Electronic Journal of Differential Equations, Vol. 2024 (2024), No. 70, pp. 1–26.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2024.70

PRACTICAL STABILITY OF STOCHASTIC DIFFERENTIAL DELAY EQUATIONS DRIVEN BY G-BROWNIAN MOTION WITH GENERAL DECAY RATE

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Abstract. This article is concerned with the quasi sure practical stability of nonlinear stochastic differential delay equations driven by G-Brownian motion (G-SDDEs) with a general decay rate. Sufficient conditions are established by constructing appropriate G-Lyapunov functionals. Moreover, we provide some numerical examples to demonstrate the effectiveness of the obtained results.

1. Introduction

Since Peng [24, 25] set up the G-expectation and G-Brownian motion, many papers have been published on stochastic calculus based upon G-Brownian motion, see [11, 17] and the references therein.

On that basis, Gao [13] and Peng [24] studied the existence and uniqueness of solution to G-stochastic differential equations (G-SDE) under a standard Lipschitz condition. Moreover, Lin [19] obtained the existence and uniqueness of solution to G-SDE with reflecting boundary. Later on, several authors have been working on stochastic differential equations driven by G-Brownian motion, see [1, 13, 18, 19, 20, 25]. Stochastic models under G-framework proved to be powerful to analyze interesting applications in many branches of problems with uncertainty, risk measures, the superhedging in finance, etc.

Many applied problems are modeled by non-delay systems. These are governed by the assumption that the future evolution of the system is determined just by the present state, being independent of the past states. In reality, such an assumption can be considered only as a first approximation to the real system. A more realistic model assumes that the evolution of the future states depends not only on the current state but also on the past history. Delay differential equations (DDEs) (also called hereditary systems, systems with aftereffect, functional differential equations, retarded differential equations) provide an appropriate model for physical processes whose time evolution depends on their history. Stochastic differential delay equations (SDDEs, in short) have been widely investigated over the last decades, see [14, 21, 23].

2020 *Mathematics Subject Classification.* 93E03, 60H10.
Key words and phrases. G-Stochastic delay systems; quasi sure practical stability, decay function, G-Brownian motion, G-Lyapunov functional, G-Itô formula.

Submitted March 29, 2024. Published November 11, 2024.

Recently, several works have been published on stochastic differential delay equations driven by G-Brownian motions (G-SDDEs, in short). Young et al. [27] proved the existence and uniqueness of solution for a class of G-SDDEs. The problem of stability of G-SDDEs is more complicated, and there have been published only in a few works, see [22, 27, 29].

When the origin is not a trivial solution, we investigate the stability of the SDEs with respect to a small neighborhood of the origin. Several results on the stability of the nontrivial solution of stochastic systems are proposed in [6, 8, 9]. In the investigation of the asymptotic behavior of solutions to SDEs, one can find that a solution is asymptotically stable but may not necessarily be exponentially stable. Further, in the nonlinear and/or nonautonomous situations, it may happen that the stability cannot always be exponential but can be sub- or super-exponential, see [2, 3, 7]. For this reason, the main aim of this paper is to discuss the quasi sure practical stability with a general decay rate of G-SDDEs.

Lyapunov's technique is available to state sufficient conditions for the stability of solutions to SDDEs by using the construction of some Lyapunov functions or functionals. The latter method provides better conditions than using Lyapunov functions, although the construction of Lyapunov functionals is more complicated. Different works tackled the problem of the construction of Lyapunov functionals for a wide range of equations containing some hereditary properties, see [4, 5, 28].

The general method of Lyapunov functionals construction was proposed by Kolmanovskii and Shaikhet [15, 16, 28]. This approach has already been successfully used for functional differential equations, for difference equations with discrete time, for difference equations with continuous time, etc.

Recently, the concept of practical stability with general decay rate of stochastic differential delay equations was introduced by Caraballo et al. [10]. Our main objective in this paper is to extend the results in [10] to the case of G-Brownian motion. Using the method of Lyapunov functionals and recently developed Itô calculus for SDDE driven by G-Brownian motion, we introduce and develop the practical stability with a general decay rate of stochastic differential equations with constant and time-varying delay driven by G-Brownian motion.

To the best of our knowledge, no work has been done on the practical stability for delayed stochastic differential equations driven by G-Brownian motion in the literature. Motivated by these considerations, in this paper we will investigate the practical convergence to a small ball centered at the origin with a general decay rate in terms of the existence and construction of G-Lyapunov functionals. Furthermore, we construct G-Lyapunov functionals for stochastic differential equations with constant and time-varying delay driven by G-Brownian motions, to obtain sufficient conditions ensuring the practical convergence to a small ball centered at the origin with a general decay rate.

The arrangement of the paper is presented as follows. In Section 2, we establish some preliminaries on sublinear expectations and G-Brownian motions. In Section 3, we state sufficient conditions for quasi sure practical stability of the G-SDDEs with a general decay rate by using G-Lyapunov's functionals. In Section 4, we analyze the quasi sure practical stability with a general decay rate of stochastic differential equations with constant and time-varying delay by constructing suitable G-Lyapunov functionals. Moreover, we exhibit some examples to illustrate the theoretical findings. Finally, some conclusions appear in Section 5.

2. Preliminaries

This section reviews the basic concepts and notation within the G-framework which are needed in our analysis. The reader interested in a more detailed description of the notions are referred, for instance, to [24, 25, 26].

Notation on G-stochastic calculus.
$\mathbb{R}^n$: Space of n-dimensional real column vectors,
$\langle x, y\rangle$: Scalar product of two vectors $x, y \in \mathbb{R}^n$,
If $x \in \mathbb{R}^n$, $\|x\|$ denotes its Euclidean norm,
$\Omega_t := \{\omega_{\cdot\wedge t} : \omega \in \Omega\}$, $\mathcal{F}_t = \mathcal{B}(\Omega_t)$,
$\mathcal{B}(\Omega)$: Borel σ-algebra of Ω,
$\mathcal{C}_{b,Lip(\mathbb{R}^n)}$: the space of all bounded real-valued Lipschitz continuous functions,
$L^0(\Omega)$: Space of all $\mathcal{B}(\Omega)$-measurable real functions,
$L^0(\Omega_t)$: Space of all $\mathcal{B}(\Omega_t)$-measurable real functions,
$\mathcal{B}_b(\Omega)$: all bounded elements in $L^0(\Omega), \mathcal{B}_b(\Omega_t) := \mathrm{B}_b(\Omega) \cap L^0(\Omega_t)$,
$L^p_G(\Omega)$: Banach space under the natural norm $\|x\|^p = \widehat{\mathrm{E}}\left(|x|^p\right)^{1/p}$,

$$M^{p,0}_G([0,T]) = \big\{\zeta := \zeta_t(\omega) = \sum_{i=0}^{N-1} \zeta_j \mathbf{1}_{[t_i,t_{i+1})}(t),\ \forall N > 0,\ 0 = t_0 < \cdots < t_N = T,\\ \zeta_i \in L^p_G(\omega_{t_i}),\ i = 0, 1, 2, \ldots, N-1\big\},$$

$M^p_G([0,T])$: Completion of $M^{p,0}_G$ under $\|\eta\|_{M^p_G} = \big(\int_0^T \widehat{\mathrm{E}}\big(|\eta(t)|^p\big)dt\big)^{1/p}$.

Let Ω be a given set and let $\mathcal{H}$ be a linear space of real valued functions defined on Ω. We suppose that $\mathcal{H}$ satisfies $b \in \mathcal{H}$ for each constant b and $\|Y\| \in \mathcal{H}$ if $Y \in \mathcal{H}$.

Definition 2.1. [24] A sublinear expectation $\widehat{\mathrm{E}}$ on $\mathcal{H}$ is a functional $\widehat{\mathrm{E}} : \mathcal{H} \to \mathbb{R}$ satisfying the following properties: for all $Y, Z \in \mathcal{H}$,

- (i) Monotonicity: if $Y \geq Z$, then $\widehat{\mathrm{E}}(Y) \geq \widehat{\mathrm{E}}(Z)$.
- (ii) Constant preserving: $\widehat{\mathrm{E}}(b) = b$ for all $b \in \mathbb{R}$.
- (iii) Sub-additivity: $\widehat{\mathrm{E}}(Y + Z) \leq \widehat{\mathrm{E}}(Y) + \widehat{\mathrm{E}}(Z)$.
- (iv) Positive homogeneity: $\widehat{\mathrm{E}}(\alpha Y) = \alpha\widehat{\mathrm{E}}(Y)$ for $\alpha \geq 0$.

The triple $(\Omega, \mathcal{H}, \widehat{\mathrm{E}})$ is called a sublinear expectation space. $Y \in \mathcal{H}$ is called a random variable in $(\Omega, \mathcal{H}, \widehat{\mathrm{E}})$. $Y = (Y_1, \ldots, Y_n)$, where $Y_j \in \mathcal{H}$ is called an n-dimensional random vector in $(\Omega, \mathcal{H}, \widehat{\mathrm{E}})$.

Definition 2.2. [24] Weakly compact sets are defined to be sets which are compact with respect to the weak topology of a Banach space.

The representation of a sublinear expectation can be expressed as a supremum of linear expectations.

Theorem 2.3 ([25])**.** *There exists a weakly compact family $\mathcal{P}$ of probability measures defined on $(\Omega, \mathcal{B}(\Omega))$, such that*

$$\widehat{\mathrm{E}}(Y) = \sup_{\mathrm{p}\in\mathcal{P}} \mathrm{E}_{\mathrm{p}}(Y), \quad Y \in L^1_G(\Omega).$$

Definition 2.4 ([24])**.** In a sublinear expectation space $(\Omega, \mathcal{H}, \widehat{\mathrm{E}})$, an n-dimensional random vector $Z = (Z_1, \ldots, Z_n) \in \mathcal{H}$ is said to be independent from an m-dimensional random vector $Y = (Y_1, \ldots, Y_m) \in \mathcal{H}$ under the sublinear expectation

$\widehat{\mathrm{E}}$, if for any test function $\varphi \in \mathcal{C}_{b,Lip}(\mathbb{R}^{m+n})$

$$\widehat{\mathrm{E}}(\varphi(Z,Y)) = \widehat{\mathrm{E}}\left(\widehat{\mathrm{E}}\left(\varphi(z,Y)\right)|_{z=Z}\right).$$

Definition 2.5 ([24]). Let Y_1 and Y_2 be two n-dimensional random vectors defined on sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \widehat{\mathrm{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \widehat{\mathrm{E}}_2)$, respectively. They are called identically distributed, denoted by $Y_1 \overset{d}{=} Y_2$, if

$$\widehat{\mathrm{E}}_1(\psi(Y_1)) = \widehat{\mathrm{E}}_2(\psi(Y_2)), \quad \forall \psi \in \mathcal{C}_{b,Lip(\mathbb{R}^n)}.$$

$\bar{Y}$ is said to be an independent copy of Y, if $\bar{Y} \overset{d}{=} Y$ and $\bar{Y}$ is independent from Y.

Definition 2.6 ([24]). A random variable Y on a sublinear expectation space $(\Omega, \mathcal{H}, \widehat{\mathrm{E}})$ is called G-normal distributed, denoted by $Y \sim \mathcal{N}\left(0, [\underline{\sigma}^2, \bar{\sigma}^2]\right)$ for a given pair $0 \leq \bar{\sigma} \leq \underline{\sigma}$, if for any $c, d \geq 0$,

$$cY + d\widetilde{Y} \overset{d}{=} \sqrt{c^2 + d^2}Y,$$

where $\widetilde{Y}$ is an independent copy of Y.

