Electronic Journal of Differential Equations, Vol. 2024 (2024), No. 71, pp. 1–12.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2024.71

CONSTRUCTION OF SOLUTIONS TO PDES USING HOLOMORPHIC FUNCTIONS OF SEVERAL VARIABLES

VITALII SHPAKIVSKYI

Abstract. The work develops Humbert's method of solving PDEs. It applies method for constructing solutions of the three-dimensional Laplace, Helmholtz, and Poisson equations in the form of components of holomorphic functions of several variables (complex and hypercomplex).

1. Introduction

In 1929, Humbert [3] proposed an original idea for constructing solutions of the two-dimensional Laplace equation

$$\Delta_2 U(x,y) := \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0. \tag{1.1}$$

Namely, introducing the change of variables

$$\begin{aligned} u &= x+iy,\\ v &= x-iy, \end{aligned} \tag{1.2}$$

it is easy to verify that in the new variables u, v equation (1.1) takes the form

$$\frac{\partial^2 U}{\partial u \partial v} = 0. \tag{1.3}$$

The general solution of equation (1.3) is the function $U = f(u) + g(v)$, where f and g are "arbitrary" functions of their arguments from a certain class of functions.

The goal of Humbert was to generalize equation (1.3) to the spatial case. Thus, he considers the equation

$$\frac{\partial^3 U}{\partial u \partial v \partial w} = 0. \tag{1.4}$$

By introducing the change of variables

$$\begin{aligned} u &= x+y+z,\\ v &= x+jy+j^2z,\\ w &= x+j^2y+jz, \end{aligned} \tag{1.5}$$

2020 *Mathematics Subject Classification.* 32A10, 30G35, 35A25.
Key words and phrases. Holomorphic functions of several complex variables; holomorphic functions of several hypercomplex variables; harmonic algebra; elliptic PDE.

Submitted September 12, 2024. Published November 11, 2024.

where $\{1, j, j^2\}$ is a basis of a commutative associative algebra such that $j^3 = 1$, Humbert arrives to the equation

$$\frac{\partial^3 U}{\partial x^3} + \frac{\partial^3 U}{\partial y^3} + \frac{\partial^3 U}{\partial z^3} - 3\frac{\partial^3 U}{\partial x \partial y \partial z} = 0, \tag{1.6}$$

which he calls the spatial generalization of equation (1.1). Equation (1.6) is usually called Humbert's equation.

Note that (1.4) has the general solution

$$U = F(u, v) + G(v, w) + H(w, u),$$

where F, G, H are "arbitrary" functions of their arguments from a certain class of functions. In particular, a solution of (1.6) is the function

$$U_0 = \ln u + \ln v + \ln w = \ln uvw.$$

Since $uvw = x^3 + y^3 + z^3 - 3xyz$, it follows that $U_0 = \ln(x^3 + y^3 + z^3 - 3xyz)$.

The dissertation by Devisme [2] is devoted to the study of the properties of solutions (1.6). Also, (1.6) and the function theory of the hypercomplex variable $v = x+jy+j^2z$ were considered by Roşculeţ [12]. He constructed a theory of monogenic functions similar to the theory of analytic functions of a complex variable.

Our goal is to develop Humbert's idea and make it suitable for a predetermined PDEs. The main one the problem here is to choose a suitable algebra (in our case, commutative and associative) and the appropriate substitution of type (1.2), (1.5). The basic principles of algebra selection and suitable replacement of variables will be demonstrated on the example of the three-dimensional Laplace, Helmholtz, and Poisson equations.

2. Change of variables for a differential equation in the real case

For simplicity, we study the equation in the three variables x, y, z. In the domain $\Omega \subset \mathbb{R}^3$ we consider the differential equation

$$F\Big(x, y, z, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \dots, \frac{\partial^n f}{\partial x^\alpha \partial y^\beta \partial z^\gamma}\Big) = 0, \quad \alpha + \beta + \gamma = n. \tag{2.1}$$

In this article, we use the notation $\frac{\partial f}{\partial x} = f_x$, $\frac{\partial^2 f}{\partial x \partial y} = f_{xy}$ etc.

One of the methods for solving equation (2.1) is as follows:

(1) for the given equation, we find such a change of variables so that the new equation has a simpler form in the new coordinates;
(2) we solve the obtained equation in the new coordinates;
(3) we return to the solutions of given equation (2.1).

Humbert used this approach in his reasoning, but, unlike the classical method, he used the change of variables with values in a commutative associative algebra.

For equation (2.1) we define the change of variables

$$\begin{aligned} \varphi &= \varphi(x, y, z), \\ \psi &= \psi(x, y, z), \\ \eta &= \eta(x, y, z), \end{aligned} \tag{2.2}$$

where the real functions φ, ψ, η are defined in a domain Ω and are continuously differentiable in it with respect to x, y, z a sufficient number of times. Assume also that Q is a domain and a function $f : Q \to \mathbb{R}$, where $Q := \{(\varphi, \psi, \eta) \in \mathbb{R}^3 :$

$(x, y, z) \in \Omega\}$, is continuously differentiable in Q with respect to φ, ψ, η a sufficient number of times.

