Electronic Journal of Differential Equations, Vol. 2024 (2024), No. 69, pp. 1–14.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2024.69

LONG-TIME BEHAVIOR OF SOLUTIONS FOR TIME-PERIODIC REACTION-DIFFUSION EQUATIONS AND APPLICATIONS

LINLIN LI, ZHUO CHEN

Abstract. This article concerns the asymptotic behavior of solutions for time-periodic reaction-diffusion equations with a drift term in one dimensional space. Assuming that drift term is decaying, we analyze the effect of the decaying rate of the drift term on the propagation speed of solutions. Also we show some applications of our results in high-dimensional domains.

1. Introduction

In this article, we consider the initial value problem of the time-periodic reaction-diffusion equation,

$$\begin{gathered} u_t = u_{xx} + k(x)u_x + f(t,u), \quad x \in \mathbb{R},\ t > 0, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}, \end{gathered} \tag{1.1}$$

where $0 \le u_0(x) \le 1$. Throughout this article, we assume that $f(t,u)$ is periodic in t, that is, there is $T \in \mathbb{R}$ such that $f(t+T,u) = f(t,u)$ for all $u \in \mathbb{R}$ and $t \in \mathbb{R}$. We also assume that $f(t,\cdot)$ is bistable for $u \in [0,1]$, that is, there is a periodic function $\theta_t \in (0,1)$ such that

$$\begin{gathered} f(t,0) = f(t,1) = 0, \quad f(t,\theta_t) = 0, \\ f(t,\cdot) < 0 \text{ on } (0,\theta_t), \quad f(t,\cdot) > 0 \text{ on } (\theta_t, 1), \\ f_u(t,0), \quad f_u(t,1) < 0 \text{ for all } t \in \mathbb{R}. \end{gathered}$$

which means that 0 and 1 are stable zeros of f. This implies that there exist $\rho \in (0,1/2)$ and $\tau > 0$ such that

$$-f_u(t,u) \ge \tau \text{ for } t \in \mathbb{R} \quad \text{and} \quad u \in [0,\rho] \cup [1-\rho, 1]. \tag{1.2}$$

The drift term $k(x)$ is assumed to be decaying to 0 as $x \to +\infty$. More precise assumptions on $k(x)$ and the initial value $u_0(x)$ will be given later. We aim to analyze the effect of the decaying rate of $k(x)$ on the propagation speed of the solution.

Before showing our results, we recall some well-known results of reaction-diffusion equations. In the pioneering work [5], Fife and McLeod studied the one-dimensional

---

2020 *Mathematics Subject Classification.* 35B09, 35B40, 35E15, 35K15, 35K57.
Key words and phrases. Reaction-diffusion equation; asymptotic behavior; propagation speed.

Submitted January 6, 2024. Published November 11, 2024.

homogeneous reaction-diffusion equation

$$u_t = u_{xx} + f(u), \quad t \in \mathbb{R},\ x \in \mathbb{R}, \tag{1.3}$$

where $f(u)$ is of bistable type, that is, there is $\theta \in (0,1)$ such that

$$f(0) = f(1) = f(\theta) = 0,\ f < 0 \text{ on } (0,\theta) \quad \text{and} \quad f > 0 \text{ on } (\theta,1). \tag{1.4}$$

They proved that (1.3) admits a unique traveling front which is an entire solution having the form $\phi(x - ct)$ and satisfying

$$\begin{gathered}\phi'' + c\phi' + f(\phi) = 0, \quad \text{in } \mathbb{R},\\ \phi(-\infty) = 1, \quad \phi(+\infty) = 0.\end{gathered}$$

The function ϕ and the constant c are called profile and propagation speed of the traveling front respectively. By [5], the profile ϕ and the speed c are uniquely determined by the reaction term f, and the sign of c is the same as the sign of $\int_0^1 f(u)du$. For the initial value problem of (1.3) with the initial value $u(0,x) = u_0(x)$, they proved that if $u_0(x)$ satisfies $\liminf_{x\to-\infty} u_0(x) > \theta$ and $\limsup_{x\to+\infty} u_0(x) < \theta$, then

$$\sup_{x\in\mathbb{R}} |u(t,x) - \phi(x - ct + x_0)| \to 0, \quad \text{as } t \to +\infty,$$

for some constant x_0. This implies that the traveling front $\phi(x-ct)$ is asymptotically stable. Moreover, if $c > 0$ and the initial value $u_0(x)$ satisfies

$$\limsup_{|x|\to+\infty} u_0(x) < \theta \ \text{ and } u_0(x) > \theta \quad \text{for } |x| < L,$$

where L is a sufficiently large constant, then

$$\sup_{x\in\mathbb{R}} |u(t,x) - (\phi(x - ct + \xi_1) + \phi(-x - ct + \xi_2) - 1)| \to 0 \quad \text{as } t \to +\infty,$$

for some constants ξ_1 and ξ_2.

For the high dimensional version of (1.3),

$$u_t = \Delta u + f(u), \quad t > 0,\ x \in \mathbb{R}^N \quad (\text{with } N \ge 2), \tag{1.5}$$

Jones [7] proved that if $c > 0$ and the solution of the compactly supported initial value grows in the sense that $u(t,x) \to 1$ locally uniformly in $\mathbb{R}^N$ as $t \to +\infty$, its level sets will go around and around. Roughly speaking, the solution is approximately radially symmetric solution as $t \to +\infty$. From the results of [7], we also know that the solution $u(t,x)$ satisfies

$$\begin{gathered}\forall 0 < c_1 < c, \quad \inf_{|x|\le c_1 t} u(t,x) \to 1 \text{ as } t \to +\infty,\\ \forall c_2 > c, \quad \sup_{|x|\ge c_2 t} u(t,x) \to 0 \text{ as } t \to +\infty.\end{gathered}$$

The constant c is also called propagation speed of the solution u by the fact that the area where u close to 1 extends almost at the speed c as $t \to +\infty$. The propagation speeds of solutions of initial value problems for other types of reaction terms and environment have been extensively investigated, see [2, 4, 9] and references therein.

