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GLOBAL SOLUTION FOR COUPLED PARABOLIC SYSTEMS WITH DEGENERATE COEFFICIENTS AND TIME-WEIGHTED SOURCES

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Abstract. In this article we obtained the so-called Fujita exponent for the degenerate parabolic coupled system

$$u_t - \operatorname{div}(\omega(x)\nabla u) = t^r v^p$$
$$v_t - \operatorname{div}(\omega(x)\nabla v) = t^s u^p$$

in $\mathbb{R}^N \times (0,T)$ with initial data belonging to $[L^\infty(\mathbb{R}^N)]^2$, where $p, q > 0$ with $pq > 1$; $r, s > -1$, and either $\omega(x) = |x_1|^a$ or $\omega(x) = |x|^b$ with $a, b > 0$.

1. Introduction

Several authors have studied models associated with elliptic and parabolic partial differential equations, which presents a diffusion operator of the form $\operatorname{div}(\omega(x)\nabla\cdot)$, where div is the divergent, ∇ is the gradient, and the spatial function $\omega : \mathbb{R}^N \to [0,\infty)$ is a weight representing the part of thermal diffusion, which can degenerate. See for example the works of Kamin and Rosenau [21, 22, 23]; Kohn and Nirenberg [26]; Fabes, Kenig, and Serapioni [11]; Gutierrez and Nelson [15]; Fujishima, Kawakami, and Sire [12]; Dong and Phan [9]; Sire, Terracini, and Vita [28]; Zeldovich [40]; Jleli, Kirane, and Samet [19]; and Jing, Nie, and Wang [20]. See also the works of Wang and Zhao [37, 38], where it is studied parabolic problems related to biological population models.

We are interested in the degenerate coupled parabolic problem with time-weighted sources,

$$\begin{gathered} u_t - \operatorname{div}(\omega(x)\nabla u) = h_1(t)v^p \quad \text{in } \mathbb{R}^N \times (0,T), \\ v_t - \operatorname{div}(\omega(x)\nabla v) = h_2(t)u^q \quad \text{in } \mathbb{R}^N \times (0,T), \\ u(0) = u_0, \quad v(0) = v_0 \quad \text{in } \mathbb{R}^N, \end{gathered} \tag{1.1}$$

where $(u_0, v_0) \in L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N) \equiv [L^\infty(\mathbb{R}^N)]^2$; $u_0, v_0 \geq 0$; $p, q > 0$ with $pq > 1$; $h_1(t) = t^r, h_2(t) = t^s$ with $r, s > -1$; and the weighted function $\omega : \mathbb{R}^N \to [0,\infty)$ satisfies one of the the following two conditions: either

(A1) $\omega(x) = |x_1|^a$ with $a \in [0,1)$ for $N = 1, 2$, and $a \in [0, 2/N)$ for $N \geq 3$, or

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(A2) $\omega(x) = |x|^b$ with $b \in [0, 1)$.

The function ω, with these characteristics, belongs to the Muckenhoupt class of functions $A_{1+2/N}$. Moreover, the operator $\operatorname{div}(\omega(x)\nabla\cdot)$ is not self-adjoint, as noted in the observations made by Fujishima et al. [12].

In scenario (A1), the function ω exhibits a line of singularities. Consequently, problem (1.1) connects to the fractional Laplacian via the Caffarelli-Silvestre extension, as referenced in [2, 12, 30, 5]. Additionally, the fractional Laplacian is linked to nonlocal diffusion and is present in the Levy diffusion process, as illustrated in [8, 24].

Fujishima et al. [12] studied the problem

$$\begin{gathered} u_t - \operatorname{div}(\omega(x)\nabla u) = u^p \quad \text{in } \mathbb{R}^N \times (0, T), \\ u(0) = u_0 \quad \text{in } \mathbb{R}^N, \end{gathered} \tag{1.2}$$

and obtained the Fujita exponent

$$p^\star(\alpha) = 1 + \frac{2 - \alpha}{N},$$

where $\alpha = a$ in case (A1) and $\alpha = b$ in case (A2).

When $\omega = 1$, the problem defined in (1.2) has been studied by various researchers. Hirose Fujita [13] linked the critical exponent $p^\star(0)$ to the global existence of solutions for problem (1.2). He demonstrated that for $1 < p < p^*(0)$, problem (1.2) lacks any non-negative global solutions. When $p > p^*(0)$, both global and non-global solutions may arise, contingent on the size of the initial conditions; for further information, refer to [27, 31]. In the critical scenario where $p = p^*(0)$, Hayakawa [16] (for $N = 1, 2$), and subsequently Aronson and Weinberger [1] (for $N \geq 3$), proved that problem (1.2) does not possess a global solution.

Problem (1.1), with $\omega = 1$ and $h_1 = h_2 = 1$, was studied firstly by Escobedo and Herrero [10]. They showed that

$$(pq)^* = 1 + \frac{2}{N}(\max\{p, q\} + 1)$$

is the Fujita exponent for problem (1.1), that is, if $1 < pq \leq (pq)^*$, then any nontrivial nonnegative solution blows up in finite time, and when $pq > (pq)^*$, there exist both global and nonglobal solutions. The case $h_1(t) = (1+t)^r$ and $h_2(t) = (1+t)^s$ was analyzed later in Cao et al. [6] who showed the existence of the Fujita exponent

$$(pq)^* = 1 + \frac{2\max\{(r+1)q + s + 1, (s+1)p + r + 1\}}{N},$$

for problem (1.1). See also [3, 4, 18] and the references therein for other related results.

The primary aim of this study is to ascertain the Fujita exponent for problem (1.1). To achieve this, we employ the methods outlined in [12, 10], which are adapted to address the challenges specific to the degenerate coupled system and to handle the scenario where $pq > 1$ with $0 < p < 1$ (or $0 < q < 1$). Notably, we rely solely on the properties (A3)-(A7) that are confirmed by the fundamental solution Γ linked to the linear problem (2.1) (detailed in Section 2). Consequently, the conventional approaches for addressing problem (1.1) (where $h_1 = h_2 = \omega = 1$) require refinement.

The approach that we use can also be applied to determine the critical Fujita exponent of the following problems:

$$\begin{gathered}(u_i)_t - \operatorname{div}(\omega(x)\nabla u_i) = t^{r_i} u_{i+1}^{q_i}, \quad i = 1, \dots, m-1 \quad \text{in } \mathbb{R}^N \times (0,T), \\ (u_m)_t - \operatorname{div}(\omega(x)\nabla u_m) = t^{r_m} u_1^{q_m} \quad \text{in } \mathbb{R}^N \times (0,T),\end{gathered} \tag{1.3}$$

and

$$\begin{gathered}u_t - \operatorname{div}(\omega(x)\nabla u) = t^{r_1} u^p + t^{r_2} v^q \quad \text{in } \mathbb{R}^N \times (0,T), \\ v_t - \operatorname{div}(\omega(x)\nabla v) = t^{r_3} u^r + t^{r_4} v^s \quad \text{in } \mathbb{R}^N \times (0,T).\end{gathered} \tag{1.4}$$

When $\omega = 1$, problem (1.3) was investigated in [32, 35, 4], while problem (1.4) was examined in [7, 34, 3]. Moreover, similar outcomes can be achieved by considering the operator $\omega(x)^{-1} \operatorname{div}(\omega(x)\nabla u_i)$ in place of the operator $\operatorname{div}(\omega(x)\nabla u_i)$, as demonstrated in the recent findings in [17, 25].

Solutions for problem (1.1) with initial data $(u_0, v_0) \in [L^\infty(\mathbb{R}^N)]^2$ are understood in the following sense.

Definition 1.1. Let u and v be a.e. finite, measurable functions defined on $\mathbb{R}^N \times (0.T)$ for some $T > 0$. A pair (u, v) is called a solution of (1.1) with initial condition $(u_0, v_0) \in [L^\infty(\mathbb{R}^N)]^2$, if $(u, v) \in [L^\infty((0,T); L^\infty(\mathbb{R}^N))]^2$ and satisfies

$$\begin{aligned}u(x,t) &= \int_{\mathbb{R}^N} \Gamma(x,y,t) u_0(y)\,dy + \int_0^t \int_{\mathbb{R}^N} \Gamma(x,y,t-\sigma) h_1(\sigma) v(y,\sigma)^p \,dy\,d\sigma < \infty, \\ v(x,t) &= \int_{\mathbb{R}^N} \Gamma(x,y,t) v_0(y)\,dy + \int_0^t \int_{\mathbb{R}^N} \Gamma(x,y,t-\sigma) h_2(\sigma) u(y,\sigma)^q \,dy\,d\sigma < \infty,\end{aligned} \tag{1.5}$$

for almost all $x \in \mathbb{R}^N$ and $t \in (0,T)$. If $T = \infty$, we say that (u, v) is a global-in-time solution of (1.1). Here

$$S(t)\phi(x) := [S(t)\phi](x) := \int_{\mathbb{R}^N} \Gamma(x,y,t)\phi(y)\,dy$$

where $\Gamma(x, y, t)$ is the fundamental solution of the linear problem $u_t - \operatorname{div}(\omega\nabla u) = 0$ in $\mathbb{R}^N \times (0, \infty)$.