Now, with the assumptions on a real function f, let us write the first few formulas for the transition from the variables x, y, z to the variables φ, ψ, η.

$$f_x = f_\varphi \varphi_x + f_\psi \psi_x + f_\eta \eta_x \,, \tag{2.3}$$

$$\begin{aligned} f_{xx} &= f_{\varphi\varphi}\varphi_x^2 + f_{\psi\psi}\psi_x^2 + f_{\eta\eta}\eta_x^2 + 2f_{\varphi\psi}\varphi_x\psi_x + 2f_{\varphi\eta}\varphi_x\eta_x \\ &\quad + 2f_{\psi\eta}\psi_x\eta_x + f_\varphi\varphi_{xx} + f_\psi\psi_{xx} + f_\eta\eta_{xx} \,, \end{aligned} \tag{2.4}$$

$$\begin{aligned} f_{xy} &= f_{\varphi\varphi}\varphi_x\varphi_y + f_{\psi\psi}\psi_x\psi_y + f_{\eta\eta}\eta_x\eta_y + f_{\varphi\psi}(\varphi_y\psi_x + \varphi_x\psi_y) \\ &\quad + f_{\varphi\eta}(\varphi_y\eta_x + \varphi_x\eta_y) + f_{\psi\eta}(\psi_y\eta_x + \psi_x\eta_y) + f_\varphi\varphi_{xy} + f_\psi\psi_{xy} + f_\eta\eta_{xy} \,, \end{aligned}$$

$$\begin{aligned} f_{xxx} &= f_{\varphi\varphi\varphi}\varphi_x^3 + f_{\psi\psi\psi}\psi_x^3 + f_{\eta\eta\eta}\eta_x^3 + 3f_{\varphi\psi\psi}\varphi_x\psi_x^2 + 3f_{\varphi\eta\eta}\varphi_x\eta_x^2 + 3f_{\varphi\varphi\psi}\varphi_x^2\psi_x \\ &\quad + 6f_{\varphi\psi\eta}\varphi_x\psi_x\eta_x + 3f_{\varphi\varphi\eta}\varphi_x^2\eta_x + 3f_{\psi\eta\eta}\psi_x\eta_x^2 + 3f_{\psi\psi\eta}\psi_x^2\eta_x + f_{\varphi\varphi}\varphi_{xx}\varphi_x \\ &\quad + f_{\psi\psi}\psi_{xx}\psi_x + f_{\eta\eta}\eta_{xx}\eta_x + f_{\varphi\psi}(\varphi_x\psi_{xx} + \varphi_{xx}\psi_x) + f_{\varphi\eta}(\varphi_x\eta_{xx} + \varphi_{xx}\eta_x) \\ &\quad + f_{\psi\eta}(\psi_x\eta_{xx} + \psi_{xx}\eta_x) + f_\varphi\varphi_{xx} + f_\psi\psi_{xx} + f_\eta\eta_{xx} \,, \end{aligned}$$

$$\begin{aligned} f_{xxy} &= f_{\varphi\varphi\varphi}\varphi_x^2\varphi_y + f_{\psi\psi\psi}\psi_x^2\psi_y + f_{\eta\eta\eta}\eta_x^2\eta_y + f_{\varphi\psi\psi}(\psi_x^2\varphi_y + 2\varphi_x\psi_x\psi_y) \\ &\quad + f_{\varphi\eta\eta}(\eta_x^2\varphi_y + 2\varphi_x\eta_x\eta_y) + f_{\varphi\varphi\psi}(\varphi_x^2\psi_y + 2\varphi_x\psi_x\varphi_y) + f_{\varphi\varphi\eta}(\varphi_x^2\eta_y \\ &\quad + 2\varphi_x\varphi_y\eta_x) + f_{\varphi\psi\eta}(2\psi_x\eta_x\varphi_y + 2\varphi_x\eta_x\psi_y + 2\varphi_x\psi_x\eta_y) \\ &\quad + f_{\psi\eta\eta}(\eta_x^2\psi_y + 2\psi_x\eta_x\eta_y) + f_{\psi\psi\eta}(\psi_x^2\eta_y + 2\psi_x\eta_x\psi_y) \\ &\quad + f_{\varphi\varphi}\varphi_{xx}\varphi_y + f_{\psi\psi}\psi_{xx}\psi_y + f_{\eta\eta}\eta_{xx}\eta_y \\ &\quad + f_{\varphi\psi}(\psi_{xx}\varphi_y + \varphi_{xx}\psi_y) + f_{\varphi\eta}(\eta_{xx}\varphi_y + \varphi_{xx}\eta_y) \\ &\quad + f_{\psi\eta}(\eta_{xx}\psi_y + \psi_{xx}\eta_y) + f_\varphi\varphi_{yy} + f_\psi\psi_{yy} + f_\eta\eta_{yy} \,, \end{aligned}$$

$$\begin{aligned} f_{xyz} &= f_{\varphi\varphi\varphi}\varphi_x\varphi_y\varphi_z + f_{\psi\psi\psi}\psi_x\psi_y\psi_z + f_{\eta\eta\eta}\eta_x\eta_y\eta_z \\ &\quad + f_{\varphi\psi\psi}(\psi_x\psi_y\varphi_z + \varphi_y\psi_x\psi_z + \varphi_x\psi_y\psi_z) + f_{\varphi\eta\eta}(\eta_x\eta_y\varphi_z + \varphi_y\eta_x\eta_z + \varphi_x\eta_y\eta_z) \\ &\quad + f_{\varphi\varphi\psi}(\varphi_z\varphi_y\psi_x + \varphi_x\psi_y\varphi_z + \varphi_x\varphi_y\psi_z) + f_{\varphi\varphi\eta}(\varphi_y\varphi_z\eta_x \\ &\quad + \varphi_x\eta_y\varphi_z + \varphi_x\varphi_y\eta_z) + f_{\varphi\psi\eta}(\varphi_z\psi_y\eta_x + \psi_x\eta_y\varphi_z + \varphi_y\eta_x\psi_z \\ &\quad + \varphi_x\eta_y\psi_z + \varphi_y\psi_x\eta_z + \varphi_x\psi_y\eta_z) + f_{\psi\psi\eta}(\psi_y\psi_z\eta_x + \psi_x\eta_y\psi_z + \psi_x\psi_y\eta_z) \\ &\quad + f_{\psi\eta\eta}(\eta_x\eta_y\psi_z + \psi_y\eta_x\eta_z + \psi_x\eta_y\eta_z) + f_{\varphi\varphi}\varphi_{xy}\varphi_z + f_{\psi\psi}\psi_{xy}\psi_z \\ &\quad + f_{\eta\eta}\eta_{xy}\eta_z + f_{\varphi\psi}(\psi_{xy}\varphi_z + \varphi_{xy}\psi_z) + f_{\varphi\eta}(\eta_{xy}\varphi_z + \varphi_{xy}\eta_z) \\ &\quad + f_{\psi\eta}(\eta_{xy}\psi_z + \psi_{xy}\eta_z) + f_\varphi\varphi_{zz} + f_\psi\psi_{zz} + f_\eta\eta_{zz} \,. \end{aligned}$$

