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QUASILINEAR BIHARMONIC EQUATIONS ON $\mathbb{R}^4$ WITH EXPONENTIAL SUBCRITICAL GROWTH

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Abstract. This article studies the fourth-order equation

$$\Delta^2 u - \Delta u + V(x)u - \frac{1}{2}u\Delta(u^2) = f(x,u) \quad \text{in } \mathbb{R}^4,$$
$$u \in H^2(\mathbb{R}^4),$$

where $\Delta^2 := \Delta(\Delta)$ is the biharmonic operator, $V \in C(\mathbb{R}^4,\mathbb{R})$ and $f \in C(\mathbb{R}^4 \times \mathbb{R},\mathbb{R})$ are allowed to be sign-changing. With some assumptions on V and f we prove existence and multiplicity of nontrivial solutions in $H^2(\mathbb{R}^4)$, obtained via variational methods. Three main theorems are proved, the first two assuming that V is coercive to obtain compactness, and the third one requires only that V be bounded. We work carefully with the sub-criticality of f to get a (PS) condition for a related equation.

1. Introduction

In this article, we consider the fourth-order equation

$$\begin{gathered}\Delta^2 u - \Delta u + V(x)u - \frac{1}{2}u\Delta(u^2) = f(x,u) \quad \text{in } \mathbb{R}^4,\\ u \in H^2(\mathbb{R}^4),\end{gathered} \tag{1.1}$$

where $\Delta^2 := \Delta(\Delta)$ is the biharmonic operator, V and f are continuous functions that are allowed to be sign-changing.

In recent years, bi-harmonic and nonlocal operators arise in the description of various phenomena in the pure mathematical research and real-world applications, for example, for studying the traveling waves in suspension bridges [7, 10]. Recently in [8], the authors studied the existence and multiplicity results for fourth-order elliptic equations on $\mathbb{R}^N$ involving $u\Delta(u^2)$ and sign-changing potentials. The results generalize some recent results on this kind of problems. To study this type of problem, first consider the case where the potential V is coercive so that the working space can be compactly embedded into Lebesgue spaces. Next, we study the case where the potential V is bounded so that the workspace is exactly $H^2(\mathbb{R}^N)$, which can not be compactly embedded into Lebesgue spaces. In [8], for sub-critical

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nonlinearity in the Sobolev sense, the authors defined

$$W(x) = V(x) + W_0 \geq 1, \quad x \in \mathbb{R}^N,\ N \in \mathbb{N},$$

to deal with the potential allowed to be sign-changing. They then treated of the following equivalent problem with the potential $W > 0$:

$$\begin{gathered} \Delta^2 u - \Delta u + W(x)u - \frac{1}{2}u\Delta(u^2) = f(x,u) \quad \text{in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{gathered}$$

Here, among other requirements, our nonlinearity $f(x,t)$ satisfies subcritical exponential growth in the sense of Adams' Inequality, which is a Trudinger-Moser type inequality for high dimensions, i.e., $f(x,s)$ behaves like $\pm e^{\alpha s^2}$ as $t \to \pm\infty$ uniformly in $x \in \mathbb{R}^4$, but slower than that.

2. Preliminaries

We now formulate assumptions for V and f:

- (A1) $V \in C(\mathbb{R}^4)$ is bounded from below, $|V^{-1}(-\infty, M]| < \infty$ for all $M > 0$, where $|\cdot|$ is the Lebesgue measure on $\mathbb{R}^4$.
- (A2) $V \in C(\mathbb{R}^4)$ is a bounded function such that the quadratic form $\mathfrak{B} : X \to \mathbb{R}$,
$$\mathfrak{B}(u) = \frac{1}{2}\int_{\mathbb{R}^4} (|\Delta u|^2 + |\nabla u|^2 + V(x)u^2)\,\mathrm{d}x \tag{2.1}$$
is non-degenerate and the negative space of $\mathfrak{B}$ is finite-dimensional.
- (A3) $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(s) = o(s)$ near origin;
- (A4) for $(x,s) \in \mathbb{R}^4 \times \mathbb{R}$ we have $0 \leq 4F(x,s) \leq sf(x,s)$, moreover, for almost all $x \in \mathbb{R}^4$;
$$\lim_{|s|\to\infty} \frac{F(x,s)}{s^4} = +\infty, \quad \text{where } F(x,s) = \int_0^s f(x,\mu)\,\mathrm{d}\mu; \tag{2.2}$$
- (A5) f has subcritical exponential growth, that is,
$$\lim_{|s|\to+\infty} \frac{|f(s)|}{e^{\alpha s^2}} = 0 \quad \forall \alpha > 0;$$
- (A6) For any $r > 0$, we have
$$\lim_{|x|\to\infty} \sup_{0<|t|\leq r} |\frac{f(x,t)}{t}| = 0.$$

Let $H^2(\mathbb{R}^4)$ be the standard Sobolev space. If $V \in C(\mathbb{R}^4)$ is bounded from below, we can choose a constant $\lambda > 0$ such that $\tilde{V}(x) = V(x) + \lambda \geq 1$ for $x \in \mathbb{R}^4$. On the linear subspace

$$X := \{u \in H^2(\mathbb{R}^4) : \int_{\mathbb{R}^4} V(x)|u|^2\,\mathrm{d}x < \infty\}$$

which is equip with the inner product

$$(u,v) = \int_{\mathbb{R}^4} (\Delta u \Delta v + \nabla u \cdot \nabla v + \tilde{V}(x)uv)\,\mathrm{d}x$$

and the corresponding norm $\|\cdot\|_x$. Note that if $V \in C(\mathbb{R}^4)$ is bounded, then X is precisely the standard Sobolev space $H^2(\mathbb{R}^4)$.

By the spectral theory of self-adjoint compact operators we have that the eigenvalue problem

$$\Delta^2 u - \Delta u + V(x)u = \lambda u, \quad u \in X. \tag{2.3}$$

possesses a complete sequence of eigenvectors and eigenvectors, such that

$$-\infty < \lambda_1 \leq \lambda_2 \leq \dots, \lambda_k \to +\infty,$$

where λ_k has been repeated according to its finite multiplicity. We denote by ϕ_k the eigenfunction of λ_k with $|\phi_k|_2 = 1$, where $|\cdot|_s$ is the $L^s(\mathbb{R}^4)$-norm. The main results in this article can be stated as follows.

Theorem 2.1. *Suppose* (A1), (A4)–(A6) *are satisfied. If* 0 *is not an eigenvalue of* (2.3)*, then* (1.1) *has a nontrivial solution* $u \in X$*.*

Theorem 2.2. *Suppose* ((A1), (A4)–(A6) *are satisfied. If* $f(x, \cdot)$ *is odd for all* $x \in \mathbb{R}^4$*, then* (1.1) *has a sequence of solutions* $\{u_n\}$ *such that* $J(u_n) \to +\infty$*.*

Similarly to [8], when that V satisfies (A2) and X is the standard Sobolev space H^2, we do not have the compact embedding $X \hookrightarrow L^s(\mathbb{R}^4)$ for $s \in [2, \infty)$ any more. But we still have the following result.

Theorem 2.3. *Suppose* (A2)–(A6) *are satisfied. Then* (1.1) *has a nontrivial solution* $u \in X$*.*

3. Proof of theorem 2.1

In this section and the next section, we assume that (A1) holds. Now, let us present some preliminary results necessary to demonstration of Theorem 2.1 and that can be similarly used to the others main theorems.