Let Ω be the space of $\mathbb{R}^d$-valued continuous paths $(\omega_t)_{t\geq 0}$ with $\omega_0 = 0$. Further, we assume that Ω is a metric space equipped with the distance

$$\varrho(\omega^1, \omega^2) := \sum_{N=1}^{\infty} 2^{-N}\Big(\max_{0\leq t\leq N}(\|\omega_t^1 - \omega_t^2\|) \wedge 1\Big),$$

and consider the canonical process $\mathrm{B}_t(\omega) = \omega_t$, $t \in [0,\infty)$ for $\omega \in \Omega$; then for each fixed $T \in [0,\infty)$, we have

$$L_{ip}^0(\Omega_T) := \{\psi\left(\mathrm{B}_{t_1}, \mathrm{B}_{t_2}, \dots, \mathrm{B}_{t_n}\right) : n \geq 1,\ 0 \leq t_1 \leq \dots \leq t_n \leq T,\ \psi \in \mathcal{C}_{b,lip}(\mathbb{R}^{d\times n})\}.$$

Definition 2.7 ([24]). On the sublinear expectation space $(\Omega, L_{ip}^0(\Omega_T), \widehat{\mathrm{E}})$, the canonical process $(\mathrm{B}_t)_{t\geq 0}$ is called a G-Brownian motion, if the ensuing properties are satisfied:

(i) $\mathrm{B}_0 = 0$;
(ii) for $t, s \geq 0$, the increment $\mathrm{B}_{t+s} - \mathrm{B}_t \overset{d}{=} \sqrt{s}Y$, where Y is G-normal distributed;
(iii) for $t, s \geq 0$, the increment $\mathrm{B}_{t+s} - \mathrm{B}_t$ is independent from $(\mathrm{B}_{t_1}, \mathrm{B}_{t_2}, \dots, \mathrm{B}_{t_n})$ for each $n \in \mathbb{N}$, and $0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq t$.

Moreover, the sublinear expectation $\widehat{\mathrm{E}}(\cdot)$ is called G-expectation.

For $\bar{\sigma}^2 = \underline{\sigma}^2 = 1$, $(\mathrm{B}_t)_{t\geq 0}$ is the classical Brownian motion.

For simplicity, let $(\mathrm{B}_t)_{t\geq 0}$ be a 1-dimensional G-Brownian motion. The letter G denotes the function

$$\mathrm{G}(b) := \frac{1}{2}\widehat{\mathrm{E}}(b\mathrm{B}_1^2) = \frac{1}{2}(\overline{\sigma}^2 b^+ - \underline{\sigma}^2 b^-), \quad b \in \mathbb{R},$$

with $\underline{\sigma}^2 := -\widehat{\mathrm{E}}(-\mathrm{B}_1^2) \leq \widehat{\mathrm{E}}(\mathrm{B}_1^2) := \overline{\sigma}^2,\ 0 \leq \underline{\sigma} \leq \overline{\sigma} < \infty$. Recall that $b^+ = \max\{0, b\}$ and $b^- = -\min\{0, b\}$.

Definition 2.8 ([24]). *Let π_t^N, $N = 1, 2, \dots,$ be a sequence of partitions of $[0,t]$, $(\mathrm{B}_t)_{t\geq 0}$ be an n-dimensional G-Brownian motion. For each fixed $b \in \mathbb{R}^n$, $(\mathrm{B}_t^b)_{t\geq 0}$*

is a 1*-dimensional G-Brownian motion, we define*

$$\langle \mathrm{B}^b\rangle_t := \langle b, \mathrm{B}_t\rangle = \lim_{\mu(\pi_t^N)\to 0} \sum_{i=0}^{N-1} (\mathrm{B}^b_{t^N_{j+1}} - \mathrm{B}^b_{t^N_j})^2 = (\mathrm{B}^b_t)^2 - 2\int_0^t \mathrm{B}^b_s d\mathrm{B}^b_s.$$

$\langle \mathrm{B}^b\rangle$ *is called the quadratic variation process of G-Brownian motion.*

Let $\bar{b} \in \mathbb{R}^n$*, we define the mutual variation process by*

$$\langle \mathrm{B}^b, \mathrm{B}^{\bar{b}}\rangle_t := \frac{1}{4}\Big(\langle \mathrm{B}^b + \mathrm{B}^{\bar{b}}\rangle_t - \langle \mathrm{B}^b - \mathrm{B}^{\bar{b}}\rangle_t\Big) = \frac{1}{4}\Big(\langle \mathrm{B}^{b+\bar{b}}\rangle_t - \langle \mathrm{B}^{b-\bar{b}}\rangle_t\Big).$$

Proposition 2.9 ([24])**.** *Let* $(\mathrm{B}_t)_{t\geq 0}$ *be an n-dimensional G-Brownian motion on a sublinear expectation space* $(\Omega, \mathcal{H}, \widehat{\mathrm{E}})$*. Then,* $(\mathrm{B}^b_t)_{t\geq 0}$ *is a* 1*-dimensional G-Brownian motion for each* $b \in \mathbb{R}^n$*, where*

$$\begin{gathered} \mathrm{G}_b(\beta) = \frac{1}{2}\left(\sigma^2_{bb^T}\beta^+ - \sigma^2_{-bb^T}\beta^-\right), \\ \sigma^2_{bb^T} = 2\mathrm{G}(bb^T) = \widehat{\mathrm{E}}\left(\langle b, \mathrm{B}_1\rangle^2\right), \\ \sigma^2_{-bb^T} = -2\mathrm{G}(-bb^T) = -\widehat{\mathrm{E}}\left(-\langle b, \mathrm{B}_1\rangle^2\right). \end{gathered}$$

In particular, for each $t, s \geq 0$*,* $\mathrm{B}^b_{t+s} - \mathrm{B}^b_t \overset{d}{=} \mathcal{N}\left(0, [s\sigma^2_{-bb^T}, s\sigma^2_{bb^T}]\right)$*.*

Definition 2.10 ([26])**.** For $p \geq 1$ and $T \in \mathbb{R}_+$ fixed, we consider the type of simple processes,

$$\begin{gathered} M_{b,0}([0,T]) = \Big\{\eta := \eta_t(\omega) = \sum_{i=0}^{N-1} \xi_i \mathbf{1}_{[t_i,t_{i+1})}(t), \quad \forall N > 0, \\ 0 = t_0 < \dots < t_N = T, \quad \xi_i \in \mathrm{B}_b(\Omega_{t_i}), \quad i = 0,1,2,\dots,N-1\Big\}. \end{gathered}$$

For each $p \geq 1$, we denote by $M^p_\star([0,T])$ the completion of $M_{b,0}([0,T])$ under the norm:

$$\|\eta\|_{M^p([0,T])} = \Big(\widehat{\mathrm{E}}\Big(\int_0^T \|\eta_t\|^p dt\Big)\Big)^{1/p}.$$

Now, we introduce the natural Choquet capacity.

Definition 2.11 ([24])**.** Let $\mathcal{B}(\Omega)$ the Borel σ-algebra and $\mathcal{P}$ be a weakly compact collection of probability measures P defined on $(\Omega, \mathcal{B}(\Omega))$, then the capacity $\hat{\mathrm{C}}(\cdot)$ associated to $\mathcal{P}$ is defined as follows:

$$\hat{\mathrm{C}}(\mathcal{A}) := \sup_{\mathrm{P}\in\mathcal{P}} \mathrm{P}(\mathcal{A}), \quad \mathcal{A} \in \mathcal{B}(\Omega).$$

Definition 2.12 ([24])**.** A set $\mathcal{A} \subset \mathcal{B}(\Omega)$ is polar, if $\hat{\mathrm{C}}(\mathcal{A}) = 0$. A property holds "quasi-surely" (q.s.), if it holds outside a polar set.

Next we recall the following Borel-Cantelli lemma in the G-framework.

Lemma 2.13 ([11])**.** *Let* $\{\mathcal{A}_k\} \subset \mathcal{B}(\Omega)$*, such that*

$$\sum_{k=1}^{\infty} \hat{\mathrm{C}}(\mathcal{A}_k) < \infty.$$

Then, $\limsup_{k\to\infty} \mathcal{A}_k$ *is polar.*

Lemma 2.14 ([30]). *Let B_t be a one-dimensional G-Brownian motion, we suppose that there exist constants $\epsilon > 0$ and $\nu > 0$, such that*

$$\widehat{\mathrm{E}}\Big(\exp\Big(\frac{\nu^2}{2}(1+\epsilon)\int_0^T f^2(s)d\langle \mathrm{B}\rangle_s\Big)\Big) < \infty.$$

Then, for any $T > 0$ and $\eta > 0$,

$$\hat{\mathrm{C}}\Big(\sup_{0\le t\le T}\Big(\int_0^t f(s)d\mathrm{B}_s - \frac{\nu}{2}\int_0^t f^2(s)d\langle \mathrm{B}\rangle_s\Big) > \eta\Big) \le \exp(-\nu\eta).$$

3. Practical stability of stochastic delay equation driven by G-Brownian motion

Let $\tau > 0$ and $\mathcal{C}([-\tau,0],\mathbb{R}^n)$ denote the family of all continuous $\mathbb{R}^n$-valued function φ defined on $[-\tau,0]$ with the norm $\|\varphi\| = \sup_{-\tau\le\theta\le 0}\|\varphi(\theta)\|$. If $x(t)$ is a continuous $\mathbb{R}^n$-valued stochastic process on $[-\tau,\infty)$, for every $t \ge 0$ we define $x_t : [-\tau,0] \to \mathbb{R}^n$ by $x_t(\theta) = x(t+\theta)$, $-\tau \le \theta \le 0$, which is considered as $\mathcal{C}([-\tau,0],\mathbb{R}^n)$-valued stochastic process.

Now, we consider the nonlinear stochastic differential delay equations driven by a G-Brownian motion in the form

$$dx(t) = f(t,x_t)dt + h(t,x_t)d\langle \mathrm{B}\rangle_t + g(t,x_t)d\mathrm{B}_t, \quad t \ge t_0, \tag{3.1}$$

where B_t is a one-dimensional G-Brownian motion, with $\mathrm{B}_t \sim \mathcal{N}(0,[\underline{\sigma}^2 t, \bar{\sigma}^2 t])$, and $(\langle \mathrm{B}\rangle)_{t\ge 0}$ is the quadratic variation process of the G-Brownian, and $f : [t_0,\infty)\times\mathcal{C}([-\tau,0],\mathbb{R}^n) \to \mathbb{R}^n$, $g : [t_0,\infty)\times\mathcal{C}([-\tau,0],\mathbb{R}^n) \to \mathbb{R}^n$, $h : [t_0,\infty)\times\mathcal{C}([-\tau,0],\mathbb{R}^n) \to \mathbb{R}^n$ satisfy appropriate assumptions described below.

To solve equation (3.1), we need to know an initial datum, so we assume that it is given as follows

$$x_{t_0} = \xi(\text{ in other words } x_{t_0}(\theta) = x(t_0+\theta) = \xi(\theta), -\tau \le \theta \le 0), \tag{3.2}$$

where ξ is a $\mathcal{C}([-\tau,0],\mathbb{R}^n)$-valued random variable.

For the well-posedness of system (3.1), we impose the following hypotheses.