Henceforth, we consider the following values:

$$\gamma_1 := \frac{(r+1) + (s+1)p}{pq-1}, \tag{1.6}$$

$$\gamma_2 := \frac{(s+1) + (r+1)q}{pq-1}, \tag{1.7}$$

$$r_{1\star} := \frac{N}{(2-\alpha)\gamma_1}, \tag{1.8}$$

$$r_{2\star} := \frac{N}{(2-\alpha)\gamma_2}. \tag{1.9}$$

Our main result is the following.

Theorem 1.2. *Let $r, s > -1$, $p, q > 0$, with $pq > 1$. Suppose that $\alpha = a$ in the case that ω satisfies the condition* (A1), *and $\alpha = b$ in the case that ω satisfies the condition rm (A2).*

(i) *If $\gamma := \max\{\gamma_1, \gamma_2\} \geq N/(2-\alpha)$, then problem* (1.1) *has no nontrivial global- in-time solution.*

(ii) *If* $\gamma := \max\{\gamma_1, \gamma_2\} < N/(2-\alpha)$, *then there are nontrivial global-in-time solutions to* (1.1). *Moreover, there exists a constant* $\delta > 0$ *such that for any*

$$(u_0, v_0) \in [L^\infty(\mathbb{R}^N) \cap L^{r_{1\star},\infty}(\mathbb{R}^N)] \times [L^\infty(\mathbb{R}^N) \cap L^{r_{2\star},\infty}(\mathbb{R}^N)]$$

with $\max\{\|u_0\|_{r_{1\star},\infty}, \|v_0\|_{r_{2\star},\infty}\} < \delta$, *then problem* (1.1) *has a global-in-time solution* (u, v) *satisfying:*

$$\sup_{t>0}(1+t)^{\frac{N}{2-\alpha}\left(\frac{1}{r_{1\star}}-\frac{1}{\mu}\right)}\|u(t)\|_{\mu,\infty} < \infty,$$
$$\sup_{t>0}(1+t)^{\frac{N}{2-\alpha}\left(\frac{1}{r_{2\star}}-\frac{1}{\mu}\right)}\|v(t)\|_{\mu,\infty} < \infty$$

for $\max\{r_{1\star}, r_{2\star}\} < \mu \leq \infty$.

Remark 1.3. Here are some comments on Theorem 1.2.

(i) When $\alpha = 0$, Theorem 1.2 coincides with the result in [6, Theorem 1].
(ii) When $\alpha = 0$ and $r = s = 0$, this theorem coincides with the results in [10]. Moreover, the values $r_{1\star} = N(pq-1)/2(p+1)$ and $r_{2\star} = N(pq-1)/2(q+1)$ are the same used in [10] to determine the global existence.
(iii) The result is sharp and shows that the critical value of Fujita is given by

$$(pq)^*(\alpha) = 1 + \frac{(2-\alpha)\max\{(s+1)p + r + 1, (r+1)q + s + 1\}}{N}.$$

This work is organized as follows. In section 2, we present the necessary preliminaries. Then in section 3, we prove the non-global existence. Finally, in section 4, we prove the global existence.

2. Preliminaries and technical results

In that follows, C denotes a generic positive constant that may vary in different places, and its change is not essential to the analysis. The positive part of $\phi(x)$ is defined by $\phi^+(x) = \max\{\phi(x), 0\}$. The negative part of ϕ is defined analogously.

For $x = (x_1, \dots, x_N) \in \mathbb{R}^N$, $|x| = \left(\sum_{i=1}^N x_i^2\right)^{1/2}$ is the Euclidean norm of $\mathbb{R}^N$. The spaces $L^\infty(\mathbb{R}^N)$ and $L^\zeta(\mathbb{R}^N)(\zeta \geq 1)$ are defined as usual, and their norms are denoted by $\|\cdot\|_\infty$ and $\|\cdot\|_\zeta$, respectively.

For $1 \leq \zeta \leq \infty$ and $1 \leq \sigma \leq \infty$, the Lorentz space $L^{\zeta,\sigma}(\mathbb{R}^N)$ is defined as

$$L^{\zeta,\sigma} := \{\psi : \mathbb{R}^N \to \mathbb{R}; \psi \text{ is measurable and } \|\psi\|_{L^{\zeta,\sigma}(\mathbb{R}^N)} < \infty\},$$

where

$$\|\psi\|_{L^{\zeta,\sigma}(\mathbb{R}^N)} := \|\psi\|_{L^{\zeta,\sigma}} = \begin{cases} \left(\int_0^\infty [s^{\frac{1}{\zeta}}\psi^\star(s)]^\sigma \frac{ds}{s}\right)^{1/\sigma} & \text{if } 1 \leq \sigma < \infty, \\ \sup_{s>0} s^{1/\zeta}\psi^\star(s) & \text{if } \sigma = \infty, \end{cases}$$
$$\psi^\star(s) := \inf\{\lambda > 0; \mu_\psi(\lambda) \leq s\},$$

$$\mu_\psi(\lambda) := \{x : |\psi(x)| > \lambda\}|, \quad \lambda \geq 0,$$

is the distribution function of ψ. By definition, $L^{\infty,\infty}(\mathbb{R}^N) = L^\infty(\mathbb{R}^N)$. The Lorentz space $L^{\zeta,\sigma}(\mathbb{R}^N)$ is a Banach space; see [14, 41] for details.

Definition 2.1. The Muckenhoupt class A_p, with $1 < p < \infty$, is the set of locally integrable nonnegative functions w that satisfy

$$\Big(\fint_Q w\, dx \Big) \Big(\fint_Q w^{-\frac{1}{(p-1)}}\, dx \Big)^{p-1} < K,$$

for every cube Q and some constant $K > 0$. For $p = 1$, the function w belongs to the Muckenhoupt class A_1 if there exists a constant $K > 0$ such that

$$\fint_Q w\, dx \le K \inf_Q w,$$

for all cube Q.

We will denote by $\Gamma := \Gamma(x, y, t)$ the fundamental solution of the homogeneous problem

$$u_t - \operatorname{div}(\omega(x)\nabla u) = 0 \tag{2.1}$$

in $\mathbb{R}^N \times (0, T)$, with a pole at point $(y, 0)$, and ω verifying either (A1) or (A2) condition. Since ω belongs to the classes $A_{1+2/N}$ and A_2 (see[29]), we have that the fundamental solution $\Gamma = \Gamma(x, y, t)$ satisfies the following properties (see [15, 12] for more details):

- (A3) $\int_{\mathbb{R}^N} \Gamma(x, y, t)dx = \int_{\mathbb{R}^N} \Gamma(x, y, t)\, dy = 1$ for $x, y \in \mathbb{R}^N$ and $t > 0$;
- (A4) $\Gamma(x, y, t) = \int_{\mathbb{R}^N} \Gamma(x, \xi, t - s)\Gamma(\xi, y, s)d\xi$ for $x, y \in \mathbb{R}^N$ and $t > s > 0$;
- (A5) Let $c_0 := \sup_Q \big(\fint_Q \omega(x)dx \big) \big(\fint_Q \omega(x)^{-1}dx \big) < \infty$, where the supremum is taken over all cubes $Q \in \mathbb{R}^N$, and

$$h_x(r) = \Big(\int_{B_r(x)} \omega(y)^{-N/2}\, dy \Big)^{2/N}.$$

Then there exist constants $C_{0\star}, c_{0\star} > 0$, depending only on N and c_0, such that

$$\begin{aligned} & c_{0\star}^{-1} \Big(\frac{1}{[h_x^{-1}(t)]^N} + \frac{1}{[h_y^{-1}(t)]^N} \Big) \exp\Big[- c_{0\star} \Big(\frac{h_x(|\ x - y\ |)}{t} \Big)^{1/(1-\alpha)} \Big] \\ & \le \Gamma(x, y, t) \\ & \le C_{0\star}^{-1} \Big(\frac{1}{[h_x^{-1}(t)]^N} + \frac{1}{[h_y^{-1}(t)]^N} \Big) \exp\Big[- C_{0\star} \Big(\frac{h_x(|\ x - y\ |)}{t} \Big)^{1/(1-\alpha)} \Big] \end{aligned}$$

for $x, y \in \mathbb{R}^N$, $t > 0$, and $\alpha \in \{a, b\}$, where h_x^{-1} denotes the inverse function of h_x.