The negative space of $\mathfrak{B}$ is given by

$$X^- = \operatorname{span}\{\phi_1, \dots, \phi_\ell\}$$

and X^+ is the orthogonal complement of X^- in X, such that $X = X^- \oplus X^+$. It is well known that for $u \in X^\pm$, there is a constant $\tilde{a} > 0$ such that

$$\mathfrak{B}(u) \geq \tilde{a} \|u\|_X^2. \tag{3.1}$$

Let us apply the linking theorem to find critical points of the functionals with indefinite quadratic part, like J, with

$$\partial B_\rho \cap W = \{u \in X^+ : \|u\|_X = \rho\}, \quad Q = \{u \in X^- \oplus \mathbb{R}^+ \phi : \|u\|_X \leq R\},$$

where $\phi \in X^+ \backslash 0$. To prove Theorem 2.1 we need the following definition.

Definition 3.1. Let X be a Banach space, we say that functional $J \in C^1(X, \mathbb{R})$ satisfies Palais-Smale condition at the level $c \in \mathbb{R}$, ((PS)$_c$ for short notation) if any sequence $\{u_n\} \subset X$ satisfying $J(u_n) \to c$, $J'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence. J satisfies (PS) condition if J satisfies (PS)$_c$ condition at all $c \in \mathbb{R}$.

Having established the (PS) condition for the functional J, now we present some concepts and results from infinite-dimensional Morse theory [14]. Let X be a Banach space, $J : X \to \mathbb{R}$ be a C^1-functional, u is an isolated critical point of J and $J(u) = c$. Then

$$C_m(J, u) := H_m(J_c, J_c \backslash \{0\}), \quad m \in \mathbb{N} = \{0, 1, 2, \dots\}$$

is called the m-th critical group of J at u, where $J_c := J^{-1}(-\infty, c]$ and H_* stands for the singular homology with coefficients in $\mathbb{Z}$.

If J satisfies the (PS) condition and the critical values of J are bounded from below by κ, then following Bartsch-Li [3], we define the m-th critical group of J at infinity by

$$C_m(J,\infty) := H_m(X, J_\kappa), \quad m \in \mathbb{N}.$$

It is well known that the homology on the right hand-side does not depend on the choice of κ.

Proposition 3.2 ([11, Theorem 5.3]). *Let E be a real Banach space with $E = V \oplus W$, where V is finite dimensional. Suppose $J \in C^1(E,\mathbb{R})$, satisfies (PS), and*

- (i) *there are constants $\rho, d > 0$ such that $J|_{\partial B_{r_1} \cap W} \geq d$, and*
- (ii) *there is an $e \in \partial B_1 \cap W$ and $R > \rho$ such that if $Q \equiv (\bar{B}_R \cap V) \oplus \{re : 0 < r < R\}$, then $J|_{\partial Q} \leq 0$. Then J possesses a critical value $\tilde{c} \geq d$ which can be characterized as*

$$\tilde{c} \equiv \inf_{h\in\Gamma} \max_{u\in Q} J(h(u)),$$

where

$$\Gamma = \{h \in C(\bar{Q}, E) : h = \mathrm{id} \ \text{ on } \partial Q\}.$$

Proposition 3.3 ([3, Proposition 3.6]). *If $J \in C^1(X,\mathbb{R})$ satisfies the condition (PS) and $C_m(J,0) \neq C_m(J,\infty)$ for some $m \in \mathbb{N}$, then J has a nonzero critical point.*

Proposition 3.4 ([9, Theorem 2.1]). *Suppose $J \in C^1(X,\mathbb{R})$ has a local linking at 0 with respect to the decomposition $X = X^- \bigoplus X^+$, i.e., for some $\varepsilon > 0$,*

$$J(u) \leq 0 \quad \text{for} u \in X^- \cap B_\varepsilon,$$
$$J(u) > 0 \quad \text{for} u \in (X^+ \setminus \{0\}) \cap B_\varepsilon,$$

where $B_\varepsilon = \{u \in X : \|u\|_X \leq \varepsilon\}$. If $m = \dim X^- < \infty$, then $C_m(J,0) \neq 0$.

Lemma 3.5. *Assume that* (A1), (A4), (A5) *are satisfied,* 0 *is not an eigenvalue of* (2.3). *Then J has a local linking at* 0 *with respect to the decomposition $X = X^- \oplus X^+$.*

Proof. For $u \in X$, we see that

$$\int_{\mathbb{R}^4} u^2|\nabla u|^2 u \,\mathrm{d}x \leq Big(\int_{\mathbb{R}^4} |u|^6 u \,\mathrm{d}x\Big)^{1/3}\Big(\int_{\mathbb{R}^4} |\nabla u|^3 u \,\mathrm{d}x\Big)^{2/3}.$$

It follows from [1, Thm. 4.12 (Sobolev Imbedding Theorem)] that $H^2(\mathbb{R}^4) = W^{2,2}(\mathbb{R}^4) \hookrightarrow W^{1,s}(\mathbb{R}^4)$ for $2 \leq s \leq 2^* = 8/(4-2)$ and $H^2(\mathbb{R}^4) \hookrightarrow L^q(\mathbb{R}^4)$ for $2 \leq q < \infty$, we have

$$\int_{\mathbb{R}^4} u^2|\nabla u|^2 u \,\mathrm{d}x \leq |u|_6^2 \|u\|_{W^{1,3}}^2 \leq S\|u\|_X^4. \tag{3.2}$$

By (A3) and (A4), we see that as $\|u\|_X \to 0$,

$$\int_{\mathbb{R}^4} u^2|\nabla u|^2 = o(\|u\|_X^2), \quad \int_{\mathbb{R}^4} F(x,u) = o(\|u\|_X^2).$$

Thus, as $\|u\|_X \to 0$,

$$J(u) = \frac{1}{2}(\|u^+\|_V^2 - \|u^-\|_V^2) + \frac{1}{2}\int_{\mathbb{R}^4} u^2|\nabla u|^2 u \,\mathrm{d}x - \int_{\mathbb{R}^4} F(x,u)u \,\mathrm{d}x$$

$$
\begin{aligned}
&= \mathfrak{B}(u) + \frac{1}{2}\int_{\mathbb{R}^4} u^2|\nabla u|^2 u \,\mathrm{d}x - \int_{\mathbb{R}^4} F(x,u)u \,\mathrm{d}x \\
&= \mathfrak{B}(u) + o(\|u\|_X^2).
\end{aligned}
$$

It follows from the above estimate and (3.1) that the proof of our lemma is complete. □

Setting $g(x,s) = f(x,s) + \gamma s$, by (A4) we can see that

$$
G(x,t) := \int_0^t g(x,s)\,\mathrm{d}s = F(x,t) + \frac{\gamma}{2}t^2 \le \frac{t}{4}g(x,t) + \frac{\tau}{4}t^2, \tag{3.3}
$$

where $\tau = b + \gamma$.

