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CAUCHY PROBLEM FOR THE LANE-EMDEN HEAT FLOW WITH SIGN-CHANGING INITIAL DATA

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Abstract. This article concerns the blow-up phenomenon of sign-changing solutions to the Lane-Emden heat flow. We construct sign-changing weak sub-solutions and localization of the positive and negative parts of the sign-changing solutions. We also extend the results to a nonlinear and finite diffusion equations.

1. Introduction

We consider the initial-boundary value problem for n-dimensional evolutionary Lane-Emden heat flow

$$\begin{gathered} \frac{\partial u}{\partial t} = \Delta u + |u|^{p-1}u, \quad x \in \Omega,\ t > 0, \\ u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{gathered} \tag{1.1}$$

where $p > 1$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, $u_0 \in L^\infty(\Omega)$ is non-negative or sign-changing. The closely related stationary problem of (1.1) is

$$\begin{gathered} -\Delta u = |u|^{p-1}u, \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega, \quad u(x) > 0,\ x \in \Omega. \end{gathered} \tag{1.2}$$

For a convex domain or a star-shaped domain Ω, it was known by Pohozaev [7] that the positive solution of (1.2) exists if and only if $1 < p < 2^*$, where

$$2^* := \begin{cases} +\infty, & n = 1, 2, \\ \frac{n+2}{n-2}, & n \geq 3. \end{cases}$$

Especially for the radially symmetric case such that $\Omega = B_R$, the stationary problem (1.2) is called the Lane-Emden problem arising from the study of stellar interiors [1, 6]. If the domain Ω is not star-shaped, an annulus $B_{2R} \backslash B_R$ for example, positive solutions may still exist for super-critical case $p \geq 2^*$, see [2, 10] and the references therein. For periodic problems with periodic coefficients related to (1.1), we refer the readers to Esteban [3, 4], Quittner [8], Yin and Jin [12].

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Hereafter, we always assume that $p > 1$ and the stationary problem (1.2) admits a positive solution, denoted by u_Ω as it depends on Ω. Therefore, if Ω is star-shaped, we assume that $1 < p < 2^*$.

It is worth noticing that, from Quittner [9, Theorem 17.8], there is no globally existent solution "above" the equilibrium u_Ω, namely the solutions with $u_0(x) \geq u_\Omega(x)$ and $u_0(x) \not\equiv u_\Omega(x)$. This paper aims to show that the blow-up phenomenon is also valid for sign-changing solutions. For any sign-changing function $f(x)$, $x \in \Omega$, we denote

$$f^+(x) := \max\{f(x), 0\}, \quad f^-(x) := \max\{-f(x), 0\}, \tag{1.3}$$

such that $f(x) = f^+(x) - f^-(x)$ and $0 \leq f^\pm(x) \leq |f(x)|$. Further, we define

$$\begin{gathered} \Omega_f^+ := \{x \in \Omega; f(x) > 0\}, \quad \Omega_f^- := \{x \in \Omega; f(x) < 0\}, \\ \Gamma_f := \{x \in \Omega; f(x) = 0\}. \end{gathered} \tag{1.4}$$

Clearly, Ω_f^+, Ω_f^-, and Γ_f are disjoint subsets of Ω such that $\Omega = \Omega_f^+ \cup \Omega_f^- \cup \Gamma_f$, and supp $f^\pm = \overline{\Omega}_f^\pm$.

It may happen that Γ_f has interior points and both Ω_f^+ and Ω_f^- are non-empty. If we take such kind of functions as initial data, meaning that $u_0(x) \equiv 0$ in some open subset $G \subset \Omega$ and $u_0^\pm \not\equiv 0$, according to the smoothing effect of the parabolic operator we know that $\Omega_{u(x,t)}^\pm$ are non-empty but $\Gamma_{u(x,t)}$ has no interior points for small time interval $t \in (0, \delta)$. Therefore, the interface between $\Omega_{u(x,t)}^+$ and $\Omega_{u(x,t)}^-$ is not continuous as $t \to 0^+$, which shows one of the main difficulties arising in the study of asymptotic behaviors of sign-changing solutions to parabolic problems.

Another example illustrating the complexity of large time behaviors of sign-changing solutions is a special case with geometric symmetry: If Ω is symmetric with respect to a hyperplane Π (taking $\Pi := \{x \in \mathbb{R}^n; x_1 = 0\}$ for simplicity), i.e., $x = (x_1, x_2, \cdots, x_n) \in \Omega$ if and only if $\hat{x} := (-x_1, x_2, \cdots, x_n) \in \Omega$, and the initial data $u_0(x)$ is antisymmetric with respect to the same hyperplane Π, i.e., $u_0(\hat{x}) = -u_0(x)$, then the solution $u(x,t)$ is also antisymmetric with respect to Π. Further if $u_0(x) \geq, \not\equiv u_{\Omega^+}(x)$ for $x \in \Omega^+$, where $\Omega^\pm := \{x \in \Omega; \pm x_1 > 0\}$, then the solution $u(x,t)$ blows up to positive infinity on Ω^+ and blows up to negative infinity on Ω^- at the same time.

Observing the above two phenomena, we present the following asymptotic behavior of sign-changing solutions to the initial-boundary-value problem (1.1) with conditions on the initial data such that the solution cannot blow up to negative infinity.

Theorem 1.1. *Assume that $u_0 \in L^\infty(\Omega)$ is sign-changing and satisfies the following conditions: there exist disjoint non-empty open subsets $\hat{\Omega}_{u_0}^\pm \subset \Omega$ and relatively closed subset $\hat{\Gamma}_{u_0} \subset \Omega$ such that*

$$\Omega_{u_0}^- \subset \hat{\Omega}_{u_0}^-, \quad \hat{\Omega}_{u_0}^+ \subset \Omega_{u_0}^+, \quad \hat{\Gamma}_{u_0} = \partial\hat{\Omega}_{u_0}^+ \cap \partial\hat{\Omega}_{u_0}^- \cap \Omega, \quad \Omega = \hat{\Omega}_{u_0}^+ \cup \hat{\Omega}_{u_0}^- \cup \hat{\Gamma}_{u_0}, \tag{1.5}$$