Part I

Solutions of Laplace, Helmholtz, and Poisson equations in the form of components of holomorphic functions of several complex variables.

3. Change of variables in a differential equation in the complex case

Now let functions (2.2) be complex-valued: $\varphi, \psi, \eta : \Omega \to \mathbb{C}$, $\Omega \subset \mathbb{R}^3$, and the functions φ, ψ, η are continuously differentiable with respect to x, y, z in Ω a sufficient number of times. Suppose also that the function $f : Q \to \mathbb{C}$, where $Q := \{(\varphi, \psi, \eta) \in \mathbb{C}^3 : (x, y, z) \in \Omega\}$. Now note that for the holomorphic function f of three complex variables φ, ψ, η the formulas for the transition from the derivatives by x, y, z to the derivatives by φ, ψ, η, which are given in section 2, remain valid.

By dividing the increment of the holomorphic function

$$\Delta f(\varphi, \psi, \eta) = f_\varphi \, \Delta\varphi + f_\psi \Delta\psi + f_\eta \Delta\eta + o\Big(\sqrt{|\Delta\varphi|^2 + |\Delta\psi|^2 + |\Delta\eta|^2}\Big)$$

on Δx and passing to the limit as $\Delta x \to 0$, we obtain equality (2.3). Taking into that the holomorphic function has derivatives of all orders by all variables, which are again holomorphic functions, we get the rest of the transition formulas. The class of functions $f : Q \to \mathbb{C}$, holomorphic in the domain $Q \subset \mathbb{C}^3$, is denoted by $\mathcal{H}_{\mathbb{C}}(Q)$.

4. Solutions of the Laplace equation in the form of components of holomorphic functions of three complex variables

We consider the Laplace equation

$$\Delta_3 f(x, y, z) := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{4.1}$$

in a domain $\Omega \subset \mathbb{R}^3$. For this, in equation (4.1) we will make twice continuously differentiable change of variables (2.2). In this case, equation (4.1) will turn into a certain equation that we will solve in the class $\mathcal{H}_{\mathbb{C}}(Q)$, where $Q := \{(\varphi, \psi, \eta) \in \mathbb{C}^3 : (x, y, z) \in \Omega\}$.

Let us transition to the new coordinates (2.2), using the transition formulas (2.4). We have

$$\begin{aligned}
&\Delta_3 f(x, y, z) \\
&= f_{\varphi\varphi}(\varphi_x^2 + \varphi_y^2 + \varphi_z^2) + f_{\psi\psi}(\psi_x^2 + \psi_y^2 + \psi_z^2) + f_{\eta\eta}(\eta_x^2 + \eta_y^2 + \eta_z^2) \\
&\quad + 2f_{\varphi\psi}(\varphi_x\psi_x + \varphi_y\psi_y + \varphi_z\psi_z) + 2f_{\varphi\eta}(\varphi_x\eta_x + \varphi_y\eta_y + \varphi_z\eta_z) \\
&\quad + 2f_{\psi\eta}(\psi_x\eta_x + \psi_y\eta_y + \psi_z\eta_z) + f_\varphi(\varphi_{xx} + \varphi_{yy} + \varphi_{zz}) \\
&\quad + f_\psi(\psi_{xx} + \psi_{yy} + \psi_{zz}) + f_\eta(\eta_{xx} + \eta_{yy} + \eta_{zz}) = 0.
\end{aligned} \tag{4.2}$$

Now we need to choose the functions φ, ψ, η so that equation (4.2) turns into an equation that can be solved explicitly. Or, much better, let the equation turn into an identity. For this purpose, let us consider the linear change of variables

$$\begin{aligned}
\varphi &= a_1 x + b_1 y + c_1 z, \\
\psi &= a_2 x + b_2 y + c_2 z, \\
\eta &= a_3 x + b_3 y + c_3 z.
\end{aligned} \tag{4.3}$$

In this case, equation (4.2) takes the form

$$\begin{aligned}
\Delta_3 f(x, y, z) = {} & f_{\varphi\varphi}(a_1^2 + b_1^2 + c_1^2) + f_{\psi\psi}(a_2^2 + b_2^2 + c_2^2) + f_{\eta\eta}(a_3^2 + b_3^2 + c_3^2) \\
& + 2f_{\varphi\psi}(a_1a_2 + b_1b_2 + c_1c_2) + 2f_{\varphi\eta}(a_1a_3 + b_1b_3 + c_1c_3) \\
& + 2f_{\psi\eta}(a_2a_3 + b_2b_3 + c_2c_3) = 0.
\end{aligned} \tag{4.4}$$

For equation (4.4) to turn into an identity, it is sufficient that the substitution coefficients (4.3) satisfy the system of equations

$$\begin{aligned} a_1^2+b_1^2+c_1^2&=0,\\ a_2^2+b_2^2+c_2^2&=0,\\ a_3^2+b_3^2+c_3^2&=0,\\ a_1a_2+b_1b_2+c_1c_2&=0,\\ a_1a_3+b_1b_3+c_1c_3&=0,\\ a_2a_3+b_2b_3+c_2c_3&=0. \end{aligned} \tag{4.5}$$

Note that the system of equations (4.5) does not have non-trivial solutions on the set of real numbers. But this system has solutions on the set of complex numbers. Thus, we have proved the following theorem.