Also, by [12, estimates (2.11), (2.12)], we have

- (A6) $\int_{|x| \le t^{1/(2-\alpha)}} \Gamma(x, y, t)dx \ge C$, for all $|y| \le t^{1/(2-\alpha)}$, and some constant $C > 0$.
- (A7) $\Gamma(x, y, t) \ge Ct^{-N/(2-\alpha)}$, for $|x|, |y| \le t^{1/(2-\alpha)}$, $t > 0$, and some constant $C > 0$.

Remark 2.2. From (A5), we deduce that the fundamental solution Γ is nonnegative. Moreover, if $u_0 \in L^\infty(\mathbb{R}^N)$ is such that $u_0 \ge 0$ and $u_0 \ne 0$, then according to (A6) (or (A7), there exists a $\tau_0 := \tau(u_0) > 0$ for which

$$S(t)u_0(x) = \int_{\mathbb{R}^N} \Gamma(x, y, t)u_0(y)\ dy > 0,$$

for almost every $x \in \mathbb{R}^N$ and all $t > \tau_0$.

The subsequent results will be utilized to demonstrate the existence of solutions global-in-time for equation (1.1).

Proposition 2.3 ([12]). *(i) Let $\phi \in L^{q_1}(\mathbb{R}^N)$ and $1 \le q_1 \le q_2 \le \infty$, then*

$$\|S(t)\phi\|_{q_2} \le c_1 t^{-\frac{N}{2-\alpha}(\frac{1}{q_1}-\frac{1}{q_2})} \|\phi\|_{q_1}, \tag{2.2}$$

for $t > 0$. The constant $c_1 > 0$ can be taken so that it depends only on N, $\alpha \in \{a, b\}$.

(ii) Let $\phi \in L^{q_1,\infty}(\mathbb{R}^N)$ with $1 < q_1 \le q_2 \le \infty$, then

$$\|S(t)\phi\|_{q_2,\infty} \le c_2 t^{-\frac{N}{2-\alpha}(\frac{1}{q_1}-\frac{1}{q_2})} \|\phi\|_{q_1,\infty}, \tag{2.3}$$

for $t > 0$. The constant $c_2 > 0$ can be taken so that it depends only on q_1, N, and $\alpha \in \{a, b\}$. In particular, c_2 is bounded in $q_1 \in (1+\varepsilon, \infty)$ for any fixed $\varepsilon > 0$ and $c_2 \to \infty$ as $q_1 \to 1$.

Another tool used is the following interpolation result in Lorentz space.

Proposition 2.4 ([14]). *Let $1 \le r_0 \le r_2 \le r_1 \le \infty$ be such that $\frac{1}{r_2} = \frac{\theta}{r_0} + \frac{1-\theta}{r_1}$, for $\theta \in [0, 1]$. Then*

$$\|f\|_{r_2,\infty} \le \|f\|_{r_0,\infty}^{\theta} \|f\|_{r_1,\infty}^{1-\theta}, \tag{2.4}$$

for $f \in L^{r_0,\infty} \cap L^{r_1,\infty}$.

The subsequent results will be utilized to demonstrate the existence of non-global solutions to equation (1.1).

Lemma 2.5 ([12]). *Assume that ω satisfies either* (A1) *or* (A2). *Let $\phi \in L^\infty(\mathbb{R}^N)$, $\phi \ge 0$, and $\phi \ne 0$. Then there exists a positive constant $C(\alpha, N)$, depending only on α and N, such that*

$$S(t)\phi(x) \ge C(\alpha, N)^{-1} t^{-\frac{N}{2-\alpha}} \int_{|y| \le t^{\frac{1}{2-\alpha}}} \phi(y) dy,$$

for $|x| \le t^{\frac{1}{2-\alpha}}$ and $t > 0$, where α is defined by $\alpha = a$ in the case (A1) *and $\alpha = b$ in the case* (A2)*.*

Lemma 2.6. *Assume that ω satisfies either* (A1) *or rm (A2). If $u_0 \in L^\infty(\mathbb{R}^N)$ is a nonnegative function and $q \ge 1$, then*

$$\int_{\mathbb{R}^N} \Gamma(x, y, t)[u_0(y)]^q \, dy \ge \Big(\int_{\mathbb{R}^N} \Gamma(x, y, t) u_0(y) \, dy \Big)^q.$$

If $0 < q < 1$, then

$$\Big(\int_{\mathbb{R}^N} \Gamma(x, y, t) u_0(y) \, dy \Big)^q \ge \int_{\mathbb{R}^N} \Gamma(x, y, t)[u_0(y)]^q dy.$$

Proof. Since the fundamental solution Γ is nonnegative, by (A5), $\int_{\mathbb{R}^N} \Gamma(x, y, t) \, dy = 1$, and by (A3), we can use Jensen's inequality for $q > 1$ in the estimate

$$\int_{\mathbb{R}^N} \Gamma(x, y, t)[u_0(y)]^q \, dy \ge \Big(\int_{\mathbb{R}^N} \Gamma(x, y, t) u_0(y) \, dy \Big)^q.$$

For $0 < q < 1$, we observe that $u_0^q \in L^\infty(\mathbb{R}^N)$. Thus, the conclusion follows as the anterior case replacing q by $1/q$. □

3. Nonglobal existence

To demonstrate the non-global existence aspect of Theorem 1.2, we require the subsequent result. The method employed is traditional, albeit with necessary adjustments (refer to [39]).

Proposition 3.1. *Assume that ω satisfies either* (A1) *or* (A2), *and $u_0, v_0 \in L^\infty(\mathbb{R}^N)$ with $u_0, v_0 \geq 0$. Suppose that $(u, v) \in [L^\infty((0,T), L^\infty(\mathbb{R}^N))]^2$ is a solution of problem* (1.1) *with $0 < T \leq \infty$, and $p, q > 0$ with $pq > 1$. Then there exists a constant $C^\star > 0$ (which depends only on p, q, r, and s), such that*

$$\begin{aligned} t^{\gamma_1}\|S(t)u_0\|_\infty &\leq C^\star, \quad \text{if } q > 1, \\ t^{q\gamma_1}\|S(t)u_0^q\|_\infty &\leq C^\star, \quad \text{if } 0 < q < 1, \\ t^{\gamma_2}\|S(t)v_0\|_\infty &\leq C^\star, \quad \text{if } p > 1, \\ t^{p\gamma_2}\|S(t)v_0^p\|_\infty &\leq C^\star, \quad \text{if } 0 < p < 1, \end{aligned} \tag{3.1}$$

for all $t \in [0,T)$, where γ_1, γ_2 are given by (1.6) *and* (1.7).

Proof. Since $u_0 \in L^\infty(\mathbb{R}^N)$, from (2.2) we have $S(t)u_0(x) < \infty$ for a.e. $x \in \mathbb{R}^N$. We will prove the first inequality of (3.1). To do this, we will show the estimate

$$u(x,t) \geq C_k t^{(\beta^k - 1)\gamma_1}[S(t)u_0(x)]^{\beta^k} \quad (k \in \mathbb{N} \cup \{0\}), \tag{3.2}$$

for a.e. $x \in \mathbb{R}^N$ and $t \in (0,T)$, where $C_0 = 1$, $\beta = pq$ and

$$C_k = C_{k-1}^\beta[(\beta^{k-1}-1)q\gamma_1 + s + 1]^{-p}[(\beta^{k-1}-1)\gamma_1\beta + p(s+1) + (r+1)]^{-1}, \tag{3.3}$$

for $k \in \mathbb{N} \cup \{0\}$. We proceed by induction on k. From (1.5) and property (A5), it follows that $u(x,t) \geq S(t)u_0(x)$ for almost every $x \in \mathbb{R}^N$ and all $t > 0$; thus, (3.2) is satisfied for $k = 0$. Now, assuming that estimate (3.2) is valid for $k \geq 1$, we apply (1.5), properties (A3), (A4), (A5), and Lemma 2.6 to obtain

$$\begin{aligned} &v(x,t) \\ &\geq \int_0^t \int_{\mathbb{R}^N} \Gamma(x,y,t-\sigma)h_1(\sigma)[u(y,\sigma)]^q \, dy d\sigma \\ &\geq \int_0^t \int_{\mathbb{R}^N} \Gamma(x,y,t-\sigma)\sigma^s[C_k\sigma^{(\beta^k-1)\gamma_1}[S(\sigma)u_0(y)]^{\beta^k}]^q d\sigma \\ &\geq C_k^q \int_0^t \sigma^{(\beta^k-1)\gamma_1 q + s}\Big[\int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}\Gamma(x,y,t-\sigma)\Gamma(y,z,\sigma)\,dy\Big)u_0(z)dz\Big]^{q\beta^k} d\sigma \\ &\geq C_k^q[S(t)u_0(x)]^{q\beta_k}\int_0^t \sigma^{(\beta^k-1)\gamma_1 q+s}d\sigma \\ &= C_{k,1}t^{(\beta^k-1)\gamma_1 q+s+1}[S(t)u_0(x)]^{q\beta^k} \end{aligned} \tag{3.4}$$

for a.e. $x \in \mathbb{R}^N$ and $t > 0$, where $C_{k,1} = C_k^q/((\beta^k-1)\gamma_1 q + s + 1)$.