(1) Linear growth condition: There exists a positive constant K_1, such that for all $\varphi \in \mathcal{C}([-\tau,0],\mathbb{R}^n)$, and all $t \in [t_0,T]$,
$$|f(t,\varphi)|^2 + |h(t,\varphi)|^2 + |g(t,\varphi)|^2 \le K_1(1+|\varphi|^2).$$
(2) Lipschitz condition: There exists a positive constant K_2, such that for all $\varphi,\ \widetilde{\varphi} \in \mathcal{C}([-\tau,0],\mathbb{R}^n)$, and for all $t \in [t_0,T]$,
$$\|f(t,\varphi)-f(t,\widetilde{\varphi})\|^2 + \|h(t,\varphi)-h(t,\widetilde{\varphi})\|^2 + \|g(t,\varphi)-g(t,\widetilde{\varphi})\|^2 \le K_2\|\varphi-\widetilde{\varphi}\|^2.$$

Then, under these assumptions, the G-SDDE (3.1) with initial value (3.2) has a unique solution $x(t)$, see [27] for details. The solution $x(t)$ of (3.1) with initial value (3.2) satisfies the integral equation

$$x(t) = \xi(0) + \int_{t_0}^t f(s,x_s)ds + \int_{t_0}^t h(s,x_s)d\langle \mathrm{B}\rangle_s + \int_{t_0}^t g(s,x_s)d\mathrm{B}_s, \quad \text{q.s.},$$
$$x(t) = \xi(t-t_0), \quad t \in [t_0-\tau, t_0].$$

To calculate the stochastic differential of the process $\vartheta(t) = v(t,x(t))$, where $x(t)$ is a solution of the G-SDDE (3.1) and $v : [0,\infty)\times\mathbb{R}^n \to \mathbb{R}_+$, we define an operator L (called G-Lyapunov function) as

$$Lv(t,x(t)) := v_t(t,x(t)) + v_x f(t,x_t)$$

$$+ G\Big(\langle v_x(t,x(t)), 2h(t,x_t)\rangle + \langle v_{xx}(t,x(t))g(t,x_t), g(t,x_t)\rangle\Big),$$

where

$$v_t(t,x) = \frac{\partial v}{\partial t}(t,x) \quad v_x(t,x) = (\frac{\partial v}{\partial x_1}(t,x), \dots, \frac{\partial v}{\partial x_n}(t,x));$$
$$v_{xx}(t,x) = \Big(\frac{\partial^2 v}{\partial x_i \partial x_j}(t,x)\Big)_{n\times n}.$$

The G-Lyapunov function L can be implemented too for some functionals $V(\cdot,\cdot) : [0,\infty)\times\mathcal{C}([-\tau,0],\mathbb{R}^n) \to \mathbb{R}_+$. We assume that a functional $V(t,\varphi)$ can be described in the form $V(t,\varphi(0),\varphi(\theta))$, $\theta < 0$, and for $\varphi = x_t$, we put

$$\begin{gathered} V_\varphi(t,x) = V(t,\varphi) = V(t,x_t) = V(t,x,x(t+\theta)), \quad \theta < 0, \\ x = \varphi(0) = x(t). \end{gathered} \tag{3.3}$$

Let $\mathcal{D}$ represent the set of functionals for which the function $V_\varphi(t,x)$, defined by (3.3), has a continuous derivative with respect to t and two continuous derivatives with respect to x_i, $i = 1,\dots,n$. For functionals from $\mathcal{D}$, the operator L of the G-SDDE (3.1) has the form

$$\begin{aligned} LV(t,x_t) &= V_{\varphi t}(t,x(t)) + V_{\varphi x}(t,x(t))f(t,x_t) + G\Big(\langle V_{\varphi x}(t,x(t)), 2h(t,x_t)\rangle \\ &\quad + \langle V_{\varphi xx}(t,x(t))g(t,x_t), g(t,x_t)\rangle\Big). \end{aligned}$$

From the G-Itô formula it follows that for a functional V from $\mathcal{D}$,

$$dV(t,x_t) = LV(t,x_t)dt + V_{\varphi x}(t,x(t))g(t,x_t)d\mathrm{B}_t.$$

We assume that there exits $t \in \mathbb{R}_+$, such that $f(t,0) \neq 0$ or $h(t,0) \neq 0$ or $g(t,0) \neq 0$, i.e., the G-stochastic differential delay equation (3.1) does not have the trivial solution $x \equiv 0$.

Now, we state the definition of practical exponential stability of a stochastic delay equation driven by G-Brownian motion (3.1) when the origin is no longer an equilibrium point. In this case we study the stability of solutions with respect to a small neighborhood of the origin.

The study of the asymptotic behavior of solutions leads to investigate the stability behavior of a small ball centered at the origin, $\mathcal{B}_r := \{x \in \mathbb{R}^n : \|x\| \leq r\}$, $r > 0$.

Definition 3.1. (i) The ball $\mathcal{B}_r := \{x \in \mathbb{R}^n : \|x\| \leq r\}$, $r > 0$ is said to be quasi surely globally uniformly exponentially stable, if for each initial data $\xi \in \mathcal{C}([-\tau,0],\mathbb{R}^n)$, such that $0 < \|x(t,t_0,\xi)\| - r$, for all $t \geq 0$,

$$\limsup_{t\to\infty} \frac{1}{t} \ln(\|x(t,t_0,\xi)\| - r) < 0, \quad \text{q.s.}$$

(ii) System (3.1) is said to be quasi surely practically uniformly exponentially stable, if there exists $r > 0$ such that $\mathcal{B}_r$ is quasi surely uniformly exponentially stable.

Next, we state the definition of practical convergence to the ball $\mathcal{B}_r$ with a general decay function $\lambda(t)$.

Definition 3.2. Let $\lambda(\cdot)$ be a positive function defined for sufficiently large $t > 0$, such that $\lambda(t) \to \infty$ as $t \to \infty$. A solution $x(\cdot)$ to system (3.1) is said to decay to the ball $\mathcal{B}_r$ quasi surely with decay function $\lambda(t)$ and order at least $\gamma > 0$, if its generalized Lyapunov exponent is less than or equal to $-\gamma$, i.e.,

$$\limsup_{t\to\infty} \frac{\ln(\|x(t,t_0,\xi)\| - r)}{\ln\lambda(t)} \le -\gamma, \quad \text{q.s.}$$

If in addition, 0 is a solution to system (3.1), the zero solution is said to be quasi surely practically asymptotically stable with decay function $\lambda(t)$ and order at least γ, if every solution to system (3.1) tends to the ball $\mathcal{B}_r$ quasi surely with decay function $\lambda(t)$ and order at least γ, for all $r > 0$ sufficiently small.

Replacing the decay function $\lambda(t)$ by $O(\exp(t))$ in the above definition leads to the quasi sure practical exponential stability.

Our aim now is to study the practical stability of stochastic differential delay equations driven by G-Brownian motion with a general decay rate based upon the method of G-Lyapunov functionals.

Theorem 3.3. *Let $V : \mathbb{R}_+ \times \mathcal{C}([-\tau,0],\mathbb{R}^n) \to \mathbb{R}_+$ be a functional from $\mathcal{D}$. Assume that $\ln\lambda(t)$ is uniformly continuous on $t \ge 0$ and there exists a constant $\delta \ge 0$, such that*

$$\lim_{t\to\infty} \frac{\ln\ln t}{\ln\lambda(t)} \le \delta.$$

Let $x(\cdot) = x(\cdot,0,\xi)$ be a solution to (3.1) and assume that there exist constants $q \in \mathbb{N}^\star$, $m \ge 0$, $b_1 \ge 0$, $b_2 \in \mathbb{R}$, a non-increasing function $\varphi_1(t) > 0$ and a continuous non-negative function $\varphi_2(t)$, such that for all $t \ge t_0 \ge 0$, the following inequalities hold:

(H1) $\lambda^m(t)\|x(t)\|^q \le V(t,x_t)$.

(H2)
$$\begin{aligned}&\int_{t_0}^t LV(s,x_s)ds + \bar\sigma^2 \int_{t_0}^t \varphi_1(s)\|V_x(s,x_s)g(s,x_s)\|^2 ds,\\ &\le \int_{t_0}^t \varphi_2(s)\lambda^m(s)\|x(s)\|^q ds + r(t),\end{aligned}$$

where $r(\cdot)$ is a continuous non-negative function.

(H3)
$$\limsup_{t\to\infty} \frac{\int_{t_0}^t \varphi_2(s)ds}{\ln\lambda(t)} \le b_2, \quad \liminf_{t\to\infty} \frac{\ln\varphi_1(t)}{\ln\lambda(t)} \ge -b_1, \quad \lim_{t\to\infty} \frac{r(t)}{\lambda^m(t)} = \widetilde{r} > 0.$$

(H4) *The solution $x(t,t_0,\xi)$ satisfies*

$$\|x(t,t_0,\xi)\| > \Big(\frac{r(t)}{\lambda^m(t)}\Big)^{1/q}, \ \forall t \ge t_0.$$

Then

$$\limsup_{t\to\infty} \frac{\ln\big(\|x(t,t_0,\xi)\| - \big(\frac{r(t)}{\lambda^m(t)}\big)^{1/q}\big)}{\ln\lambda(t)} \le -\big(m - (b_1 + (b_2+\delta)\vee m)\big), \quad q.s.$$

Proof. Notice that

$$\lambda^m(t)\|x(t)\|^q - r(t) = \lambda^m(t)\Big(\|x(t)\|^q - \frac{r(t)}{\lambda^m(t)}\Big)$$

$$= \lambda^m(t)\Big(\|x(t)\|^q - \Big(\big(\frac{r(t)}{\lambda^m(t)}\big)^{1/q}\Big)^q\Big).$$

From the inequality

$$a_1{}^q - a_2{}^q = (a_1 - a_2)\left(a_1{}^{q-1} + a_1{}^{q-2}a_2 + a_1{}^{q-3}a_2{}^2 + \dots + a_1{}^0 a_2{}^{q-1}\right),$$

it follows that

$$\begin{aligned}
&\lambda^m(t)\|x(t)\|^q - r(t)\\
&= \lambda^m(t)\Big(\|x(t)\|^q - \Big(\big(\frac{r(t)}{\lambda^m(t)}\big)^{1/q}\Big)^q\Big)\\
&= \lambda^m(t)\Big(\|x(t)\| - \Big(\frac{r(t)}{\lambda^m(t)}\Big)^{1/q}\Big)\Big(\|x(t)\|^{q-1} + \|x(t)\|^{q-2}\Big(\frac{r(t)}{\lambda^m(t)}\Big)^{1/q}\\
&\quad + \dots + \big(\frac{r(t)}{\lambda^m(t)}\big)^{\frac{q-1}{q}}\Big)\\
&= \lambda^m(t)\Big(\|x(t)\| - \Big(\frac{r(t)}{\lambda^m(t)}\Big)^{1/q}\Big)\sum_{k=1}^{q}\|x(t)\|^{q-k}\Big(\frac{r(t)}{\lambda^m(t)}\Big)^{\frac{k-1}{q}}.
\end{aligned}$$

Since $\lim_{t\to\infty}\frac{r(t)}{\lambda^m(t)} = \widetilde{r} > 0$, it follows that for $0 < \widetilde{r}_0 < \widetilde{r}$, there exits $\widetilde{T} \geq t_0$, such that $\frac{r(t)}{\lambda^m(t)} \geq \widetilde{r}_0$ for all $t \geq \widetilde{T}$. As we are assuming that $\|x(t)\| > \big(\frac{r(t)}{\lambda^m(t)}\big)^{1/q}$, for all $t \geq 0$, we obtain

$$\begin{aligned}
&\sum_{k=1}^{q}\|x(t)\|^{q-k}\Big(\frac{r(t)}{\lambda^m(t)}\Big)^{\frac{k-1}{q}}\\
&= \|x(t)\|^{q-1} + \|x(t)\|^{q-2}\Big(\frac{r(t)}{\lambda^m(t)}\Big)^{1/q} + \dots + \Big(\frac{r(t)}{\lambda^m(t)}\Big)^{\frac{q-1}{q}}\\
&\geq \widetilde{r}^* = q\,(\widetilde{r}_0)^{(q-1)/q}, \quad \forall t \geq \widetilde{T} \geq t_0.
\end{aligned}$$

Hence, we see that

$$\lambda^m(t)\|x(t)\|^q - r(t) \geq \lambda^m(t)\Big(\|x(t)\| - \Big(\frac{r(t)}{\lambda^m(t)}\Big)^{1/q}\Big)\widetilde{r}^*, \quad \forall t \geq \widetilde{T} \geq t_0.$$

This yields

$$V(t,x_t) \geq \lambda^m(t)\|x(t)\|^q \geq \lambda^m(t)\|x(t)\|^q - r(t) \geq \lambda^m(t)\Big(\|x(t)\| - \Big(\frac{r(t)}{\lambda^m(t)}\Big)^{1/q}\Big)\widetilde{r}^*.$$

That is,

$$\widetilde{r}^*\lambda^m(t)\Big(\|x(t)\| - \Big(\frac{r(t)}{\lambda^m(t)}\Big)^{1/q}\Big) \leq V(t,x_t).$$

Therefore,

$$\ln(\widetilde{r}^*) + m\ln\lambda(t) + \ln\Big(\|x(t)\| - \Big(\frac{r(t)}{\lambda^m(t)}\Big)^{1/q}\Big) \leq \ln\left(V(t,x_t)\right), \quad \forall t \geq \widetilde{T} \geq t_0.$$