The functional J is equivalent to

$$
J(u) = \frac{1}{2}\|u\|_X^2 + \frac{1}{2}\int_{\mathbb{R}^4} u^2|\nabla u|^2 u \,\mathrm{d}x - \int_{\mathbb{R}^4} G(x,u)u \,\mathrm{d}x, \tag{3.4}
$$

with derivative given by

$$
J'(u)v = (u,v) + \int_{\mathbb{R}^4} (uv|\nabla u|^2 + u^2\nabla u\cdot\nabla v)u \,\mathrm{d}x - \int_{\mathbb{R}^4} g(x,u)vu \,\mathrm{d}x.
$$

Lemma 3.6. *Under the conditions of Theorem 2.1, there exist $\rho > 0$, $\xi \in X$ with $\|\xi\|_X > \rho$ such that $J(\xi) < 0$.*

Proof. Combining 3.5 with (2.2), there exists a large $K > 0$ such that for any $e \in X$, with $\|e\|_X = 1$, we have

$$
\lim_{t\to\infty} J(te) \le \lim_{t\to\infty} \left[t^2\mathfrak{B}(e) + \frac{t^4}{2}S\|e\|_X^4 - Kt^4|e|_4^4\right] = -\infty.
$$

So, for some $t_0 > 0$ there exists $\rho > 0$ such that $J(t_0 e) < 0$ with $\|t_0 e\|_X > \rho$. □

Lemma 3.7. *Under* (A1)*, the embedding of X into $L^p(\mathbb{R}^4)$, for any $p \in [2,+\infty)$, is compact.*

Proof. Firstly, we may see that

$$
\int_{\mathbb{R}^4} u^2 u \,\mathrm{d}x \le \int_{\mathbb{R}^4} \tilde{V}(x)u^2 u \,\mathrm{d}x = \int_{\mathbb{R}^4} V(x)u^2 u \,\mathrm{d}x + \int_{\mathbb{R}^4} \gamma u^2 u \,\mathrm{d}x < \infty.
$$

Then, X is continuously embedded into $H^2(\mathbb{R}^4)$. Now let us to show that the embedding of X into $L^p(\mathbb{R}^4)$, with $2 \le p < \infty$, is compact. Let $u_n \rightharpoonup 0$ in X. Hence, $\|u_n\|$ is bounded and by the embedding continuous of X into $L^p(\mathbb{R}^4)$ there exists a constant $C > 0$ such that

$$
|u_n|_2 \le C \quad \forall n \ge 1.
$$

Notice that

$$
|u_n|_2^2 = \int_{\mathbb{R}^4\setminus B(0,R)} u_n^2 u \,\mathrm{d}x + \int_{B(0,R)} u_n^2 u \,\mathrm{d}x \quad \forall n \ge 1,
$$

where R is large positive constant to be determined during the proof. We know that $u_n \to 0$ in $L^2(B(0,R))$, for any $R > 0$. So, for any $\varepsilon > 0$ there exists $N_0(\varepsilon) \in \mathbb{N}$ such that

$$
|u_n|_2^2 \le \frac{\varepsilon}{2} + \int_{\mathbb{R}^4\setminus B(0,R)} u_n^2 u \,\mathrm{d}x \quad \forall n \ge N_0(\varepsilon),
$$

Hence, we need to show that for any $\varepsilon > 0$ there exist $R = R(\varepsilon) > 0$ and $N(\varepsilon) \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^4 \setminus B(0,R)} u_n^2 u \,\mathrm{d}x \le \frac{\varepsilon}{2} \quad \forall n \ge N(\varepsilon).$$

From (A1), it follows that there exists $R > 0$ such that

$$V(x) \le \frac{2C}{\varepsilon} \quad \forall x \in \mathbb{R}^4 \setminus B(0,R).$$

Thus,

$$\begin{aligned} \int_{\mathbb{R}^4 \setminus B(0,R)} u_n^2 u \,\mathrm{d}x &\le \frac{\varepsilon}{2C} \int_{\mathbb{R}^4 \setminus B(0,R)} V(x) u_n^2 u \,\mathrm{d}x \\ &\le \frac{\varepsilon}{2C} \int_{\mathbb{R}^4 \setminus B(0,R)} Z(x) u_n^2 u \,\mathrm{d}x \\ &\le \frac{\varepsilon}{2C} C = \frac{\varepsilon}{2}. \end{aligned}$$

Therefore, $u_n \to 0$ in $L^2(\mathbb{R}^4)$. By interpolation, $u_n \to 0$ in $L^t(\mathbb{R}^4)$ for any $t \in [2, +\infty)$. □

Lemma 3.8. *Suppose that* (A1), (A3)–(A5) *hold. Then J satisfies the (PS) condition.*

Proof. Its clear that J satisfies the mountain pass geometry, that is, there exist $\tilde{\alpha}$, $\tilde{R} > 0$ and $e \in X$ such that $J(u) \le \tilde{\alpha}$ with $\|u\| = \tilde{R}$ and $J(e) < 0$ for $e \in X$ with $\|e\| \ge \tilde{R}$. Observe that there is a sequence $\{u_n\} \in X$ such that

$$\infty > C := \sup_n |J(u_n)|, \quad J'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{3.5}$$

Firstly, we show that $\{u_n\}$ is bounded in X. Otherwise, we have, up to a subsequence, $\|u_n\| \to \infty$. Then, using the inequalities (3.3) and (3.5), we obtain

$$\begin{aligned} 4C + \|u_n\|_X &\ge 4J(u_n) - J'(u_n)u_n \\ &= \|u_n\|_X^2 - \int_{\mathbb{R}^4} (4G(x,u_n) - g(x,u_n)u_n) u \,\mathrm{d}x \\ &\ge \|u_n\|_X^2 - \tau \int_{\mathbb{R}^4} u_n^2 u \,\mathrm{d}x. \end{aligned} \tag{3.6}$$

Let $\omega_n = u_n / \|u_n\|$. Then there exists $\omega \in X$, going if necessary to a subsequence, by the Lemma 3.7 such that

$$\begin{gathered} \omega_n \rightharpoonup \omega \quad \text{in } X, \quad \omega_n \to \omega \quad \text{in } L^2(\mathbb{R}^4) \\ \omega_n \to \omega \quad \text{a.e. in } \mathbb{R}^4 \quad \text{as } n \to \infty. \end{gathered}$$

Multiplying by $1/\|u_n\|_X^2$ on both sides of (3.6) we have

$$\tau \int_{\mathbb{R}^4} \omega_n^2 u \,\mathrm{d}x \ge 1 + o_n(1)$$

and then

$$\tau \int_{\mathbb{R}^4} \omega^2 u \,\mathrm{d}x \ge 1 \tag{3.7}$$

as $n \to \infty$. So, $\omega \neq 0$. By (2.2) and (3.3), since $|u_n(x)| \to \infty$ on $\{x \in \mathbb{R}^4 : \omega(x) \neq 0\}$ we see that

$$\lim_{n\to\infty} \frac{G(x,u_n(x))}{\|u_n\|_X^4} = \lim_{n\to\infty} \frac{G(x,u_n(x))}{u_n^4(x)}\omega_n^4(x) = +\infty. \tag{3.8}$$

From (3.7), we have $|\{x \in \mathbb{R}^4 : \omega(x) \neq 0\}| > 0$. So, by Fatou's lemma and (3.8), we obtain that

$$\liminf_{n\to+\infty} \int_{\mathbb{R}^4} \frac{G(x,u_n)u\,\mathrm{d}x}{\|u_n\|_X^4} \geq \liminf_{n\to+\infty} \int_{\omega\neq 0} \frac{G(x,u_n)u\,\mathrm{d}x}{\|u_n\|_X^4} = +\infty. \tag{3.9}$$

Thus, by (3.4), (3.5) and (3.2), we have

$$\begin{aligned} o(1) = \frac{J(u_n)}{\|u_n\|_X^4} &= \frac{1}{\|u_n\|_X^4}\Big(\frac{1}{2}\|u_n\|_X^2 + \frac{1}{2}\int_{\mathbb{R}^4} u_n^2|\nabla u_n|_X^2 - \int_{\mathbb{R}^4} G(x,u_n)u\,\mathrm{d}x\Big) \\ &\leq \frac{1}{2}\frac{1}{\|u_n\|_X^2} + \frac{S}{2} - \int_{\mathbb{R}^4} \frac{G(x,u_n)}{\|u_n\|_X^4}u\,\mathrm{d}x \to -\infty \quad \text{as } n \to +\infty. \end{aligned}$$

This is a contradiction. Therefore, the sequence $\{u_n\}$ is bounded in X. Next, we proof the existence of a subsequence of $\{u_n\}$ which converges strongly in X. By (A1) and Lemma 3.7, we have that $u_n \to u$ in $L^p(\mathbb{R}^4)$ for any $p \in [2,+\infty)$. For a fixed $n \in \mathbb{N}$,

$$\|u-u_n\|_X^2 = [J'(u_n) - J'(u)](u-u_n) + \int_{\mathbb{R}^4} [g(x,u_n) - g(x,u)]\,(u-u_n)\,\mathrm{d}x$$

and

$$[J'(u_n) - J'(u)]\,(u-u_n) \to 0,$$

because $\{u_n\}$ is a bounded Palais-Smale sequence.