we further assume that the boundary of $\hat{\Omega}_{u_0}^\pm$ is piecewise smooth and the unit normal vector at $x \in \hat{\Gamma}_{u_0}$ is denoted by ν pointing in the direction from $\hat{\Omega}_{u_0}^+$ to $\hat{\Omega}_{u_0}^-$, the stationary Lane-Emden problem (1.2) on $\hat{\Omega}_{u_0}^\pm$ admits positive solutions $u_{\hat{\Omega}_{u_0}^\pm}$ (zero extended to Ω). Moreover,

$$u_0^-(x) \leq, \not\equiv u_{\hat{\Omega}_{u_0}^-}(x), \ x \in \hat{\Omega}_{u_0}^-, \quad u_0^+(x) \geq, \not\equiv u_{\hat{\Omega}_{u_0}^+}(x), \ x \in \hat{\Omega}_{u_0}^+, \tag{1.6}$$

and

$$\Big(\frac{\partial}{\partial\nu}u_{\hat{\Omega}^+_{u_0}}\Big)^- \le \Big(\frac{\partial}{\partial\nu}u_{\hat{\Omega}^-_{u_0}}\Big)^+, \quad x\in\hat{\Gamma}_{u_0}, \tag{1.7}$$

where $\big(\frac{\partial f}{\partial\nu}\big)^\pm$ *denotes the one-side directional derivative of* $f(x)$ *with respect to* ν. *Under the above conditions, we have that*

(i) *the solution* $u(x,t)\ge u_{\hat{\Omega}^+_{u_0}}(x)-u_{\hat{\Omega}^-_{u_0}}(x)$ *for all* $t\in(0,T_{\max})$, *where* $T_{\max}\in(0,+\infty)$ *is the maximal existence time of* $u(x,t)$*;*

(ii) *the solution* $u(x,t)$ *blows up to positive infinity in finite time*

$$\lim_{t\to T^-_{\max}}\sup_{x\in\Omega}u(x,t)=+\infty.$$

According to the blow-up rate estimates established by Giga-Matsui-Sasayama [5] in convex domain with $1<p<2^*$, it holds

$$\|u(x,t)\|_{L^\infty(\Omega)}\le C(T_{\max}-t)^{-\frac{1}{p-1}}, \quad \text{for } t\in(0,T_{\max}).$$

Our result in Theorem 1.1 shows that the Lane-Emden problem does admit a solution blowing up in finite time with sign-changing initial data.

The main idea of the proof is the localization of positive and negative parts of the sign-changing solutions based on the comparison with weak sub-solutions.

We present a simple one-dimensional example of initial data satisfying the conditions in Theorem 1.1: let $\Omega=(a,b)$, $c\in[\frac{a+b}{2},b)$, and

$$u_0(x):=\begin{cases}\lambda(x-a)(x-c), & x\in(a,c),\\ -\mu(x-c)(x-b), & x\in(c,b),\end{cases}$$

where $\lambda>0$ is sufficiently large and $\mu>0$ is sufficiently small. Then, according to Theorem 1.1, the solution blows up to positive infinity in finite time.

The rest of this articleis organized as follows. In Section 2, we construct weak sub-solutions, localize the positive and negative parts of the sign-changing solutions, and utilize them to provide a proof of the main theorem. In Section 3, we will extend the method developed in Section 2 to study nonlinear diffusion equations with finite propagation speed and obtain their sign-changing blow-up solutions.

2. Proof of the main results

The key ingredient is the construction and verification of weak sub-solutions, which makes it possible to localize the positive and negative parts of the sign-changing solution.

Lemma 2.1. *Under the assumptions in Theorem 1.1, the sign-changing function*

$$\underline{u}(x):=u_{\hat{\Omega}^+_{u_0}}(x)-u_{\hat{\Omega}^-_{u_0}}(x)$$

is a weak sub-solution to the initial-boundary-value problem (1.1)*, where* $u_{\hat{\Omega}^\pm_{u_0}}(x)$ *is the positive solution (with zero extension to* Ω*) of the stationary Lane-Emden problem* (1.2) *on* $\hat{\Omega}^\pm_{u_0}$.

Proof. According to condition (1.6),

$$\underline{u}(x)=u_{\hat{\Omega}^+_{u_0}}(x)-u_{\hat{\Omega}^-_{u_0}}(x)\le u_0^+(x)-u_0^-(x)=u_0(x)$$

for $x \in \Omega$, and $\underline{u}(x)=0$ for $x \in \partial\Omega$. We only need to show the differential inequality

$$\int_\Omega \nabla \underline{u}(x)\cdot\nabla\varphi(x)\mathrm{d}x \le \int_\Omega \underline{u}^p(x)\varphi(x)\mathrm{d}x, \tag{2.1}$$

for any $0 \le \varphi(x) \in C_0^\infty(\Omega)$. We employ the partition of unity such that

$$\varphi(x) \equiv \chi^+(x)\varphi(x)+\chi^-(x)\varphi(x)+\chi^0(x)\varphi(x) =: \varphi^+(x)+\varphi^-(x)+\varphi^0(x), \tag{2.2}$$

where $\operatorname{supp}\chi^\pm \subset \hat\Omega^\pm_{u_0}$, $0\le \chi^+(x),\chi^-(x),\chi^0(x)\le 1$ are smooth functions. For the differential inequality (2.1) supported in $\hat\Omega^\pm_{u_0}$, we have

$$\begin{aligned}\int_\Omega \nabla\underline{u}(x)\cdot\nabla\varphi^\pm(x)\mathrm{d}x &= \int_{\hat\Omega^\pm_{u_0}} -\Delta\underline{u}(x)\cdot\varphi^\pm(x)\mathrm{d}x\\ &= \int_{\hat\Omega^\pm_{u_0}} \underline{u}^p(x)\cdot\varphi^\pm(x)\mathrm{d}x = \int_\Omega \underline{u}^p(x)\cdot\varphi^\pm(x)\mathrm{d}x.\end{aligned} \tag{2.3}$$

If we take $\chi^\pm(x)$ sufficiently large such that $\hat\Omega^\pm_{u_0}\setminus\operatorname{supp}\chi^\pm$ is sufficiently "narrow", then $\operatorname{supp}\chi^0$ is sufficiently close to $\hat\Gamma_{u_0}$, meaning that there exists $\varepsilon>0$ (sufficiently small) such that

$$\operatorname{supp}\chi^0 \subset \hat\Gamma^\varepsilon_{u_0} := \{x\in\Omega : \operatorname{dist}(x,\hat\Gamma_{u_0})<\varepsilon\}.$$