Invoking the G-Itô formula, it follows that

$$V(t,x_t) = V(0,x_0) + \int_{t_0}^{t} LV(s,x_s)ds + \int_{t_0}^{t} V_x(s,x_s)g(s,x_s)d\mathrm{B}_s. \tag{3.4}$$

By using that $\ln\lambda(t)$ is uniformly continuous on $t \geq 0$, we can obtain that, for each $\varepsilon > 0$, there exist two positive integers $N = N(\varepsilon)$ and $K_1(\varepsilon)$, such that if $\frac{K-1}{2^N} \leq t \leq \frac{K}{2^N}$ and $K \geq K_1(\varepsilon)$, then

$$|\ln\lambda\big(\frac{K}{2^N}\big) - \ln\lambda(t)| \leq \varepsilon.$$

Thanks to Lemma 2.14, we deduce that

$$\widehat{C}\Big\{\omega : \sup_{t_0\leq t\leq\omega}\Big(M(t) - \frac{\nu}{2}\int_{t_0}^t \|V_x(s,x_s)g(s,x_s)\|^2 d\langle \mathrm{B}\rangle_s\Big) > \eta\Big\} \leq \exp(-\nu\eta),$$

for any positive constants α, β and ω, with

$$M(t) = \int_{t_0}^t V_x(s,x_s)g(s,x_s)d\mathrm{B}_s.$$

For $\varepsilon > 0$, we set

$$\nu = 2\varphi_1\big(\frac{K}{2^N}\big), \quad \eta = \varphi_1\big(\frac{K}{2^N}\big)^{-1}\ln\frac{K-1}{2^N}, \quad \omega = \frac{K}{2^N}, \quad K = 2, 3, \dots.$$

Applying the well-known Borel-Cantelli lemma (Lemma 2.13) for capacity, we can conclude that, for almost all $\omega \in \Omega$, there exists an integer $K_0 = K(\varepsilon,\omega) > 0$, such that

$$\begin{aligned}
M(t) &\leq \varphi_1\big(\frac{K}{2^N}\big)^{-1}\ln\frac{K-1}{2^N} + \varphi_1\big(\frac{K}{2^N}\big)\int_{t_0}^t \|V_x(s,x_s))g(s,x_s)\|^2 d\langle \mathrm{B}\rangle_s \\
&\leq \varphi_1\big(\frac{K}{2^N}\big)^{-1}\ln\frac{K-1}{2^N} + \int_{t_0}^t \varphi_1(s)\|V_x(s,x_s)g(s,x_s)\|^2 d\langle \mathrm{B}\rangle_s\,,
\end{aligned}$$

for $t_0 \leq t \leq \frac{K}{2^N}$ and $K \geq K_0(\varepsilon,\omega)$. Substituting the above inequality into (3.4), we obtain

$$\begin{aligned}
V(t,x_t) \leq V(0,x_0) &+ \varphi_1\big(\frac{K}{2^N}\big)^{-1}\ln\frac{K-1}{2^N} + \int_{t_0}^t LV(s,x_s)ds \\
&+ \int_{t_0}^t \varphi_1(s)\|V_s(s,x_s)g(s,x_s)\|^2 d\langle \mathrm{B}\rangle_s,
\end{aligned}$$

for $t_0 \leq t \leq \frac{K}{2^N}$ and $K \leq K_0(\varepsilon,\omega)$. From Peng [24, Chapter III], we have that for each $0 \leq s \leq t \leq T$,

$$\underline{\sigma}^2(t-s) \leq \langle \mathrm{B}\rangle_t - \langle \mathrm{B}\rangle_s \leq \bar{\sigma}^2(t-s).$$

Based on this fact, we deduce that

$$\begin{aligned}
V(t,x_t) \leq V(0,x_0) &+ \varphi_1\big(\frac{K}{2^N}\big)^{-1}\ln\frac{K-1}{2^N} + \int_{t_0}^t LV(s,x_s)ds \\
&+ \bar{\sigma}^2\int_{t_0}^t \varphi_1(s)\|V_s(s,x_s)g(s,x_s)\|^2 ds,
\end{aligned}$$

for $t_0 \leq t \leq K/2^N$ and $K \leq K_0(\varepsilon,\omega)$.

It follows from conditions (H1) and (H2), that

$$\begin{aligned}
V(t,x_t) &\leq V(0,x_0) + \varphi_1\big(\frac{K}{2^N}\big)^{-1}\ln\frac{K-1}{2^N} + r(t) + \int_{t_0}^t \varphi_2(s)\lambda^m(s)\|x(s)\|^q ds \\
&\leq V(0,x_0) + \varphi_1\big(\frac{K}{2^N}\big)^{-1}\ln\frac{K-1}{2^N} + r(t) + \int_{t_0}^t \varphi_2(s)V(s,x_s)ds,
\end{aligned}$$

for $t_0 \le t \le \frac{K}{2^N}$ and $K \ge K_0(\varepsilon,\omega)$. Using Gronwall's Lemma [12], we derive

$$V(t,x_t) \le \Big(V(0,x_0) + \varphi_1\big(\frac{K}{2^N}\big)^{-1} \ln \frac{K-1}{2^N} + r(t)\Big) \exp\Big(\int_{t_0}^t \varphi_2(s)ds\Big).$$

From (H3) we have that for any $\varepsilon > 0$,

$$\lim_{t\to\infty} \sup \frac{\int_{t_0}^t \varphi_2(s)ds}{\ln \lambda(t)} < b_2 + \varepsilon,$$

and $\lim_{t\to\infty} \inf \frac{\ln \varphi_1(t)}{\ln \lambda(t)} > -b_1 - \varepsilon$. Thanks to the uniform continuity of $\ln \lambda(t)$, there exists a positive integer $K_1(\varepsilon)$, such that whenever $t \ge K_1(\varepsilon)$, we have

$$\int_{t_0}^t \varphi_2(s)ds \le (b_2+\varepsilon)\ln \lambda(t), \quad \varphi_1\Big(\frac{K-1}{2^N}\Big)^{-1} \le \varphi_1(t) \le \lambda(t)^{b_1+\varepsilon},$$

for $\frac{K-1}{2^N} \le t \le \frac{K}{2^N}$ and $K \ge K_1(\varepsilon)$. Also observe that

$$\ln \frac{k-1}{2^N} \le \ln t \le \ln \frac{k}{2^N} \quad \text{for } \frac{k-1}{2^N} \le t \le \frac{k}{2^N}.$$

Therefore, for almost all $\omega \in \Omega$, we obtain

$$\ln V(t,x_t) \le \ln\big(V(0,x_0) + \lambda(t)^{b_1+\delta+2\varepsilon} + r(t)\big) + (b_2+\varepsilon)\ln \lambda(t),$$

for $\frac{K-1}{2^N} \le t \le \frac{K}{2^N}$ and $K \ge K_1(\varepsilon)$. Thus, we conclude that

$$\lim_{t\to\infty} \sup \frac{\ln V(t,x_t)}{\ln \lambda(t)} \le (b_1+\delta+2\varepsilon) \vee m + (b_2+\varepsilon)\,, \quad \text{q.s.}$$

Recall that for $t \ge \widetilde{T} \ge t_0$ and $q \in \mathbb{N}^\star$, we have

$$\ln\Big(\|x(t)\| - \Big(\frac{r(t)}{\lambda^m(t)}\Big)^{1/q}\Big) \le \ln(V(t,x_t)) - m\ln\lambda(t) - \ln(\widetilde{r}^*).$$

Letting $\varepsilon \to 0$,

$$\lim_{t\to\infty} \sup \frac{\ln\big(\|x(t)\| - \big(\frac{\rho(t)}{\lambda^m(t)}\big)^{1/q}\big)}{\ln \lambda(t)} \le -(m - (b_2 + (b_1+\delta)\vee m)), \quad \text{q.s.},$$

as required. □

Next, we will infer the practical convergence toward the ball $\mathcal{B}_r$ with a general decay rate of our stochastic differential delay equations driven by G-Brownian motion.

Corollary 3.4. *Let $V : \mathbb{R}_+ \times \mathcal{C}([-\tau,0],\mathbb{R}^n) \to \mathbb{R}_+$ be a functional from $\mathcal{D}$. Assume that $\ln \lambda(t)$ is uniformly continuous on $t \ge 0$, and there exists a constant $\delta \ge 0$, such that*

$$\lim_{t\to\infty} \frac{\ln \ln t}{\ln \lambda(t)} \le \delta.$$

Let $x(\cdot) = x(\cdot,0,\xi)$ be a solution to system (3.1) *and assume that there exist constants $q \in \mathbb{N}^\star$, $m \ge 0$, $b_1 \ge 0$, $b_2 \in \mathbb{R}$, a non-increasing function $\varphi_1(t) > 0$ and a continuous non-negative function $\varphi_2(t)$, such that, for all $t \ge t_0 \ge 0$, and for any solution $x(\cdot)$ to Eq.*(3.1)*, defined in the future, assumptions $(\mathcal{H}_1) - (\mathcal{H}_3)$ hold, and the following assumption is also satisfied*

(H4') *There exists $\widetilde{r'} > \widetilde{r} > 0$, such that the solution $x(t,t_0,\xi)$ satisfies*

$$\|x(t,t_0,\xi)\| > \big(\widetilde{r'}\big)^{1/q}, \quad \forall t \ge t_0.$$

Then

$$\limsup_{t\to\infty} \frac{\ln\left(\|x(t,t_0,\xi)\| - \left(\widetilde{r'}\right)^{1/q}\right)}{\ln\lambda(t)} \le -\gamma, \quad q.s.,$$

where $\gamma = m - (b_2 + (b_1+\delta)\vee m)$.

In particular, if $m > (b_2 + (b_1+\delta)\vee m)$, *the solution to system* (3.1) *tends to the ball* $\mathcal{B}_r$, *with* $r = \left(\widetilde{r'}\right)^{1/q}$ *quasi surely with decay function* $\lambda(t)$ *and order at least* γ.

Remark 3.5. *Notice that the condition* $m > b_2 + (b_1+\delta)\vee m$ *(or equivalently* $\gamma > 0$*) in the corollary holds in the next cases:*

- *If* $b_1+\delta \le m$, *then the condition becomes* $m > b_2 + m$. *Therefore, this needs* $b_2 < 0$.
- *If* $b_1+\delta > m$, *then the condition turns into* $m > b_2 + b_1 + \delta$ *which also needs* $b_2 < 0$.

As a conclusion, to ensure that γ *is positive requires that* $b_2 < 0$, *and this implies that when* $b_1+\delta \le m$, *then* $\gamma > 0$, *and when* $b_2+\delta > m$, *then* b_2 *must be smaller than* $m - b_1 - \delta$.

Proof of Corollary 3.4. By Theorem 3.3, it follows that

$$\limsup_{t\to\infty} \frac{\ln\left(\|x(t)\| - \left(\frac{r(t)}{\lambda^m(t)}\right)^{1/q}\right)}{\ln\lambda(t)} \le -\gamma, \quad \text{q.s.}$$

Since, we have $\lim_{t\to\infty}\frac{r(t)}{\lambda^m(t)} = \widetilde{r} < \widetilde{r'}$, there exists $\widetilde{T} \ge t_0$ such that $\frac{r(t)}{\lambda^m(t)} \le \widetilde{r'}$, for all $t \ge \widetilde{T} \ge t_0$. Consequently,

$$\limsup_{t\to\infty} \frac{\ln\left(\|x(t)\| - \left(\widetilde{r'}\right)^{1/q}\right)}{\ln\lambda(t)} \le \limsup_{t\to\infty} \frac{\ln\left(\|x(t)\| - \left(\frac{r(t)}{\lambda^m(t)}\right)^{1/q}\right)}{\ln\lambda(t)} \le -\gamma, \quad \text{q.s.},$$

where $\gamma = m - (b_2 + (b_1+\delta)\vee m)$. Hence, if $m > b_2 + (b_1+\delta)\vee m$, then the solution to system (3.1) tends to the ball $\mathcal{B}_r$, with $r = \left(\widetilde{r'}\right)^{1/q}$ quasi surely with decay function $\lambda(t)$ and order at least γ. □

We analyze the following example to show how the previous theorem can be implemented.