If we look at the radially symmetric solutions $u(t,r) = u(t,|x|)$ of (1.5), it is not hard to show that $u(t,r)$ satisfies the one-dimensional equation

$$u_t = u_{rr} + \frac{N-1}{r} u_r + f(u). \tag{1.6}$$

By comparing with (1.3), the above equation contains an additional drift term. The influence of more general drifts to the long-time behavior of solutions has been investigated by Uchiyama [10], through considering the following equation

$$u_t = u_{xx} + k(x)u_x + f(u), \quad t > 0,\ x \in \mathbb{R},$$

with the initial value $u(0,x) = u_0(x)$. Here, f is still bistable, that is, satisfying (1.4), and the drift $k(x)$ satisfies

$$\lim_{x\to+\infty} k(x) = 0 \quad \text{and} \quad k'(x) = O(\frac{1}{x\ln^3 x}) \quad \text{as } x \to +\infty.$$

Then, if $\limsup_{x\to+\infty} u_0(x) < \theta$ and $u(t,x)$ grows in the sense that $u(t,x) \to 1$ locally uniformly in $\mathbb{R}$ as $t \to +\infty$, it holds

$$\sup_{x>0} |u(t,x) - \phi(x - ct + m(t) + x_0)| \to 0 \quad \text{as } t \to +\infty, \tag{1.7}$$

where x_0 is a constant and $m(t)$ is the solution of $m'(t) = k(ct - m(t))$ for $t \ge 0$ and $m(0) = -L$ for some $L > 0$. As mentioned in [10], if $k'(x) = O(x^{-\frac{3}{2}} \setminus \ln x)$, then the asymptotic form of $m(t)$ can be given. Especially if $k(x) = (N-1)/x$ as in (1.6), then $m(t) = (N-1)/c \ln t + O(1)$. By applying this result to the high-dimensional equation (1.5), it implies that the high-dimensional diffusion will cause a logarithmic delay to the propagation speed of the solution.

For the time-periodic reaction-diffusion equation (1.1) without drifts, Alikakos, Bates and Chen [1] proved that there exists a unique time-periodic traveling front $\phi(t, x - ct)$ satisfying

$$\begin{gathered} \phi_t - c\phi_\xi - \phi_{\xi\xi} - f(t,\phi) = 0, \quad t \in \mathbb{R},\ \xi \in \mathbb{R}, \\ \phi(t+T,\cdot) = \phi(t,\cdot),\ \phi(t,-\infty) = 1, \quad \phi(t,+\infty) = 0,\ t \in \mathbb{R}. \end{gathered} \tag{1.8}$$

The profile ϕ and the speed c are uniquely determined by the reaction term f, and the sign of c has the same sign of $\int_0^T \int_0^1 f(t,s)dsdt$. They also studied the stability of the time-periodic traveling front $\phi(t, x-ct)$. Precisely, they proved that for (1.3) with $k(x) \equiv 0$, if the initial value $u_0(x)$ satisfies $\liminf_{x\to-\infty} u_0(x) > 1 - \rho$ and $\limsup_{x\to+\infty} u_0(x) < \rho$ where ρ is given by (1.2), then the solution $u(t,x)$ satisfies

$$\sup_{x\in\mathbb{R}} |u(t,x) - \phi(t, x - ct + x_0)| \to 0, \quad \text{as } t \to +\infty, \tag{1.9}$$

for a constant x_0.

In this article, we first investigate the influence of the drift term on the long-time behavior of the solution of (1.1), especially to the propagation speed. Some a priori assumptions will be needed throughout this paper. We always assume that $\int_0^T \int_0^1 f(t,s)dsdt > 0$ which means $c > 0$ and

$$\lim_{x\to+\infty} k(x) = 0 \quad \text{and} \quad k'(x) = O(\frac{1}{x\ln^3 x}), \quad \text{as } x \to +\infty. \tag{1.10}$$

Moreover, we assume that the solution $u(t,x)$ of (1.1) grows in the sense that

$$u(t,x) \to 1 \quad \text{locally uniformly in } \mathbb{R} \text{ as } t \to +\infty. \tag{1.11}$$

This assumption is not empty. For example, according to the results of [1], (1.11) will hold if $u_0(x) = 1$ for $x \in [-L, L]$ and $\sup_{x\in\mathbb{R}} k(x)$ is small enough for large enough L.

We aim to generalize the results of Uchiyama [10] to the time-periodic case. However, by the effect of the time periodicity, such generalization is not trivial. For example, the Lyapunov functional in [10] can not be used to prove the convergence of the solution to the traveling front. Now, we present our main results.

Theorem 1.1. *Assume that* (1.10) *holds. If* $\limsup_{x\to+\infty} u_0(x) < \rho$ *and the solution* $u(t,x)$ *of* (1.1) *grows in the sense of* (1.11), *then*

$$\sup_{x>0}|u(t,x) - \phi(t, x - ct + m(t) + x_0)| \to 0, \qquad \text{as } t \to +\infty, \tag{1.12}$$

for some constant x_0, *where* $m(t)$ *is the solution of* $m'(t) = k(ct - m(t))$ *for* $t \geq 0$ *with* $m(0) = -L$ *for some large* L.

By changing $u(t,x)$ to $u(t,-x)$, one can easily show that similar results hold for $x < 0$.

Corollary 1.2. *Assume that* $k(x) \to 0$ *and* $k'(x) = O(1/(|x|\ln^3|x|))$ *as* $x \to -\infty$. *If* $\limsup_{x\to-\infty} u_0(x) < \rho$ *and the solution* $u(t,x)$ *of* (1.1) *grows in the sense of* (1.11), *then*

$$\sup_{x<0}|u(t,x) - \phi(t, -x - ct + m(t) + x_0)| \to 0, \quad \text{as } t \to +\infty \tag{1.13}$$

for a constant x_0, *where* $m(t)$ *is the solution of* $m'(t) = -k(-ct + m(t))$ *for* $t \geq 0$ *with* $m(0) = -L$ *for some large* L.

From Theorem 1.1, we have that if $k(x) = 1/x$ for large x, then $m(t) = \frac{1}{c}\ln t + O(1)$ which means that there is a logarithmic delay for the propagation speed of u. If $k(x) = 1/x^r$ for large x with $r > 1$, then $m(t) = o(1/t^{r-1}) + O(1)$ which means that if the drift delays very fast, then the influence to the propagation speed of u is small. On the other hand, when $k(x) \equiv 0$, Theorem 1.1 also leads to the asymptotic stability of the time-periodic traveling front proved in [1].

The above results can be applied to initial value problems in high-dimensional domains. Precisely, we consider the following initial value problem

$$\begin{gathered} v_t = \Delta v + f(t,v), \quad x \in \Omega,\ t > 0, \\ v(0,x) = v_0(x), \quad x \in \Omega, \\ \partial_\nu v = 0, \quad x \in \partial\Omega. \end{gathered} \tag{1.14}$$

where Ω is an unbounded connected set of $\mathbb{R}^N$ with smooth boundary and $0 \leq v_0(x) \leq 1$ is compactly supported. The first application is to the whole space $\mathbb{R}^N$.