Therefore, near any point $x\in\hat\Gamma_{u_0}$, noticing that $u_{\hat\Omega^\pm_{u_0}}(x)=0$ for $x\in\hat\Gamma_{u_0}$, the gradient $\nabla\underline{u}(x)$ in the differential inequality (2.1) can be approximated by $(\frac{\partial}{\partial\nu}u_{\hat\Omega^+_{u_0}})^-\cdot\nu$ for the negative side of $\hat\Gamma_{u_0}$ and approximated by $(\frac{\partial}{\partial\nu}u_{\hat\Omega^-_{u_0}})^+\cdot\nu$ for positive side of $\hat\Gamma_{u_0}$, where ν is the unit normal vector at $x\in\hat\Gamma_{u_0}$ pointing in the direction from $\hat\Omega^+_{u_0}$ to $\hat\Omega^-_{u_0}$. That is,

$$\nabla\underline{u}(x) = \Big(\frac{\partial}{\partial\nu}u_{\hat\Omega^+_{u_0}}\Big)^-\cdot\nu\cdot\chi_{\hat\Omega^+_{u_0}}(x) + \Big(\frac{\partial}{\partial\nu}u_{\hat\Omega^-_{u_0}}\Big)^+\cdot\nu\cdot\chi_{\hat\Omega^-_{u_0}}(x)+o(1), \quad x\in\hat\Gamma^\varepsilon_{u_0},$$

where $\chi_{\hat\Omega^\pm_{u_0}}(x)$ is the characteristic function of the set $\hat\Omega^\pm_{u_0}$, and $o(1)$ is an infinitesimal as $\varepsilon\to0^+$. Then we have

$$\begin{aligned}&\int_\Omega \nabla\underline{u}(x)\cdot\nabla\varphi^0(x)\mathrm{d}x\\ &= \int_{\hat\Gamma^\varepsilon_{u_0}} \nabla\underline{u}(x)\cdot\nabla\varphi^0(x)\mathrm{d}x\\ &= \int_{\hat\Gamma^\varepsilon_{u_0}\cap\hat\Omega^+_{u_0}} \nabla\underline{u}(x)\cdot\nabla\varphi^0(x)\mathrm{d}x + \int_{\hat\Gamma^\varepsilon_{u_0}\cap\hat\Omega^-_{u_0}} \nabla\underline{u}(x)\cdot\nabla\varphi^0(x)\mathrm{d}x\\ &= \int_{\hat\Gamma^\varepsilon_{u_0}\cap\hat\Omega^+_{u_0}} \Big(\frac{\partial}{\partial\nu}u_{\hat\Omega^+_{u_0}}\Big)^-\cdot\frac{\partial}{\partial\nu}\varphi^0(x)\mathrm{d}x\\ &\quad+\int_{\hat\Gamma^\varepsilon_{u_0}\cap\hat\Omega^-_{u_0}} \Big(\frac{\partial}{\partial\nu}u_{\hat\Omega^-_{u_0}}\Big)^+\cdot\frac{\partial}{\partial\nu}\varphi^0(x)\mathrm{d}x+o(1)\\ &= \int_{\hat\Gamma_{u_0}} \Big(\frac{\partial}{\partial\nu}u_{\hat\Omega^+_{u_0}}\Big)^-\cdot\varphi^0(x)\mathrm{d}x - \int_{\hat\Gamma_{u_0}} \Big(\frac{\partial}{\partial\nu}u_{\hat\Omega^-_{u_0}}\Big)^+\cdot\varphi^0(x)\mathrm{d}x+o(1).\end{aligned} \tag{2.4}$$

On the other hand,

$$\int_\Omega \underline{u}^p(x)\varphi^0(x)\mathrm{d}x = \int_{\hat\Gamma^\varepsilon_{u_0}} \underline{u}^p(x)\varphi^0(x)\mathrm{d}x = O(\varepsilon), \quad \text{as } \varepsilon\to0^+, \tag{2.5}$$

since the measure $\mathrm{meas}(\hat{\Gamma}^{\varepsilon}_{u_0}) = O(\varepsilon)$. Combining the above estimates (2.3), (2.4), (2.5), and noticing that

$$\Big(\frac{\partial}{\partial\nu}u_{\hat{\Omega}^+_{u_0}}\Big)^- \le \Big(\frac{\partial}{\partial\nu}u_{\hat{\Omega}^-_{u_0}}\Big)^+,$$

for $x \in \hat{\Gamma}_{u_0}$ according to the condition (1.7), letting ε tends to zero, we see that the differential inequality (2.1) is valid and thus $\underline{u}(x)$ is a weak sub-solution to problem (1.1). Similar computation of second order generalized derivatives $\Delta\underline{u}(x)$ of piecewise continuous function $\underline{u}(x)$ can be found in [11]. □

Compared with the above weak sub-solution, we can localize the positive part $u^+(x,t)$ and the negative part $u^-(x,t)$ of the sign-changing solution $u(x,t)$ to the subsets $\Omega^{\pm}_{u(x,t)}$ respectively (see notations (1.3) and (1.4) for the meaning of symbols $u^{\pm}(x,t)$ and $\Omega^{\pm}_{u(x,t)}$). Comparison principle of the heat equation implies that $u(x,t) \ge \underline{u}(x)$ for $t \in (0, T_{\max})$, which means that $\hat{\Omega}^+_{u_0} \subset \Omega^+_{u(x,t)}$ and $\Omega^-_{u(x,t)} \subset \hat{\Omega}^-_{u_0}$. Therefore, we define the localized part $u_{\pm}(x,t)$ of $u(x,t)$ as follows:

$$u_+(x,t) := u(x,t)\cdot\chi_{\hat{\Omega}^+_{u_0}}(x), \quad u_-(x,t) := -u(x,t)\cdot\chi_{\hat{\Omega}^-_{u_0}}(x), \tag{2.6}$$

where $\chi_{\hat{\Omega}^{\pm}_{u_0}}(x)$ is the characteristic function of the set $\hat{\Omega}^{\pm}_{u_0}$. Then

$$u(x,t) = u_+(x,t) - u_-(x,t) = u^+(x,t) - u^-(x,t),$$

and $u^+(x,t) \ge u_+(x,t)$, $u^-(x,t) \ge u_-(x,t)$, $u^{\pm}(x,t)$ coincides with $u_{\pm}(x,t)$ in $\hat{\Omega}^{\pm}_{u_0} \cap \Omega^{\pm}_{u(x,t)}$. However, since $\hat{\Omega}^+_{u_0} \subset \Omega^+_{u(x,t)}$ and $\Omega^-_{u(x,t)} \subset \hat{\Omega}^-_{u_0}$, we have

$$0 = u_+(x,t) < u^+(x,t) = u(x,t), \quad -u(x,t) = u_-(x,t) < u^-(x,t) = 0,$$

for all $x \in \Omega^+_{u(x,t)} \backslash \hat{\Omega}^+_{u_0}$.

