{-1}(\tau)))'\omega(\xi,\theta^{-1}(\tau)) \\ &- (\ln a(\theta^{-1}(\tau)))' \int_0^{\theta^{-1}(\tau)} k(\xi',\alpha)\vartheta(\xi,\theta^{-1}(\tau)-\alpha)d\alpha\Big] \\ &\times G(x-\xi,\theta(t)-\tau)d\xi \\ &+ \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \int_0^{\theta^{-1}(\tau)} k(\xi',\alpha)\omega(\xi,\theta^{-1}(\tau)-\alpha)G(x-\xi,\theta(t)-\tau)d\alpha d\xi \\ &+ \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} k(\xi',\theta^{-1}(\tau))\varphi_{\xi_n\xi_n}(\xi)G(x-\xi,\theta(t)-\tau)d\xi \\ &+ \frac{\lambda}{\Gamma(\alpha)} \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \int_0^{\theta^{-1}(\tau)} (\theta^{-1}(\tau)-\beta)^{\alpha-1}\omega(\tilde{\xi},\beta)G(x-\xi,\theta(t)-\tau)d\beta d\xi \\ &- \frac{\lambda}{\Gamma(\alpha)} \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} (\ln a(\theta^{-1}(\tau)))' \int_0^{\theta^{-1}(\tau)} (\theta^{-1}(\tau)-\beta)^{\alpha-1}\vartheta(\tilde{\xi},\beta)Gd\beta d\xi,\end{aligned} \tag{3.14}$$

$$\begin{aligned}&k(x',t) \\ &= \psi_{03}(x,t) + \frac{a(t)}{\varphi(x',0)}\Big\{ \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \\ &\times \int_{\mathbb{R}^n} ((\ln a(\theta^{-1}(\tau)))' \int_0^{\theta^{-1}(\tau)} k(\xi',\alpha)\vartheta(\xi,\theta^{-1}(\tau)-\alpha)d\alpha)G(x'-\xi',\xi_n,\theta(t)-\tau)d\xi \\ &- \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} (\ln a(\theta^{-1}(\tau)))'\omega(\xi,\theta^{-1}(\tau))G(x'-\xi',\xi_n,\theta(t)-\tau)d\xi \\ &- \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \int_0^{\theta^{-1}(\tau)} k(\xi',\alpha)\omega(\xi,\theta^{-1}(\tau)-\alpha)G(x'-\xi',\xi_n,\theta(t)-\tau)d\alpha d\xi \\ &- \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} k(\xi',\theta^{-1}(\tau))\varphi_{\xi_n\xi_n}(\xi)G(x'-\xi',\xi_n,\theta(t)-\tau)d\xi \\ &+ \frac{\lambda}{\Gamma(\alpha)} \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \\ &\times \int_{\mathbb{R}^n} ((\ln a(\theta^{-1}(\tau)))' \int_0^{\theta^{-1}(\tau)} (\theta^{-1}(\tau)-\beta)^{\alpha-1}\vartheta(\tilde{\xi},\beta)d\beta G(x'-\xi',\xi_n,\theta(t)-\tau)d\xi\end{aligned}$$

$$
\begin{aligned}
&- \int_0^{\theta^{-1}(\tau)} (\theta^{-1}(\tau)-\beta)^{\alpha-1}\omega(\tilde{\xi},\beta)G(x'-\xi',\xi_n,\theta(t)-\tau)d\beta d\xi)\Big\} \\
&+ \frac{(\ln a(t))'}{\varphi(x',0)}\int_0^t k(x',\tau)f(x',t-\tau)d\tau - \frac{1}{\varphi(x',0)}\int_0^t k(x',\tau)f_t(x',t-\tau)d\tau,
\end{aligned} \tag{3.15}
$$

where

$$
\psi_{01}(x,t) = \int_{\mathbb{R}^n} \varphi_{\xi_n\xi_n}(\xi)G(x-\xi,\theta(t))d\xi,
$$