Example 3.6. Consider the following one-dimensional stochastic differential delay equation with constant time delay driven by G-Brownian motion.

$$\begin{gathered} dx(t) = -\frac{\beta+1}{2(1+t)}x(t)dt + \frac{1}{1+t}x(t-\tau)d\langle \mathrm{B}\rangle_t + (1+t)^{-\frac{1}{2}}d\mathrm{B}_t, \quad t\ge 0, \\ x(t) = \xi(t), \quad t\in[-\tau,0], \end{gathered} \tag{3.5}$$

where $\beta \in \mathbb{R}_+$, B_t is a one-dimensional G-Brownian motion with $\mathrm{B}_t \sim \mathcal{N}(0,[\frac{1}{3},\frac{1}{2}])$ and τ is a positive constant.

For $\Phi \in \mathcal{C}([-\tau,0],\mathbb{R})$ and $t \ge 0$, we define

$$f(t,\Phi) = -\frac{\beta+1}{2(1+t)}\Phi(0), \quad g(t,\Phi) = (1+t)^{-\frac{1}{2}}, \quad h(t,\Phi) = \frac{1}{1+t}\Phi(-\tau)\,.$$

Now, we aim at investigating the practical stability with a general decay rate of system (3.5) by using a G-Lyapunov functional. Consider the functional

$$V(t,x_t) := (1+t)\|x(t)\|^2 + \frac{1}{4}\int_{t-\tau}^{t} \|x(u)\|^2 du.$$

Then, it is easy to check that for arbitrary $\alpha > 1$ and $\varphi_1(t) = \frac{\beta}{4(1+t)^\alpha}$, we obtain

$$\begin{aligned}
&\int_0^t LV(s,x_s)ds + \int_0^t \frac{\beta}{16(1+s)^\alpha}\|V_x(s,x_s)g(s,x_s)\|^2 ds \\
&\le \int_0^t \|x(s)\|^2 ds + \int_0^t -(\beta+1)\|x(s)\|^2 ds + 2\int_0^t \|x(s)\|\|x(s-\tau)\| d\langle \mathrm{B}\rangle_s \\
&+ \int_0^t d\langle \mathrm{B}\rangle_s + \int_0^t \|x(s)\|^2 ds - \int_0^t \|x(s-\tau)\|^2 ds + \int_0^t \frac{\beta}{4(1+s)^{\alpha-2}}\|x(s)\|^2 ds \\
&\le \int_0^t \|x(s)\|^2 ds + \int_0^t -(\beta+1)\|x(s)\|^2 ds + 2\int_0^t \frac{1}{4}\|x(s)\|\|x(s-\tau)\| ds + \int_0^t \frac{1}{4} ds \\
&+ \int_0^t \|x(s)\|^2 ds - \int_0^t \|x(s-\tau)\|^2 ds + \int_0^t \frac{\beta}{4(1+s)^{\alpha-2}}\|x(s)\|^2 ds \\
&\le \int_0^t \|x(s)\|^2 ds + \int_0^t -(\beta+1)\|x(s)\|^2 ds + \int_0^t \frac{1}{4}\|x(s)\|^2 ds \\
&\quad + \int_0^t \frac{1}{4}\|x(s-\tau)\|^2 ds + \int_0^t \frac{1}{4} ds \\
&\quad + \int_0^t \|x(s)\|^2 ds - \int_0^t \|x(s-\tau)\|^2 ds + \int_0^t \frac{\beta}{4(1+s)^{\alpha-2}}\|x(s)\|^2 ds.
\end{aligned}$$

That is,

$$\begin{aligned}
&\int_0^t LV(s,x_s)ds + \int_0^t \frac{\beta}{16(1+s)^\alpha}\|V_x(s,x_s)g(s,x_s)\|^2 ds, \\
&\le \frac{1}{4}t + \int_0^t \Big(\frac{\frac{1}{2}-\beta}{1+s} + \frac{\beta}{(1+s)^{\alpha-1}}\Big)(1+s)\|x(s)\|^2 ds.
\end{aligned}$$

Hence, we see that

$$\varphi_2(t) = \frac{\beta}{(1+t)^{\alpha-1}} + \frac{\frac{1}{2}-\beta}{1+t}, \quad r(t) = \frac{1}{4}t.$$

Taking $\lambda(t) = (1+t)$ and doing easy computations, we can check that

$$\delta = 0, \quad b_1 = \alpha, \quad b_2 = \frac{1}{2} - \beta, \quad \widetilde{r} = \frac{1}{4}, \quad m = 1.$$

Finally, using Corollary 3.4 we deduce that

$$\lim_{t\to\infty} \sup \frac{\ln\big(\|x(t)\| - \frac{1}{4}\big)}{\ln(1+t)} \le -\gamma, \quad \text{q.s.},$$

where $\gamma = \beta - \alpha + \frac{1}{2}$. Hence, the solution to system (3.5) tends to the ball $\mathcal{B}_r$ quasi surely with decay function $\lambda(t) = (1+t)$, $r = \frac{1}{4}$ and order at least γ whenever $\beta > \alpha - \frac{1}{2}$.

4. Practical stability

In this section we construct G-Lyapunov functionals for practical stability of stochastic delay differential equations driven by G-Brownian motion.

Corollary 3.4 shows that the quasi sure practical stability with a general decay rate of G-SDDEs (3.1) can be reduced to the construction of appropriate G-Lyapunov functionals. In the following, we propose a procedure to construct G-Lyapunov functionals for G-SDDEs, which consists of four steps.

Step 1: Let us represent (3.1) in the form

$$\begin{aligned} dz(t,x_t) &= (f_1(t,x(t)) + f_2(t,x_t))\,dt + (h_1(t,x(t)) + h_2(t,x_t))\,d\langle \mathrm{B}\rangle_t \\ &\quad + (g_1(t,x(t)) + g_2(t,x_t))\,d\mathrm{B}_t, \end{aligned} \tag{4.1}$$

where $z(t,x_t)$ is some functional of x_t, the functions $f_1(t,x(t)), h_1(t,x(t))$ and $g_1(t,x(t))$, depend on t and $x(t)$ only and do not depend on the previous values $x(t+\theta)$, $\theta < 0$, of the solution. Assume that there exists $t \in \mathbb{R}_+$, such that $f_1(t,\cdot) \neq 0$ or $h_1(t,\cdot) \neq 0$ or $g_1(t,\cdot) \neq 0$.

Step 2: Consider the auxiliary differential equation without memory

$$dy(t) = f_1(t,y(t))dt + h_1(t,y(t))d\langle \mathrm{B}\rangle_t + g_1(t,y(t))d\mathrm{B}_t. \tag{4.2}$$

Assume that (4.2) is quasi sure practical stable with a general decay rate and there exists a G-Lyapunov function $v(t,y(t))$, which satisfies the conditions of Corollary 3.4.

Step 3: A G-Lyapunov functional $V(t,x_t)$ for (3.1) is constructed in the form $V = V_1 + V_2$, where $V_1(t,x_t) = v(t,z(t,x_t))$. Here the argument y of the function $v(t,y)$ is replaced on the functional $z(t,x_t)$ from the left-hand side of (4.1).

Step 4: Usually, the functional $V_1(t,x_t)$ almost fulfills the conditions of Corollary 3.4. To fully satisfy these conditions, it is necessary to calculate $LV_1(t,x_t)$ and estimate it. Then, we choose the additional functional $V_2(t,x_t)$ in a standard way.

The representation (4.1) is not unique. This fact allows, using different representations of the type of (4.1) or different ways to estimate $LV_1(t,x_t)$, to construct different G-Lyapunov functionals and, as a result, to obtain different sufficient conditions for the practical stability with general decay rate.

The above procedure is a general method of Lyapunov functionals construction, which was proposed by Kolmanovskii and Shaikhet [15, 16, 28], and it has already been successfully used for functional differential equations, for difference equations with discrete time, for difference equations with continuous time. This method is used here for stochastic differential equations with delay driven by G-Brownian motion. Our interest now is to investigate the quasi sure practical stability with a general decay rate of stochastic differential equations with a constant and time-varying delay driven by G-Brownian motion exploiting the method of Lyapunov functionals construction.

Now we construct G-Lyapunov functionals for stochastic differential equations with constant delay driven by G-Brownian motion. Consider the following stochastic differential equation with constant delay driven by G-Brownian motion,

$$\begin{aligned} dx(t) &= (F(t,x(t)) + f(t,x(t),x(t-\tau_1)))\,dt \\ &\quad + h(t,x(t),x(t-\tau_2))d\langle \mathrm{B}\rangle_t + g(t,x(t),x(t-\tau_3))d\mathrm{B}_t, \\ x(t) &= \xi(t-t_0), \quad t \in (t_0-\widetilde{\tau},t_0), \end{aligned} \tag{4.3}$$

where

$$\bar{\tau} = \max[\tau_1, \tau_2], \quad \widetilde{\tau} = \max[\bar{\tau}, \tau_3], \quad F : \mathbb{R}_+ \times \mathbb{R}^n \to \times\mathbb{R}^n,$$
$$f : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \quad g : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m},$$
$$h : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}.$$

Here B_t is an m-dimensional G-Brownian motion, $\langle B\rangle_{t\geq 0}$ is the quadratic variation process of the G-Brownian B. Remark that (4.3) is a particular case of (3.1).

We will apply the method described above to construct G-Lyapunov functionals for (4.3), and, as a consequence, to deduce sufficient conditions ensuring the quasi sure practical stability with decay function $\lambda(t)$, where $\lambda(\cdot) \in \mathcal{C}^1(\mathbb{R}_+)$.

Theorem 4.1. *Assume that $\ln\lambda(t)$ is uniformly continuous on $t \geq 0$, there exists a constant $\delta \geq 0$ such that*

$$\lim_{t\to\infty} \frac{\ln\ln t}{\ln\lambda(t)} \leq \delta.$$

Let $\psi(t)$ be a continuous non-negative function, and $r(t)$ a non-negative continuous differentiable function such that for all $t \geq t_0 \geq 0$, the following inequalities hold:

(1)
$$\begin{gathered} 2\langle x, F(t,x)\rangle \leq (\psi(t) - U)\|x\|^2 + \frac{r'(t)}{\lambda^m(t)}, \quad U > 0, \\ \|\widetilde{f}(t,\Phi)\| \leq a_1\|\Phi(-\tau_1)\|, \\ \|\widetilde{h}(t,\Phi)\| \leq a_2\|\Phi(-\tau_2)\|, \\ \|\widetilde{g}(t,\Phi)\| \leq a_3\|\Phi(-\tau_3)\|, \\ \|\Phi(0)\widetilde{g}(t,\Phi)\| \leq a_4\|\Phi(-\tau_3)\|, \end{gathered} \tag{4.4}$$

where $\widetilde{f}(t,\Phi) = f(t,\Phi(0),\Phi(-\tau_1))$, $\widetilde{g}(t,\Phi) = g(t,\Phi(0),\Phi(-\tau_3))$, $\widetilde{h}(t,\Phi) = h(t,\Phi(0),\Phi(-\tau_2))$.

(2)
$$\limsup_{t\to\infty} \frac{\int_{t_0}^t \psi(s)ds}{\ln\lambda(t)} \leq a, \quad a \in \mathbb{R},$$
$$\limsup_{t\to\infty} \frac{t}{\ln\lambda(t)} = C \geq 0, \quad \lim_{t\to\infty} \frac{r(t)}{\lambda^m(t)} = \widetilde{r} > 0.$$

(3) *There exists $\widetilde{r'} \geq \widetilde{r} > 0$, such that the solution $x(t,t_0,\xi)$ satisfies*
$$\|x(t,t_0,\xi)\| > \big(\widetilde{r'}\big)^{1/2}, \quad \forall t \geq t_0.$$

Then

$$\lim_{t\to\infty} \frac{\ln\big(\|x(t,t_0,\xi)\| - \big(\widetilde{r'}\big)^{1/2}\big)}{\ln\lambda(t)} \leq -\gamma, \quad q.s.,$$

where $\gamma = UC - \big(m + a + \delta + (2a_1 + 2\bar{\sigma}^2 a_2 + \bar{a})C\big)$, $\bar{a} = \bar{\sigma}^2(a_3^2 + a_4^2)$.