Lemma 2.2. *The localized part $u_+(x,t)$ satisfies*

$$\begin{gathered}\frac{\partial u_+}{\partial t} = \Delta u_+ + |u_+|^{p-1}u_+, \quad x \in \hat{\Omega}^+_{u_0},\ t > 0,\\ u_+(x,t) \ge 0, \quad x \in \partial\hat{\Omega}^+_{u_0},\ t > 0,\\ u_+(x,0) = u_0^+(x) \ge, \not\equiv u_{\hat{\Omega}^+_{u_0}}(x), \quad x \in \hat{\Omega}^+_{u_0},\end{gathered} \tag{2.7}$$

and the localized part $u_-(x,t)$ satisfies

$$\begin{gathered}\frac{\partial u_-}{\partial t} = \Delta u_- + |u_-|^{p-1}u_-, \quad x \in \hat{\Omega}^-_{u_0},\ t > 0,\\ u_-(x,t) \le 0, \quad x \in \partial\hat{\Omega}^-_{u_0},\ t > 0,\\ u_-(x,0) = u_0^-(x) \le, \not\equiv u_{\hat{\Omega}^-_{u_0}}(x), \quad x \in \hat{\Omega}^-_{u_0}.\end{gathered} \tag{2.8}$$

Proof. The properties of the localized part $u_{\pm}(x,t)$ follow from the definition (2.6) and the comparison principle of the heat equation compared with the weak sub-solution $\underline{u}(x)$ proved in Lemma 2.1. □

Now that we have localized the solution $u(x,t)$ to disjoint subsets $\hat{\Omega}^{\pm}_{u_0}$, we can analyze the asymptotic behavior of different localized parts $u_{\pm}(x,t)$ separately, following the similar line as Quittner and Souplet [9, Theorem 17.8].

Lemma 2.3. *The localized part $u_+(x,t)$ blows up in finite time, i.e., $T_{\max} < +\infty$.*

Proof. We denote $v(x) := u_{\hat{\Omega}^+_{u_0}}(x)$ for simplicity in this proof. We define

$$z(t) := \int_{\hat{\Omega}^+_{u_0}} u_+(x,t) \cdot u_{\hat{\Omega}^+_{u_0}}(x)\mathrm{d}x = \int_{\hat{\Omega}^+_{u_0}} u_+(x,t) \cdot v(x)\mathrm{d}x.$$

Noticing that $v(x) = 0$ and $\frac{\partial v(x)}{\partial \nu} < 0$ for $x \in \partial\hat{\Omega}^+_{u_0}$, we have

$$\begin{aligned} z'(t) &= \int_{\hat{\Omega}^+_{u_0}} \frac{\partial u_+(x,t)}{\partial t} \cdot v(x)\mathrm{d}x \\ &= \int_{\hat{\Omega}^+_{u_0}} \Delta u_+(x,t) \cdot v(x)\mathrm{d}x + \int_{\hat{\Omega}^+_{u_0}} u_+^p(x,t) \cdot v(x)\mathrm{d}x \\ &= -\int_{\hat{\Omega}^+_{u_0}} \nabla u_+(x,t) \cdot \nabla v(x)\mathrm{d}x + \int_{\hat{\Omega}^+_{u_0}} u_+^p(x,t) \cdot v(x)\mathrm{d}x \\ &= \int_{\hat{\Omega}^+_{u_0}} u_+(x,t) \cdot \Delta v(x)\mathrm{d}x - \int_{\partial\hat{\Omega}^+_{u_0}} u_+(x,t)\frac{\partial v(x)}{\partial \nu}\mathrm{d}x \\ &\quad + \int_{\hat{\Omega}^+_{u_0}} u_+^p(x,t) \cdot v(x)\mathrm{d}x \\ &\geq -\int_{\hat{\Omega}^+_{u_0}} u_+(x,t) \cdot v^p(x)\mathrm{d}x + \int_{\hat{\Omega}^+_{u_0}} u_+^p(x,t) \cdot v(x)\mathrm{d}x \\ &= \int_{\hat{\Omega}^+_{u_0}} \Big(1 - \Big(\frac{u_+(x,t)}{v(x)}\Big)^{1-p}\Big) u_+^p(x,t) \cdot v(x)\mathrm{d}x. \end{aligned} \tag{2.9}$$

Similar to the proof of [9, Lemma 17.9], for each fixed $\tau \in (0, T_{\max})$, there exists a constant $\alpha > 1$ such that

$$u_+(x,t) \geq \alpha v(x), \quad x \in \hat{\Omega}^+_{u_0},\ t \in (\tau, T_{\max}),$$

since $u_+(x,0) \geq, \not\equiv v(x)$. Applying Jensen's inequality to (2.9), we have

$$\begin{aligned} z'(t) &\geq (1 - \alpha^{1-p}) \int_{\hat{\Omega}^+_{u_0}} u_+^p(x,t) \cdot v(x)\mathrm{d}x \\ &\geq (1 - \alpha^{1-p}) \Big(\int_{\hat{\Omega}^+_{u_0}} u_+(x,t) \cdot v(x)\mathrm{d}x\Big)^p \Big(\int_{\hat{\Omega}^+_{u_0}} v(x)\mathrm{d}x\Big)^{1-p} \\ &= (1 - \alpha^{1-p}) \cdot \Big(\int_{\hat{\Omega}^+_{u_0}} v(x)\mathrm{d}x\Big)^{1-p} z^p(t), \quad t \in (\tau, T_{\max}), \end{aligned} \tag{2.10}$$

which implies that $z(t)$ blows up in finite time since $p > 1$ and $z(\tau) > 0$. □