$$
\begin{aligned}
\psi_{02}(x,t) &= \int_{\mathbb{R}^n} a(0)\Delta\varphi_{\xi_n\xi_n}(\xi)G(x-\xi,\theta(t))d\xi \\
&\quad + \frac{\lambda}{\Gamma(\alpha)}\int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))}\int_{\mathbb{R}^n}(\theta^{-1}(\tau))^{\alpha-1}\varphi_{\xi_n\xi_n}(\tilde{\xi})G(x-\xi,\theta(t)-\tau)d\xi, \\
\psi_{03}(x,t) &= \frac{a(t)}{\varphi(x',0)}(F_t(x',t) - \int_{\mathbb{R}^n} a(0)\Delta\varphi_{\xi_n\xi_n}(\xi)G(x'-\xi',\xi_n,\theta(t))d\xi) \\
&\quad + \frac{a(t)}{\varphi(x',0)}(-\frac{\lambda}{\Gamma(\alpha)}\int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))}\int_{\mathbb{R}^n}(\theta^{-1}(\tau))^{\alpha-1}\varphi_{\xi_n\xi_n}(\tilde{\xi}) \\
&\quad \times G(x'-\xi',\xi_n,\theta(t)-\tau)d\xi).
\end{aligned}
$$

The proof of the lemma 3.5 can obviously be obtained, from (3.4), (3.5), (3.9), and (3.10), we did the integral relations (3.13) and (3.14), as the integral equation (2.4). And also, from equations (3.11) and (3.14), we easily obtain relation (3.15).

4. Existence and uniqueness for the inverse problem

In this section, we demonstrate one of the main results of the inverse problem, the existence and uniqueness theorem for solutions to the system of integral equations (3.13)-(3.15), from which the unique solvability of the problem (3.1)-(3.3) follows.

Theorem 4.1. *Let $a(t) \in E$, $\varphi(x) \in H^{l+6}(\mathbb{R}^n)$, and $f(x',t) \in H^{l+4,(l+4)/2}(\bar{R}_T^{n-1})$, moreover*

$$
f(x',0) = \varphi(x',0)\varphi_{x_nx_n}(x',0) = \frac{1}{a(0)}f_t(x',0) - \sum_{k=1}^{n-1}\frac{\partial^2}{\partial x_k^2}f(x',0),
$$

and the terms of the agreement be reasonable. Then there exists a sufficiently small number $T_0 > 0$ such that for any $T \in (0,T_0]$, there is a unique solution to the system of integral equations (3.13)–(3.15)*, in the domains*

$$
\{\vartheta(x,t),\omega(x,t)\} \in H^{l+2,(l+2)/2}(\bar{R}_T^n),\ k(x',t) \in H^{l,l/2}(\bar{R}_T^{n-1}).
$$

Proof. Without loss of generality, we can prove this theorem by the classical integral method. First, we write the system of integral equations (3.13)-(3.15) in the form of an operator equation

$$
\psi = L\psi, \tag{4.1}
$$

where $\psi = (\psi_1,\psi_2,\psi_3)^* = (\vartheta(x,t),\omega(x,t),k(x',t))^*$, with $*$ denote the transposition. Equations (3.13)-(3.15) as operator in equations become

$$
L\psi = [(L\psi)_1,(L\psi)_2,(L\psi)_3]^*.
$$

Operators $(L\psi)_i$, $i=1,2,3$ can be written in the form

$$\begin{aligned}(L\psi)_1 &= \psi_{01}(x,t)+\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\int_{\mathbb{R}^n}\int_0^{\theta^{-1}(\tau)}\psi_3(\xi',\alpha)\psi_1(\xi,\theta^{-1}(\tau)-\alpha)\\ &\quad\times G(x-\xi,\theta(t)-\tau)d\alpha d\xi+\frac{\lambda}{\Gamma(\alpha)}\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\\ &\quad\times\int_{\mathbb{R}^n}\int_0^{\theta^{-1}(\tau)}(\theta^{-1}(\tau)-\beta)^{\alpha-1}\psi_1(\tilde{\xi},\beta)Gd\beta d\xi,\end{aligned}\tag{4.2}$$