Let $\alpha > 0$ and $q > 0$ to be determined during the proof. Hence, for some $C(\alpha,q)$, applying [13, Lemma 2.3] for $\|u_n\|_X \leq M$ and $\alpha < 64\pi^2/(5M^2)$, yields

$$\begin{aligned} &|\int_{\mathbb{R}^4} [g(x,u_n) - g(x,u)]\,(u-u_n)\,\mathrm{d}x| \\ &\leq \int_{\mathbb{R}^4} [|g(x,u_n)| + |g(x,u)|]\,|u-u_n|\,\mathrm{d}x \\ &\leq C(\alpha,q)\int_{\mathbb{R}^4} [(|u_n| + |u|) + (|u_n|^q(e^{\alpha u_n^2}-1) + |u|^q(e^{\alpha u^2}-1))]|u-u_n|\,\mathrm{d}x \\ &\leq C_1(|u_n|_2 + |u|_2)|u_n-u|_2 + C_2\Big(\int_{\mathbb{R}^4}(|u_n|^q(e^{\alpha u_n^2}-1))^{5/4}dx\Big)^{4/5}|u_n-u|_5 \\ &\quad + C_3\Big(\int_{\mathbb{R}^4}(|u|^q(e^{\alpha u^2}-1))^{5/4}dx\Big)^{4/5}|u_n-u|_5) \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

□

To study the functional J let us rewrite $\mathfrak{B}$ in a more convenient representation. Let us note that, if (A1) holds and 0 is not an eigenvalue of (2.3), or if (A2) holds, then the quadratic form $\mathfrak{B}$ is nondegenerate and the negative space of $\mathfrak{B}$ is finite-dimensional, and so we may choose an equivalent norm $\|\cdot\|_V$ on X such that

$$\mathfrak{B}(u) = \frac{1}{2}\big(\|u^+\|_V^2 - \|u^-\|_V^2\big),$$

where $u^{\pm}$ is the orthogonal projection of u on $X^{\pm}$ being $X^{\pm}$ the positive/negative space of $\mathfrak{B}$. Let us use the equivalent norm on X and rewrite J as

$$J(u) = \frac{1}{2}(\|u^+\|_V^2 - \|u^-\|_V^2) + \frac{1}{2}\int_{\mathbb{R}^4} u^2|\nabla u|^2 u \,\mathrm{d}x - \int_{\mathbb{R}^4} F(x,u)u \,\mathrm{d}x. \tag{3.10}$$

It follows from (3.2) that there exists some constant $A_0 > 0$ such that

$$\int_{\mathbb{R}^4} u^2|\nabla u|^2 \le A_0\|u\|_V^4. \tag{3.11}$$

Lemma 3.9. *If* (A1) *or* (A2)*, and* (A3)–(A5) *hold,* 0 *is not an eigenvalue of* (2.3)*, then there exists $L > 0$ such that if $J(u) \le -L$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=1} J(tu) < 0.$$

Proof. We argue by contradiction. Suppose that for any $n \in \mathbb{N}$ there exists u_n such that $J(u_n) \le -n$ but

$$J'(u_n)u_n = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=1} J(tu_n) \ge 0.$$

Then, arguing by contradiction, we can see that $\|u_n\|_V \to +\infty$ and

$$\begin{aligned}
-4n &\ge 4J(u_n) - J'(u_n)u_n \\
&= (\|u_n^+\|_V^2 - \|u_n^-\|_V^2) - \int_{\mathbb{R}^4} 4F(x,u_n) - f(x,u_n)u_n u \,\mathrm{d}x \\
&\ge \|u_n^+\|_V^2 - \|u_n^-\|_V^2
\end{aligned} \tag{3.12}$$

Let $\omega_n = \frac{u_n}{\|u_n\|_v}$ and $\omega_n^{\pm}$ be the orthogonal projection on $X^{\pm}$. Since $\{\omega_n\}$ is bounded and dim $X^- < \infty$, we have that X^- is closed and

$$\omega_n^- \to \omega^- \quad \text{in } X^-,$$

with $\omega^- \in X^-$. If $\omega^- = 0$, then $\|\omega_n^+\|_V \to 1$, because $\|\omega_n\|_V^2 = \|\omega_n^+\|_V^2 + \|\omega_n^-\|_V^2 = 1$. By the definition of $\omega_n^{\pm}$, for large n we have

$$\|u_n^+\|_V^2 = \|u_n\|_V^2\|\omega_n^+\|_V^2 = \|u_n\|_V^2(1 + o_n(1))^2 \ge \|u_n\|_V^2\|\omega_n^-\|_V^2 = \|u_n^-\|_V^2.$$

Hence, using (3.12) we obtain

$$0 \le \|u_n^+\|_V^2 - \|u_n^-\|_V^2 \le -4n$$

which cannot happen. If $\omega^- \ne 0$, then

$$\omega_n \rightharpoonup \omega \quad \text{in } X,$$

where $\omega \ne 0$. As a direct consequence of (A4) and

$$+\infty = \lim_{l\to+\infty} \frac{4F(x,l)}{l^4} \le \lim_{l\to+\infty} \frac{f(x,l)l}{l^4},$$

we have

$$\begin{aligned}
\infty &\le \liminf_{n\to\infty} \int_{\omega^- \ne 0} \frac{4F(x,u_n)}{\|u_n\|_V^4} \,\mathrm{d}x \\
&\le \liminf_{n\to\infty} \int_{\mathbb{R}^4} \frac{4F(x,u_n)}{\|u_n\|_V^4} \,\mathrm{d}x \qquad \le \liminf_{n\to\infty} \int_{\mathbb{R}^4} \frac{f(x,u_n)u_n}{\|u_n\|_V^4} \,\mathrm{d}x.
\end{aligned} \tag{3.13}$$

But, it follows from (3.11) that

$$\frac{J'(u_n)u_n}{\|u_n\|_V^4}$$

$$
\begin{aligned}
&= \frac{1}{\|u_n\|_V^4}(\|u_n^+\|_V^2 - \|u_n^-\|_V^2) + 2\frac{1}{\|u_n\|_V^4}\int_{\mathbb{R}^4} u_n^2|\nabla u_n|^2 - \frac{1}{\|u_n\|_V^4}\int_{\mathbb{R}^4} f(x,u_n)u_n \\
&\leq o_n(1) + 2A_0 - \frac{1}{\|u_n\|_V^4}\int_{\mathbb{R}^4} f(x,u_n)u_n \to -\infty \quad \text{as } n \to +\infty,
\end{aligned}
$$

which is a contradiction, because we are assuming that $J'(u_n)u_n \geq 0$.