In particular, if $UC > m + (a+\delta) + (2a_1 + 2\bar{\sigma}^2 a_2 + \bar{a})C$, then the solution to system (4.3) tends to the ball $\mathcal{B}_r$, with $r = (\widetilde{r'})^{1/2}$ quasi surely, with decay function $\lambda(t)$ and order at least γ.

Proof. Based upon the procedure of G-Lyapunov functionals construction, we consider the auxiliary equation without memory of the type (4.2) as

$$\dot{y}(t) = F(t, y(t)). \tag{4.5}$$

Our interest now is to prove that the solution to system (4.5) tends to the ball $\mathcal{B}_r$, with $r = (\widetilde{r'})^{1/2}$ quasi surely with decay function $\lambda(t)$. We consider the function $v(t,y) = \lambda^m(t)\|y\|^2$, $m \geq 0$ as a Lyapunov function for Eq.(4.5). Then, we have to prove that $v(t,y)$ satisfies all conditions of Corollary 3.4.

Based upon (4.4), we have

$$\begin{aligned}
&\int_{t_0}^t v_s(s,y(s))ds + \int_{t_0}^t v_x(s,y(s))F(s,y(s))ds \\
&\leq \int_{t_0}^t m\lambda'(s)\lambda^{m-1}(s)\|y(s)\|^2 ds + \int_{t_0}^t 2\lambda^m(s)\langle y(s), F(s,y(s))\rangle ds \\
&\leq \int_{t_0}^t m\lambda'(s)\lambda^{m-1}(s)\|y(s)\|^2 ds + \int_{t_0}^t \big(\lambda^m(s)\left(\psi(s) - U\right)\|y(s)\|^2 + r'(s)\big)\, ds \\
&\leq \int_{t_0}^t \Big(m\frac{\lambda'(s)}{\lambda(s)} + \psi(s) - U\Big)\lambda^m(s)\|y(s)\|^2 ds + r(t) - r(t_0).
\end{aligned}$$

That is,

$$\begin{aligned}
&\int_{t_0}^t v_s(s,y(s))ds + \int_{t_0}^t v_x(s,y(s))F(s,y(s))ds, \\
&\leq \int_{t_0}^t \Big(m\frac{\lambda'(s)}{\lambda(s)} + \psi(s) - U\Big)\lambda^m(s)\|y(s)\|^2 ds + r(t),
\end{aligned}$$

we set $\varphi_2(t) = m\frac{\lambda'(t)}{\lambda(t)} + \psi(t) - U$.

Based on assumption $(\mathcal{A}_2)$, we obtain

$$\lim_{t\to\infty}\sup \frac{\int_{t_0}^t \varphi_2(s)ds}{\ln\lambda(t)} \leq m + a - UC.$$

In view of Corollary 3.4, we deduce that

$$\lim_{t\to\infty}\sup \frac{\ln\big(\|y(t)\| - (\widetilde{r'})^{1/2}\big)}{\ln\lambda(t)} \leq -\gamma, \quad \text{q.s.},$$

where $\gamma = UC - (a + \delta \vee m)$. Hence, if $UC > (a + \delta \vee m)$, the solution to system (4.4) tends to the ball $\mathcal{B}_r$, with $r = (\widetilde{r'})^{1/2}$ quasi surely with decay function $\lambda(t)$ and order at least γ.

Now we construct a G-Lyapunov functional V for (4.3) in the form

$$V = V_1 + V_2, \quad \text{where } V_1(t,x_t) = \lambda^m(t)\|x(t)\|^2.$$

Considering $\varphi_1(t) = \frac{1}{4\lambda^m(t)}$ for $t \geq 0$, we obtain

$$\begin{aligned}
&\int_{t_0}^t LV_1(s,x_s)ds + \bar{\sigma}^2\int_{t_0}^t \varphi_1(s)\|V_{1x}(s,x_s)\widetilde{g}(s,x(s),x(s-\tau_3))\|^2 ds \\
&= \int_{t_0}^t m\lambda'(s)\lambda^{m-1}(s)\|x(s)\|^2 ds + \int_{t_0}^t 2\lambda^m(s)\langle F(s,x(s)), x(s)\rangle ds \\
&\quad + \int_{t_0}^t 2\lambda^m(s)\langle \widetilde{f}(s,x(s),x(s-\tau_1)), x(s)\rangle ds \\
&\quad + \int_{t_0}^t 2\lambda^m(s)\langle \widetilde{h}(s,x(s),x(s-\tau_2)), x(s)\rangle d\langle \mathrm{B}\rangle_s
\end{aligned}$$

$$+ \int_{t_0}^{t} \lambda^m(s) \|\widetilde{g}(s, x(s), x(s-\tau_3))\|^2 d\langle \mathrm{B}\rangle_s$$
$$+ \int_{t_0}^{t} \bar{\sigma}^2 \lambda^m(s) \|x(s)\widetilde{g}(s, x(s), x(s-\tau_3))\|^2 ds.$$

Based on Peng [24, Chapter III], we have that for each $0 \le s \le t \le T$,

$$\underline{\sigma}^2(t-s) \le \langle \mathrm{B}\rangle_t - \langle \mathrm{B}\rangle_s \le \bar{\sigma}^2(t-s).$$

Then

$$\begin{aligned}
&\int_{t_0}^{t} LV_1(s, x_s) ds + \int_{t_0}^{t} \bar{\sigma}^2 \varphi_1(s) \|V_{1x}(s, x_s)\widetilde{g}(s, x(s), x(s-\tau))\|^2 ds \\
&= \int_{t_0}^{t} m\lambda(s)\lambda^{m-1}(s)\|x(s)\|^2 ds + \int_{t_0}^{t} 2\lambda^m(s)\langle F(s, x(s)), x(s)\rangle ds \\
&\quad + \int_{t_0}^{t} 2\lambda^m(s)\langle \widetilde{f}(s, x(s), x(s-\tau_1)), x(s)\rangle ds \\
&\quad + \int_{t_0}^{t} 2\bar{\sigma}^2\lambda^m(s)\langle \widetilde{h}(s, x(s), x(s-\tau_2)), x(s)\rangle ds \\
&\quad + \int_{t_0}^{t} \bar{\sigma}^2\lambda^m(s)\|\widetilde{g}(s, x(s), x(s-\tau_3))\|^2 ds \\
&+ \int_{t_0}^{t} \bar{\sigma}^2\lambda^m(s)\|x(s)\widetilde{g}(s, x(s), x(s-\tau_3))\|^2 ds.
\end{aligned}$$

Taking into account assumption (4.4),

$$\begin{aligned}
&\int_{t_0}^{t} LV_1(s, x_s) ds + \int_{t_0}^{t} \frac{1}{4\lambda^m(s)} \|V_{1x}(s, x_s)\widetilde{g}(s, x(s), x(s-\tau))\|^2 ds \\
&\le \int_{t_0}^{t} \lambda^m(s) \left(m\frac{\lambda'(s)}{\lambda(s)} + \psi(s) - U \right) \|x(s)\|^2 ds \\
&\quad + \int_{t_0}^{t} 2a_1\lambda^m(s)\|x(s)\| \, \|x(s-\tau_1)\| ds + \int_{t_0}^{t} 2\bar{\sigma}^2 a_2\lambda^m(s)\|x(s)\| \, \|x(s-\tau_2)\| ds \\
&\quad + \int_{t_0}^{t} \bar{\sigma}^2 a_3^2\lambda^m(s)\|x(s-\tau_3)\|^2 ds + \int_{t_0}^{t} \bar{\sigma}^2 a_4^2\lambda^m(s)\|x(s-\tau_3)\|^2 ds + r(t) \\
&\le \int_{t_0}^{t} \lambda^m(s)\Big(\Big(m\frac{\lambda'(s)}{\lambda(s)} + \psi(s) - U\Big) + a_1 + \bar{\sigma}^2 a_2\Big)\|x(s)\|^2 ds \\
&\quad + \int_{t_0}^{t} a_1\lambda^m(s)\|x(s-\tau_1)\|^2 ds + \int_{t_0}^{t} \bar{\sigma}^2 a_2\lambda^m(s)\|x(s-\tau_2)\|^2 ds \\
&\quad + \int_{t_0}^{t} \bar{a}\lambda^m(s)\|x(s-\tau_3)\|^2 ds + r(t),
\end{aligned}$$

where $\bar{a} = \bar{\sigma}^2(a_3^2 + a_4^2)$.

Let

$$\begin{aligned}
V_2(t, x_t) &= a_1 \int_{t-\tau_1}^{t} \lambda^m(u+\tau_1)\|x(u)\|^2 du + \bar{\sigma}^2 a_2 \int_{t-\tau_2}^{t} \lambda^m(u+\tau_2)\|x(u)\|^2 du \\
&\quad + \bar{a} \int_{t-\tau_3}^{t} \lambda^m(u+\tau_3)\|x(u)\|^2 du.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\int_{t_0}^t LV_2(s,x_s)ds &= a_1\int_{t_0}^t \lambda^m(s+\tau_1)\|x(s)\|^2ds - a_1\int_{t_0}^t \lambda^m(s)\|x(s-\tau_1)\|^2ds \\ &\quad + \bar{\sigma}^2 a_2\int_{t_0}^t \lambda^m(s+\tau_2)\|x(s)\|^2ds - \bar{\sigma}^2 a_2\int_{t_0}^t \lambda^m(s)\|x(s-\tau_2)\|^2ds \\ &\quad + \bar{a}\int_{t_0}^t \lambda^m(s+\tau_3)\|x(s)\|^2ds - \bar{a}\int_{t_0}^t \lambda^m(s)\|x(s-\tau_3)\|^2ds \\ &\simeq a_1\int_{t_0}^t \lambda^m(s)\|x(s)\|^2ds - a_1\int_{t_0}^t \lambda^m(s)\|x(s-\tau_1)\|^2ds \\ &\quad + \bar{\sigma}^2 a_2\int_{t_0}^t \lambda^m(s)\|x(s)\|^2ds - \bar{\sigma}^2 a_2\int_{t_0}^t \lambda^m(s)\|x(s-\tau_2)\|^2ds \\ &\quad + \bar{a}\int_{t_0}^t \lambda^m(s)\|x(s)\|^2ds - \bar{a}\int_{t_0}^t \lambda^m(s)\|x(s-\tau_3)\|^2ds.\end{aligned}$$

For $V = V_1 + V_2$, we have

$$\begin{aligned}&\int_{t_0}^t LV(s,x_s)ds + \int_{t_0}^t \frac{\bar{\sigma}^2}{4\lambda^m(s)}\|V_x(s,x_s)\widetilde{g}(s,x(s),x(s-\tau_3))\|^2ds \\ &\le \int_{t_0}^t \lambda^m(s)\Big(m\frac{\lambda'(s)}{\lambda(s)} + \psi(s) + 2a_1 + 2\bar{\sigma}^2 a_2 + \bar{a} - U\Big)\|x(s)\|^2ds + r(t).\end{aligned}$$

Thus,

$$\varphi_2(t) = m\frac{\lambda'(t)}{\lambda(t)} + \psi(t) + 2a_1 + 2\bar{\sigma}^2 a_2 + \bar{a} - U, \quad \varphi_1(t) = \frac{1}{4\lambda^m(t)}.$$