Theorem 1.3. *Assume that* $\Omega = \mathbb{R}^N$ *and the solution* $v(t,x)$ *of* (1.14) *grows in the sense that* $v(t,x) \to 1$ *locally uniformly in* $\mathbb{R}^N$ *as* $t \to +\infty$. *Then, for any* $\varepsilon > 0$, *there exist positive constants* $L(\varepsilon)$ *and* T *such that*

$$v(t,x) \geq 1 - \varepsilon, \quad \text{for } |x| \leq ct - \frac{N-1}{c}\ln t - L(\varepsilon) \text{ and } t \geq T, \tag{1.15}$$

$$v(t,x) \leq \varepsilon, \quad \text{for } |x| \geq ct - \frac{N-1}{c}\ln t + L(\varepsilon) \text{ and } t \geq T. \tag{1.16}$$

The second application is to exterior domains. Here, an exterior domain is defined by $\Omega = \mathbb{R}^N \setminus K$, where K is a compact set.

Theorem 1.4. *Assume that* $\Omega = \mathbb{R}^N \setminus K$ *and the solution* $v(t,x)$ *of* (1.14) *grows in the sense that* $v(t,x) \to 1$ *locally uniformly in* Ω *as* $t \to +\infty$. *Then, for any*

$\varepsilon > 0$, there exist positive constants $L(\varepsilon)$ and T such that

$$v(t,x) \geq 1-\varepsilon, \quad \text{for } x \in \Omega \text{ such that } |x| \leq ct - \frac{N-1}{c}\ln t - L_1(\varepsilon) \text{ and } t \geq T, \tag{1.17}$$

$$v(t,x) \leq \varepsilon, \quad \text{for } x \in \Omega \text{ such that } |x| \geq ct - \frac{N-1}{c}\ln t + L_2(\varepsilon) \text{ and } t \geq T. \tag{1.18}$$

Remark 1.5. From the results in [3], one knows that if $v_0(x)$ is close to 1 in a sufficiently large ball and K is star-shaped or directionally convex with respect to a hyperplane, the solution $v(t,x)$ grows.

This paper is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3, we show the applications, that is, we prove Theorems 1.3 and 1.4.

2. Asymptotic behavior of solutions

This section is devoted to the proof of Theorem 1.1. We first need some parameters and some auxiliary functions. Let $L > 0$ be a sufficiently large constant and $m_L(t)$ be the solution of

$$m_L'(t) = k(ct - m_L(t)), \text{ for } t \geq 0 \quad \text{and} \quad m_L(0) = -L. \tag{2.1}$$

Then, by taking L sufficiently large and since $k(x) \to 0$ as $x \to +\infty$, one has that $m_L(t) < ct$ for all $t > 0$ and $ct - m(t) = O(t) \to +\infty$ as $t \to +\infty$. Take a function

$$\gamma(t) = \frac{2}{3}\ln^{\frac{3}{2}}(t+1), \quad \text{for } t > 0.$$

Then, it is increasing and has the following properties

$$\gamma(0) = 0, \quad \sup_{t\geq 0}\gamma'(t) \leq 1, \quad \int_0^{+\infty} e^{-b\gamma(t)}dt < +\infty \quad \text{for all } b > 0.$$

For each $M \geq 0$, we let

$$\lambda_{L,M}(t) = \sup_{x>1,\,|x-ct|\leq M+\gamma(t)} |k(x-m_L(t)) - k(ct - m_L(t))|. \tag{2.2}$$

Since $k'(x) = O(1/(x\ln^3 x))$ as $x \to +\infty$, one knows that $\lambda_{L,M}(t)$ is integrable on $(0,+\infty)$ and

$$\lim_{L\to+\infty}\int_0^{+\infty}\lambda_{L,M}(t)dt = 0.$$

Remember that there exist $\rho \in (0,\frac{1}{2})$ and $\tau > 0$ such that

$$-f_u(t,u) \geq \tau \quad \text{for } t \in \mathbb{R} \text{ and } u \in [0,\rho]\cup[1-\rho,1]. \tag{2.3}$$

Since $\phi(t,-\infty) = 1$ and $\phi(t,+\infty) = 0$, there is $R > 0$ such that

$$\begin{aligned} 0 < \phi(t,\xi) \leq \frac{\rho}{2} \quad &\text{for } t \in \mathbb{R} \text{ and } \xi \geq R, \\ 1 - \frac{\rho}{2} \leq \phi(t,\xi) < 1 \quad &\text{for } t \in \mathbb{R} \text{ and } \xi \leq -R. \end{aligned} \tag{2.4}$$

Since $\phi_\xi(t,\xi) < 0$ by [1], there is $a > 0$ such that

$$-\phi_\xi(t,\xi) \geq a \quad \text{for } t \in \mathbb{R} \text{ and } -R \leq \xi \leq R. \tag{2.5}$$

By [1], one also knows that there exist $C_1 > 0$ and $\eta > 0$ such that

$$|\phi_\xi(t,\xi)| \leq C_1 e^{-\eta|\xi|}, \quad \text{for } t \in \mathbb{R} \text{ and } \xi \in \mathbb{R}, \tag{2.6}$$

and

$$1-\phi(t,\xi) \le C_1 e^{\eta\xi}, \quad \text{for } t\in\mathbb{R} \text{ and } \xi<0, \tag{2.7}$$

Let

$$q=\min\{\frac{\tau}{2},\eta\}.$$

We show some upper and lower bounds for the solution $u(t,x)$ under some initial conditions and boundary conditions.

Lemma 2.1. *For each $0<r<\frac{\rho}{2}$, there are constants M and $L_0>0$ such that*

(i) *if $L\ge L_0$, $u_0(x)\ge \phi(0,x-L)-r$ for $x>0$, and $u(t,1)\ge 1-re^{-q\gamma(t)}$ for $t>0$, then*

$$u(t,x)\ge \phi(t,x-ct+m_L(t)+\alpha(t))-re^{-q\gamma(t)}, \quad \text{for } t>0 \text{ and } x\ge 1. \tag{2.8}$$

(ii) *if $L\ge L_0$ and $u_0(x)\le \phi(0,x-L)+r$ for $x>0$, then*

$$u(t,x)\le \phi(t,x-ct+m_L(t)-\alpha(t))+re^{-q\gamma(t)}, \quad \text{for } t>0 \text{ and } x\ge 1, \tag{2.9}$$

where

$$\alpha(t)=\int_0^t(\lambda_{M,L}(s)+rCe^{-q\gamma(s)})ds$$

and C is a large constant.