Similarly, from (3.4), we obtain

$$\begin{aligned} u(x,t) &\geq \int_0^t \int_{\mathbb{R}^N} \Gamma(x,y,t-\sigma)\sigma^s\big[C_{k,1}\sigma^{(\beta^k-1)\gamma_1 q+s+1}[S(\sigma)u_0(y)]^{q\beta^k}\big]^p dy d\sigma \\ &\geq C_{k,1}^p[S(t)u_0(x)]^{\beta^{k+1}}\int_0^t \sigma^{(\beta^k-1)\gamma_1\beta+(s+1)p+r}d\sigma \\ &= C_{k,2}t^{(\beta^k-1)\gamma_1\beta+(s+1)p+(r+1)}[S(t)u_0(x)]^{\beta^{k+1}} \end{aligned}$$

for a.e. $x \in \mathbb{R}^N$ and $t > 0$, where $C_{k,2} = C_{k,1}^p/[(\beta^k - 1)\gamma_1\beta + (s+1)p + (r+1)]$. Since

$$(\beta^k - 1)\gamma_1\beta + (s+1)p + (r+1) = (\beta^{k+1} - 1)\gamma_1,$$

we have

$$u(x,t) \geq C_{k,2} t^{(\beta^{k+1}-1)\gamma_1} [S(t)u_0(x)]^{\beta^{k+1}},$$

for a.e. $x \in \mathbb{R}^N$ and $t > 0$. Setting $C_{k+1} = C_{k,2}$ and inserting the value of $C_{k,1}$, we obtain (3.3). Thus, the induction process is complete.

Now we show that there exists $\kappa_0 > 0$ such that $C_k \geq \kappa_0^{\beta^k}$ for all $k \geq 2$. Defining $\theta_k = -\beta^{-k}\ln(C_k)$ it is sufficient to prove that the sequence $\{\theta_k\}_{k\in\mathbb{N}}$ is bounded from above. From relation (3.3), we have

$$\begin{aligned}
\theta_i - \theta_{i-1} &= \beta^{-i}\ln\Big(\frac{C_{i-1}^{\beta}}{C_i}\Big) \\
&= \beta^{-i}\ln\big([(\beta^{i-1}-1)q\gamma_1 + s + 1]^p[(\beta^{i-1}-1)\gamma_1\beta + p(s+1) + (r+1)]\big) \\
&\leq \begin{cases} \beta^{-i}\ln[\gamma_1(\beta^i - 1)]^{p+1} & \text{if } p > 1, \\ \beta^{-i}\ln[q[\gamma_1(\beta^i - 1)]^2 & \text{if } 0 < p \leq 1 \end{cases} \\
&\leq C\beta^{-i}(i+1).
\end{aligned}$$

This implies that $\theta_k - \theta_1 = \sum_{i=1}^k(\theta_i - \theta_{i-1}) \leq C\sum_{i=1}^k \beta^{-i}(i+1) < \infty$.

From (3.2) and the estimate $C_k \geq \kappa_0^{\beta^k}$ we have that

$$u(x,t)^{1/\beta^k} \geq \kappa_0 t^{\gamma_1(1-1/\beta^k)} S(t)u_0(x),$$

for a.e $x \in \mathbb{R}^N$ and $t \in (0,T)$. Since $\beta > 1$, letting $k \to \infty$, we obtain the first inequality of (3.1).

For the proof of the second inequality of (3.1), we argue similarly to the previous case. We use properties (A3)–(A5), and Lemma 2.6 iteratively, starting with

$$v(x,t) \geq t^{s+1}S(t)[u_0(x)]^q, \tag{3.5}$$

until the inequality

$$u(x,t) \geq D_k t^{(\beta^k-1)\gamma_1}[S(t)[u_0(x)]^q]^{p\beta^{k-1}}, \tag{3.6}$$

for a.e. $x \in \mathbb{R}^N$, $t \in (0,T)$, and $k \in \mathbb{N}$, where $\beta = pq$, and $D_k \geq \eta_1^{\beta_k}$ ($\eta_1 > 0$). So, from (3.6), we obtain

$$u(x,t)^{q/\beta^k} \geq \eta_1 t^{q\gamma_1(1-1/\beta^k)} S(t)[u_0(x)]^q,$$

for a.e. $x \in \mathbb{R}^N$, $t \in (0,T)$ and some positive constant η_1. Letting k tends to infinity, we obtain the desired estimate.

By the symmetry the problem, the other inequalities can be proved analogously. □

The following result is a direct consequence of the above proposition.

Corollary 3.2. *Assume that ω satisfies either* (A1) *or* (A2) *condition, and $u_0, v_0 \in L^\infty(\mathbb{R}^N)$ with $u_0, v_0 \geq 0$. If $(u,v) \in [L^\infty((0,\infty), L^\infty(\mathbb{R}^N))]^2$ is a global-in-time solution of* (1.1) *then there exists a constant $C^{\star\star} > 0$ (which depends only on p, q, r, and s) such that*

$$t^{\gamma_1}\|S(t)u(t)\|_\infty \leq C^{\star\star}, \quad \text{if } q > 1,$$
$$t^{q\gamma_1}\|S(t)[u(t)]^q\|_\infty \leq C^{\star\star}, \quad \text{if } 0 < q < 1,$$

$$t^{\gamma_2}\|S(t)v(t)\|_\infty \le C^{\star\star}, \quad \text{if } p>1,$$
$$t^{p\gamma_2}\|S(t)[v(t)]^p\|_\infty \le C^{\star\star}, \quad \text{if } 0<p<1,$$

for all $t\in(0,\infty)$.

Proof. Since (u,v) is a global-in-time solution to equation (1.1), the pair $(u(\cdot+\sigma), v(\cdot+\sigma))$ for $\sigma>0$ also constitutes a global-in-time solution to the same problem with the initial condition $(u(\sigma),v(\sigma))$. Consequently, the estimate in equation (3.1) applies with $(u(\sigma),v(\sigma))$ replacing (u_0,v_0). Therefore, the result is obtained by setting $\sigma=t$ in this estimate. □

Lemma 3.3. *Under the assumptions of Proposition 3.1, let* (u,v) *be a global-in-time solution of* (1.1) *with initial condition* $(0,0)\neq(u_0,v_0)\in[L^\infty(\mathbb{R}^N)]^2$. *Then there exist* $\tau_0=\tau_0(u_0,v_0)>0$ *such that* $u(x,t)>0$ *and* $v(x,t)>0$ *a.e.* $x\in\mathbb{R}^N$ *and* $t>\tau_0$.

Proof. Assuming $u_0\neq 0$, Remark 2.2 implies that $[S(t)u_0(x)]>0$ for almost every $x\in\mathbb{R}^N$ and $t>\tau_0$ with some $\tau_0>0$. Following the reasoning used to derive (3.4), we obtain

$$u(x,t)\ge [S(t)u_0](x)>0, \quad \text{and} \quad v(x,t)\ge (s+1)^{-1}[(S(t)u_0)(x)]^q t^{s+1}>0,$$

for almost every $x\in\mathbb{R}^N$ and $t>\tau_0$. A similar approach applies when $v_0\neq 0$. □

Proof of the nonglobal existence (Theorem 1.2(i)). Assuming without loss of generality that $\gamma=\gamma_1$, we proceed by contradiction. Suppose there exists a global-in-time solution (u,v) to problem (1.1) with the initial condition $(u_0,v_0)\neq(0,0)$. We will consider two cases:

Case I: $q>1$. Let us assume first that $\gamma_1>N/(2-\alpha)$. By Lemma 3.3, there exists τ_0 such that

$$u(x,t)>0 \quad \text{and} \quad v(x,t)>0, \tag{3.7}$$

for a.e. $x\in\mathbb{R}^N$ and $t>\tau_0$.