Hence, we arrive at

$$\lim_{t\to\infty}\sup\frac{\int_{t_0}^t \varphi_2(s)ds}{\ln\lambda(t)} \le m + a + (2a_1 + 2\bar{\sigma}^2 a_2 + \bar{a} - U)C,$$

$$\lim_{t\to\infty}\inf\frac{\ln\varphi_1(t)}{\ln\lambda(t)} \ge -m.$$

Therefore, Corollary 3.4 allows us to conclude that

$$\lim_{t\to\infty}\frac{\ln\big(\|x(t,t_0,\xi)\| - \big(\widetilde{r'}\big)^{1/2}\big)}{\ln\lambda(t)} \le -\gamma, \quad \text{q.s.},$$

where $\gamma = UC - \big(m + a + \delta + (2a_1 + 2\bar{\sigma}^2 a_2 + \bar{a})\big)$. Consequently, if $UC > m + (a+\delta) + (2a_1 + 2\bar{\sigma}^2 a_2 + \bar{a})C$, the solution to system (4.3) tends to the ball $\mathcal{B}_r$, with $r = (\widetilde{r'})^{1/2}$ quasi surely with decay function $\lambda(t)$. □

Now we construct G-Lyapunov functionals for stochastic differential equations with time-varying delay driven by G-Brownian motion. We consider the following stochastic differential equation with time-varying delay driven by G-Brownian motion,

$$\begin{aligned} dx(t) = [F(t, x(t)) + f(t, x(t), x(t-\tau_1(t)))]\, dt \\ + h(t, x(t), x(t-\tau_2(t)))d\langle \mathrm{B}\rangle_t + g(t, x(t), x(t-\tau_3(t)))d\mathrm{B}_t, \\ \tau_1(t) \in [0, \tau_{10}], \quad \tau_2(t) \in [0, \tau_{20}], \quad \tau_3(t) \in [0, \tau_{30}], \\ \bar{\tau} = \max[\tau_{10}, \tau_{20}], \quad \tau = \max[\bar{\tau}, \tau_{30}], \\ x(t) = \xi(t - t_0), \quad t \in [t_0 - \tau, t_0], \end{aligned} \tag{4.6}$$

where

$$F : \mathbb{R}_+ \times \mathbb{R}^n \to \times\mathbb{R}^n, \quad f : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,$$
$$h : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}, \quad g : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}.$$

Here B_t is an m-dimensional G-Brownian motion, $\langle B\rangle_{t\geq 0}$ is the quadratic variation process of the G-Brownian motion B. Notice that (4.6) is a particular case of (3.1).

Now, we apply the procedure of constructing G-Lyapunov functionals for (4.6), to state sufficient conditions ensuring the quasi sure practical uniform exponential stability, with decay function $\lambda(t) = \exp(t)$. The construction of G-Lyapunov functionals for general decay functions will be analyzed elsewhere.

Theorem 4.2. *Let $\phi_1(t)$ be a continuous non-negative function, $\phi_2(t), \phi_3(t) > 0$ non-increasing functions and $r(t)$ a continuous non-negative differentiable function such that, for all $t \geq t_0 \geq 0$, (H3) holds, and the following assumptions as well,*

(1)

$$\begin{gathered} 2\langle x, F(t, x)\rangle \leq (\phi_1(t) - U)\|x\|^2 + \frac{r'(t)}{\exp(mt)}, \quad U > 0, \\ \|\widetilde{f}(t, \Phi)| \leq \phi_2(t)\|\Phi(-\tau_1(t))\|, \\ \|\widetilde{h}(t, \Phi)\| \leq \phi_3(t)\|\Phi(-\tau_2(t))\|, \\ \|\widetilde{g}(t, \Phi)\| \leq c_4\|\Phi(-\tau_3(t))\|, \\ \|\Phi(0)\widetilde{g}(t, \Phi)\| \leq c_5\|\Phi(-\tau_3(t))\|, \end{gathered} \tag{4.7}$$

where

$$\widetilde{f}(t, \Phi) = f(t, \Phi(0), \Phi(-\tau_1(t))), \quad \widetilde{h}(t, \Phi) = h(t, \Phi(0), \Phi(-\tau_2(t))),$$
$$\widetilde{g}(t, \Phi) = g(t, \Phi(0), \Phi(-\tau_3(t))),$$

and

$$\begin{aligned} \tau_1(t) \in [0, \tau_{1_0}], \quad \dot{\tau_1}(t) \leq \tau_1 \leq 1, \\ \tau_2(t) \in [0, \tau_{20}], \quad \dot{\tau_2}(t) \leq \tau_2 \leq 1, \\ \tau_3(t) \in [0, \tau_{30}], \quad \dot{\tau_3}(t) \leq \tau_3 \leq 1. \end{aligned} \tag{4.8}$$

(2)

$$\begin{gathered} \lim_{t\to\infty} \sup \frac{\int_{t_0}^t \phi_1(s)ds}{t} \leq c_1, \quad c_1 > 0, \\ \lim_{t\to\infty} \sup \frac{\int_{t_0}^t \phi_2(s)ds}{t} \leq c_2, \quad c_2 > 0, \\ \lim_{t\to\infty} \sup \frac{\int_{t_0}^t \phi_3(s)ds}{t} \leq c_3, \quad c_3 > 0, \end{gathered}$$

$$\lim_{t\to\infty} \frac{r(t)}{\exp(mt)} = \widetilde{r}, \quad \widetilde{r} > 0.$$

Then

$$\lim_{t\to\infty} \frac{\ln\left(\|x(t,t_0,\xi)\| - (\widetilde{r'})^{1/2}\right)}{\ln\lambda(t)} \le -\gamma, \quad q.s.,$$

where

$$\gamma = U - \Big(m + c_1 + \Big(1 + \frac{\exp(m\tau_{10})}{1-\tau_1}\Big)c_2 + \bar{\sigma}^2\Big(1 + \frac{\exp(m\tau_{20})}{1-\tau_2}\Big)c_3 + \bar{c}\frac{\exp(m\tau_{30})}{1-\tau_3}\Big),$$
$$\bar{c} = \bar{\sigma}^2\left(c_4^2 + c_5^2\right).$$

In particular, if

$$U > m + c_1 + \Big(1 + \frac{\exp(m\tau_{10})}{1-\tau_1}\Big)c_2 + \bar{\sigma}^2\Big(1 + \frac{\exp(m\tau_{20})}{1-\tau_2}\Big)c_3 + \bar{c}\frac{\exp(m\tau_{30})}{1-\tau_3},$$

the solution to (4.6) *tends to the ball $\mathcal{B}_r$, with $r = (\widetilde{r'})^{1/2}$ quasi surely uniformly practically exponentially stable, i.e., with decay function $\lambda(t) = \exp(t)$, and order at least γ.*

Proof. Proceeding as in the proof of Theorem 4.1, we consider the auxiliary equation without memory of the type (4.2),

$$\dot{y}(t) = F(t, y(t)). \tag{4.9}$$

We have to prove that the solution to (4.9) tends to the ball $\mathcal{B}_r$, with $r = (\widetilde{r'})^{1/2}$ and decay function $\lambda(t)$. To this end, we consider the function $v(t,y) = \exp(mt)\|y\|^2$ with $m \ge 0$ as a Lyapunov function for (4.9). Therefore, we prove that $v(t,y)$ satisfies all conditions of Corollary 3.4. Using (4.7), we have

$$\begin{aligned}
&\int_{t_0}^t v_s(s,y(s))ds + \int_{t_0}^t v_x(s,y(s))F(s,y(s))ds,\\
&\le \int_{t_0}^t (m + \phi_1(s) - U)\exp(ms)\|y(s)\|^2 ds + r(t).
\end{aligned}$$

Thus, setting $\varphi_2(t) = m + \phi_1(t) - U$, by Corollary 3.4, we obtain

$$\lim_{t\to\infty}\sup \frac{\ln\left(\|y(t)\| - (\widetilde{r'})^{1/2}\right)}{t} \le -\gamma, \quad \text{q.s.},$$

where $\gamma = U - (c_1 + m)$, then if $U > c_1 + m$, and the solution to (4.9) tends to the ball $\mathcal{B}_r$, with $r = (\widetilde{r'})^{1/2}$ practically uniformly exponentially stable with order at least $\gamma = U - (c_1 + m)$.

Based on this procedure, now we construct a G-Lyapunov functional V for (4.6) in the form $V = V_1 + V_2$, where $V_1(t,x_t) = \exp(mt)\|x(t)\|^2$. Consider $\varphi_1(t) = \frac{1}{4\exp(mt)}$, $t \ge 0$, we then deduce

$$\begin{aligned}
&\int_{t_0}^t LV_1(s,x_s)ds + \bar{\sigma}^2\int_{t_0}^t \varphi_1(s)\|V_{1x}(s,x_s)\widetilde{g}(s,x_s)\|^2 ds\\
&= \int_{t_0}^t \exp(ms)\|x(s)\|^2 ds + \int_{t_0}^t 2\exp(ms)\langle F(s,x(s)), x(s)\rangle ds\\
&\quad + \int_{t_0}^t 2\exp(ms)\langle \widetilde{f}(s,x_s), x(s)\rangle ds + \int_{t_0}^t 2\exp(ms)\langle \widetilde{h}(s,x_s), x(s)\rangle d\langle \mathrm{B}\rangle_s
\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_0}^t \exp(ms)\|\widetilde{g}(s,x_s)\|^2 d\langle \mathrm{B}\rangle_s+\int_{t_0}^t \bar{\sigma}^2\exp(ms)\|x(s)\widetilde{G}(s,x_s)\|^2ds\\
&\le \int_{t_0}^t \exp(ms)\|x(s)\|^2ds+\int_{t_0}^t 2\exp(ms)\langle F(s,x(s)),x(s)\rangle ds\\
&+\int_{t_0}^t 2\exp(ms)\langle \widetilde{f}(s,x_s),x(s)\rangle ds+\int_{t_0}^t 2\bar{\sigma}^2\exp(ms)\langle \widetilde{h}(s,x_s),x(s)\rangle ds\\
&+\int_{t_0}^t \bar{\sigma}^2\exp(ms)\|\widetilde{g}(s,x_s)\|^2ds+\int_{t_0}^t \bar{\sigma}^2\exp(ms)\|x(s)\widetilde{G}(s,x_s)\|^2ds.
\end{aligned}
$$

Taking into account assumption (4.7), we have

$$
\begin{aligned}
&\int_{t_0}^t LV_1(s,x_s)ds+\int_{t_0}^t \frac{\bar{\sigma}^2}{4\exp(ms)}\|V_{1x}(s,x_s)\widetilde{g}(s,x_s)\|^2ds\\
&\le \int_{t_0}^t \exp(ms)(m+\phi_1(s)-U)\ \|x(s)\|^2ds+r(t)\\
&\quad+\int_{t_0}^t 2\phi_2(s)\exp(ms)\|x(s)\|\ \|x(s-\tau_1(s))\|ds\\
&\quad+\int_{t_0}^t 2\bar{\sigma}^2\phi_3(s)\exp(ms)\|x(s)\|\|x(s-\tau_2(s))\|ds\\
&\quad+\int_{t_0}^t \bar{\sigma}^2c_4^2\exp(ms)\|x(s-\tau_3(s))\|^2ds+\int_{t_0}^t \bar{\sigma}^2c_5^2\exp(ms)\|x(s-\tau_3(s))\|^2ds\\
&\le \int_{t_0}^t \exp(ms)\big((m+\phi_1(s)-U)+\phi_2(s)+\bar{\sigma}^2\phi_3(s)\big)\ \|x(s)\|^2ds\\
&\quad+\int_{t_0}^t \phi_2(s)\exp(ms)\|x(s-\tau_1(s))\|ds+\int_{t_0}^t \bar{\sigma}^2\phi_3(s)\exp(ms)\|x(s-\tau_2(s))\|^2ds\\
&\quad+\int_{t_0}^t \bar{c}\exp(ms)\|x(s-\tau_3(s))\|^2ds+r(t),
\end{aligned}
$$