Proof. (i) For $t>0$ and $x\in\mathbb{R}$, we define

$$u^-(t,x)=\max\{\phi(t,x-ct+m_L(t)+\alpha(t))-re^{-q\gamma(t)},0\}$$

We are going to show that $u^-(t,x)$ is a subsolution of (1.1) for $t>0$ and $x\ge 1$.

Firstly, we check the initial and boundary conditions. For $t=0$, one has that

$$u^-(0,x)=\max\{\phi(0,x-L)-r,0\}\le u_0(x) \text{ for } x>0.$$

For $x=1$, one has that

$$\begin{aligned} u^-(t,1)&=\max\{\phi(t,1-ct+m_L(t)+\alpha(t))-re^{-q\gamma(t)},0\}\\ &\le 1-re^{-q\gamma(t)}\le u(t,1)\end{aligned}$$

for $t>0$.

Then, we have to show only that

$$Q[u^-]:=u^-_t-u^-_{xx}-k(x)u^-_x-f(t,u^-)\le 0, \tag{2.10}$$

for $t>0$ and $x\ge 1$ such that $u^-(t,x)>0$. By (1.8), it follows that

$$\begin{aligned} Q[u^-]&=\phi_\xi(m_L'(t)+\alpha'(t)-k(x))+rq\gamma'(t)e^{-q\gamma(t)}+f(t,\phi)-f(t,u^-)\\ &=\phi_\xi(k(ct-m_L(t))-k(x)+\lambda_{L,M}(t)+rCe^{-q\gamma(t)})+rq\gamma'(t)e^{-q\gamma(t)}\\ &\quad+f(t,\phi)-f(t,u^-)\end{aligned} \tag{2.11}$$

where ϕ and ϕ_ξ take values at $(t,x-ct+m_L(t)+\alpha(t))$.

For $t>0$ and $x\ge 1$ such that $x-ct+m_L(t)+\alpha(t)\le -R$, one has that

$$1-\frac{\rho}{2}\le \phi(t,x-ct+m_L(t)+\alpha(t))<1 \quad \text{and} \quad 1-\rho\le u^-(t,x)<1,$$

since $r<\rho/2$. Then, by (2.3) and the mean value theorem, it follows that

$$f(t,\phi)-f(t,u^-)\le -\tau re^{-q\gamma(t)}.$$

By $\phi_\xi < 0$, $q \le \tau/2$, $\gamma'(t) \le 1$ and (2.11), one has that

$$Q[u^-] \le \phi_\xi(k(ct - m_L(t)) - k(x) + \lambda_{M,L}(t)) - \frac{\tau}{2} r e^{-q\gamma(t)}. \tag{2.12}$$

Let

$$K = \sup_{x>0} |k(x)|.$$

Notice that $\alpha(t)$ is bounded for all $t \ge 0$. Take M sufficiently large such that $M - \alpha(t) > 0$ and

$$2KC_1 e^{-\eta(M-\alpha(t))} \le \frac{\tau r}{2} \tag{2.13}$$

where C_1 and η are defined by (2.6). Then, if $1 \le x \le ct - m_L(t) - \gamma(t) - M$, one has that $x - ct + m_L(t) + \alpha(t) \le -\gamma(t) - M + \alpha(t) < 0$ and

$$|\phi_\xi(t, x - ct + m_L(t) + \alpha(t))| \le C_1 e^{-\eta(\gamma(t)+M-\alpha(t))} \le \frac{\tau r}{4K} e^{-\eta\gamma(t)}.$$

In this case, it follows from (2.12) that

$$Q[u^-] \le -2K\phi_\xi - \frac{\tau}{2} r e^{-q\gamma(t)} \le 0,$$

by $\lambda_{M,L}(t) \ge 0$ and $q \le \eta$. If $x \ge 1$ and $x \ge ct - m_L(t) - \gamma(t) - M$, it follows from $x - ct + m_L(t) + \alpha(t) \le -R$ and (2.2) that

$$k(ct - m_L(t)) - k(x) + \lambda_{M,L}(t) \ge 0.$$

Thus, in this case, $Q[u^-] \le 0$ by (2.12).

For $t > 0$ and $x \ge 1$ such that $-R \le x - ct + m_L(t) + \alpha(t) \le R$. One has that $-\phi_\xi \ge a$. Moreover, by (2.2),

$$k(ct - m_L(t)) - k(x) + \lambda_{M,L}(t) \ge 0.$$

Then, it follows from (2.11) that

$$Q[u^-] \le -arCe^{-q\gamma(t)} + rqe^{-q\gamma(t)} + \|f_u(t,u)\|_{L^\infty} re^{-q\gamma(t)} \le 0,$$

by taking C sufficiently large.

For $t > 0$ and $x \ge 1$ such that $x - ct + m_L(t) + \alpha(t) \ge R$, one has that

$$0 < \phi(t, x - ct + m_L(t) + \alpha(t)) \le \frac{\rho}{2} \quad \text{and} \quad 0 \le u^-(t,x) \le \rho.$$

Then, by (2.3), it follows that

$$f(t,\phi) - f(t,u^-) \le -\tau r e^{-q\gamma(t)}.$$

By $\phi_\xi < 0$, $q \le \frac{\tau}{2}$, $\gamma'(t) \le 1$ and (2.11), one has that

$$Q[u^-] \le \phi_\xi(k(ct - m_L(t)) - k(x) + \lambda_{M,L}(t)) - \frac{\tau}{2} r e^{-q\gamma(t)}.$$

If $x \ge ct - m_L(t) + \gamma(t) + M$, one has that

$$x - ct + m_L(t) + \alpha(t) \ge \gamma(t) + M + \alpha(t) > 0,$$

and

$$|\phi_\xi(t, x - ct + m_L(t) + \alpha(t))| \le C_1 e^{-\eta(\gamma(t)+M+\alpha(t))} \le \frac{\tau r}{4K} e^{-\eta\gamma(t)},$$

by (2.13). Then, $Q[u^-] \le 0$ since $\lambda_{M,L}(t) \ge 0$ and $q \le \eta$. If $x \le ct - m_L(t) + \gamma(t) + M$, it follows from $x - ct + m_L(t) + \alpha(t) \ge R$ and (2.2) that

$$k(ct - m_L(t)) - k(x) + \lambda_{M,L}(t) \ge 0.$$

Then $Q[u^-] \le 0$.

This completes the proof of (2.10). Then, (2.8) follows from the comparison principle.