$$\begin{aligned}(L\psi)_2 &= \psi_{02}(x,t)+\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\int_{\mathbb{R}^n}\Big[(\ln a(\theta^{-1}(\tau)))'\psi_2(\xi,\theta^{-1}(\tau))\\ &\quad-(\ln a(\theta^{-1}(\tau)))'\int_0^{\theta^{-1}(\tau)}\psi_3(\xi',\alpha)\psi_1(\xi,\theta^{-1}(\tau)-\alpha)d\alpha\Big]\\ &\quad\times G(x-\xi,\theta(t)-\tau)d\xi+\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\\ &\quad\times\int_{\mathbb{R}^n}\int_0^{\theta^{-1}(\tau)}\psi_3(\xi',\alpha)\psi_2(\xi,\theta^{-1}(\tau)-\alpha)G(x-\xi,\theta(t)-\tau)d\alpha d\xi\\ &\quad+\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\int_{\mathbb{R}^n}\psi_3(\xi',\theta^{-1}(\tau))\varphi_{\xi_n\xi_n}(\xi)G(x-\xi,\theta(t)-\tau)d\xi\\ &\quad+\frac{\lambda}{\Gamma(\alpha)}\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\int_{\mathbb{R}^n}\int_0^{\theta^{-1}(\tau)}(\theta^{-1}(\tau)-\beta)^{\alpha-1}\psi_2(\tilde{\xi},\beta)Gd\beta d\xi\\ &\quad-\frac{\lambda}{\Gamma(\alpha)}\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\int_{\mathbb{R}^n}(\ln a(\theta^{-1}(\tau)))'\times\\ &\quad\times\int_0^{\theta^{-1}(\tau)}(\theta^{-1}(\tau)-\beta)^{\alpha-1}\psi_1(\tilde{\xi},\beta)G(x-\xi,\theta(t)-\tau)d\beta d\xi,\end{aligned}\tag{4.3}$$

$$\begin{aligned}(L\psi)_3 &= \psi_{03}(x,t)+\frac{a(t)}{\varphi(x',0)}\Big(\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\int_{\mathbb{R}^n}\ln a(\theta^{-1}(\tau))\Big)'\\ &\quad\times\int_0^{\theta^{-1}(\tau)}\psi_3(\xi',\alpha)\psi_1(\xi,\theta^{-1}(\tau)-\alpha)Gd\alpha d\xi\\ &\quad-\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\int_{\mathbb{R}^n}\Big(\ln a(\theta^{-1}(\tau))\Big)'\psi_2(\xi,\theta^{-1}(\tau))Gd\xi\\ &\quad-\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\int_{\mathbb{R}^n}\int_0^{\theta^{-1}(\tau)}\psi_3(\xi',\alpha)\psi_2(\xi,\theta^{-1}(\tau)-\alpha)Gd\alpha d\xi\\ &\quad-\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\int_{\mathbb{R}^n}\psi_3(\xi',\theta^{-1}(\tau))\varphi_{\xi_n\xi_n}(\xi)G(x'-\xi',\xi_n,\theta(t)-\tau)d\xi\\ &\quad-\frac{\lambda}{\Gamma(\alpha)}\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\int_{\mathbb{R}^n}\int_0^{\theta^{-1}(\tau)}(\theta^{-1}(\tau)-\beta)^{\alpha-1}\psi_2(\tilde{\xi},\beta)Gd\beta d\xi\\ &\quad+\frac{\lambda}{\Gamma(\alpha)}\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\int_{\mathbb{R}^n}\Big(\ln a(\theta^{-1}(\tau))\Big)'\end{aligned}$$

$$
\begin{aligned}
&\times \int_0^{\theta^{-1}(\tau)} (\theta^{-1}(\tau)-\beta)^{\alpha-1}\psi_1(\tilde{\xi},\beta)G d\beta d\xi) \\
&+ \frac{\big(\ln a(t)\big)'}{\varphi(x',0)} \int_0^t \psi_3(x',\tau) f(x',t-\tau)d\tau \\
&- \frac{1}{\varphi(x',0)} \int_0^t \psi_3(x',\tau) f_t(x',t-\tau)d\tau. \qquad (4.4)
\end{aligned}
$$

Therefore, in relations (4.2)–(4.4), taking into account the notation of Lemma 3.5, we denote

$$
|\psi|_T^{l,l/2} = \max(|\psi_1|_{T_0}^{l.l/2}, |\psi_2|_{T_0}^{l.l/2}, |\psi_3|_{T_0}^{l.l/2}),
$$

and on $H^{l,l/2}(\mathbb{R}^n_T)$, we include the conditions

$$
S(T) = |\psi_1 - \psi_0|_T^{l,l/2} \le |\psi_0|_{T_0}^{l,l/2}, \qquad (4.5)
$$

where $\psi_0 = (\psi_{01}, \psi_{02}, \psi_{03})$ and $|\psi_0|_{T_0}^{l,l/2} = \max(|\psi_{01}|_{T_0}^{l,l/2}, |\psi_{02}|_{T_0}^{l,l/2}, |\psi_{03}|_{T_0}^{l,l/2})$. □