Define $w(t):=u(t+\tau)$ and $z(t):=v(t+\tau)$ for all $t\ge 0$ and some $\tau>\min\{1,\tau_0\}$. It follows from (3.7) that $w_0:=w(0)\neq 0$ and $z_0:=z(0)\neq 0$. Given that (w,z) forms a global-in-time solution to (1.1) with the initial condition $(w_0,z_0)=(u(\tau),v(\tau))$, Proposition 3.1 ensures that

$$t^{\gamma_1}\|S(t)w_0\|_\infty\le C^\star \quad \text{for all } t\ge 0. \tag{3.8}$$

On the other hand, since $w_0>0$ there exists a non-trivial function $0\le U_1\in L^\infty(\mathbb{R}^N)$ such that $\operatorname{supp} U_1\subset B(t_0^{1/(2-\alpha)})$ (the ball of center 0 and radius $t_0^{1/(2-\alpha)}$) for some $t_0\ge 1$, and $0\le U_1\le w_0$. By Lemma 2.5, we obtain

$$S(t)U_1(x)\ge CMt^{-\frac{N}{2-\alpha}}, \quad M:=\int_{B(t_0^{1/(2-\alpha)})} U_1(y)\,dy, \tag{3.9}$$

for $|x|\le t^{1/(2-\alpha)}$, $t\ge t_0$ and $C>0$. Consequently, by property (A5), it follows that

$$t^{\gamma_1}\|S(t)w_0\|_\infty\ge t^{\gamma_1}\|S(t)U_1\|_\infty\ge CMt^{\gamma_1-\frac{N}{2-\alpha}},$$

for all $t\ge t_2$, which contradicts (3.8).

Now, reconsider the previously mentioned global-in-time solution $(w(t),z(t))$ with $\gamma_1=\frac{N}{2-\alpha}$. Following a computation similar to that in the derivation of (3.4), we obtain

$$z(x,t)\ge Ct^{s+1}[(S(t)w_0)(x)]^q, \tag{3.10}$$

for almost every $x \in \mathbb{R}^N$ and for all $t > 0$, and some constant $C > 0$. Conversely, from (3.9), it follows that

$$S(t)w_0(x) \geq Ct^{-\frac{N}{2-\alpha}} = Ct^{-\gamma_1}, \tag{3.11}$$

for all $t \geq t_0$, and for $|x| \leq t^{\frac{1}{2-\alpha}}$.

Note that, $t + 1 - \sigma \leq t$ and $\sigma \leq t + 1 - \sigma$ for $1 \leq \sigma \leq t/2$. Thus, from (1.5), (A5), (A6), (A7), (3.10), and (3.11), we obtain

$$\begin{aligned}
&\int_{|x|\leq (t+1)^{1/(2-\alpha)}} w(x,t+1)dx \\
&\geq \int_{|x|\leq t^{1/(2-\alpha)}} w(x,t+1)dx \\
&\geq \int_{|x|\leq t^{1/(2-\alpha)}} \int_1^{t/2} \int_{|y|\leq (t+1-\sigma)^{1/(2-\alpha)}} \sigma^r \Gamma(x,y,t+1-\sigma) z(y,\sigma)^p \, dy d\sigma dx \\
&\geq \int_{t_1}^{t/2} \int_{|y|\leq (t+1-\sigma)^{1/(2-\alpha)}} \sigma^r \Big(\int_{|x|\leq (t+1-\sigma)^{\frac{1}{2-\alpha}}} \Gamma(x,y,t+1-\sigma) dx \Big) z(y,\sigma)^p \, dy \, d\sigma \\
&\geq C \int_{t_1}^{t/2} \int_{|y|\leq (t+1-\sigma)^{1/(2-\alpha)}} \sigma^r (\sigma^{s+1} [S(\sigma) w_0(y)]^q)^p \, dy \, d\sigma \\
&\geq C \int_{t_1}^{t/2} \int_{|y|\leq (t+1-\sigma)^{1/(2-\alpha)}} \sigma^{r+(s+1)p} [S(\sigma) w_0(y)]^{pq-1} [S(\sigma) w_0(y)] \, dy \, d\sigma \\
&\geq C \int_{t_1}^{t/2} \sigma^{r+(s+1)p} \cdot \sigma^{-(pq-1)\gamma_1} \Big(\int_{|y|\leq \sigma^{1/(2-\alpha)}} \sigma^{-\gamma_1} \, dy \Big) d\sigma \\
&\geq C \int_{t_1}^{t/2} \sigma^{-1} d\sigma \\
&= C \ln (t/(2t_1)) > 0,
\end{aligned} \tag{3.12}$$

for $t/2 > t_1 = \max\{t_0, 2\}$ and some constant $C > 0$.

From (3.12), we deduce that for any $R > 0$, there exists $t_2 - 1 > 2t_1$ such that the function U_2, defined by $U_2(x) := w(x, t_2) \in L^\infty(\mathbb{R}^N)$, satisfies

$$\int_{|x|\leq t_2^{1/(2-\alpha)}} U_2(x) \, dx \geq C \ln \Big(\frac{t_2 - 1}{2t_1} \Big) > R. \tag{3.13}$$

Define $(w_1(t), z_1(t)) = (w(t + t_2), z(t + t_2))$. Note that (w_1, z_1) constitutes a global-in-time solution of (1.5) with the initial condition $(w_1(0), z_1(0)) = (U_2(x), z(t_2))$. Consequently, by Proposition 3.1, it follows that

$$t^{\gamma_1} \|S(t)U_2\|_\infty \leq C^\star \quad \text{for all } t \geq 0. \tag{3.14}$$

However, from (3.13) and Lemma 2.5, it is established that

$$S(t)U_2(x) \geq C(\alpha, N)^{-1} R t^{-\frac{N}{2-\alpha}},$$

for $|x| \leq t^{1/(2-\alpha)}$ and $t > t_2$. Consequently,

$$t^{\gamma_1} \|S(t)U_2\|_\infty = t^{\frac{N}{2-\alpha}} \|S(t)U_2\|_\infty \geq C(\alpha, N)^{-1} R,$$

for all $t > t_2$. This contradicts (3.14) because of the arbitrariness of $R > 0$.

Case II: $0 < q < 1$**.** From Lemma 3.3, we can assume without loss of generality, that $u(t) > 0$ and $v(t) > 0$ for all $t \geq 0$. Thus, Corollary 3.2 implies

$$t^{q\gamma_1}\|S(t)u^q(t)\|_\infty \leq C^{\star\star}, \quad \text{for all } t > 0. \tag{3.15}$$

First, assume that $\gamma_1 > \frac{N}{2-\alpha}$. We can then find a non-trivial function $0 \leq U_3 \in L^\infty(\mathbb{R}^N)$ such that $\operatorname{supp} U_3 \subset B(t_0^{\frac{1}{2-\alpha}})$ for some $t_0 > 1$ and $0 \leq U_3 \leq u_0$. Following a similar argument to the derivation of (3.9), we obtain

$$u(x,t) \geq S(t)u_0(x) \geq Ct^{-\frac{N}{2-\alpha}}\mathcal{X}_{t^{\frac{1}{2-\alpha}}}(x), \tag{3.16}$$

for $t \geq t_0$ and some constant $C > 0$, where $\mathcal{X}_{t^{1/(2-\alpha)}}$ is the characteristic function on the ball centered at 0 with radius $t^{1/(2-\alpha)}$. Consequently,

$$[u(x,t)]^q \geq Ct^{-q\frac{N}{2-\alpha}}\mathcal{X}_{t^{1/(2-\alpha)}}(x),$$

for $t \geq t_0$ and some constant $C > 0$. This leads to

$$t^{q\gamma_1}\|S(t)[u(t)]^q\|_\infty \geq Ct^{q(\gamma_1-\frac{N}{2-\alpha})}S(t)\mathcal{X}_{t^{1/(2-\alpha)}}(x), \tag{3.17}$$

for $t \geq t_0$. Moreover, by (A7), we have

$$S(t)\mathcal{X}_{t^{\frac{1}{2-\alpha}}}(x) \geq \int_{|y|<t^{\frac{1}{2-\alpha}}}\Gamma(x,y,t)\,dy \geq Ct^{-\frac{N}{2-\alpha}}t^{\frac{N}{2-\alpha}}, \tag{3.18}$$

for all $|x| \leq t^{1/(2-\alpha)}$ and $t > 0$. Hence, estimate (3.17) contradicts (3.15).

Now, let us assume that $\gamma_1 = \frac{N}{2-\alpha}$. Given that (u,v) is a global-in-time solution to equation (1.1), it follows that for any $\tau > 0$:

$$\begin{aligned}
u(x,t+\tau) &= \int_{\mathbb{R}^N}\Gamma(x,y,t)u(y,\tau)\,dy + \int_0^t\int_{\mathbb{R}^N}\Gamma(x,y,t-\sigma)\sigma^r v^p(y,\sigma+\tau)\,dy\,d\sigma,\\
v(x,t+\tau) &= \int_{\mathbb{R}^N}\Gamma(x,y,t)v(y,\tau)\,dy + \int_0^t\int_{\mathbb{R}^N}\Gamma(x,y,t-\sigma)\sigma^s u^q(y,\sigma+\tau)\,dy\,d\sigma,
\end{aligned}$$

and $(u(\cdot+\tau), v(\cdot+\tau))$ is also a global-in-time solution of (1.1) with initial condition $(u(\tau), v(\tau))$. Then, recalling that $0 < q < 1$ and proceeding similarly as in (3.5), we obtain

$$v(x,t+\tau) \geq Ct^{s+1}S(t)[u(\tau)]^q(x), \tag{3.19}$$

for a.e. $x \in \mathbb{R}^N$ and $t > 0$. Thus, taking $t = \tau$ in (3.19) and arguing similarly as in (3.16)-(3.18), we have

$$\begin{aligned}
v(x,2t) &\geq Ct^{s+1}S(t)[u(t)]^q(x),\\
&\geq Ct^{s+1}S(t)[t^{-\frac{N}{2-\alpha}}\mathcal{X}_{t^{\frac{1}{2-\alpha}}}]^q(x)\\
&\geq Ct^{s+1}\cdot t^{-q\gamma_1},
\end{aligned} \tag{3.20}$$

for $|x| \leq t^{1/(2-\alpha)}$ and $t > t_0 > 1$.