Theorem 4.1. *In the class $\mathcal{H}_{\mathbb{C}}(Q)$ equation* (4.1) *is satisfied by the function*

$$f(x,y,z)=F(a_1x+b_1y+c_1z,a_2x+b_2y+c_2z,a_3x+b_3y+c_3z), \tag{4.6}$$

where F is an arbitrary holomorphic function of three complex variables in $Q=\{(\varphi,\psi,\eta)\in\mathbb{C}^3:(x,y,z)\in\Omega\}$, φ,ψ,η are defined by equalities (4.3)*, and the coefficients a_k,b_k,c_k at $k=1,2,3$, are arbitrary complex numbers that satisfy system* (4.5)*.*

Note that the real and imaginary part of function (4.6) satisfies equation (4.1) in the domain Ω.

5. Solutions of the Helmholtz equation in the form of components of holomorphic functions of three complex variables

Consider the three-dimensional Helmholtz equation

$$(\Delta_3+\lambda^2)f(x,y,z)=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}+\lambda^2 f=0,\quad \lambda\in\mathbb{C} \tag{5.1}$$

in a domain $\Omega\subset\mathbb{R}^3$. For this, in equation (5.1), we will replace variables (4.3). In this case, equation (5.1) will turn into a certain equation that we will solve in the class $\mathcal{H}_{\mathbb{C}}(Q)$, where $Q:=\{(\varphi,\psi,\eta)\in\mathbb{C}^3:(x,y,z)\in\Omega\}$.

Under such assumptions in equation (5.1), we will move to new coordinates (4.3). Using equality (4.4), we have

$$\begin{aligned} \Delta_3 f(x,y,z)&=f_{\varphi\varphi}(a_1^2+b_1^2+c_1^2)+f_{\psi\psi}(a_2^2+b_2^2+c_2^2)+f_{\eta\eta}(a_3^2+b_3^2+c_3^2)\\ &\quad+2f_{\varphi\psi}(a_1a_2+b_1b_2+c_1c_2)+2f_{\varphi\eta}(a_1a_3+b_1b_3+c_1c_3)\\ &\quad+2f_{\psi\eta}(a_2a_3+b_2b_3+c_2c_3)+\lambda^2 f=0. \end{aligned} \tag{5.2}$$

On the coefficients a_k,b_k,c_k, $k=1,2,3$, we impose the following conditions

$$\begin{aligned} a_1^2+b_1^2+c_1^2&=\lambda^2,\\ a_2^2+b_2^2+c_2^2&=0,\\ a_3^2+b_3^2+c_3^2&=0,\\ a_1a_2+b_1b_2+c_1c_2&=0,\\ a_1a_3+b_1b_3+c_1c_3&=0,\\ a_2a_3+b_2b_3+c_2c_3&=0. \end{aligned} \tag{5.3}$$

In this case, equation (5.2) will turn into the equation

$$f_{\varphi\varphi}(\varphi,\psi,\eta)+f(\varphi,\psi,\eta)=0. \tag{5.4}$$

The general solution of equation (5.4) is the function

$$f(\varphi,\psi,\eta)=(\alpha\cos\varphi+\beta\sin\varphi)G(\psi,\eta),$$

where α,β are arbitrary complex constants, and $G(\psi,\eta)$ is an arbitrary holomorphic function in $Q_2:=\{(\psi,\eta)\in\mathbb{C}^2:(x,y,z)\in\Omega\}$. Thus, we have proved the following theorem.

Theorem 5.1. *In the class $\mathcal{H}_{\mathbb{C}}(Q)$ equation* (5.1) *is satisfied by the function*

$$\begin{aligned} f(x,y,z)=&\Big(\alpha\cos(a_1x+b_1y+c_1z)+\beta\sin(a_1x+b_1y+c_1z)\Big)\\ &\times\times G(a_2x+b_2y+c_2z,a_3x+b_3y+c_3z), \end{aligned} \tag{5.5}$$

where G is an arbitrary holomorphic function in the domain $Q_2:=\{(\psi,\eta)\in\mathbb{C}^2:(x,y,z)\in\Omega\}$, ψ,η are defined by equalities (4.3)*, coefficients α,β are arbitrary complex numbers, and numbers a_k,b_k,c_k for $k=1,2,3$, are arbitrary complex numbers, which satisfy the system* (5.3)*.*

Note that the system of equations (5.3) does not have non-trivial solutions on the set of real numbers, but this system has solutions on the set of complex numbers.

Note that the real and imaginary parts of function (5.5) satisfy equation (5.1) in the domain Ω.