Now, since the proof does not depend of the compact embedding $X \hookrightarrow L^2(\mathbb{R}^4)$, the result is true if we assume (A2) instead of (A1). $\square$

Lemma 3.10. *Let $B := B(0,1)$ the unit ball in X. Then, there exists $u \in \partial B$ such that $t_u u \in X \setminus B$ and $J(t_u u) < 0$, for some $t_u > 0$.*

Proof. For $t > 0$ and $u \in \partial B$, we have

$$
\frac{J(tu)}{t^4} = \frac{1}{t^2}(\|u_n^+\|_V^2 - \|u_n^-\|_V^2) + \frac{1}{2}\int_{\mathbb{R}^4} u^2|\nabla u|^2 u\,\mathrm{d}x - \int_{\mathbb{R}^4} \frac{F(x,tu)}{t^4} u\,\mathrm{d}x. \tag{3.14}
$$

By (3.14), (2.2) and (3.2), we obtain

$$
J(tu) \to -\infty \quad \text{as} \quad t \to +\infty.
$$

Hence, there exists $t_u > 0$ such that $J(t_u u) < 0$. $\square$

Now, let us introduce some concepts and results from infinite-dimensional Morse theory [4]. Let X be a real Banach space, $J \in C^1(X,\mathbb{R})$, u be an isolated critical point of J and $J(u) = c$. Then the *ith* critical group of J at u is defined by

$$
C_i(J,u) := H_i(J_c, J_c\backslash\{0\}), \quad i = 0,1,2,\dots,
$$

where $J^c = \{u \in X : J(u) \leq c\}$, and $H_i(\cdot,\cdot)$ denotes a singular relative homology group of pair $(\cdot,\cdot)$ with integer coefficients.

If J satisfies the (PS) condition and the critical values of J are bounded from below by some a, then, following Bartsch and Li [3], the critical groups of J at infinity

$$
C_i(J,\infty) := H_i(X, J_c), \quad i = 0,1,2,\dots,
$$

do not depend on the choice of a, because the homology on the right satisfies this.

Lemma 3.11. *Assume that the conditions* (A1) *or* (A2)*, and* (A3)–(A5) *are satisfied. Then $C_i(J,\infty) = 0$ fori $= 0,1,2,\dots$.*

Proof. By Lemma 3.10, for $\tilde{A} \geq A > 0$ large enough, for any $v \in S$ there exists a unique $t_v > 0$ such that $J(t_v v) = -\tilde{A}$. So, letting $u = t_v v$ we have $J'(u)u < 0$. By the Implicit Function Theorem, there is a unique continuous $T : W_0 \subset S \to W_1 \subset \mathbb{R}$, for some W_0 and W_1 open neighborhoods such that for $F(s,v) = J(sv)$, we have $F(T(v),v) = J(T(v)v) = -\tilde{A}$. Then, $T(v) = t_v$ and we obtain a unique application $\varphi \in C(S,\mathbb{R})$ such that $J(\varphi(v)v) = -\tilde{A}$. Moreover, if $J(u) = -\tilde{A}$, then $\varphi(u) = 1$. Using the function φ we can construct a strong deformation retract $\eta : X\backslash B \to J_{-\tilde{A}}$

$$
\eta(u) = \begin{cases} u, & \text{if } J(u) \leq -\tilde{A}, \\ \varphi(\frac{u}{\|u\|_X})\frac{u}{\|u\|_X}, & \text{if } J(u) > -\tilde{A} \end{cases}
$$

and we obtain

$$
C_i(J,\infty) = H_i(X, J_{-\tilde{A}}) \cong H_i(X, X\backslash B) = 0, \quad i = 0,1,2,\dots. \qquad \square
$$

Proof ofTheorem2.1. By Lemma 3.8, J satisfies the Palais-Smale condition and by Lemma 3.5, J has a local linking at 0 with respect to the decomposition $X^- \bigoplus X^+$. Hence, since $m = \dim X^- < \infty$, we have $C_m(J,0) \neq 0 = C_m(J,\infty)$. Then, it follows from Proposition 3.3 that J has a critical point u, which is a nontrivial solution of (1.1). □

4. Proof of theorem 2.2

In the proof of Theorem 2.2 we apply the following symmetric mountain pass theorem.

Proposition 4.1 ([2, Theorem 9.12]). *Let X be an infinite dimensional Banach space, $J \in C^1(X,\mathbb{R})$ be even, satisfies (PS) condition and $J(0) = 0$. If $X = Y \oplus Z$ with $\dim Y < \infty$, and J satisfies*

1. *there are constants $\rho, \alpha > 0$ such that $J|_{\partial B_\rho \cap Z} \geq \alpha$,*
2. *for any finite dimensional subspace $W \subset X$, there is an $R = R(W)$ such that $J \leq 0$ on $W \backslash B_{R(W)}$ then J has a sequence of critical values $c_j \to +\infty$.*

Lemma 4.2. *For $m \in \mathbb{N}$, let $Z_m = \overline{\operatorname{span}}\{\phi_m, \phi_{m+1}, \dots\}$ and set*

$$\sigma_m = \sup_{u \in Z_m, \|u\|=1} |u|_2$$

Then $\sigma_m \to 0$ as $m \to \infty$.

Proof. For $m \in \mathbb{N}$ large, let us take $u \in Z_m$ with $\|u\| = 1$. Thus, we obtain

$$\lambda_m \int_{\mathbb{R}^4} u^2 u \, \mathrm{d}x \leq \int_{\mathbb{R}^4} (|\Delta u|^2 + |\nabla u|^2 + V(x)u^2)u \, \mathrm{d}x$$

or equivalently

$$(\lambda_m + \gamma) \int_{\mathbb{R}^4} u^2 u \, \mathrm{d}x \leq \int_{\mathbb{R}^4} (|\Delta u|^2 + |\nabla u|^2 + \tilde{V}(x)u^2)u \, \mathrm{d}x = \|u\|_X^2 = 1.$$

Therefore, as $m \to \infty$,

$$|\sigma_m| \leq \frac{1}{\sqrt{\lambda_m + \gamma}} \to 0.$$

□

Proof of Theorem 2.2. Note that here we are considering that the functional J is even and satisfies the (PS) condition. Then, it suffices to show that the Proposition 4.1 is applicable to J.

(1) It follows from (A3) and (A4) that for fixed $\alpha > 32\pi$ and $q > 2$, the existence of two constants c_1, $c_2 > 0$ such that

$$|G(x,s)| \leq c_1|s|^2 + c_2|s|^q(e^{\alpha s^2} - 1) \quad \forall s \in \mathbb{R}. \tag{4.1}$$

We have that the embedding $X \hookrightarrow H^2(\mathbb{R}^4)$ is continuous, namely there exists a constant $\mathfrak{L}$ such that

$$\|u\|_{H^2} \leq \mathfrak{L}\|u\|_X \quad \forall u \in X.$$

Now, let us choose $m \in \mathbb{N}$, Z_m and σ_m such that by Lemma 4.2 we have

$$\bar{c} = \frac{1}{2} - c_1\sigma_m^2 > 0$$

and we set

$$Y = \operatorname{span}\{\phi_1, \dots, \phi_{m-1}\}, \quad Z = \overline{\operatorname{span}}\{\phi_m, \phi_{m+1}, \dots\}.$$

Then, we have $X = Y \oplus Z$.