Let $t > 4t_0$. Since $\gamma_1 = N/(2-\alpha)$, from (3.20) and proceeding as in the derivation of (3.12), we have

$$\begin{aligned}
&\int_{|x|\leq(t+1)^{\frac{1}{2-\alpha}}}u(x,t+1)dx\\
&\qquad \geq C\int_1^{t/2}\int_{|y|\leq(t+1-\sigma)^{\frac{1}{2-\alpha}}}\sigma^r(v(y,\sigma))^p\,dy\,d\sigma
\end{aligned}$$

$$
\begin{aligned}
&\geq C\int_{2t_0}^{t/2}\int_{|y|\leq\sigma^{\frac{1}{2-\alpha}}}\sigma^r(v(y,2\cdot 2^{-1}\sigma))^p\,dy\,d\sigma\\
&\geq C\int_{2t_0}^{t/2}\int_{|y|\leq\sigma^{\frac{1}{2-\alpha}}}\sigma^r\Big([\frac{\sigma}{2}]^{s+1}[\frac{\sigma}{2}]^{-q\gamma_1}\Big)^p\,dy\,d\sigma\\
&\geq C\int_{2t_0}^{t/2}\int_{|y|\leq\sigma^{\frac{1}{2-\alpha}}}\sigma^{r+(s+1)p}[\sigma^{-\gamma_1}]^{pq-1}[\sigma^{-\gamma_1}]\,dy\,d\sigma\\
&\geq C\int_{2t_0}^{t/2}\sigma^{r+(s+1)p}\sigma^{-(pq-1)\gamma_1}\Big(\int_{|y|\leq\sigma^{\frac{1}{2-\alpha}}}\sigma^{-\gamma_1}\,dy\Big)\,d\sigma\\
&\geq C\int_{2t_0}^{t/2}\sigma^{-1}\,d\sigma\\
&= C\ln\big(\frac{t}{4t_0}\big).
\end{aligned}
$$

Thus, we can use the same argument given in the previous case, using Corollary 3.2 in place of Proposition 3.1, to obtain a contradiction.

4. Global Existence

4.1. **Local existence.**

Lemma 4.1 (Comparison principle). *Assume that either* (A1) *or* (A2) *is verified, and* $(u_{0,i},v_{0,i})\in[L^\infty(\mathbb{R}^N)]^2$, *for* $i=1,2$. *Let* $f,g:[0,\infty)\to[0,\infty)$ *be nondecreasing and locally Lipschitz functions;* $r,s>-1$; *and*

$$(u_i,v_i)\in[L^\infty((0,T),L^\infty(\mathbb{R}^N)]^2,$$

such that

$$
\begin{aligned}
u_i(x,t)&=\int_{\mathbb{R}^N}\Gamma(t,x,y)u_{0,i}(y)dy+\int_0^t\int_{\mathbb{R}^N}\Gamma(t-\sigma,x,y)\sigma^r f(v_i(y,\sigma))\,dyd\sigma,\\
v_i(x,t)&=\int_{\mathbb{R}^N}\Gamma(t,x,y)v_{0,i}(y)dy+\int_0^t\int_{\mathbb{R}^N}\Gamma(t-\sigma,x,y)\sigma^s g(u_i(y,\sigma))\,dy\,d\sigma,
\end{aligned}
\tag{4.1}
$$

for a.e. $x\in\mathbb{R}^N$ *and* $t>0$. *If* $u_{0,1}\leq u_{0,2}$ *and* $v_{0,1}\leq v_{0,2}$, *then* $u_1(t)\leq u_2(t)$ *and* $v_1(t)\leq v_2(t)$ *for all* $t\in(0,T)$.

Proof. Note that it is sufficient to show that $[u_1(t)-u_2(t)]^+=[v_1(t)-v_2(t)]^+=0$ for $t\in(0,T)$. Let $M_0=\max\{\|u_i(t)\|_\infty,\|v_i(t)\|_\infty;t\in[0,T],\,i=1,2\}$. Since $u_{0,1}\leq u_{0,2}$ and $v_{0,1}\leq v_{0,2}$, from property (A4) and (4.1) we have

$$
\begin{aligned}
u_1(t)-u_2(t)&\leq\int_0^t S(t-\sigma)\sigma^r[f(v_1(\sigma))-f(v_2(\sigma))]\,d\sigma,\\
v_1(t)-v_2(t)&\leq\int_0^t S(t-\sigma)\sigma^s[g(u_1(\sigma))-g(u_2(\sigma))]\,d\sigma.
\end{aligned}
\tag{4.2}
$$

Since f and g are nondecreasing and locally Lipschitz, we obtain

$$
\begin{aligned}
g(u_1(t))-g(u_2(t))&\leq[g(u_1(t))-g(u_2(t))]^+\leq L_{M_1}[u_1(t)-u_2(t)]^+,\\
f(v_1(t))-f(v_2(t))&\leq[f(v_1(t))-f(v_2(t))]^+\leq L_{M_2}[v_1(t)-v_2(t)]^+,
\end{aligned}
\tag{4.3}
$$

where L_{M_1} and L_{M_2} are the Lipschitz constants on the interval $[0;M_0]$.

It follows from estimate (2.2), (4.2), and (4.3) that

$$\|[u_1(t)-u_2(t)]^+\|_\infty \le L_{M_2}\int_0^t \sigma^r \|[v_1(\sigma)-v_2(\sigma)]^+\|_\infty \, d\sigma,$$
$$\|[v_1(t)-v_2(t)]^+\|_\infty \le L_{M_1}\int_0^t \sigma^s \|[u_1(\sigma)-u_2(\sigma)]^+\|_\infty \, d\sigma,$$

The results is now a direct consequence of Gronwall's inequality (see for example [36]). □

Theorem 4.2. *Suppose $p,q>0$ with $pq>1$, ω satisfies either* (A1) *or* (A2)*, and $(u_0,v_0)\in[L^\infty(\mathbb{R}^N)]^2$, $u_0,v_0\ge 0$. Then there exists $T>0$ and a constant $C_0>0$ such that problem* (1.1) *possesses a unique solution (u,v) on $(0,T)$ satisfying*

$$\sup_{0<t<T}(\|u(t)\|_\infty+\|v(t)\|_\infty)\le C_0(\|u_0\|_\infty+\|v_0\|_\infty).$$

Proof. For $(u_0,v_0)\in[L^\infty(\mathbb{R}^N)]^2$, $u_0,v_0\ge 0$. We define the sequences $\{u_n\}_{n\ge1}$ and $\{v_n\}_{n\ge1}$ by

$$u_1(x,t)=\int_{\mathbb{R}^N}\Gamma(x,y,t)u_0(y)\,dy,\quad v_1(x,t)=\int_{\mathbb{R}^N}\Gamma(x,y,t)v_0(y)\,dy$$

and

$$u_{n+1}(x,t)=u_1(x,t)+\int_0^t\sigma^r\int_{\mathbb{R}^N}\Gamma(x,y,t-\sigma)v_n(y,s)^p\,dy\,d\sigma,$$
$$v_{n+1}(x,t)=v_1(x,t)+\int_0^t\sigma^s\int_{\mathbb{R}^N}\Gamma(x,y,t-\sigma)u_n(y,s)^q\,dy\,d\sigma,$$

for a.e. $x\in\mathbb{R}^N$, $n\ge1$ and $t>0$. The sequences $\{u_n\}_{n\ge1}$ and $\{v_n\}_{n\ge1}$ satisfy

$$0\le u_n(x,t)\le u_{n+1}(x,t)\quad\text{and}\quad 0\le v_n(x,t)\le v_{n+1}(x,t) \tag{4.4}$$

for a.e. $x\in\mathbb{R}^N$, $t>0$. This is clear since Γ, u_0, and v_0 are non-negative functions (Γ is nonnegative by (A5) property). Thus, we define

$$u_\infty(x,t)=\lim_{n\to\infty}u_n(x,t),\quad v_\infty=\lim_{n\to\infty}v_n(x,t). \tag{4.5}$$

Furthermore, we see that $u_\infty(x,t),v_\infty(x,t)\in[0,\infty]$.