6. Solutions of Poisson equation in the form of components of holomorphic functions of three complex variables

Consider the three-dimensional Poisson equation

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=g(x,y,z) \tag{6.1}$$

in a domain $\Omega\subset\mathbb{R}^3$. For this, in equation (6.1), we will replace variables (4.3). In this case, equation (6.1) will turn into a certain equation that we will solve in the class $\mathcal{H}_{\mathbb{C}}(Q)$, where $Q:=\{(\varphi,\psi,\eta)\in\mathbb{C}^3:(x,y,z)\in\Omega\}$. The conditions for the function g will be given below.

Under such assumptions in equation (6.1), we will move to new coordinates (4.3). Using equality (4.4), equation (6.1) takes the form

$$\begin{aligned} &f_{\varphi\varphi}(a_1^2+b_1^2+c_1^2)+f_{\psi\psi}(a_2^2+b_2^2+c_2^2)+f_{\eta\eta}(a_3^2+b_3^2+c_3^2)\\ &+2f_{\varphi\psi}(a_1a_2+b_1b_2+c_1c_2)+2f_{\varphi\eta}(a_1a_3+b_1b_3+c_1c_3)\\ &+2f_{\psi\eta}(a_2a_3+b_2b_3+c_2c_3)=g(\varphi,\psi,\eta). \end{aligned} \tag{6.2}$$

On the coefficients a_k,b_k,c_k, $k=1,2,3$, we impose the following conditions

$$\begin{aligned} a_1^2+b_1^2+c_1^2&=1,\\ a_2^2+b_2^2+c_2^2&=0,\\ a_3^2+b_3^2+c_3^2&=0,\\ a_1a_2+b_1b_2+c_1c_2&=0,\\ a_1a_3+b_1b_3+c_1c_3&=0,\\ a_2a_3+b_2b_3+c_2c_3&=0. \end{aligned} \tag{6.3}$$

In this case, equation (6.2) turns into the equation

$$f_{\varphi\varphi}(\varphi,\psi,\eta)=g(\varphi,\psi,\eta). \tag{6.4}$$

The general solution of equation (6.4) is the function

$$f(\varphi,\psi,\eta)=\iint g(\varphi,\psi,\eta)d\varphi d\varphi+\varphi h_1(\psi,\eta)+h_2(\psi,\eta),$$

where h_1,h_2 are arbitrary holomorphic functions in the domain $Q_2:=\{(\psi,\eta)\in\mathbb{C}^2:(x,y,z)\in\Omega\}$. Thus, we have proved the following theorem.

Theorem 6.1. *Let in equation* (6.1) *the function* $g(\varphi,\psi,\eta)$ *for some set of parameters* (6.3) *be a holomorphic in the domain* $Q:=\{(\psi,\psi,\eta)\in\mathbb{C}^3:(x,y,z)\in\Omega\}$. *Then in the class* $\mathcal{H}_{\mathbb{C}}(Q)$ *equation* (6.1) *is satisfied by the function*

$$f(\varphi,\psi,\eta)=\iint g(\varphi,\psi,\eta)d\varphi d\varphi+\varphi h_1(\psi,\eta)+h_2(\psi,\eta), \tag{6.5}$$

where h_1,h_2 *are arbitrary holomorphic functions in the domain* $Q_2:=\{(\psi,\eta)\in\mathbb{C}^2:(x,y,z)\in\Omega\}$, φ,ψ,η *are defined by equalities* (4.3), *and coefficients* a_k,b_k,c_k, $k=1,2,3$, *are determined by equalities* (6.3).

Note that the system of equations (6.3) does not have non-trivial solutions on the set of real numbers, but this system has solutions on the set of complex numbers. Also note that the real and imaginary parts of function (6.5) satisfy equation (6.1) in the domain Ω.

Part 2

Solutions of the Laplace equation in the form of components of holomorphic functions of three hypercomplex variables.

7. Holomorphic functions of three hypercomplex variables

Now let functions (2.2) take values in some commutative associative algebra $\mathbb{A}$ over the field $\mathbb{R}$ or $\mathbb{C}$, i.e. $\varphi,\psi,\eta:\Omega\to\mathbb{A}$, $\Omega\subset\mathbb{R}^3$. Let the functions φ,ψ,η be continuously differentiable in x,y,z in Ω a sufficient number of times. Suppose that the function f takes values in the algebra $\mathbb{A}$: $f:D\to\mathbb{A}$, where $D:=\{(\varphi,\psi,\eta)\in\mathbb{A}^3:(x,y,z)\in\Omega\}$. Let us ask the question: under what conditions for the function f do the formulas for the transition from the derivatives by x,y,z to the derivatives by φ,ψ,η, which are given in section 2? For this, first of all, it is necessary to explain how we will understand the derivative.

Definition 7.1. We say that the function $f:D\to\mathbb{A}$, $D\subset\mathbb{A}^3$, of three hypercomplex variables $\varphi,\psi,\eta\in\mathbb{A}$ has a partial derivative with respect to the variable φ in the domain D, if f is differentiable with respect to this variable in the sense of Lorch [6] at each point of the domain D, i.e. if there exists an element of algebra f_φ such that for an arbitrary $\varepsilon>0$ there exists $\delta>0$ such that for all $h_1\in\mathbb{A}$ with $\|h_1\|<\delta$ the inequality

$$\|f(\varphi+h_1,\psi,\eta)-f(\varphi,\psi,\eta)-h_1f_\varphi(\varphi,\psi,\eta)\|\le\|h_1\|\varepsilon$$

holds. The element f_φ is called the Lorch partial derivative of the function f with respect to the variable φ.

Similar to the complex case [13, p. 270], we introduce the following definition.