Suppose that ψ is an arbitrary in $S(T)$, here $T < T_0$. In this case, the following inequalities are valid:

$$
|\psi_i|_T^{l,l/2} \le 2|\psi_0|_{T_0}^{l,l/2}, \quad i = 1, 2, 3.
$$

Therefore, it is similar to setting the Cauchy problem for the classical heat equation when we consider a function in the class $\varphi(x) \in H^{l+6}$, and introduce the following notation:

$$
a_2 := \max_{t\in[0,T]} |(\ln a(t))'|, \quad \varphi_1 := |\varphi|^{l+6}, \quad f_0 := |f|_T^{l+4,(l+4)/2}.
$$

Let the operator L be defined in a closed S, which is part of the Banach space.

Definition 4.2. An operator L is called a contraction operator in S, when the following two conditions are satisfied:

(1) if $y \in S$, then $Ly \in S$, i.e. the operator L maps the set S in itself;
(2) if there exists a real number $\rho \in [0, 1)$ such that

$$
\|Ly - Lz\| \le \rho \, \|y - z\| \,, \quad \forall y, z \in S \,.
$$

If for the mapping $L : X \to X$ there exists a point $x \in X$ such that $Lx = x$, then the point x is called the fixed point of L, [4].

In this section, we state and prove the contraction mapping principle [18], which is one of the most useful methods for the construction of the solution of the differential equations [20].

Contraction mapping principle. Every contraction mapping defined in a complete metric space R has one, and only one fixed point, that is, the equation, $Lx = x$ has a unique solution $x_0 \in S$.

First of all, we show that the operator L satisfies the first condition for the definition of compressibility in S, i.e.:

$$
\begin{aligned}
&|(L\psi)_1 - \psi_{01}|_T^{l,l/2} \\
&= | \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \int_0^{\theta^{-1}(\tau)} \psi_3(\xi',\alpha)\psi_1(\xi, \theta^{-1}(\tau)-\alpha)
\end{aligned}
$$

$$
\begin{aligned}
&\times G(x-\xi,\theta(t)-\tau)d\alpha d\xi\big|_T^{l,l/2}+\big|\frac{\lambda}{\Gamma(\alpha)}\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\\
&\times\int_{\mathbb{R}^n}\int_0^{\theta^{-1}(\tau)}(\theta^{-1}(\tau)-\beta)^{\alpha-1}\psi_1(\tilde{\xi},\beta)G(x-\xi,\theta(t)-\tau)d\beta d\xi\big|_T^{l,l/2},
\end{aligned}
$$

here after replacing $|y=\theta^{-1}(\tau)|$ as a result, we obtain

$$
\begin{aligned}
&\big|\int_0^t\frac{d\tau}{a(\theta^{-1}(\tau))}\int_{\mathbb{R}^n}\int_0^y\psi_3(\xi',\alpha)\psi_1(\xi,y-\alpha)G(x-\xi,\theta(t)-\theta(y))d\alpha d\xi\big|_T^{l,l/2}\\
&+\big|\frac{\lambda}{\Gamma(\alpha)}\int_0^t\frac{d\tau}{a(\theta^{-1}(\tau))}\int_{\mathbb{R}^n}\int_0^y(y-\beta)^{\alpha-1}\psi_1(\tilde{\xi},\beta)\\
&\times G(x-\xi,\theta(t)-\theta(y))d\beta d\xi\big|_T^{l,l/2}\\
&\le\bar{\beta}_0(T)\big|(\psi_3(\xi',t_0)\psi_1(\xi,y-t_0))\big|_T^{l,l/2}+\bar{\beta}_1(T)\big|(\psi_1(\tilde{\xi},t_0))\big|_T^{l,l/2}\\
&\le 4\beta_0(T)(|\psi_0|_{T_0}^{l,l/2})^2+2\beta_1(T)(|\psi_0|_{T_0}^{l,l/2}),
\end{aligned}
$$