Proof of Theorem 1.1. By the comparison principle and the weak sub-solution proved in Lemma 2.1, we know that the solution is bounded from below such that $u(x,t) \geq \underline{u}(x)$ for all $t \in (0, T_{\max})$. Lemma 2.3 shows that the localized part $u_+(x,t)$ blows up in finite time, which also implies that $u(x,t)$ blows up in finite time and $T_{\max} < +\infty$. □

In this section, we extend our results to the Lane-Emden heat flow with general nonlinear terms of the form

$$\begin{gathered}\frac{\partial u}{\partial t}=\Delta u+f(u), \quad x\in\Omega,\ t>0,\\ u(x,t)=0, \quad x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), \quad x\in\Omega,\end{gathered} \tag{2.11}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded domain with smooth boundary, $u_0\in L^\infty(\Omega)$ is nonnegative or sign-changing, and f is a C^1-function with a superlinear growth.

Theorem 2.4. *Assume that f is a convex C^1-function with $f(0)=0$, f' is nonconstant near 0. Then, under the assumptions in Theorem 1.1, we have*

(i) *the solution $u(x,t)\geq u_{\hat\Omega^+_{u_0}}(x)-u_{\hat\Omega^-_{u_0}}(x)$ for all $t\in(0,T_{\max})$, where $T_{\max}\in(0,+\infty)$ is the maximal existence time of $u(x,t)$;*

(ii) *the solution $u(x,t)$ blows up to positive infinity in finite time*

$$\lim_{t\to T^-_{\max}}\sup_{x\in\Omega}u(x,t)=+\infty.$$

Proof. The proof of this theorem is similar to that of Theorem 1.1. For simplicity, we only present the key and distinct parts. We firstly note that the conditions on the function f in Theorem 2.4 ensure that equation (2.11) has a positive (classical) equilibrium v_0 (see [9, Theorem 17.10]), which will help us construct a sub-solution similar to that in Lemma 2.1. Then, by the comparison principle compared with the sub-solution, we can localize the sign-changing solution $u(x,t)$ to disjoint subsets $\hat\Omega^\pm_{u_0}$, getting the positive part $u^+(x,t)$ and the negative part $u^-(x,t)$. Finally, similar to Lemma 2.3, we prove that the positive part $u^+(x,t)$ blows up in finite time. The key inequality is as follows.

$$\begin{aligned} z'(t)&=\int_{\hat\Omega^+_{u_0}}\frac{\partial u_+(x,t)}{\partial t}\cdot v(x)\mathrm{d}x\\ &=\int_{\hat\Omega^+_{u_0}}\Delta u_+(x,t)\cdot v(x)\mathrm{d}x+\int_{\hat\Omega^+_{u_0}}f(u_+(x,t))\cdot v(x)\mathrm{d}x\\ &=-\int_{\hat\Omega^+_{u_0}}\nabla u_+(x,t)\cdot\nabla v(x)\mathrm{d}x+\int_{\hat\Omega^+_{u_0}}f(u_+(x,t))\cdot v(x)\mathrm{d}x\\ &=\int_{\hat\Omega^+_{u_0}}u_+(x,t)\cdot\Delta v(x)\mathrm{d}x-\int_{\partial\hat\Omega^+_{u_0}}u_+(x,t)\frac{\partial v(x)}{\partial\nu}\mathrm{d}x\\ &\quad+\int_{\hat\Omega^+_{u_0}}f(u_+(x,t))\cdot v(x)\mathrm{d}x\\ &\geq-\int_{\hat\Omega^+_{u_0}}u_+(x,t)\cdot f(v(x))\mathrm{d}x+\int_{\hat\Omega^+_{u_0}}f(u_+(x,t))\cdot v(x)\mathrm{d}x\\ &=\int_{\hat\Omega^+_{u_0}}\Big(1-\frac{f(v(x))u_+(x,t)}{f(u_+(x,t))v(x)}\Big)f(u_+(x,t))v(x)\mathrm{d}x. \end{aligned} \tag{2.12}$$

Based on the convexity of function f and inequality ([9, Lemma 17.9])

$$u_+(x,t)\geq\alpha v(x), \quad x\in\hat\Omega^+_{u_0},\ t\in(\tau,T_{\max}),$$

we can obtain

$$\begin{aligned} z'(t) &\geq \big(1 - \frac{1}{\alpha}\big) \int_{\hat{\Omega}_{u_0}^+} f(u_+(x,t)) \cdot v(x) \mathrm{d}x \\ &= \big(1 - \frac{1}{\alpha}\big) \frac{1}{\int_{\hat{\Omega}_{u_0}^+} v(x)\mathrm{d}x} f(z(t)), \quad t \in (\tau, T_{\max}), \end{aligned} \tag{2.13}$$

with $\alpha > 1$. This completes the proof. □

3. Extension

In the previous section, we considered the blow-up phenomenon of sign-changing solutions to the linear diffusion equations. In this section, we will focus on the case of nonlinear diffusion. As an example, we consider the following porous media equation with finite diffusion speed,

$$\begin{gathered} \frac{\partial u}{\partial t} = \Delta u^m + |u|^{p-1}u, \quad x \in \Omega,\ t > 0, \\ u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{gathered} \tag{3.1}$$

where $p > 1$, $m > 1$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, $u_0 \in L^\infty(\Omega)$ is non-negative or sign-changing. Porous medium equations serve as fundamental models for describing fluid flow, heat transfer, and mass diffusion processes within porous materials, such as soil, rocks, and biological tissues. The classical heat equation exhibits infinite propagation speed, which may not align with real-world scenarios. In contrast, porous medium equations with $m > 1$ offer a more realistic representation as they exhibit finite propagation speed. These equations are widely studied due to their relevance in fields like hydrology, petroleum engineering, environmental science, and biomedicine.