Definition 7.2. The function $f : D \to \mathbb{A}$, $D \subset \mathbb{A}^3$, of three hypercomplex variables $\varphi, \psi, \eta \in \mathbb{A}$ is called holomorphic in the domain D, if in this domain there exist continuous Lorch partial derivatives $f_\varphi, f_\psi, f_\eta$ and in D the increment of the function $\Delta f := f(\varphi + \Delta\varphi, \psi + \Delta\psi, \eta + \Delta\eta) - f(\varphi, \psi, \eta)$ is given in the form

$$\Delta f = f_\varphi \, \Delta\varphi + f_\psi \Delta\psi + f_\eta \Delta\eta + o\left(\sqrt{\|\Delta\varphi\|^2 + \|\Delta\psi\|^2 + \|\Delta\eta\|^2}\right), \tag{7.1}$$

where $o(\cdot) \to 0$ as $\Delta\varphi \to 0, \Delta\psi \to 0, \Delta\eta \to 0$.

Remark 7.3. In equality (7.1) we pass to the limit as $\Delta\varphi \to 0, \Delta\psi \to 0, \Delta\eta \to 0$. It is obvious that in this case $\Delta f \to 0$. And this means that every holomorphic function in the domain $D \subset \mathbb{A}^3$ is continuous in this domain.

It is obvious that every function holomorphic of three hypercomplex variables is differentiable in the sense of Lorch with respect to each variable separately.

Note that all elementary functions of the three variables $\varphi, \psi, \eta \in \mathbb{A}$ are holomorphic.

Now we divide equality (7.1) by Δx and pass to the limit as $\Delta x \to 0$, we obtain equality (2.3). To obtain PDEs faster, without resorting to the study of the holomorphic functions theory, let us additionally assume that the functions f_φ, f_ψ, f_η are also holomorphic in D. Under this assumption, from equality (2.3) we immediately obtain equality (2.4).

The class of functions $f : D \to \mathbb{A}$, holomorphic in the domain $D \subset \mathbb{A}^3$ together with Lorch partial derivatives f_φ, f_ψ, f_η, denote by $\mathcal{H}^2_{\mathbb{A}}(D)$.

8. Solutions of the Laplace equation in the form of components of holomorphic functions of three hypercomplex variables

In this section, we look for solutions of equation (4.1) in a domain $\Omega \subset \mathbb{R}^3$ using the change of variables (4.3) under the assumption that the coefficients a_k, b_k, c_k at $k = 1, 2, 3$, are elements of some three-dimensional commutative associative algebra over the field $\mathbb{C}$. In this case, equation (4.1) will turn into a certain equation that we will solve in class $\mathcal{H}^2_{\mathbb{A}}(D)$, where $D := \{(\varphi, \psi, \eta) \in \mathbb{A}^3 : (x, y, z) \in \Omega\}$.

So, in system (4.5) we put $a_1 = c_2 = b_3 = e_1$, $b_1 = a_2 = c_3 = e_2$, $c_1 = b_2 = a_3 = e_3$, where e_1, e_2, e_3 are basis elements of some algebra. That is,

$$\begin{aligned} \varphi &= xe_1 + ye_2 + ze_3\,, \\ \psi &= xe_2 + ye_3 + ze_1\,, \\ \eta &= xe_3 + ye_1 + ze_2\,. \end{aligned} \tag{8.1}$$

Then the first three equations of system (4.5) are equivalent to one equation – this is the condition of harmonicity of vectors (see [4, 10]):

$$e_1^2 + e_2^2 + e_3^2 = 0.$$

If we consider $e_1 = 1$ as a unit of algebra, then the fourth, fifth and sixth equations of system (4.5) are equivalent to one condition:

$$e_2 e_3 = -e_2 - e_3. \tag{8.2}$$

Our problem was reduced to finding a three-dimensional (though not necessarily three-dimensional) harmonic algebra with multiplication condition (8.2).

In [1], a multiplication table of all harmonic bases for $e_1 = 1$ in three-dimensional algebras was obtained:

$$\begin{gathered}
e_2^2 = \Big(p(p+t) \mp m\sqrt{-1-(p+t)^2}\Big)e_1 + \Big(m \pm \sqrt{-1-(p+t)^2}\Big)e_2 + pe_3,\\
e_2e_3 = \Big(-m(p+t) \mp p\sqrt{-1-(p+t)^2}\Big)e_1 + te_2 + me_3,\\
e_3^2 = \Big(-1-p(p+t) \pm m\sqrt{-1-(p+t)^2}\Big)e_1\\
+\Big(-m \mp \sqrt{-1-(p+t)^2}\Big)e_2 - pe_3\,,
\end{gathered} \tag{8.3}$$

where m, t, p are arbitrary complex numbers, and in double signs $\mp$, $\pm$, either upper or lower signs are taken simultaneously. Thus, condition (8.2) leads us to the equalities $t = m = -1$ and to the equation

$$(p-1) \mp p\sqrt{-1-(p-1)^2} = 0.$$

The solutions of the above equation are numbers $p = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. By substituting the obtained values of p and $t = m = -1$, we obtain two families of harmonic bases. The first family of harmonic bases is multiplied by the following multiplication rules:

$$\begin{gathered}
e_2^2 = (-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i)e_1 + (-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i)e_2 + (\frac{1}{2} \pm \frac{\sqrt{3}}{2}i)e_3,\\
e_2e_3 = -e_2 - e_3,\\
e_3^2 = (-\frac{1}{2} \mp \frac{\sqrt{3}}{2}i)e_1 + (\frac{1}{2} \mp \frac{\sqrt{3}}{2}i)e_2 + (-\frac{1}{2} \mp \frac{\sqrt{3}}{2}i)e_3\,,
\end{gathered} \tag{8.4}$$

and the second family multiplies as follows:

$$\begin{gathered}
e_2^2 = (-\frac{3}{2} \mp \frac{\sqrt{3}}{2}i)e_1 + (-\frac{3}{2} \mp \frac{\sqrt{3}}{2}i)e_2 + (\frac{1}{2} \pm \frac{\sqrt{3}}{2}i)e_3,\\
e_2e_3 = -e_2 - e_3,\\
e_3^2 = (\frac{1}{2} \pm \frac{\sqrt{3}}{2}i)e_1 + (\frac{3}{2} \pm \frac{\sqrt{3}}{2}i)e_2 + (-\frac{1}{2} \mp \frac{\sqrt{3}}{2}i)e_3\,,
\end{gathered} \tag{8.5}$$

where in double signs $\mp$, $\pm$, either upper or lower signs are taken simultaneously.