Similarly, we obtain estimates for other components of the vector L:

$$
\begin{gathered}
\big|(L\psi)_2-\psi_{02}\big|_T^{l,l/2}\le 4\beta_1(T)(a_2+1)(|\psi_0|_{T_0}^{l,l/2})^2+2\beta_2(T)(2a_2+\varphi_1+1)(|\psi_0|_{T_0}^{l,l/2}),\\
\big|(L\psi)_3-\psi_{03}\big|_T^{l,l/2}\le 2(\beta_1(T)a_1\varphi_0^{-1}(2a_2+\varphi_1+1)+f_0\varphi_0^{-1}T_0(a_2+1))|\psi_0|_{T_0}^{l,l/2}\\
+4\beta_2(T)a_1\varphi_0^{-1}(a_2+1)(|\psi_0|_{T_0}^{l,l/2})^2.
\end{gathered}
$$

Here, If $T\to 0$ then, $\beta_i(T)$, $(i=0,1,2)$ tends to zero. If we choose T_0 such that

$$
\begin{gathered}
4\beta_0(T_0)(|\psi_0|_{T_0}^{l,l/2})^2+2\beta_1(T_0)(|\psi_0|_{T_0}^{l,l/2})\le 1,\\
4\beta_1(T_0)(a_2+1)(|\psi_0|_{T_0}^{l,l/2})^2+2\beta_2(T_0)(2a_2+\varphi_1+1)(|\psi_0|_{T_0}^{l,l/2})\le 1,\\
2(\beta_1(T_0)a_1\varphi_0^{-1}(2a_2+\varphi_1+1)+f_0\varphi_0^{-1}T_0(a_2+1))|\psi_0|_{T_0}^{l,l/2}\\
+4\beta_2(T_0)a_1\varphi_0^{-1}(a_2+1)(|\psi_0|_{T_0}^{l,l/2})^2\le 1,
\end{gathered}
\tag{4.6}
$$

then the operator L satisfies the first mapping condition by reducing $T<T_0$, i.e., $LS\subset S$. In the same way, we can pass the second condition to the contraction principle. We suppose that

$$
\psi^{(1)}=(\psi_1^{(1)},\psi_2^{(1)},\psi_3^{(1)})\in S(T),\psi^{(2)}=(\psi_1^{(2)},\psi_2^{(2)},\psi_3^{(2)})\in S(T).
$$

To estimate the distance between the images of the functions $\psi^{(1)}$ and $\psi^{(2)}$ as a result of mapping L, with regards to

$$
\begin{aligned}
\big|\psi_2^{(1)}\psi_1^{(1)}-\psi_2^{(2)}\psi_1^{(2)}\big|_T^{l,l/2}&=|(\psi_2^{(1)}-\psi_2^{(2)})\psi_1^{(1)}+\psi_2^{(2)}(\psi_1^{(1)}-\psi_1^{(2)})|_T^{l,l/2}\\
&\le 2|\psi^{(1)}-\psi^{(2)}|_T^{l,l/2}\max(|\psi_1^{(1)}|_T^{l,l/2},|\psi_2^{(2)}|_T^{l,l/2})\\
&\le 4|\psi_0|_T^{l,l/2}|\psi^{(1)}-\psi^{(2)}|_T^{l,l/2},
\end{aligned}
$$

we have

$$
\begin{aligned}
&|((L\psi)^{(1)}-(L\psi)^{(2)})_1|_T^{l,l/2}\\
&=\big|\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\int_{\mathbb{R}^n}\int_0^{\theta^{-1}(\tau)}\psi_3(\xi',\alpha)\psi_1(\xi,\theta^{-1}(\tau)-\alpha)\\
&\quad\times G(x-\xi,\theta(t)-\tau)d\alpha d\xi\big|_T^{l,l/2}+\big|\frac{\lambda}{\Gamma(\alpha)}\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}
\end{aligned}
$$

$$\times \int_{\mathbb{R}^n}\int_0^{\theta^{-1}(\tau)} (\theta^{-1}(\tau)-\beta)^{\alpha-1}\psi_1(\tilde{\xi},\beta)G(x-\xi,\theta(t)-\tau)d\beta d\xi\big|_T^{l,l/2}$$
$$\le [8\beta_0(T)|\psi_0|_{T_0}^{l,l/2}+4\beta_1(T)]|\psi^{(1)}-\psi^{(2)}|_T^{l,l/2}.$$