If we consider $\|u\|_X \leq \frac{1}{\mathfrak{L}\sqrt{\alpha}}$ and then $\|u\|_{H^2} \leq \frac{1}{\sqrt{\alpha}}$, we can apply [12, Lemma 1] such that

$$\int_{\mathbb{R}^4} G(x,u)u\,\mathrm{d}x \leq c_1|u|_2^2 + \mathfrak{L}(\alpha,q,c_1)\|u\|_X^q \quad \forall u \in X,\ \|u\|_X \leq \frac{1}{\mathfrak{L}\sqrt{\alpha}}.$$

Therefore, using (4.1) for $u \in Z = Z_m$, as $\|u\|_X \to 0$ we obtain

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|_X^2 + \frac{1}{2}\int_{\mathbb{R}^4} u^2|\nabla u|^2 u\,\mathrm{d}x - \int_{\mathbb{R}^4} G(x,u)u\,\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|_X^2 - \int_{\mathbb{R}^4} G(x,u)u\,\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|_X^2 - c_1|u|_2^2 - \mathfrak{L}(\alpha,q,c_1)\|u\|_X^q \\ &\geq \big(\frac{1}{2} - c_1\sigma_m^2\big)\|u\|_X^2 - \mathfrak{L}(\alpha,q,c_1)\|u\|_X^q \\ &= \bar{c}\|u\|_X^2 + o(\|u\|_X^2). \end{aligned}$$

Thus (1) was verified.

(2) Its sufficient to show that J is anti-coercive, i.e. $J(u_n) \to -\infty$ as $\|u_n\|_X \to +\infty$. We argue by contradiction: let us suppose that there exists $\{u_n\} \subset W \subset X$ and $L < 0$ such that $\|u_n\|_X \to +\infty$ but $J(u_n) \geq L$. Let $v_n = \frac{u_n}{\|u_n\|_X}$ be the normalized sequence and, up to a sub-sequence, $v_n \to v \in W \setminus \{0\}$, because $\dim W < \infty$. Continuing as it was done in (3.9) we obtain

$$\frac{1}{\|u_n\|_X^4}\int_{\mathbb{R}^4} G(x,u_n)u\,\mathrm{d}x \to +\infty.$$

Hence, it follows from (3.2) that

$$\begin{aligned} J(u_n) &= \frac{1}{2}\|u_n\|_X^2 + \frac{1}{2}\int_{\mathbb{R}^4} u_n^2|\nabla u_n|^2 u\,\mathrm{d}x - \int_{\mathbb{R}^4} G(x,u_n)u\,\mathrm{d}x \\ &\leq \frac{1}{2}\|u_n\|_X^4\Big(\frac{1}{\|u_n\|_X^2} + S - \frac{2}{\|u_n\|_X^4}\int_{\mathbb{R}^4} G(x,u_n)u\,\mathrm{d}x\Big) \to -\infty. \end{aligned}$$

This contradicts $J(u_n) \geq L$. $\square$

5. Proof or theorem 2.3

In this section we consider the potential V satisfying the assumption (A2) instead of (A1) and so X is equivalent to standard Sobolev space $H^2(\mathbb{R}^N)$ and we do not have the compact embedding $X \hookrightarrow L^2(\mathbb{R}^N)$.

Lemma 5.1. *Under the assumptions of Theorem 2.3, $\{u_n\}$ is a (PS) sequence of J, that is, as $n \to +\infty$*

$$\sup_n |J(u_n)| < \infty, \quad J'(u_n) \to 0\,.$$

Then $\{u_n\}$ is bounded in X.

Proof. Otherwise, up to a subsequence. we assume that $\|u_n\|_V \to \infty$. Let $v_n = \|u_n\|_V^{-1}u_n$. Then $v_n = v_n^+ + v_n^- \to v = v^+ + v^- \in X$, $v_n^\pm, v^\pm \in X^\pm$. If $v = 0$, then $v_n^- \to v^- = 0$ because $\dim X^- < \infty$ and $X^- \cap X^+ = \{0\}$. Since

$$\|v_n^+\|_V^2 + \|v_n^-\|_V^2 = 1,$$

for n large enough we have

$$\|v_n^+\|_V^2 - \|v_n^-\|_V^2 \geq \frac{1}{2}.$$

Now, using (A4) for n large enough, we obtain

$$\begin{aligned}
&1 + \sup_n |J(u_n)| + \|u_n\|_V \\
&= J(u_n) - \frac{1}{4}J'(u_n)u_n \\
&= \frac{1}{4}\Big(\|u_n^+\|_V^2 - \|u_n^-\|_V^2\Big) - \int_{\mathbb{R}^4}\Big(F(x,u_n) - \frac{1}{4}f(x,u_n)u_n\Big)u\,\mathrm{d}x \\
&\geq \frac{1}{4}\|u_n\|_V^2(\|v_n^+\|_V^2 - \|v_n^-\|_V^2) - \int_{\mathbb{R}^4}\Big(F(x,u_n) - \frac{1}{4}f(x,u_n)u_n\Big)u\,\mathrm{d}x \\
&\geq \frac{1}{8}\|u_n\|_V^2,
\end{aligned}$$

a contradiction to $\|u_n\|_V \to +\infty$. Thus, we obtain that the (PS) sequence $\{u_n\}$ is bounded in X.

If we suppose $v \neq 0$, then there exists $\Omega = \{x \in \mathbb{R}^4 : v(x) \neq 0\}$ with positive Lebesgue measure such that for $x \in \Omega$ we have

$$\frac{F(x,u_n)}{u_n^4(x)}v_n^4(x) \to +\infty,$$

thanks to (2.2). On the other hand, using (3.11), we obtain

$$\begin{aligned}
\int_\Omega \frac{F(x,u_n)}{u_n^4(x)}v_n^4(x)u\,\mathrm{d}x &= \frac{1}{\|u_n\|_V^4}\int_\Omega F(x,u_n(x))u\,\mathrm{d}x \\
&\leq \frac{1}{\|u_n\|_V^4}\int_{\mathbb{R}^4} F(x,u_n(x))u\,\mathrm{d}x \\
&\leq \frac{\|u_n^+\|_V^2 - \|u_n^-\|_V^2}{2\|u_n\|_V^4} + \frac{1}{2\|u_n\|_V^4}\int_{\mathbb{R}^4} u_n^2|\nabla u_n|^2 u\,\mathrm{d}x - \frac{J(u_n)}{\|u_n\|_V^4} \\
&\leq 1 + \frac{A_0}{2}.
\end{aligned}$$

Thus, we have that $\{u_n\}$ is bounded in X. □

Now, let us to investigate the C^1-functional $\mathcal{V} : H^2(\mathbb{R}^4) \to \mathbb{R}$, defined by

$$\mathcal{V}(u) = \frac{1}{2}\int_{\mathbb{R}^4} u^2|\nabla u|^2 u\,\mathrm{d}x,$$

with derivative

$$\mathcal{V}'(u)v = \int_{\mathbb{R}^4}(uv|\nabla u|^2 + u^2\nabla u\cdot\nabla v)u\,\mathrm{d}x, \quad u,v \in H^2(\mathbb{R}^4),$$

to obtain the (PS) condition for J.