Thus, we have proved the following theorem.

Theorem 8.1. *In the class $\mathcal{H}^2_{\mathbb{A}}(D)$, equation (4.1) is satisfied by the function*

$$f(x,y,z) = F(xe_1 + ye_2 + ze_3, xe_2 + ye_3 + ze_1, xe_3 + ye_1 + ze_2), \tag{8.6}$$

where F is an arbitrary holomorphic function in $D = \{(\varphi,\psi,\eta) \in \mathbb{A}^3 : (x,y,z) \in \Omega\}$, φ,ψ,η are defined by equalities (8.1), and the vectors e_1, e_2, e_3 are defined by equalities (8.4) and (8.5).

If we consider that function (8.6) depends only of one variable $xe_1+ye_2+ze_3$, then the solution $f = F(xe_1+ye_2+ze_3)$ will be satisfy equation (4.1) due to the condition of harmonicity of vectors e_1, e_2, e_3. Solutions of the form $f = F(xe_1 + ye_2 + ze_3)$ are studied in works [7, 11, 9, 8].

Since the Laplace equation (4.1) is linear, all real-valued components of function (8.6) are solutions of equation (4.1).

9. Example

Let us show how some harmonic functions are presented in form (8.6). For this, consider, for example, homogeneous linearly independent harmonic polynomials of the third degree. It is known that there are 7 of them. Let us denote

$$U_1 := y^3 - 3z^2y, \quad U_2 := z^3 - 3y^2z, \quad U_3 := x^3 - 3y^2x, \quad U_4 := x^3 - 3z^2x,$$
$$U_5 := y^3 - 3x^2y, \quad U_6 := z^3 - 3x^2z, \quad U_7 := xyz.$$

In the notation (8.1) we have:

$$\begin{aligned}
U_1 &= \operatorname{Re}\Big\{\frac{1}{\sqrt{3}}\Big(-\frac{2ie_3}{1+i\sqrt{3}}+1\Big)\eta^3\Big\},\\
U_2 &= \operatorname{Re}\Big\{\frac{1}{\sqrt{3}}\Big(-\frac{2ie_3}{1+i\sqrt{3}}-i+1-\sqrt{3}\Big)\varphi^3\Big\},\\
U_3 &= \operatorname{Re}\Big\{\frac{1}{\sqrt{3}}\Big(-\frac{2ie_3}{1+i\sqrt{3}}+1\Big)\varphi^3\Big\},\\
U_4 &= \operatorname{Re}\Big\{\frac{1}{\sqrt{3}}\Big(-\frac{2ie_3}{1+i\sqrt{3}}-i+1-\sqrt{3}\Big)\eta^3\Big\},\\
U_5 &= \operatorname{Re}\Big\{\frac{1}{\sqrt{3}}\Big(-\frac{2ie_3}{1+i\sqrt{3}}-i+1-\sqrt{3}\Big)\psi^3\Big\},\\
U_6 &= \operatorname{Re}\Big\{\frac{1}{\sqrt{3}}\Big(-\frac{2ie_3}{1+i\sqrt{3}}+1\Big)\psi^3\Big\}.
\end{aligned}$$

10. Necessary and sufficient conditions for the holomorphicity of a function in an algebra

Theorem 10.1. *For the function $f : D \to \mathbb{A}$ to be holomorphic in the domain $D \subset \mathbb{A}^3$ it is necessary and sufficient that the function f be a differentiable function of three real variables in the domain $\Omega := \{(x, y, z) \in \mathbb{R}^3 : (\varphi, \psi, \eta) \in D\}$, where x, y, z are related to φ, ψ, η by equalities* (8.1)*, and that the following conditions be fulfilled:*

$$\begin{aligned}
f_x &= f_\varphi e_1 + f_\psi e_2 + f_\eta e_3\,,\\
f_y &= f_\varphi e_2 + f_\psi e_3 + f_\eta e_1\,,\\
f_z &= f_\varphi e_3 + f_\psi e_1 + f_\eta e_2\,.
\end{aligned} \tag{10.1}$$

Proof. Necessity. Let the function f be holomorphic. Then the equality (7.1) will be rewritten in the form

$$\begin{aligned}
\Delta f &= f_\varphi(\Delta x e_1 + \Delta y e_2 + \Delta z e_3) + f_\psi(\Delta x e_2 + \Delta y e_3 + \Delta z e_1)\\
&\quad + f_\eta(\Delta x e_3 + \Delta y e_1 + \Delta z e_2) + o\Big(\sqrt{\|\Delta\varphi\|^2 + \|\Delta\psi\|^2 + \|\Delta\eta\|^2}\Big)\\
&= (f_\varphi e_1 + f_\psi e_2 + f_\eta e_3)\Delta x + (f_\varphi e_2 + f_\psi e_3 + f_\eta e_1)\Delta y\\
&\quad + (f_\varphi e_3 + f_\psi e_1 + f_\eta e_2)\Delta z + o\Big(\sqrt{3}\cdot\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}\Big).
\end{aligned} \tag{10.2}$$

Equality (10.2) means that the function f is differentiable as a function of three real variables x, y, z, and the conditions (10.1) are fulfilled.