(ii) The proof is almost parallel to the proof of (i). For $t > 0$ and $x \in \mathbb{R}$, we define

$$u^+(t,x) = \min\{\phi(t, x - ct + m_L(t) - \alpha(t)) + re^{-q\gamma(t)}, 1\}$$

Let us check that $u^+(t,x)$ is a supersolution of (1.1) for $t > 0$ and $x \geq 1$.

We first verify the initial condition. For $t = 0$, one has that

$$u^+(0,x) = \min\{\phi(0, x - L) + r, 1\} \geq u_0(x), \text{ for } x > 0.$$

Since $ct - m_L(t) = O(t) \to +\infty$ and $\gamma = o(t)$ as $t \to +\infty$, it follows from (2.12) and $q \leq \eta$ that

$$\phi(t, 1 - ct + m_L(t) - \alpha(t)) + re^{-q\gamma(t)} \geq 1, \quad \text{for } t > 0,$$

even if it means increasing L and hence, $u^+(t,x) = 1 \geq u_0(1)$ for $t > 0$.

Then, we have to check only that

$$Q[u^+] := u^+_t - u^+_{xx} - k(x)u^+_x - f(t, u^+) \geq 0, \tag{2.14}$$

for $t > 0$ and $x \geq 1$ such that $u^+(t,x) < 1$. By (1.8), it follows that

$$\begin{aligned} Q[u^+] &= (m_L'(t) - \alpha'(t) - k(x))\phi_\xi - rq\gamma'(t)e^{-q\gamma(t)} + f(t,\phi) - f(t,u^+) \\ &= \phi_\xi(k(ct - m_L(t)) - k(x) - \lambda_{L,M}(t) - rCe^{-q\gamma(t)}) - rq\gamma'(t)e^{-q\gamma(t)} \\ &\quad + f(t,\phi) - f(t,u^+) \end{aligned}$$

where ϕ and ϕ_ξ take values at $(t, x - ct + m_L(t) - \alpha(t))$.

For $t > 0$, such that $x - ct + m_L(t) - \alpha(t) \leq -R$, one has that

$$1 - \frac{\rho}{2} \leq \phi(t, x - ct + m_L(t) - \alpha(t)) < 1 \quad \text{and} \quad 1 - \rho \leq u^+(t,x) < 1,$$

since $r < \rho/2$. Then, by (2.3), it follows that

$$f(t,\phi) - f(t,u^+) \geq \tau re^{-q\gamma(t)}.$$

By $\phi_\xi < 0$, $q \leq \tau/2$ and $\gamma'(t) \leq 1$, one has that

$$Q[u^+] \geq \phi_\xi(k(ct - m_L(t)) - k(x) - \lambda_{L,M}(t)) + \frac{\tau}{2}re^{-q\gamma(t)}.$$

If $x \leq ct - m_L(t) - \gamma(t) - M$, one has that $x - ct + m_L(t) - \alpha(t) \leq -\gamma(t) - M - \alpha(t)$ and

$$|\phi_\xi(t, x - ct + m_L(t) - \alpha(t))| \leq C_1 e^{-\eta(\gamma(t) + M + \alpha(t))} \leq \frac{\tau r}{4K} e^{-\eta\gamma(t)},$$

by (2.13). In this case,

$$Q[u^+] \geq -2K\phi_\xi - \frac{\tau}{2}re^{-q\gamma(t)} \geq 0,$$

by $\lambda_{M,L}(t) \geq 0$ and $q \leq \eta$. If $x \geq ct - m_L(t) - \gamma(t) - M$, it follows from $x - ct + m_L(t) - \alpha(t) \leq -R$ and (2.2) that

$$k(ct - m_L(t)) - k(x) - \lambda_{M,L}(t) \leq 0.$$

Thus, $Q[u^+] \geq 0$.

For $t > 0$ such that $-R \leq x - ct + m_L(t) - \alpha(t) \leq R$. One has that $-\phi_\xi \geq a$. Moreover, by (2.2),

$$k(ct - m_L(t)) - k(x) - \lambda_{M,L}(t) \leq 0.$$

Then,

$$Q[u^+] \geq arCe^{-q\gamma(t)} - rqe^{-q\gamma(t)} - \|f_u(t,u)\|_{L^\infty} re^{-q\gamma(t)} \geq 0,$$

by taking C sufficiently large.

For $t > 0$ such that $x - ct + m_L(t) - \alpha(t) \geq R$, one has that

$$0 < \phi(t, x - ct + m_L(t) - \alpha(t)) \leq \rho \quad \text{and} \quad 0 \leq u^+(t,x) \leq \rho.$$

Then, by (2.3), it follows that

$$f(t,\phi) - f(t,u^-) \leq -\tau re^{-q\gamma(t)}.$$

By $\phi_\xi < 0$, $q \leq \tau/2$ and $\gamma'(t) \leq 1$, one has that

$$Q[u^+] \leq \phi_\xi(k(ct - m_L(t)) - k(x) - \lambda_{M,L}(t)) + \frac{\tau}{2} re^{-q\gamma(t)}.$$

If $x \geq ct - m_L(t) + \gamma(t) + M$, one has that

$$x - ct + m_L(t) - \alpha(t) \geq \gamma(t) + M - \alpha(t) > 0,$$

and

$$|\phi_\xi(t, x - ct + m_L(t) - \alpha(t))| \leq C_1 e^{-\eta(\gamma(t)+M-\alpha(t))} \leq \frac{\tau r}{4K} e^{-\eta\gamma(t)}.$$

by (2.13). Then, $Q[u^+] \geq 0$ since $\lambda_{M,L}(t) \geq 0$ and $q \leq \eta$. If $x \leq ct - m_L(t) + \gamma(t) + M$, it follows from $x - ct + m_L(t) - \alpha(t) \geq R$ and (2.2) that

$$k(ct - m_L(t)) - k(x) - \lambda_{M,L}(t) \leq 0.$$

Then, $Q[u^+] \geq 0$.

This completes the proof of (2.14). Then, (2.9) follows from the comparison principle. □

We then show that the solution $u(t,x)$ of (1.1) satisfies the initial and boundary conditions in Lemma 2.1 after some time T.