Now, we show that the sequences $\{u_n\}_{n\ge1}$ and $\{v_n\}_{n\ge1}$ are bounded in a small interval $(0,T)$, that is,

$$\sup_{0<t<T}(\|u_n(t)\|_\infty+\|v_n(t)\|_\infty)\le 2c_1(\|u_0\|_\infty+\|v_0\|_\infty) \tag{4.6}$$

for all $n\in\mathbb{N}$ and some $T>0$ sufficiently small. The constant $c_1>0$ is given by inequality (2.2). To show (4.6), we argue by induction on n. This is clear for $n=1$ due to (2.2). Suppose that (4.6) holds for some $n\in\mathbb{N}$. Then, by (2.2) we have

$$\begin{aligned}\|u_{n+1}(t)\|_\infty &\le \|u_1(t)\|_\infty+\int_0^t\sigma^r\|S(t-\sigma)v_n(\sigma)^p\|_\infty\,d\sigma\\ &\le c_1\|u_0\|_\infty+c_1\int_0^t\sigma^r\|v_n(\sigma)\|_\infty^p\,d\sigma\\ &\le c_1\|u_0\|_\infty+c_1[2c_1(\|u_0\|_\infty+\|v_0\|_\infty)]^p\int_0^t\sigma^r\,d\sigma,\end{aligned} \tag{4.7}$$

for $t \in (0,T)$. Similarly, we have

$$\begin{aligned} \|v_{n+1}(t)\|_\infty &\le \|v_1(t)\|_\infty + \int_0^t \sigma^s \|S(t-\sigma)u_n(\sigma)^q\|_\infty \, d\sigma \\ &\le c_1\|v_0\|_\infty + c_1 \int_0^t \sigma^s \|u_n(\sigma)\|_\infty^q \, d\sigma \\ &\le c_1\|v_0\|_\infty + c_1[2c_1(\|u_0\|_\infty + \|v_0\|_\infty)]^q \int_0^t \sigma^s \, d\sigma, \end{aligned} \tag{4.8}$$

for $t \in (0,T)$. Thus, the inequality (4.6) holds by adding (4.7) and (4.8) and taking $T > 0$ small enough.

Finally, by (4.4), (4.5), and (4.6), we have that the limits functions u_∞ and v_∞ satisfies (1.5) and

$$\sup_{0<t<T} (\|u_\infty(t)\|_\infty + \|v_\infty(t)\|_\infty) \le 2c_1(\|u_0\|_\infty + \|v_0\|_\infty).$$

Moreover, by the comparison principle (see Lemma 4.1), (u_∞, v_∞) is the unique solution of the problem (1.1) if $p, q \ge 1$. □

4.2. **Global existence: proof of Theorem 1.2-(ii).** Without loss of the generality, we suppose that $q > 1$. Let

$$(u_0, v_0) \in [L^\infty(\mathbb{R}^N) \cap L^{r_{1\star},\infty}(\mathbb{R}^N)] \times [L^\infty(\mathbb{R}^N) \cap L^{r_{2\star},\infty}(\mathbb{R}^N)]$$

with $\max\{\|u_0\|_{r_{1\star},\infty}, \|v_0\|_{r_{2\star},\infty}\} < \delta$, where $\delta > 0$ will be chosen small enough later.

From (1.6)-(1.9), we obtain the following estimates:

$$p\gamma_2 = \gamma_1 + (r+1), \quad q\gamma_1 = \gamma_2 + (s+1), \quad pr_{1\star} > r_{2\star}, \quad qr_{2\star} > r_{1\star}. \tag{4.9}$$

Also, since $\gamma < N/(2-\alpha)$, we have $r_{1\star}, r_{2\star} > 1$.

Let $\{(u^n, v^n)\}_{n\ge 0}$ be the sequence defined by $u^0(t) = S(t)u_0$, $v^0(t) = S(t)v_0$ and

$$\begin{aligned} u^N(t) &= S(t)u_0 + \int_0^t S(t-\sigma)h_1(\sigma)[v^{n-1}(\sigma)]^p \, d\sigma, \\ v^N(t) &= S(t)v_0 + \int_0^t S(t-\sigma)h_2(\sigma)[u^{n-1}(\sigma)]^q \, d\sigma, \end{aligned} \tag{4.10}$$

for all $t > 0$. Note that the sequences $\{u^N\}_{n\ge 0}$ and $\{v^N\}_{n\ge 0}$ are non-decreasing.

By induction, we prove that there exists a constant $\tilde{C} > 0$ such that

$$\begin{gathered} \|u^N(t)\|_{r_{1\star},\infty} \le c_{\star\star}\delta + \tilde{C}\delta^p, \\ \|u^N(t)\|_\infty \le (c_{\star\star}\delta + \tilde{C}\delta^p)t^{-\frac{N}{(2-\alpha)r_{1\star}}}, \\ \|v^N(t)\|_{r_{2\star},\infty} \le 2c_{\star\star}\delta, \\ \|v^N(t)\|_\infty \le 2c_{\star\star}\delta t^{-\frac{N}{(2-\alpha)r_{2\star}}}. \end{gathered} \tag{4.11}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $c_{\star\star} = \max\{c_1, c_2\}$ and the constants $c_i (i = 1, 2)$ are given in Proposition 2.3.

From estimate (2.3) we have

$$\begin{gathered}\|u^0(t)\|_{r_{1\star},\infty} \le c_{**}\|u_0\|_{r_{1\star},\infty},\\ \|u^0(t)\|_{\mu,\infty} \le c_{**}t^{-\frac{N}{2-\alpha}(\frac{1}{r_{1\star}}-\frac{1}{\mu})}\|u_0\|_{r_{1\star},\infty},\\ \|v^0(t)\|_{r_{2\star},\infty} \le c_{**}\|v_0\|_{r_{2\star},\infty},\\ \|v^0(t)\|_{\mu,\infty} \le c_{**}t^{-\frac{N}{2-\alpha}(\frac{1}{r_{2\star}}-\frac{1}{\mu})}\|v_0\|_{r_{2\star},\infty},\end{gathered} \tag{4.12}$$

for all $t > 0$, $\mu \in [r_{1\star}, \infty]$. This implies that (4.11) holds for $n = 0$. Assume that (4.11) holds for some $n \in \mathbb{N}$. By symmetry, we only prove that (4.11) holds for u^{n+1}. From estimates (2.4) and (4.11), we have

$$\|v^n(t)\|_{\mu,\infty} \le \|v^n(t)\|_{r_{2\star},\infty}^{\frac{r_{2\star}}{\mu}}\|v^n(t)\|_{\infty}^{1-\frac{r_{2\star}}{\mu}} \le 2c_{**}\delta t^{-\frac{N}{2-\alpha}(\frac{1}{r_{2\star}}-\frac{1}{\mu})} \tag{4.13}$$

for all $t > 0$ and $\mu \in [r_{2\star}, \infty]$. Then, from (4.9) and (4.13), we have

$$\begin{aligned}\|v^n(t)^p\|_{\eta,\infty} &= \|v^n(t)\|_{\eta p,\infty}^p\\ &\le \big[2c_{**}\delta t^{-\frac{N}{2-\alpha}(\frac{1}{r_{2\star}}-\frac{1}{\eta p})}\big]^p\\ &= [2c_{**}]^p\delta^p t^{\frac{N}{(2-\alpha)\eta}-\frac{N}{(2-\alpha)r_{1\star}}-(r+1)}\end{aligned} \tag{4.14}$$

for any $\eta > 1$ with $r_{2\star} \le \eta p$. Similarly, from (4.9) and (4.11), we obtain

$$\begin{aligned}\|v^N(t)^p\|_{\infty} &= \|v^N(t)\|_{\infty}^p\\ &\le \big(2c_{\star\star}\delta t^{-\frac{N}{(2-\alpha)r_{2\star}}}\big)^p\\ &= (2c_{\star\star}\delta)^p t^{-p\gamma_2}\\ &= [2c_{**}]^p\delta^p t^{-\frac{N}{(2-\alpha)r_{1\star}}-(r+1)}\end{aligned} \tag{4.15}$$

for all $t > 0$.