A closely related stationary problem to (3.1) is

$$\begin{gathered} -\Delta u^m = |u|^{p-1}u, \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega, \quad u(x) > 0, \quad x \in \Omega. \end{gathered} \tag{3.2}$$

By the transform $w = u^m$, (3.2) can be rewritten as

$$\begin{gathered} -\Delta w = |w|^{\frac{p}{m}-1}w, \quad x \in \Omega, \\ w(x) = 0, \quad x \in \partial\Omega, \quad w(x) > 0, \quad x \in \Omega. \end{gathered} \tag{3.3}$$

It follows from the results in [7] that a positive solution of (3.3) exists if and only if $\frac{p}{2^*} < m < p$, where

$$2^* := \begin{cases} +\infty, & n = 1,2, \\ \frac{n+2}{n-2}, & n \geq 3. \end{cases}$$

Thus, under the assumption

$$\begin{gathered} 0 < m < p < +\infty, \quad n = 1,2, \\ \frac{n-2}{n+2} < m < p < \frac{n+2}{n-2}, \quad n \geq 3, \end{gathered} \tag{3.4}$$

equation (3.3) admits positive solution.

We denote $w_0 = u_0^m$. The following theorem shows the blow-up phenomenon for the porous medium equation (3.1).

Theorem 3.1. *Assume that $u_0 \in L^\infty(\Omega)$ is sign-changing and satisfies the following conditions: there exist disjoint non-empty open subsets $\hat{\Omega}^{\pm}_{w_0} \subset \Omega$ and relatively closed subset $\hat{\Gamma}_{w_0} \subset \Omega$ such that*

$$\Omega^-_{u_0} \subset \hat{\Omega}^-_{w_0}, \quad \hat{\Omega}^+_{w_0} \subset \Omega^+_{u_0}, \quad \hat{\Gamma}_{w_0} = \partial\hat{\Omega}^+_{w_0} \cap \partial\hat{\Omega}^-_{w_0} \cap \Omega, \quad \Omega = \hat{\Omega}^+_{w_0} \cup \hat{\Omega}^-_{w_0} \cup \hat{\Gamma}_{w_0}, \tag{3.5}$$

we further assume that the boundary of $\hat{\Omega}^{\pm}_{w_0}$ is piecewise smooth and the unit normal vector at $x \in \hat{\Gamma}_{w_0}$ is denoted by ν pointing in the direction from $\hat{\Omega}^+_{w_0}$ to $\hat{\Omega}^-_{w_0}$, the stationary problem (3.3) on $\hat{\Omega}^{\pm}_{w_0}$ admits positive solutions $w_{\hat{\Omega}^{\pm}_{w_0}}$ (zero extended to Ω). Moreover,

$$u_0^-(x) \le, \not\equiv w^{\frac{1}{m}}_{\hat{\Omega}^-_{w_0}}(x), \ x \in \hat{\Omega}^-_{u_0}, \quad u_0^+(x) \ge, \not\equiv w^{\frac{1}{m}}_{\hat{\Omega}^+_{u_0}}(x), \ x \in \hat{\Omega}^+_{u_0}, \tag{3.6}$$

$$\Big(\frac{\partial}{\partial\nu} w_{\hat{\Omega}^+_{w_0}}\Big)^- \le \Big(\frac{\partial}{\partial\nu} w_{\hat{\Omega}^-_{w_0}}\Big)^+, \quad x \in \hat{\Gamma}_{w_0}, \tag{3.7}$$

where $\big(\frac{\partial f}{\partial\nu}\big)^{\pm}$ denotes the one-side directional derivative of $f(x)$ with respect to ν. Under the above conditions, we have that

- (i) *the solution $u(x,t) \ge w^{\frac{1}{m}}_{\hat{\Omega}^+_{w_0}}(x) - w^{\frac{1}{m}}_{\hat{\Omega}^-_{w_0}}(x)$ for all $t \in (0, T_{\max})$, where $T_{\max} \in (0, +\infty)$ is the maximal existence time of $u(x,t)$;*
- (ii) *the solution $u(x,t)$ blows up to positive infinity in finite time*

$$\lim_{t \to T^-_{\max}} \sup_{x \in \Omega} u(x,t) = +\infty.$$

The main idea of the proof of Theorem 3.1 is similar to Theorem 1.1. We will initially construct a weak sub-solution to equation (3.1), then localize the positive and negative parts of the sign-changing solution based on the comparison with this weak sub-solution. The main difference in the proof lies in considering the weak solutions for the problem considered in this section. Firstly, we construct the weak sub-solution.

Lemma 3.2. *Under the assumptions in Theorem 3.1, the sign-changing function*

$$\tilde{u}(x) := w^{\frac{1}{m}}_{\hat{\Omega}^+_{w_0}}(x) - w^{\frac{1}{m}}_{\hat{\Omega}^-_{w_0}}(x)$$

is a weak sub-solution to the initial-boundary-value problem (3.1), where $u_{\hat{\Omega}^{\pm}_{w_0}}(x)$ is the positive solution (with zero extension to Ω) of the stationary problem (3.3) on $\hat{\Omega}^{\pm}_{w_0}$.

Proof. It follows from condition (3.6) that

$$\tilde{u}(x) = w^{\frac{1}{m}}_{\hat{\Omega}^+_{w_0}}(x) - w^{\frac{1}{m}}_{\hat{\Omega}^-_{w_0}}(x) \le u_0^+(x) - u_0^-(x) = u_0(x)$$

for $x \in \Omega$, and $\tilde{u}(x) = 0$ for $x \in \partial\Omega$. It suffices to demonstrate the differential inequality

$$\int_\Omega \nabla\tilde{u}^m(x) \cdot \nabla\varphi(x) \mathrm{d}x \le \int_\Omega \tilde{u}^p(x)\varphi(x) \mathrm{d}x, \tag{3.8}$$

for any $0 \le \varphi(x) \in C_0^\infty(\Omega)$. Using a similar partition of unity $\varphi^+(x), \varphi^-(x), \varphi^0(x)$ in (2.2), we have

$$\begin{aligned}\int_\Omega \nabla\tilde{u}^m(x)\cdot\nabla\varphi^\pm(x)\mathrm{d}x &= \int_{\hat{\Omega}^\pm_{w_0}} -\Delta\tilde{u}^m(x)\cdot\varphi^\pm(x)\mathrm{d}x \\ &= \int_{\hat{\Omega}^\pm_{w_0}} \tilde{u}^m(x)\cdot\varphi^\pm(x)\mathrm{d}x = \int_\Omega \tilde{u}^p(x)\cdot\varphi^\pm(x)\mathrm{d}x.\end{aligned} \tag{3.9}$$