Lemma 2.2. *Let $u(t,x)$ be the solution of* (1.1) *and assume that $u(t,x)$ grows in the sense of* (1.11). *For any $0 < c_1 < c$, there is a positive constant δ such that*

$$\sup_{0<x\leq c_1 t}(1 - u(t,x)) = o(e^{-\delta t}), \quad \text{as } t \to +\infty.$$

Proof. Let $0 < c_1 < c$ and $c' = (c_1 + c)/2$. By a comparison argument, it suffices to prove that

$$1 - u(t, c_1 t) = o(e^{-\delta t}), \quad \text{as } t \to +\infty, \tag{2.15}$$

for some $\delta > 0$. Since $k(x) \to 0$ as $x \to +\infty$, one can pick $L > 0$ large enough such that

$$\alpha := c - c' - \sup_{x>L} k(x) > 0.$$

Take $r \in (0,\rho)$ such that

$$r \leq \min\Big\{\frac{a\alpha}{\|f_u(t,u)\|_{L^\infty} + \tau}, C_1\Big\}, \tag{2.16}$$

where ρ, τ and a are defined by (2.3) and (2.5) respectively. Take $\beta \in (0,\eta)$ where η is defined by (2.6) such that

$$\beta^2 + \beta \sup_{x>L} |k(x)| \leq \tau.$$

For $t > 0$ and $x \in \mathbb{R}$, define

$$u_-(t,x) := \max\{\phi(t, x - c't) - re^{-\beta x}, 0\}.$$

We are going to show that $u_-(t,x)$ is a subsolution of (1.1).

Since $\phi(0,x) \le C_1 e^{-\eta x}$ for $x > 0$ and by (1.11) and u grows, there is $T_1 > 0$ such that

$$u(T_1, x) \ge 0 \ge \phi(0,x) - re^{-\beta x}, \quad \text{for } x \ge 0,$$

and

$$u(t,0) \ge 1 - r, \quad \text{for } t \ge T_1.$$

Thus, $u(T,x) \ge u_-(0,x)$ for $x \ge 0$ and $u(t,0) \ge u_-(t,0)$ for $t \ge T_1$. Then, we only have to show that

$$Q[u_-] := (u_-)_t - (u_-)_{xx} - k(x)(u_-)_x - f(t,u_-) \le 0,$$

for $t \ge T_1$ and $x \ge 0$ such that $u_- > 0$. By (1.8), it follows that

$$\begin{aligned} Q[u_-] &= (c - c' - k(x))\phi_\xi + (r\beta k(x) - r\beta^2)e^{-\beta x} + f(t,\phi) - f(t,u_-) \\ &\le \alpha\phi_\xi + \tau re^{-\beta x} + f(t,\phi) - f(t,u_-), \end{aligned}$$

where ϕ and ϕ_ξ take values at $(t, x - c't)$. Let R be defined by (2.4). Then, for $t \ge T_1$ and $x \ge 0$ such that $x - c't \le -R$ and $x - c't \ge R$ respectively, one has that $1 - \rho \le u_-(t,x) \le 1$ and $0 \le u_-(t,x) \le \rho$ respectively. Thus, by the mean value theorem and (2.3),

$$f(t,\phi) - f(t,u_-) \le -\tau re^{-\beta x}.$$

Since $\phi_\xi < 0$, it follows that $Q[u_-] \le 0$. For $t \ge T_1$ and $x \ge 0$ such that $-R \le x - c't \le R$, one has that $-\phi_\xi \ge a$. Then, by (2.16),

$$Q[u_-] \le -a\alpha + (\tau + \|f_u(t,u)\|_{L^\infty})re^{-\beta x} \le 0.$$

Consequently, from the comparison principle it follows that

$$u(t + T_1, x) \ge u_-(t,x) \ge \phi(t, x - c't) - re^{-\beta x}, \quad \text{for } t \ge 0 \text{ and } x \ge 0.$$

Therefore, (2.15) holds. This completes the proof. □

Lemma 2.3. *Let $u(t,x)$ be the solution of* (1.1). *If $\limsup_{x\to+\infty} u_0(x) < \rho$, then $\limsup_{x\to+\infty} u(t,x)$ converges to 0, as $t \to +\infty$.*

Proof. Let $r := \limsup_{x\to+\infty} u_0(x) < \rho$. Let $v(t,x)$ be the solution of

$$v_t = v_{xx} + k(x)v_x + f(t,v), \quad \text{for } t > 0 \text{ and } x \in \mathbb{R},$$

and

$$v(0,x) = \begin{cases} 1, & \text{if } x \le 0, \\ r, & \text{if } x > 0. \end{cases}$$

Let $\Phi(t)$ be the solution of $\Phi'(t) = f(t,\Phi)$ with $\Phi(0) = r$. Then $\Phi(t) \to 0$ as $t \to +\infty$ by (2.3). It follows from the maximum principle and standard parabolic estimates that $v(t,+\infty) = \Phi(t)$. Since $\limsup_{x\to+\infty} u_0(x) = r$ and $0 \le u_0(x) \le 1$, there is $L > 0$ such that $u_0(x) \le v(0, x - L)$ and hence, $u(t,x) \le v(t, x - L)$. Then the conclusion follows. □

Proof of Theorem 1.1. Let $u(t,x)$ be the solution of (1.1). Fix $r \in (0,\rho)$. Then, by Lemma 2.3, there is $T_1 = k_1 T, k_1 \in \mathbb{Z}$, where T is the periodic such that $\limsup_{x\to+\infty} u(T_1,x) \le \frac{r}{2}$. Since $\phi(t,-\infty) = 1$ and $\phi(t,+\infty) = 0$, there is $L > 0$ such that

$$u(T_1,x) \le \phi(0, x - L) + r \quad \text{for } x \ge 0.$$

By Lemma 2.1, one has that

$$u(T_1+t,x) \le \phi(t,x-ct+m_L(t)-\alpha(t)) + re^{-q\gamma(t)}, \quad \text{for } t>0 \text{ and } x>0$$

Since $u(t,x)$ grows and $\gamma(t)=o(t)$ as $t\to+\infty$, by Lemma 2.2 there is $T_2=k_2T>0$, $k_2\in\mathbb{Z}$ such that

$$u(T_2,x) \ge \phi(0,x-L)-r, \text{ for } x\ge 0,$$

and

$$u(t,1)\ge 1-re^{-q\gamma(t)}, \quad \text{for } t\ge T_2.$$

Then, by Lemma 2.1, one has that

$$u(T_2+t,x) \ge \phi(t,x-ct+m_L(t)+\alpha(t)) - re^{-q\gamma(t)}, \quad \text{for } t>0 \text{ and } x\ge 1.$$

Since $\lim_{t\to+\infty}\alpha(t)<+\infty$ and $m_L(t)$ satisfies (2.1), there exist constants α_1 and α_2 such that

$$\begin{aligned} &\phi(t,x-ct+m_L(t)+\alpha_1) - re^{-q\gamma(t-T_2)} \\ &\le u(t,x) \\ &\le \phi(t,x-ct+m_L(t)+\alpha_2) + re^{-q\gamma(t-T_1)}, \end{aligned} \tag{2.17}$$

for $t\ge\max\{T_1,T_2\}$ and $x\ge 1$.