In the same way, the second and third components of $L\psi$ satisfy:

$$\begin{aligned}&\big|((L\psi)^{(1)}-(L\psi)^{(2)})_2\big|_T^{l,l/2}\\ &\le [2\beta_1(T)(2a_2+\varphi_1+1)+8\beta_2(T)(a_2+1)|\psi_0|_{T_0}^{l,l/2}]|\psi^{(1)}-\psi^{(2)}|_{T_0}^{l,l/2},\end{aligned}$$

$$\begin{aligned}&|((L\psi)^{(1)}-(L\psi)^{(2)})_3|_T^{l,l/2}\\ &\le [2(\beta_1(T)a_1\varphi_0^{-1}(2a_2+\varphi_1+1)+f_0\varphi_0^{-1}T_0(a_2+1))]|\psi^{(1)}-\psi^{(2)}|_{T_0}^{l,l/2}\\ &\quad+[8\beta_2(T)a_1\varphi_0^{-1}(a_2+1)(|\psi_0|_{T_0}^{l,l/2})]|\psi^{(1)}-\psi^{(2)}|_{T_0}^{l,l/2}.\end{aligned}$$

Therefore,

$$|(L\psi^{(1)}-L\psi^{(2)})|_T^{l,l/2}<\rho|\psi^{(1)}-\psi^{(2)}|_T^{l,l/2}.$$

Thus, if the following conditions are satisfied

$$\begin{gathered}[8\beta_0(T)|\psi_0|_{T_0}^{l,l/2}+2\beta_1(T)]\le\rho<1,\\ [2\beta_1(T)(2a_2+\varphi_1+1)+8\beta_2(T)(a_2+1)|\psi_0|_{T_0}^{l,l/2}]\le\rho<1,\\ [2(\beta_1(T)a_1\varphi_0^{-1}(2a_2+\varphi_1+1)+f_0\varphi_0^{-1}T_0(a_2+1))]\\ +[8\beta_2(T)a_1\varphi_0^{-1}(a_2+1)(|\psi_0|_{T_0}^{l,l/2})]\le\rho<1,\end{gathered}\tag{4.7}$$

then, operator L is compact on $S(T)$, [20].

From the fulfillment of inequality (4.7), it is not difficult to see that condition (4.6) is satisfied for T_0. It follows that the contraction principle holds for in $T<T_0$. In this case, according to the Banach fixed point theorem [4], there exists a unique solution to the equation (4.1). Therefore, from the system of integral equations (3.13)-(3.15), using the method of successive approximations, we determine the unique solution of the system that belongs to the class $H^{l+2,(l+2)/2}(\bar{R}_T^n)$.

5. Conclusion

Parabolic equations, especially diffusion and reaction-diffusion equations, are one of the most used equations of the modern theory of partial differential equations, which have been rapidly developing in recent years. This happens because the mathematical models for the dynamics of infectious diseases can show optimal control of the spread, and controllability and stabilization of the dynamics of infection. Based on this, in this paper we study the solvability of direct and inverse problems for a multidimensional fractionally loaded heat equation in Holder spaces.

Introductory section of this work is dedicated to the history and literature on problems for the fractional differential equations, time-fractional diffusion equations, integro-differential equations with fractional load, and the novelty of the work.

In the second section, we investigated the Cauchy problem for the integro-differential equation with a fractional load. We proved the unique solvability of the initial-value problem using the theory of the integral equations and the methods of successive approximations.

In the third section, we formulate inverse problems. With a change of variables, the formulated problem (3.1)-(3.3) is reduced to auxiliary problems with new functions $v(x,t)$, $w(x,t)$ and $k(x',t)$. Hence, the auxiliary problems are equivalently reduced to a system of loaded integral equations of Volterra type.

In the fourth section, we obtain one of the main results of the inverse problem (3.1)-(3.3), the existence and uniqueness of solutions to the system of integral equations (3.13)-(3.15). There, using the contraction mapping principle method, we find the solution of PDEs. Then the existence and uniqueness of the resulting system are proved. Thus, proving the unique solvability of the posed inverse problem.

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