Thus, by (2.2), (2.3), (4.9), (4.14) (with $\eta = r_{1\star}$), $r_{2\star} < r_{1\star}p$, and (4.15) we have

$$\|\int_{t/2}^t S(t-\sigma)\sigma^r v^N(\sigma)^p d\sigma\|_{\infty} \le c_1c_3[2c_{**}]^p\delta^p t^{-\frac{N}{(2-\alpha)r_{1\star}}}, \tag{4.16}$$

where $c_3 = \frac{(2-\alpha)r_{1\star}}{N}(2^{N/(2-\alpha)r_{1\star}}-1)$, and

$$\begin{aligned}\|\int_{\frac{t}{2}}^t S(t-\sigma)\sigma^r v^n(\sigma)^p\,d\sigma\|_{r_{1\star},\infty} &\le c_2\int_{\frac{t}{2}}^t \sigma^r\|v^n(\sigma)^p\|_{r_{1\star},\infty}\,d\sigma\\ &= c_2\int_{\frac{t}{2}}^t \sigma^r\|v^n(\sigma)\|_{pr_{1\star},\infty}^p\,d\sigma\\ &= c_2[2c_{**}]^p\delta^p\int_{\frac{t}{2}}^t \sigma^{-1}\,d\sigma\\ &\le c_2[2c_{**}]^p\delta^p,\end{aligned} \tag{4.17}$$

for all $t > 0$.

On the other hand, since $t-\sigma \geq t/2$ for all $\sigma \in [0,t/2]$, by (2.3) and (4.14) (with $\eta = \eta_1 > 1$, which will be chosen later), we obtain

$$\begin{aligned} &\| \int_0^{t/2} S(t-\sigma)\sigma^r v^N(\sigma)^p d\sigma\|_\infty \\ &\leq \int_0^{t/2} \|S(t-\sigma)\sigma^r v^N(\sigma)^p\|_\infty d\sigma \\ &\leq c_2 \int_0^{t/2} (t-\sigma)^{-\frac{N}{(2-\alpha)\eta_1}} \sigma^r \|v^N(\sigma)^p\|_{\eta_1,\infty} d\sigma \\ &\leq 2^{\frac{N}{(2-\alpha)\eta_1}} c_2 t^{-\frac{N}{(2-\alpha)\eta_1}} \int_0^{t/2} \sigma^r \|v^N(\sigma)^p\|_{\eta_1,\infty} d\sigma \\ &\leq 2^{\frac{N}{(2-\alpha)\eta_1}} c_2 [2c_{**}]^p \delta^p t^{-\frac{N}{(2-\alpha)\eta_1}} \int_0^{t/2} \sigma^{\frac{N}{(2-\alpha)\eta_1} - \frac{N}{(2-\alpha)r_{1\star}} - 1} d\sigma \\ &\leq c_2 c_4 [2c_{**}]^p \delta^p t^{-\frac{N}{(2-\alpha)r_{1\star}}}, \end{aligned} \tag{4.18}$$

for some $1 < \eta_1 < r_{1\star}$ so that $r_{2\star} < \eta_1 p$ (this is possible since $p\ r_{1\star} > r_{2\star} > 1$), and $c_4 = 2^{\frac{N}{(2-\alpha)r_{1\star}}} [\frac{N}{(2-\alpha)\eta_1} - \frac{N}{(2-\alpha)r_{1\star}}]^{-1}$. Analogously (using the above η_1 again), we have

$$\begin{aligned} &\| \int_0^{t/2} S(t-\sigma)\sigma^r v^n(\sigma)^p d\sigma\|_{r_{1\star},\infty} \\ &\leq \int_0^{t/2} \|S(t-\sigma)\sigma^r v^n(\sigma)^p\|_{r_{1\star},\infty}\, d\sigma \\ &\leq c_2 \int_0^{t/2} (t-\sigma)^{-\frac{N}{2-\alpha}(\frac{1}{\eta_1} - \frac{1}{r_{1\star}})} \sigma^r \|v^n(\sigma)^p\|_{\eta_1,\infty}\, d\sigma \\ &\leq c_2 2^{\frac{N}{2-\alpha}(\frac{1}{\eta_1} - \frac{1}{r_{1\star}})} t^{-\frac{N}{2-\alpha}(\frac{1}{\eta_1} - \frac{1}{r_{1\star}})} \int_0^{t/2} \sigma^r \|v^n(\sigma)^p\|_{\eta_1,\infty}\, d\sigma \\ &\leq c_2 [2c_{**}]^p \delta^p 2^{\frac{N}{2-\alpha}(\frac{1}{\eta_1} - \frac{1}{r_{1\star}})} t^{-\frac{N}{2-\alpha}(\frac{1}{\eta_1} - \frac{1}{r_{1\star}})} \int_0^{t/2} \sigma^{\frac{N}{(2-\alpha)\eta_1} - \frac{N}{(2-\alpha)r_{1\star}} - 1}\, d\sigma \\ &\leq c_2 c_5 [2c_{**}]^p \delta^p \end{aligned} \tag{4.19}$$

for all $t > 0$, where $c_5 = [\frac{N}{(2-\alpha)\eta_1} - \frac{N}{(2-\alpha)r_{1\star}}]^{-1}$.

Then, from (4.10), (4.12), (4.16), (4.17), (4.18), (4.19), and taking $\delta > 0$ sufficiently small, we obtain

$$\begin{gathered} t^{\frac{N}{(2-\alpha)r_{1\star}}} \|u^{n+1}(t)\|_\infty \leq c_{\star\star}\delta + \tilde{C}\delta^p \\ \|u^{n+1}(t)\|_{r_{1\star},\infty} \leq c_{\star\star}\delta + \tilde{C}\delta^p \end{gathered}$$

for all $t > 0$, where $\tilde{C} = [2c_{\star\star}]^p \max\{c_1c_3 + c_2c_4, c_2c_5 + c_2\}$. Arguing similarly it is possible to show that there exists a constant $\overline{C}$, independent of n, δ, and t, such that

$$\begin{gathered} t^{\frac{N}{(2-\alpha)r_{2\star}}} \|v^{n+1}(t)\|_\infty \leq c_{\star\star}\delta + \overline{C}(c_{\star\star}\delta + \tilde{C}\delta^p)^q, \\ \|v^{n+1}(t)\|_{r_{2\star},\infty} \leq c_{\star\star}\delta + \overline{C}(c_{\star\star}\delta + \tilde{C}\delta^p)^q. \end{gathered}$$

Since $q > 1$ and $pq > 1$, we have

$$t^{\frac{N}{(2-\alpha)r_{2\star}}} \|v^{n+1}(t)\|_\infty \leq c_{\star\star}\delta + 2^{q-1}\overline{C}c_{\star\star}^q\delta^q + 2^{q-1}\tilde{C}^{q+1}\delta^{pq} \leq 2c_{\star\star}\delta$$

for $\delta > 0$ sufficiently small. Analogously, $\|v^{n+1}(t)\|_{r_{2\star},\infty} \le 2c_{\star\star}\delta$ for $\delta > 0$ possibly smaller. Therefore, (u^{n+1}, v^{n+1}) satisfies the estimates of (4.11) for all $n \in \mathbb{N} \cup \{0\}$, and the induction process is finalized.

From the estimates given in (4.11) we see that there exists a global-in-time solution of (1.1) such that $(u, v) = (\lim_{n\to\infty} u^N, \lim_{n\to\infty} v^N)$ and

$$\|u(t)\|_\infty \le Ct^{-\frac{N}{(2-\alpha)r_{1\star}}}, \quad \|u(t)\|_{r_{1\star},\infty} \le C,$$
$$\|v(t)\|_\infty \le Ct^{-\frac{N}{(2-\alpha)r_{2\star}}}, \quad \|v(t)\|_{r_{2\star},\infty} \le C,$$

for some constant $C > 0$ (this solution is unique when $p > 1$ and $q > 1$). Moreover, by Theorem 4.2,

$$\|u(t)\|_\infty \le C(t+1)^{-\frac{N}{(2-\alpha)r_{1\star}}} \quad \text{and} \quad \|v(t)\|_\infty \le C(t+1)^{-\frac{N}{(2-\alpha)r_{2\star}}},$$

for $t > 0$ and some constant $C > 0$. From this and (2.4), we have

$$\|u(t)\|_{\mu,\infty} \le \|u(t)\|_{r_{1\star},\infty}^{\frac{r_{1\star}}{\mu}} \|u(t)\|_\infty^{1-\frac{r_{1\star}}{\mu}} \le C(t+1)^{-\frac{N}{2-\alpha}(\frac{1}{r_{1\star}}-\frac{1}{\mu})},$$
$$\|v(t)\|_{\mu,\infty} \le \|v(t)\|_{r_{2\star},\infty}^{\frac{r_{2\star}}{\mu}} \|v(t)\|_\infty^{1-\frac{r_{2\star}}{\mu}} \le C(t+1)^{-\frac{N}{2-\alpha}(\frac{1}{r_{2\star}}-\frac{1}{\mu})},$$

for all μ such that $\max\{r_{1\star}, r_{2\star}\} < \mu \le \infty$, $t > 0$ and some constant $C > 0$. Thus the proof is complete.

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