Lemma 5.2. *The functional $\mathcal{V} : H^2(\mathbb{R}^4) \to \mathbb{R}$ is weakly lower semi-continuous; $\mathcal{V}' : H^2(\mathbb{R}^4) \to H^{-2}(\mathbb{R}^4)$ is weakly sequentially continuous.*

Proof. Let $\{u_n\}$ be a sequence in $H^2(\mathbb{R}^4)$ such that $u_n \rightharpoonup u$ in $H^2(\mathbb{R}^4)$. Then, by the compact embedding $H^2(\mathbb{R}^4) \hookrightarrow H^1_{\rm loc}(\mathbb{R}^4)$ we have $u_n \to u$ in $H^1_{\rm loc}(\mathbb{R}^4)$. Hence, going if necessary to a subsequence, we obtain

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \mathbb{R}^4, \quad u_n \to u \quad \text{a.e. in } \mathbb{R}^4, \tag{5.1}$$

and then by Fatou's lemma,

$$\int_{\mathbb{R}^4} u^2|\nabla u|^2 u\,\mathrm{d}x \le \liminf_{n\to+\infty}\int_{\mathbb{R}^4} u_n^2|\nabla u_n|^2 u\,\mathrm{d}x,$$

that is

$$\mathcal{V}(u) \le \liminf_{n\to+\infty} V(u_n).$$

Thus, $\mathcal{V}$ is weakly lower semi-continuous. To investigate the weak lower semicontinuity of $\mathcal{V}'$ we need to see that for $u \in H^2(\mathbb{R}^4)$, since $H^2(\mathbb{R}^4) \hookrightarrow W^{1,4}(\mathbb{R}^4)$, we have

$$\begin{aligned}\int_{\mathbb{R}^4} (|\nabla u_n|^2|u_n|)^{4/3} u\,\mathrm{d}x &\le \Big(\int_{\mathbb{R}^4} |\nabla u_n|^{2\frac{4}{3}\frac{3}{2}} u\,\mathrm{d}x\Big)^{2/3}\Big(\int_{\mathbb{R}^4} |u_n|^{\frac{4}{3}\frac{3}{1}} u\,\mathrm{d}x\Big)^{1/3}\\ &\le C\|u_n\|^{8/3}_{H^2(\mathbb{R}^4)}\|u_n\|^{4/3}_{H^2(\mathbb{R}^4)}\end{aligned}$$

and then

$$\Big|\int_{\mathbb{R}^4} |\nabla u_n|^2 u_n u\,\mathrm{d}x\Big| \le C\|u_n\|^4_{H^2(\mathbb{R}^4)}. \tag{5.2}$$

Thus, $\{u_n\}$ is bounded in $L^{4/3}(\mathbb{R}^4)$ and combining with (5.1) we may apply the Brézis-Lieb lemma to obtain $|\nabla u_n|^2 u_n \rightharpoonup |\nabla u|^2 u$ in $L^{4/3}(\mathbb{R}^4)$. Hence, for any $\varphi \in H^2(\mathbb{R}^4)$, we have $\varphi \in L^4(\mathbb{R}^4)$ and

$$\int_{\mathbb{R}^4} |\nabla u_n|^2 u_n \varphi u\,\mathrm{d}x \to \int_{\mathbb{R}^4} |\nabla u|^2 u\varphi u\,\mathrm{d}x. \tag{5.3}$$

Similarly, we have

$$\begin{aligned}\int_{\mathbb{R}^4} |u^2\nabla u|^{4/3} u\,\mathrm{d}x &\le \Big(\int_{\mathbb{R}^4} |u|^4 u\,\mathrm{d}x\Big)^{2/3}\Big(\int_{\mathbb{R}^4} |\nabla u|^4 u\,\mathrm{d}x\Big)^{1/3}\\ &\le C\|u_n\|^{8/3}_{H^2(\mathbb{R}^4)}\|u_n\|^{4/3}_{H^2(\mathbb{R}^4)}.\end{aligned}$$

Thus, the sequence $\{u_n^2\nabla u_n\}$ is bounded in $L^{4/3}(\mathbb{R}^4)$ and converges point-wise to $u^2\nabla u$. Again, by Brézis-Lieb lemma we obtain

$$u_n^2\nabla u_n \to u^2\nabla u, \quad \text{in } [L^{4/3}(\mathbb{R}^4)]^4.$$

For each $\varphi \in H^2(\mathbb{R}^4)$ we have $\varphi \in W^{1,4}(\mathbb{R}^4)$ and then $\varphi \in L^4(\mathbb{R}^4)$, which implies

$$\int_{\mathbb{R}^4} u_n^2\nabla u_n\cdot\nabla\varphi u\,\mathrm{d}x \to \int_{\mathbb{R}^4} u^2\nabla u\cdot\nabla\varphi u\,\mathrm{d}x. \tag{5.4}$$

Now, using (5.3) and (5.4) we have

$$\int_{\mathbb{R}^4} |\nabla u_n|^2 u_n\varphi + u_n^2\nabla u_n\cdot\nabla\varphi u\,\mathrm{d}x \to \int_{\mathbb{R}^4} |\nabla u|^2 u\varphi + u^2\nabla u\cdot\nabla\varphi u\,\mathrm{d}x,$$

that is,

$$\mathcal{V}'(u_n)\varphi \to \mathcal{V}'(u)\varphi\,,$$

and then $\mathcal{V}'$ is weakly sequentially continuous. Moreover, if $u_n \rightharpoonup u$ in $H^2(\mathbb{R}^4)$, we obtain

$$\begin{aligned}&\liminf_{n\to+\infty}\int_{\mathbb{R}^4}(|\nabla u_n|^2 u_n(u_n-u) + u_n^2\nabla u_n\cdot\nabla(u_n-u))u\,\mathrm{d}x\\ &= \liminf_{n\to+\infty}(4\mathcal{V}(u_n) - \mathcal{V}'(u_n)u)\\ &\ge 4\mathcal{V}(u) - \mathcal{V}'(u)u = 0.\end{aligned} \tag{5.5}$$

□

Lemma 5.3. *Operator J satisfies the (PS) condition.*

Proof. Let $\{u_n\}$ be a (PS) sequence. It follows from Lemma 5.1 that $\{u_n\}$ is bounded in X and so, up to a subsequence, we obtain $u_n \rightharpoonup u$ in X. We claim that

$$\limsup_{n\to+\infty}\int_{\mathbb{R}^4} f(x,u_n)(u_n-u)u\,\mathrm{d}x \le 0. \tag{5.6}$$

Indeed, letting $\bar\varepsilon > 0$ and $\alpha > 0$ such that $2 < \frac{32\pi^2}{\alpha M^2}$ with $\|u_n\|_V \le M$, as a consequence of Lemma 5.1. Then by (A5) and [13, Theorem 2.2], for $r \ge 1$ large enough, we have

$$\begin{aligned}
&\int_{\mathbb{R}^4\cap\{|u_n|\ge r\}} f(x,u_n)(u_n-u)u\,\mathrm{d}x\\
&\le \bar\varepsilon\int_{\mathbb{R}^4\cap\{|u_n|\ge r\}} (e^{\alpha u_n^2}-1)|u_n-u|u\,\mathrm{d}x\\
&\le \bar\varepsilon\Big(\int_{\mathbb{R}^4\cap\{|u_n|\ge r\}} (e^{\alpha u_n^2}-1)^2u\,\mathrm{d}x\Big)^{1/2}\Big(\int_{\mathbb{R}^4\cap\{|u_n|\ge r\}} (u_n-u)^2u\,\mathrm{d}x\Big)^{1/2}\\
&\le \bar\varepsilon C(\frac{32\pi^2}{\alpha M^2})\Big(\int_{\mathbb{R}^4} (e^{\alpha(\frac{32\pi^2}{\alpha M^2})u_n^2}-1)u\,\mathrm{d}x\Big)^{1/2}\times|u_n-u|_2\\
&\le \bar\varepsilon C\Big(\int_{\mathbb{R}^4} (e^{32\pi^2(\frac{u_n}{M})^2}-1)u\,\mathrm{d}x\Big)^{1/2}\le\frac{\varepsilon}{3},
\end{aligned}$$

for small $\varepsilon > 0$. Moreover, by (A6) there exists $R > 0$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^4\cap\{|x|\le R\}\cap\{|u_n|\le r\}} f(x,u_n)(u_n-u)u\,\mathrm{d}x\\
&\le \sup_{|t|<r,|x|\le R}\frac{|f(x,t)|}{|t|}\int_{\mathbb{R}^4\cap\{|x|\le R\}\cap\{|u_n|\le r\}} |u_n||u_n-u|u\,\mathrm{d}x\le\frac{\varepsilon}{3}.
\end{aligned}$$