With the help of the approximation

$$\nabla w(x) = \Big(\frac{\partial}{\partial\nu}w_{\hat{\Omega}^+_{w_0}}\Big)^- \cdot\nu\cdot\chi_{\hat{\Omega}^+_{w_0}}(x) + \Big(\frac{\partial}{\partial\nu}w_{\hat{\Omega}^-_{w_0}}\Big)^+ \cdot\nu\cdot\chi_{\hat{\Omega}^-_{w_0}}(x) + o(1), \quad x\in\hat{\Gamma}^\varepsilon_{w_0},$$

we obtain

$$\begin{aligned}&\int_\Omega \nabla\tilde{u}^m(x)\cdot\nabla\varphi^0(x)\mathrm{d}x \\ &= \int_{\hat{\Gamma}^\varepsilon_{w_0}} \nabla\tilde{u}^m(x)\cdot\nabla\varphi^0(x)\mathrm{d}x \\ &= \int_{\hat{\Gamma}^\varepsilon_{w_0}\cap\hat{\Omega}^+_{w_0}} \nabla w(x)\cdot\nabla\varphi^0(x)\mathrm{d}x + \int_{\hat{\Gamma}^\varepsilon_{w_0}\cap\hat{\Omega}^-_{w_0}} \nabla w(x)\cdot\nabla\varphi^0(x)\mathrm{d}x \\ &= \int_{\hat{\Gamma}^\varepsilon_{w_0}\cap\hat{\Omega}^+_{w_0}} \Big(\frac{\partial}{\partial\nu}w_{\hat{\Omega}^+_{w_0}}\Big)^-\cdot\frac{\partial}{\partial\nu}\varphi^0(x)\mathrm{d}x \\ &\quad + \int_{\hat{\Gamma}^\varepsilon_{w_0}\cap\hat{\Omega}^-_{w_0}} \Big(\frac{\partial}{\partial\nu}w_{\hat{\Omega}^-_{w_0}}\Big)^+\cdot\frac{\partial}{\partial\nu}\varphi^0(x)\mathrm{d}x + o(1) \\ &= \int_{\hat{\Gamma}_{w_0}} \Big[\Big(\frac{\partial}{\partial\nu}w_{\hat{\Omega}^+_{w_0}}\Big)^- - \Big(\frac{\partial}{\partial\nu}w_{\hat{\Omega}^-_{w_0}}\Big)^+\Big]\cdot\varphi^0(x)\mathrm{d}x + o(1).\end{aligned} \tag{3.10}$$

Because $\mathrm{meas}(\hat{\Gamma}^\varepsilon_{w_0}) = O(\varepsilon)$, we have

$$\int_\Omega \tilde{u}^p(x)\varphi^0(x)\mathrm{d}x = \int_{\hat{\Gamma}^\varepsilon_{w_0}} \tilde{u}^p(x)\varphi^0(x)\mathrm{d}x = O(\varepsilon), \quad \text{as } \varepsilon\to 0^+, \tag{3.11}$$

Thus it follows from (3.9), (3.10), (3.11) and the condition (3.7) that the differential inequality (3.8) is valid which implies $\tilde{u}(s)$ is a weak sub-solution to problem (3.1). □

Next, we localize the positive part $u^+(x,t)$ and the negative part $u^-(x,t)$ of the sign-changing solution $u(x,t)$ to the subsets $\Omega^\pm_{u(x,t)}$ respectively.

We define the localized part $u_\pm(x,t)$ of $u(x,t)$ as follows:

$$u_+(x,t) := u(x,t)\cdot\chi_{\hat{\Omega}^+_{w_0}}(x), \quad u_-(x,t) := -u(x,t)\cdot\chi_{\hat{\Omega}^-_{w_0}}(x), \tag{3.12}$$

where $\chi_{\hat{\Omega}^\pm_{w_0}}(x)$ is the characteristic function of the set $\hat{\Omega}^\pm_{w_0}$. The comparison principle of equation (3.1) implies that $u(x,t)\ge\tilde{u}(x)$ for $t\in(0,T_{\max})$, which means that $\hat{\Omega}^+_{w_0}\subset\Omega^+_{u(x,t)}$ and $\Omega^-_{u(x,t)}\subset\hat{\Omega}^-_{w_0}$. Thus we have

$$u(x,t) = u_+(x,t) - u_-(x,t) = u^+(x,t) - u^-(x,t),$$

and $u^+(x,t)\ge u_+(x,t)$, $u^-(x,t)\ge u_-(x,t)$. $u^\pm(x,t)$ coincides with $u_\pm(x,t)$ in $\hat{\Omega}^\pm_{u_0}\cap\Omega^\pm_{u(x,t)}$. However, for all $x\in\Omega^+_{u(x,t)}\backslash\hat{\Omega}^+_{u_0}$, we have

$$0 = u_+(x,t) < u^+(x,t) = u(x,t), \quad -u(x,t) = u_-(x,t) < u^-(x,t) = 0.$$

From the definition (3.12) and the comparison principle for the equation (3.1) compared with the weak sub-solution $\tilde{u}(x)$ proved in Lemma 3.2, we derive the following lemma.

Lemma 3.3. *The localized part $u_+(x,t)$ satisfies*

$$\begin{gathered}\frac{\partial u_+}{\partial t} = \Delta u_+^m + |u_+|^{p-1}u_+, \quad x \in \hat{\Omega}_{w_0}^+,\ t > 0,\\ u_+(x,t) \geq 0, \quad x \in \partial\hat{\Omega}_{w_0}^+,\ t > 0,\\ u_+(x,0) = u_0^+(x) \geq, \not\equiv w_{\hat{\Omega}_{w_0}^+}^{\frac{1}{m}}(x), \quad x \in \hat{\Omega}_{w_0}^+,\end{gathered} \tag{3.13}$$

in the distribution sense and the localized part $u_-(x,t)$ satisfies

$$\begin{gathered}\frac{\partial u_-}{\partial t} = \Delta u_- + |u_-|^{p-1}u_-, \quad x \in \hat{\Omega}_{w_0}^-,\ t > 0,\\ u_-(x,t) \leq 0, \quad x \in \partial\hat{\Omega}_{w_0}^-,\ t > 0,\\ u_-(x,0) = u_0^-(x) \leq, \not\equiv w_{\hat{\Omega}_{w_0}^-}^{\frac{1}{m}}(x), \quad x \in \hat{\Omega}_{w_0}^-,\end{gathered} \tag{3.14}$$

in the distribution sense.