Now, take a sequence $\{t_n:=nT\}_{n\in N}$ such that $t_n\to+\infty$ as $n\to+\infty$. Let

$$u_n(t,x)=u(t+t_n,x+ct_n-m_L(t_n)).$$

Then by 2.17, one has that

$$\begin{aligned} &\phi(t,x-ct+m_L(t+t_n)+\alpha_1-m_L(t_n)) - re^{-q\gamma(t+t_n-T_2)} \\ &\le u_n(t,x) \le \phi(t,x-ct+m_L(t+t_n)-m_L(t_n)+\alpha_2) + re^{-q\gamma(t+t_n-T_1)}, \end{aligned} \tag{2.18}$$

for $t\ge -t_n+\max\{T_1+T_2\}$ and $x\ge 1-ct_n+m_L(t_n)$. Since $m_L(t)$ satisfies (2.1), it follows that $m_L(t+t_n)-m_L(t_n)\to 0$ as $t_n\to+\infty$ locally uniformly for $t\in\mathbb{R}$. By parabolic estimates, the sequence $u_n(t,x)$ converges to an entire solution $u_\infty(t,x)$ locally uniformly for $(t,x)\in\mathbb{R}\times\mathbb{R}$ of the equation

$$(u_\infty)_t=(u_\infty)_{xx}+f(t,u_\infty), \quad t\in\mathbb{R},\ x\in\mathbb{R}.$$

By (2.18), one also has that

$$\phi(t,x-ct+\alpha_1)\le u_\infty(t,x)\le \phi(t,x-ct+\alpha_2), \quad \text{for } t\in\mathbb{R},\ x\in\mathbb{R}.$$

By the stability result of $\phi(t,x-ct)$ in [1], there is $x_0\in\mathbb{R}$ such that

$$u_\infty(t,x)\equiv\phi(t,x-ct+x_0).$$

Thus, for any $r>0$, there is $N>0$ such that

$$|u(t_N,x+ct_N-m_L(t_N))-\phi(0,x+x_0)|<r,$$

that is,

$$\begin{aligned} \phi(t_N,x-ct_N+m_L(t_N)+x_0)-r &\le u(t_N,x) \\ &\le \phi(t_N,x-ct_N+m_L(t_N)+x_0)+r \end{aligned}$$

Again by lemma 2.1, one has that

$$\begin{aligned} &\phi(t,x-ct+m_L(t)+x_0+\alpha_N(t)) - re^{-q\gamma(t-t_N)} \\ &\le u(t,x) \le \phi(t,x-ct+m_L(t)+x_0-\alpha_N(t)) + re^{-q\gamma(t-t_N)} \end{aligned} \tag{2.19}$$

for $t > t_N$ and $x \geq 1$, where $\alpha_N(t) = \int_{t_N}^{t} (\lambda_{M,L}(s) + rCe^{-q\gamma(s)})ds$. For t_N large enough, $\alpha_N(t)$ can be arbitrary small. Since r is arbitrary small and $|\phi_\xi|$ is bounded. One has that

$$\sup_{x\geq 1} |u(t,x) - \phi(t, x - ct + m_L(t) + x_0)|$$

is arbitrary small for $t > t_N$ by taking t_N large enough. Therefore,

$$\sup_{x\geq 1} |u(t,x) - \phi(t, x - ct + m_L(t) + x_0)| \to 0, \quad \text{as } t \to +\infty.$$

This completes the proof. □

3. Applications

In this section, we apply our results to high-dimensional domains. More precisely, we show asymptotic speeds for solutions of initial value problems in two kinds of high-dimensional domains: the whole space $\mathbb{R}^N$ and exterior domains.

For convenience, we modify (1.1) a little bit. Let $u(t,r)$ be the solution of

$$\begin{gathered} u_t = u_{rr} + \frac{N-1}{r} u_r + f(t,u), \quad t > 0,\ r > 0 \\ u_r(t,0) = 0. \end{gathered} \tag{3.1}$$

with compactly supported initial value $u_0(x)$. Then, by the same proof as of Theorem 1.1, if $u(t,r)$ grows, then

$$\sup_{r>1} |u(t,r) - \phi(t, r - ct + m(t) + x_0)| \to 0, \quad \text{as } t \to +\infty, \tag{3.2}$$

for some constants x_0, where $m(t)$ is the solution of $m'(t) = (N-1) \setminus (ct - m(t))$ and has the form

$$m(t) = \frac{N-1}{c} \ln t + O(1). \tag{3.3}$$

Proof of Theorem 1.3. Since $v_0(x)$ is compactly supported, there is $R_1 > 0$ such that $v_0(x) = 0$ for $|x| \geq R_1$. Let $u_1(t,r)$ be the solution of (3.1) with $u_1(0,r) = 1$ for $0 < r \leq R_1$. Then, obviously $v_0(x) \leq u_1(0,|x|)$ for $x \in \mathbb{R}^N$ and $u_1(t,|x|)$ satisfies $v_t = \Delta v + f(t,v)$ for $t > 0$ and $x \in \mathbb{R}^N$. It follows from the comparison principle that

$$v(t,x) \leq u_1(t,|x|), \quad \text{for } t > 0 \text{ and } x \in \mathbb{R}^N.$$

Since (3.2) holds for $u_1(t,r)$ and $m(t)$ satisfies (3.3), we have (1.16).

On the other hand, since $v(t,x)$ grows, there are $T > 0$ and $R_2 > 0$ such that $v(T,x) \geq 1 - \epsilon$ for $|x| \leq R_2$ where $\epsilon > 0$ is a small constant. One can take T sufficiently large such that ϵ is a sufficiently small constant and R_2 is a sufficiently large constant. Let $u_2(t,x)$ be the solution of (3.1) with $u_2(0,r) = 1 - \epsilon$ for $0 < r \leq R_2$ and $u_2(0,r) = 0$ for $r > R_2$. By $v_0(x) \geq u_2(0,|x|)$ and the comparison principle, one has that

$$v(t+T,x) \geq u_2(t,|x|), \quad \text{for } t > 0 \text{ and } x \in \mathbb{R}^N.$$

Since (3.2) holds for $u_2(t,x)$ and $m(t)$ satisfies (3.3), one immediately has (1.15). This completes the proof. □

Proof of Theorem 1.4. By lemma 2.2, there is a $\delta>0$ such that

$$\sup_{0<r\le c_1 t}(1-u(t,r))=o(e^{-\delta t})\quad\text{as } t\to+\infty, \tag{3.4}$$

for some $0<c_1<c$. By parabolic estimates, one has that

$$u_r(t,r)=o(e^{-\delta t})\quad\text{for } 0<r\le R \text{ as } t\to+\infty,$$

where R is arbitrary positive constant. Assume $\delta<\tau/2$ where τ is defined by (1.2). Take a nonnegative $C^2(\mathbb{R}^N)$ function $\xi(x)$ such that $\xi(x)$ has a compact support in Ω, $\nu(x)\cdot\nabla\xi(x)=1$ on $x\in\partial\Omega$ and $|\Delta\xi\setminus\xi|_{L^\infty}<\frac{\tau}{2}$. We refer such a function to [6, 8].