Finally, since $u_n \to u$ in $L^2(B(0,R))$, by (f_0) and (A5) we have

$$\begin{aligned}
&\int_{\mathbb{R}^4\cap\{|x|\le R\}\cap\{|u_n|\le r\}} f(x,u_n)(u_n-u)u\,\mathrm{d}x\\
&\le (\int_{\mathbb{R}^4\cap\{|x|\le R\}\cap\{|u_n|\le r\}} |f(x,u_n)|^2u\,\mathrm{d}x)^{1/2}\\
&\quad\times\Big(\int_{\mathbb{R}^4\cap\{|x|\le R\}\cap\{|u_n|\le r\}} |u_n-u|^2u\,\mathrm{d}x\Big)^{1/2}\\
&\le\frac{\varepsilon}{3}
\end{aligned}$$

and then we conclude that

$$\int_{\mathbb{R}^4} f(x,u_n)(u_n-u)u\,\mathrm{d}x\le\varepsilon,$$

for small $\varepsilon > 0$. Now, by weak convergence we have

$$\begin{aligned}
&\int_{\mathbb{R}^4}(\Delta u_n\Delta u+\nabla u_n\cdot\nabla u+V(x)u_nu)u\,\mathrm{d}x\\
&\to\int_{\mathbb{R}^4}(|\Delta u|^2+|\nabla u|^2+V(x)u^2)u\,\mathrm{d}x\\
&=\|u^+\|_V^2-\|u^-\|_V^2.
\end{aligned}$$

Since X^- is a finite-dimensional vector space, we obtain $u_n^- \to u^-$ and $\|u_n^-\|_V \to \|u^-\|_V$. Then, since $J'(u_n)(u_n - u) = o(1)$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^4} f(x,u_n)(u_n-u)u\,\mathrm{d}x \\
&= \int_{\mathbb{R}^4} (\Delta u_n \Delta(u_n-u) + \nabla u_n \cdot \nabla(u_n-u) + V(x)u_n(u_n-u))u\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^4} (|\nabla u_n|^2 u_n(u_n-u) + u_n^2 \nabla u_n \cdot \nabla(u_n-u))u\,\mathrm{d}x + o(1) \\
&= (\|u_n^+\|_V^2 - \|u_n^-\|_V^2) - (\|u^+\|_V^2 - \|u^-\|_V^2) + o(1) \\
&\quad + \int_{\mathbb{R}^4} (|\nabla u_n|^2 u_n(u_n-u) + u_n^2 \nabla u_n \cdot \nabla(u_n-u))u\,\mathrm{d}x \\
&= (\|u_n^+\|_V^2 - \|u^+\|_V^2) + \int_{\mathbb{R}^4} (|\nabla u_n|^2 u_n(u_n-u) + u_n^2 \nabla u_n \cdot \nabla(u_n-u))u\,\mathrm{d}x + o(1).
\end{aligned}$$

Hence, by (5.6)

$$\begin{aligned}
0 \geq &\limsup_{n\to+\infty} (\|u_n^+\|_V^2 - \|u^+\|_V^2) \\
&+ \liminf_{n\to+\infty} \int_{\mathbb{R}^4} \big(|\nabla u_n|^2 u_n(u_n-u) + u_n^2 \nabla u_n \cdot \nabla(u_n-u)\big)u\,\mathrm{d}x
\end{aligned} \tag{5.7}$$

and so combining (5.5) with (5.7) we obtain

$$\|u^+\|_V^2 \leq \liminf_{n\to+\infty} \|u_n^+\|_V^2 \leq \limsup_{n\to+\infty} (\|u_n^+\|_V^2 - \|u^+\|_V^2) + \|u^+\|_V^2 \leq \|u^+\|_V^2.$$

Therefore,

$$\lim_{n\to\infty} \|u_n^+\|_V^2 = \|u^+\|_V^2$$

which implies

$$\lim_{n\to\infty} \|u_n\|_V^2 = \|u\|_V^2. \tag{5.8}$$

Thus, combining (5.8) and $u_n \rightharpoonup u$ in X, it follows from Radon-Riesz theorem that $u_n \to u$ in X. □

Proof of Theorem 2.3. Since the conclusion of Lemma 3.9 remains valid if instead of (A1), V satisfies (A2), there exists $L > 0$ such that if $J(u) \leq -L$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1} J(tu) < 0. \tag{5.9}$$

Applying Lemma 3.11 we obtain $C_i(J,\infty) = 0$. But analogously to Lemma 3.5 we can show that J has a local linking at 0 with respect to the decomposition $X = X^- \oplus X^+$. Since $m = dimX^- < \infty$, we have $C_m(J,0) \neq 0 = C_m(J,\infty)$. By Proposition 3.3, J has a critical point u, which concludes the proof of Theorem 2.3. □

References

[1] R. A. Adams, J. J. Fournier; *Sobolev spaces.* Elsevier, 2003.

[2] A. Ambrosetti, P.H. Rabinowitz; *Dual variational methods in critical point theory and applications,* J. Funct. Anal., 14 (1973), 349–381

[3] T. Bartsch, S. Li; *Critical point theory for asymptotically quadratic functionals and applications to problems with resonance*, Nonlinear Anal., 28 (1997), 419–441.

[4] K. C. Chang; *Infinite-dimensional Morse theory and multiple solution problems*, Progress in Nonlinear Differential Equations and their Applications 6 (Birkhäuser, Boston, 1993).

[5] Y. Chen, P. McKenna; *Traveling waves in a nonlinear suspension beam: theoretical results and numerical observations*, J. Differ. Equ. 137 (1997), 325-355.
[6] J. M. Do Ó, E. Medeiros, U. Severo; *A Nonhomogeneous elliptic problem involving critical growth in dimension two*, J. Math. Anal. Appl. 345, No. 1 (2008), 286–304.
[7] A. Lazer, P. McKenna; *Large-Amplitude Periodic Oscillations in Suspension Bridges: Some New Connections with Nonlinear Analysis.* SIAM Rev. 32 (1990), 537-578.
[8] S. B. Li, Z. H. Zhao; *Solutions for fourth order elliptic equations on $\mathbb{R}^N$ involving $u\Delta(u^2)$ and sign-changing potentials*, J. Differential Equations 267 (2019), No. 3, 1581–1599.
[9] J. Q. Liu; *The Morse index of a saddle point*, J. Systems Sci. Math. Sci. 2 (1989), 32–39.
[10] P. J. McKenna, W. Walter; *Travelling waves in a suspension bridge.* SIAM J. Appl. Math. 50 (1990), 703–715.
[11] P. Rabinowitz; *Minimax methods in critical point theory with applications to differential equations*, in: CBMS Regional Conf. Ser. in. Math., vol. 65, American Mathematical Society, Providence, RI, 1986.
[12] F. Sani; *A biharmonic equation in $\mathbb{R}^4$ involving nonlinearities with critical exponential growth*, Commun. Pure Appl. Anal., 12 (2013), 405–428
[13] F. Sani; *A biharmonic equation in $\mathbb{R}^4$ involving nonlinearities with subcritical exponential growth*, Adv. Nonlinear Stud., 11 (2011), no. 4, 889–904.
[14] Z. Wang, H.-S. Zhou; *Positive solution for a nonlinear stationary Schrödinger-Poisson system in $\mathbb{R}^3$*, Discrete Contin. Dvn. Syst., 18 (2007). 809–816.
[15] M. Willem; *Minimax theorems*, Birkhäuser, Boston, 1996.

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