where $\bar{c}=\bar{\sigma}^2(c_4^2+c_5^2)$. Let

$$
\begin{aligned}
V_2(t,x_t)&=\frac{1}{1-\tau_1}\int_{t-\tau_1(t)}^t \exp(m(u+\tau_{10}))\phi_2(u)\|x(u)\|^2du\\
&\quad+\frac{\bar{\sigma}^2}{1-\tau_2}\int_{t-\tau_2(t)}^t \exp(m(u+\tau_{20}))\phi_3(u)\|x(u)\|^2du\\
&\quad+\frac{\bar{c}}{1-\tau_3}\int_{t-\tau_3(t)}^t \exp(m(u+\tau_{30}))\|x(u)\|^2du.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\int_{t_0}^t LV_2(s,x_s)ds\\
&=\frac{1}{1-\tau_1}\int_{t_0}^t \exp(m(s+\tau_{10}))\phi_2(s)\|x(s)\|^2ds\\
&\quad-\frac{1}{1-\tau_1}\int_{t_0}^t (1-\dot{\tau}_1(s))\exp(m(s-\tau_1(s)+\tau_{10}))\phi_2(s-\tau_1(s))\|x(s-\tau_1(s))\|^2ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\bar\sigma^2}{1-\tau_2}\int_{t_0}^t \exp(m(s+\tau_{20}))\phi_3(s)\|x(s)\|^2ds\\
&-\frac{\bar\sigma^2}{1-\tau_2}\int_{t_0}^t (1-\dot\tau_2(s))\exp(m(s-\tau_2(s)+\tau_{20}))\phi_3(s-\tau_2(s))\|x(s-\tau_2(s))\|^2ds\\
&+\frac{\bar c}{1-\tau_3}\int_{t_0}^t \exp(m(s+\tau_{30}))\|x(s)\|^2ds\\
&-\frac{\bar c}{1-\tau_3}\int_{t_0}^t (1-\dot\tau_3(s))\exp(m(s-\tau_3(s)+\tau_{30}))\|x(s-\tau_3(s))\|^2ds\\
\le{}& \frac{1}{1-\tau_1}\int_{t_0}^t \exp(m(s+\tau_{10}))\phi_2(s)\|x(s)\|^2ds\\
&-\frac{1}{1-\tau_1}\int_{t_0}^t (1-\tau_1)\exp(ms)\exp(m(\tau_{10}-\tau_1(s)))\phi_2(s-\tau_1(s))\|x(s-\tau_1(s))\|^2ds\\
&+\frac{\bar\sigma^2}{1-\tau_2}\int_{t_0}^t \exp(m(s+\tau_{20}))\phi_3(s)\|x(s)\|^2ds\\
&-\frac{\bar\sigma^2}{1-\tau_2}\int_{t_0}^t (1-\tau_2)\exp(ms)\exp(m(\tau_{20}-\tau_2(s)))\phi_3(s-\tau_2(s))\|x(s-\tau_2(s))\|^2ds\\
&+\frac{\bar c}{1-\tau_3}\int_{t_0}^t \exp(m(s+\tau_{30}))\|x(s)\|^2ds\\
&-\frac{\bar c}{1-\tau_3}\int_{t_0}^t (1-\tau_3)\exp(ms)\exp(m(\tau_{30}-\tau_2(s)))\|x(s-\tau_3(s))\|^2ds.
\end{aligned}
$$

In other words,

$$
\begin{aligned}
\int_{t_0}^t LV_2(s,x_s)ds \le{}& \frac{1}{1-\tau_1}\int_{t_0}^t \exp(m(s+\tau_{10}))\phi_2(s)\|x(s)\|^2ds\\
&-\int_{t_0}^t \exp(ms)\phi_2(s-\tau_1(s))\|x(s-\tau_1(s))\|^2ds\\
&+\frac{\bar\sigma^2}{1-\tau_2}\int_{t_0}^t \exp(m(s+\tau_{20}))\phi_3(s)\|x(s)\||^2ds\\
&-\bar\sigma^2\int_{t_0}^t \exp(ms)\phi_3(s-\tau_2(s))\|x(s-\tau_2(s))\|^2ds\\
&+\frac{\bar c}{1-\tau_3}\int_{t_0}^t \exp(m(s+\tau_{30}))\|x(s)\|^2ds\\
&-\bar c\int_{t_0}^t \exp(ms)\|x(s-\tau_3(s))\|^2ds.
\end{aligned}
$$

For $V=V_1+V_2$, it follows that

$$
\begin{aligned}
&\int_{t_0}^t LV(s,x_s)ds+\int_{t_0}^t \frac{\bar\sigma^2}{4\exp(ms)}\|V_x(s,x_s)\widetilde{g}(s,x_s\|^2ds\\
&\le \int_{t_0}^t \exp(ms)\Big(m+\phi_1(s)-U+\Big(1+\frac{\exp(m\tau_{10})}{1-\tau_1}\Big)\phi_2(s)\\
&\quad+\bar\sigma^2\Big(1+\frac{\exp(m\tau_{20})}{1-\tau_2}\Big)\phi_3(s)+\bar c\frac{\exp(m\tau_{30})}{1-\tau_3}\Big)\|x(s)\|^2ds+r(t).
\end{aligned}
$$

Then, we obtain

$$\begin{aligned}\varphi_2(t) &= m+\phi_1(s)-U+\Big(1+\frac{\exp(m\tau_{10})}{1-\tau_1}\Big)\phi_2(s)\\ &\quad+\bar\sigma^2\Big(1+\frac{\exp(m\tau_{20})}{1-\tau_2}\Big)\phi_3(s)+\bar c\frac{\exp(m\tau_{30})}{1-\tau_3},\\ \varphi_1(t) &= \frac{1}{4\exp(mt)}.\end{aligned}$$

It follows that

$$\begin{aligned}\lim_{t\to\infty}\sup\frac{\int_{t_0}^t\varphi_2(s)ds}{t} &\le m+c_1-U+\Big(1+\frac{\exp(m\tau_{10})}{1-\tau_1}\Big)c_2\\ &\quad+\bar\sigma^2\Big(1+\frac{\exp(m\tau_{20})}{1-\tau_2}\Big)c_3+\bar c\frac{\exp(m\tau_{30})}{1-\tau_3},\\ \lim_{t\to\infty}\inf\frac{\ln\varphi_1(t)}{t} &\ge -m.\end{aligned}$$

Eventually, based upon Corollary 3.4, we deduce that

$$\lim_{t\to\infty}\frac{\ln\big(\|x(t,t_0,\xi)\|-(\widetilde{r'})^{1/2}\big)}{\ln\lambda(t)}\le-\gamma,\quad \text{q.s.},$$

where

$$\begin{aligned}\gamma = U-\Big(&m+c_1-U+\Big(1+\frac{\exp(m\tau_{10})}{1-\tau_1}\Big)c_2\\ &+\bar\sigma^2\Big(1+\frac{\exp(m\tau_{20})}{1-\tau_2}\Big)c_3+\bar c\frac{\exp(m\tau_{30})}{1-\tau_3}\Big).\end{aligned}$$

Hence, if $U > m+c_1+\big(1+\frac{\exp(m\tau_{10})}{1-\tau_1}\big)c_2+\bar\sigma^2\big(1+\frac{\exp(m\tau_{20})}{1-\tau_2}\big)c_3+\bar c\frac{\exp(m\tau_{30})}{1-\tau_3}$, the solution to (4.6) tends to the ball $\mathcal{B}_r$, with $r=(\widetilde{r'})^{1/2}$ quasi surely uniformly practically exponentially stable with decay function $\lambda(t)=\exp(t)$, and order at least γ. □

Now we present an illustrative example that implements the previous result.

Example 4.3. Consider the one-dimensional stochastic differential equation with time-varying delay driven by G-Brownian motion,

$$\begin{aligned}dx(t) &= \Big(\frac{1}{2}\ (\alpha+\exp(-t)-U)\,x(t)+\frac{1}{2(1+\|x(t)\|)}+\frac{1}{t+1}x(t-\tau_1(t))\Big)dt,\\ &\quad+\cos(t)x(t-\tau_2(t))d\langle \mathrm{B}\rangle_t+g(x(t))\frac{x(t-\tau_3(t))}{1+\|x(t)\|}d\mathrm{B}_t,\quad t\ge0,\end{aligned}\tag{4.10}$$

where $x(t)=\xi(t)$ and $t\in[-\tau,0]$, with the conditions

$$\begin{aligned}\tau_1(t)\in[0,\tau_{10}],&\quad \dot\tau_1(t)\le\tau_{10}\le1,\\ \tau_2(t)\in[0,\tau_{20}],&\quad \dot\tau_2(t)\le\tau_{20}\le1,\\ \tau_3(t)\in[0,\tau_{30}],&\quad \dot\tau_3(t)\le\tau_{30}\le1.\end{aligned}$$

Here $\alpha,U\in\mathbb{R}_+$, $g(\cdot):\mathbb{R}\to\mathbb{R}$ is a bounded Lipshitz continuous function, such that $g(0)\neq0$, and $\|g(x)\|\le l$, $l>0$. B_t is a one-dimensional G-Brownian motion with $\mathrm{B}_t\sim\mathcal{N}(0,[\frac12,1])$, and $\bar\tau=\max[\tau_{10},\tau_{20}]$, $\tau=\max[\bar\tau,\tau_{30}]$.

Now we set this problem in our formulation by taking

$$\begin{gathered}F(t,x)=\frac{1}{2}\left(\alpha+\exp(-t)-U\right)x+\frac{1}{2(1+\|x(t)\|)},\\ \widetilde{f}(t,\Phi)=\frac{1}{1+t}\Phi(-\tau_1(t)),\\ \widetilde{h}(t,\Phi)=\cos(t)\Phi(-\tau_2(t)),\\ \widetilde{g}(t,\Phi)=g(\Phi(0))\frac{\Phi(-\tau_3(t))}{1+\|\Phi(0)\|},\end{gathered}$$

$x\in\mathbb{R}$, $\Phi\in\mathcal{C}([-\tau,0],\mathbb{R})$.

For $m=2$, we can check that

$$\begin{gathered}2\langle x,F(t,x)\rangle\le(\alpha+\exp(-t)-U)\|x\|^2+\frac{\exp(t)}{\exp(2t)},\\ \|\widetilde{f}(t,\Phi)\|\le\frac{1}{t+1}\|\Phi(-\tau_1(t))\|,\\ \|\widetilde{h}(t,\Phi)\|\le\|\Phi(-\tau_2(t))\|,\\ \|\widetilde{g}(t,\Phi)\|\le l\|\Phi(-\tau_3(t))\|,\\ \|\Phi(0)\widetilde{g}(t,\Phi)\|\le l\|\Phi(-\tau_3(t))\|.\end{gathered}$$

Therefore,

$$\phi_1(t)=(\alpha+\exp(-t)),\quad \phi_2(t)=\frac{1}{1+t},\quad \phi_3(t)=1,\quad r(t)=\exp(t).$$

Then, we can choose constants in Theorem 4.2 as follows: $c_1=\alpha$, $c_2=0$, $c_3=1$, $c_4=c_5=l$, $\widetilde{r}=1$. Finally, Theorem 4.2 allows us to conclude that

$$\lim_{t\to\infty}\sup\frac{\ln(\|x(t)\|-1)}{t}\le-\gamma,\quad \text{q.s.}$$

where

$$\gamma=U-\Big(\alpha+3+\frac{\exp(2\tau_{20})}{1-\tau_2}+2l^2\frac{\exp(2\tau_{30})}{1-\tau_3}\Big).$$

Hence, if

$$U>\alpha+3+\frac{\exp(2\tau_{20})}{1-\tau_2}+2l^2\frac{\exp(2\tau_{30})}{1-\tau_3},$$

we deduce that the solution to (4.10) is quasi surely practically exponentially stable, i.e., with decay function $\lambda(t)=\exp(t)$, and order at least γ.

5. Conclusion

This article studies the practical convergence to a small ball centered at the origin with a general decay rate of G-SDDES. By using G-Lyapunov functionals, some sufficient conditions of practical stability with a general decay rate of G-SDDEs is stated. Meanwhile, we construct suitable Lyapunov functionals for G-SDDES with constant and time-varying delay to obtain sufficient conditions ensuring the practical exponential stability with a general decay rate. Finally, some examples to illustrate the effectiveness of the proposed techniques are presented.

Acknowledgments. Tomás Caraballo was partially supported by the Spanish Ministerio de Ciencia e Innovación (MCI), Agencia Estatal de Investigación (AEI) and Fondo Europeo de Desarrollo Regional (FEDER) under the project PID2021-122991NB-C21.

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