Sufficiency. Since the function f is differentiable then the equality

$$\Delta f = f_x\Delta x + f_y\Delta y + f_z\Delta z + o\Big(\sqrt{3}\cdot\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}\Big) \tag{10.3}$$

holds. We will substitute relation (10.1) into equality (10.3), and we have

$$\begin{aligned}\Delta f &= (f_\varphi e_1 + f_\psi e_2 + f_\eta e_3)\Delta x + (f_\varphi e_2 + f_\psi e_3 + f_\eta e_1)\Delta y \\ &\quad + (f_\varphi e_3 + f_\psi e_1 + f_\eta e_2)\Delta z + o\Big(\sqrt{3}\cdot\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}\Big) \\ &= f_\varphi \Delta\varphi + f_\psi \Delta\psi + f_\eta \Delta\eta + o\Big(\sqrt{\|\Delta\varphi\|^2 + \|\Delta\psi\|^2 + \|\Delta\eta\|^2}\Big).\end{aligned}$$

□

Equalities (10.1) are analogs of the Cauchy-Riemann conditions. Let us transform these equalities. We multiply the first equality with (10.1) by e_1 and add to the second equality multiplied by e_2 and add to the third equality multiplied by e_3. At the same time we get equality

$$f_x e_1 + f_y e_2 + f_z e_3 = 0.$$

Similarly, multiplying equalities (10.1) respectively by e_2, e_3, e_1 and adding the resulting expressions, we will have

$$f_x e_2 + f_y e_3 + f_z e_1 = 0.$$

Finally, multiplying equalities (10.1) respectively by e_3, e_1, e_2 and adding the resulting expressions, we obtain

$$f_x e_3 + f_y e_1 + f_z e_2 = 0.$$

So the consequence of conditions (10.1) is the conditions

$$\begin{aligned} f_x e_1 + f_y e_2 + f_z e_3 &= 0, \\ f_x e_2 + f_y e_3 + f_z e_1 &= 0, \\ f_x e_3 + f_y e_1 + f_z e_2 &= 0. \end{aligned} \tag{10.4}$$

That is, the holomorphic function of three hypercomplex variables φ, ψ, η *necessarily* is hyperholomorphic (see, for example, [5]) for each variable separately.

Acknowledgments. This work was supported by a grant from the Simons Foundation (1030291,V.S.Sh.).

References

[1] Beckh-Widmanstetter, H. A. v.; *Laßt sich die Eigenschaft der analytischen Funktionen einer gemeinen komplexen veranderlichen, Potentiale als Bestandteile zu liefern, auf Zahlsysteme mit drei einheiten verallgemeinern?* Monatshefte für Mathematik und Physik, **23** (1912), 257–260.

[2] Devisme, J.; *Sur l'équation de M. Pierre Humbert*, Theses de l'entre-deux-guerres, **148** (1933), 108 p.

[3] Humbert, P.; *Sur une généralisation de l'équation de Laplace*, Journal de Mathématiques Pures et Appliquées, Serie 9, **8** (1929), 145–159.

[4] Ketchum, P. W.; *Analytic Functions of Hypercomplex Variables*, Trans. Amer. Math. Soc., **30** (1928), 641–667.

[5] Kravchenko, V. V.; Shapiro, M. V.; *Integral representations for spetial models of mathematical physics.* Pitman Research Notes in Mathematics Series 351, Addison Wesley Longman, 1996, 247 p.

[6] Lorch, E. R.; *The theory of analytic function in normed abelin vector rings*, Trans. Amer. Math. Soc., **54** (1943), 414–425.

[7] Mel'nichenko, I. P.; *Algebras of functionally invariant solutions of the three-dimensional Laplace equation*, Ukr. Math. J., **55** (9) (2003), 1551–1557.

[8] Plaksa, S. A.; Pukhtaevich, R. P.; *Constructive description of monogenic functions in a three-dimensional harmonic algebra with one-dimensional radical*, Ukr. Math. J., **65** (5) (2013), 740–751.

[9] Plaksa, S. A.; Pukhtaievych, R. P.; *Monogenic functions in a finite-dimensional semi-simple commutative algebra*, An. St. Univ. Ovidius Constanta, **22** (1) (2014), 221–235.

[10] Plaksa, S. A.; Shpakivskyi, V. S.; *Monogenic Functions in Spaces with Commutative Multiplication and Applications.* Frontiers in Mathematics, Birkhäuser Cham, 2023, 550 p.

[11] Plaksa, S. A.; Shpakovskii, V. S.; *Constructive description of monogenic functions in a harmonic algebra of the third rank*, Ukr. Math. J., **62** (8) (2011), 1251–1266.

[12] Roşculeţ, M. N.; *O theorie a funcţiilor de o variabilă hipercomplexă în spaţiul cu trei dimensiuni*, Studii şi Cercetări Matematice, **5**, 3–4 (1954), 361–401.

[13] Shabat, B. V.; *Introduction to complex analysis.* In 2 volumes. Moscow, Nauka, 1976, 720 p. (in Russian).

Vitalii Shpakivskyi
Department of Complex Analysis and Potential Theory, Institute of Mathematics of the National Academy of Sciences of Ukraine, 3, Tereschenkivska st., Kyiv-4, 01024, Ukraine
Email address: shpakivskyi86@gmail.com