Following the approach in the proof of [9, Lemma 17.9], we derive the subsequent result for (3.1), which will be used in the next lemma.

Lemma 3.4. *Assume that $u_0^{(1)}, u_0^{(2)} \in L^\infty(\Omega)$ and $u_0^{(1)} \geq, \not\equiv u_0^{(2)}$. Let $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ be the corresponding solutions of* (3.1)*, then for any fixed $\tau \in (0, T_{\max})$, there exists a constant $\alpha > 1$ such that*

$$u^{(1)}(x,t) \geq \alpha u^{(2)}(x,t), \quad t \in (\tau, T_{\max}).$$

Finally, with the localized solution $u(x,t)$ to disjoint subsets $\hat{\Omega}_{w_0}^{\pm}$, we establish the following result about the asymptotic behavior of the localized part $u_+(x,t)$, which will complete the proof the Theorem 3.1.

Lemma 3.5. *The localized part $u_+(x,t)$ defined in* (3.12) *blows up in finite time, i.e., $T_{\max} < +\infty$.*

Proof. Inspired by [9], we define

$$z(t) := \int_{\hat{\Omega}_{w_0}^+} u_+(x,t) \cdot u_{\hat{\Omega}_{w_0}^+}(x)\mathrm{d}x = \int_{\hat{\Omega}_{u_0}^+} u_+(x,t) \cdot v(x)\mathrm{d}x,$$

where we denote $v(x) := w_{\hat{\Omega}^+_{w_0}}(x)$ for simplicity. Noticing that $v(x) = 0$ and $\frac{\partial v(x)}{\partial \nu} < 0$ for $x \in \partial\hat{\Omega}^+_{w_0}$, we have

$$
\begin{aligned}
z'(t) &= \int_{\partial\hat{\Omega}^+_{w_0}} \frac{\partial u_+^m(x,t)(x)}{\partial \nu} \cdot v(x)\mathrm{d}x - \int_{\hat{\Omega}^+_{w_0}} \nabla u_+^m(x,t) \cdot \nabla v(x)\mathrm{d}x \\
&\quad + \int_{\hat{\Omega}^+_{w_0}} u_+^p(x,t) \cdot v(x)\mathrm{d}x \\
&= -\int_{\hat{\Omega}^+_{w_0}} \nabla u_+^m(x,t) \cdot \nabla v(x)\mathrm{d}x + \int_{\hat{\Omega}^+_{w_0}} u_+^p(x,t) \cdot v(x)\mathrm{d}x \\
&= \int_{\hat{\Omega}^+_{w_0}} u_+(x,t) \cdot \Delta v(x)\mathrm{d}x - \int_{\partial\hat{\Omega}^+_{w_0}} u_+(x,t)\frac{\partial v(x)}{\partial \nu}\mathrm{d}x \\
&\quad + \int_{\hat{\Omega}^+_{w_0}} u_+^p(x,t) \cdot v(x)\mathrm{d}x \\
&\geq -\int_{\hat{\Omega}^+_{w_0}} u_+^m(x,t) \cdot v^{\frac{p}{m}}(x)\mathrm{d}x + \int_{\hat{\Omega}^+_{w_0}} u_+^p(x,t) \cdot v(x)\mathrm{d}x \\
&= \int_{\hat{\Omega}^+_{w_0}} \Big(u_+^p(x,t) \cdot v(x) - u_+^m(x,t) \cdot v^{\frac{p}{m}}\Big)\mathrm{d}x.
\end{aligned} \tag{3.15}
$$

Taking $u_0^{(1)} = u_+(x,0), u_0^{(2)} = v^{\frac{1}{m}}(x)$ and applying lemma 3.4, we have that for any fixed $\tau \in (0, T_{\max})$, there exists a constant $\alpha > 1$ such that

$$
u_+(x,t) \geq \alpha v^{\frac{1}{m}}(x), \quad \forall x \in \hat{\Omega}^+_{w_0},\ t \in (\tau, T_{\max}),
$$

since $u_+(x,0) \geq, \not\equiv v^{\frac{1}{m}}(x)$. Thus we have from (3.15) that

$$
\begin{aligned}
z'(t) &\geq \int_{\hat{\Omega}^+_{w_0}} \Big(1 - \Big(\frac{u_+(x,t)}{v^{\frac{1}{m}}(x)}\Big)^{m-p}\Big)u_+^p(x,t)v(x)\mathrm{d}x. \\
&\geq (1-\alpha^{m-p})\int_{\hat{\Omega}^+_{w_0}} u_+^p(x,t)v(x)\mathrm{d}x, \quad t \in (\tau, T_{\max}).
\end{aligned}
$$

Employing Jensen's inequality, we derive

$$
\begin{aligned}
z'(t) &\geq (1-\alpha^{m-p})\Big(\int_{\hat{\Omega}^+_{w_0}} u_+(x,t) \cdot v(x)\mathrm{d}x\Big)^p \Big(\int_{\hat{\Omega}^+_{w_0}} v(x)\mathrm{d}x\Big)^{1-p} \\
&= (1-\alpha^{m-p})\Big(\int_{\hat{\Omega}^+_{w_0}} v(x)\mathrm{d}x\Big)^{1-p} z^p(t), \quad t \in (\tau, T_{\max}),
\end{aligned} \tag{3.16}
$$

which implies that $z(t)$ blows up in finite time since $p > 1$ and $z(\tau) > 0$. □

Proof of Theorem 3.1. Based on the comparison principle and the weak sub-solution derived in Lemma 3.2, we know that the solution is bounded from below, ensuring $u(x,t) \geq \underline{u}(x)$ for all $t \in (0, T_{\max})$. Furthermore, Lemma 3.5 demonstrates that the localized part $u_+(x,t)$ blows up in finite time, which consequently indicates that the solution $u(x,t)$ to porous medium equation (3.1) blows up in finite time and $T_{\max} < +\infty$. □

According to (3.16), when $m > 1$, the blow-up rate of the porous medium equation (3.1) decreases relative to the classical heat equation (1.1).

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