Let $R_1>0$ such that $v_0(x)=0$ for $x\in\Omega$ such that $|x|\ge R_1$ and $K\subset B(0,R_1)$. Let $u_1(t,r)$ be the solution of (3.1) with $u_1(0,r)=1$ for $0<r\le R_1$ and $u_1(0,r)=0$ for $r>R_1$. We define

$$\overline{u}_1(t,x)=u_1(t,|x|)+\beta\xi(x)e^{-\delta t},$$

where $\beta>0$ is to be given. One can calculate that

$$\partial_\nu\overline{u}_1(t,x)=(u_1)_r\frac{x}{|x|}\cdot\nu+\beta\nabla\xi\cdot\nu e^{-\delta t}\quad\text{for } x\in\partial\Omega.$$

Since $K\subset B(0,R_1)$ and $(u_1)_r=o(e^{-\delta t})$ for $x\in\partial\Omega$ as $t\to+\infty$, one can pick $\beta>0$ such that $\partial_\nu\overline{u}_1(t,x)>0$ for $x\in\partial\Omega$. Then, we show that $\overline{u}_1(t,x)$ is a supersolution of (1.14). Obviously, $v_0(x)\le\overline{u}_1(0,x)$ for $x\in\Omega$. Assume that $\xi(x)\equiv 0$ in $\mathbb{R}^N\setminus B(0,L)$ for some $L>0$. Assume without loss of generality that $u_1(t,|x|)\ge 1-\rho$ for $x\in B(0,L)$ and $t>0$. Otherwise, one can consider $u_1(t+T,|x|)$ for large $T>0$ by (3.4). Then, one only has to check that

$$(\overline{u}_1)_t-\Delta\overline{u}_1-f(t,\overline{u}_1)\ge 0,$$

for $t>0$ and $x\in\Omega\cap B(0,L)$. One can compute that

$$\begin{aligned}(\overline{u}_1)_t-\Delta\overline{u}_1-f(t,\overline{u}_1)&=-\beta\delta\xi(x)e^{-\delta t}-\beta\Delta\xi e^{-\delta t}+f(t,u_1(t,|x|))-f(t,\overline{u}_1)\\&\ge-\beta\delta\xi(x)e^{-\delta t}-\beta\Delta\xi e^{-\delta t}-\tau\beta\xi(x)e^{-\delta t}\ge 0,\end{aligned}$$

for $x\in\Omega\cap B(0,L)$. Then, by the comparison principle, one has that

$$v(t,x)\le\overline{u}_1(t,x),\quad\text{for } t>0 \text{ and } x\in\Omega.$$

By (3.2), one has (1.18).

On the other hand, since $v(t,x)$ grows, there are $T>0$ and $R_2>0$ such that $v(T,x)\ge 1-\epsilon$ for $x\in\Omega\cap B(0,R_2)$. One can take T sufficiently large such that ϵ is small enough and R_2 is large enough such that $K\subset B(0,R_2)$. Let $u_2(t,r)$ be the solution of (3.1) with $u_2(0,r)=1-\epsilon$ for $0<r\le R_2$ and $u_2(0,r)=0$ for $r>R_2$. We define

$$\underline{u}_2(t,x)=u_2(t,x)-\beta\xi(x)e^{-\delta t}$$

Similar as above arguments, one can show that $\underline{u}_2(t,x)$ is a subsolution of (1.14). By the comparison principle, one has that

$$v(t,x)\ge\underline{u}_2(t,x),\quad\text{for } t>0 \text{ and } x\in\Omega.$$

By (3.1), one has (1.17). This completes the proof. □

Acknowledgments. Linlin Li was supported by the NSF of China under Grant 12301202.

References

[1] N. Alikakos, P. W. Bates, X. Chen; Periocid traveling waves and locating oscillating patterns in multidimensional domains, *Trans. Amer. Math. Soc.*, **351** (1999), 2777-2805.

[2] D. G. Aronson, H. F. Weinberger; Multidimensional nonlinear diffusion arising in population genetics, *Adv. Math.*, **30** (1978), 33–76.

[3] H. Berestycki, F. Hamel, H. Matano; Bistable travelling waves around an obstacle, *Comm. Pure Appl. Math.*, **62** (2009), 729-788.

[4] H. Berestycki, G. Nadin; Asymptotic spreading for general heterogeneous Fisher-KPP type equations, *Memoirs Amer. Math. Soc.*, **280** (2022): 1381.

[5] P. C. Fife, J. B. McLeod; The approach of solutions of nonlinear diffusion equations to traveling front solutions, *Arch. Ration. Mech. Anal.*, **65** (1977), 335-361.

[6] H. Guo, H. Monobe; V-shaped fronts around an obstacle, *Math. Ann.*, **379** (2021), 661–689.

[7] C. K. R. T. Jones; Asymptotic behaviour of a reaction-diffusion equation in higher space dimensions, *Rocky Mountain J. Math.*, **13** (1983), 355-364.

[8] L. Li; Time-periodic planar fronts around an obstacle, *J. Nonlinear Sci.*, **31** (2021), 90.

[9] L. Rossi; The Freidlin-Gärtner formula for general reaction terms, *Adv. Math.*, **317** (2017), 267-298.

[10] K. Uchiyama; Asymptotic behavior of solutions of reaction-diffusion equations with carying drift coefficients, *Arch. Rational Mech. Anal.*, **90** (1985), 291-311.

Linlin Li
College of Science, University of Shanghai for Science and Technology, Shanghai, China
Email address: `lilinlin@usst.edu.cn`

Zhuo Chen
College of Science, University of Shanghai for Science and Technology, Shanghai, China
Email address: `imchenz@163.com`