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MULTIPLICITY RESULTS FOR SCHRÖDINGER TYPE FRACTIONAL p-LAPLACIAN BOUNDARY VALUE PROBLEMS

EMER LOPERA, LEANDRO RECÔVA, ADOLFO RUMBOS

Dedicated to Djairo G. de Figueiredo on his 90th birthday

Abstract. In this work, we study the existence and multiplicity of solutions to the problem

$$\begin{gathered} -(\Delta)^s_p u + V(x)|u|^{p-2}u = \lambda f(u), \quad x \in \Omega; \\ u = 0, \quad x \in \mathbb{R}^N \backslash \Omega, \end{gathered}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded set with Lipschitz boundary $\partial\Omega$, $N \geqslant 2$, $V \in L^\infty(\mathbb{R}^N)$, and $(-\Delta)^s_p$ denotes the fractional p-Laplacian with $s \in (0,1)$, $1 < p$, $sp < N$, $\lambda > 0$, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. We extend the results of Lopera et al. [22] by proving the existence of a second weak solution to this problem. We apply a variant of the mountain-pass theorem due to Hofer [15] and infinite-dimensional Morse theory to obtain the existence of at least two solutions.

1. Introduction

Let Ω be an open bounded set in $\mathbb{R}^N$, $N \geqslant 2$, with Lipschitz boundary $\partial\Omega$. In this work, we study the existence and multiplicity of solutions for the problem

$$\begin{gathered} -(\Delta)^s_p u(x) + V(x)|u(x)|^{p-2}u(x) = \lambda f(u(x)), \quad x \in \Omega; \\ u(x) = 0, \quad x \in \mathbb{R}^N \backslash \Omega, \end{gathered} \tag{1.1}$$

where $V \in L^\infty(\mathbb{R}^N)$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $(-\Delta)^s_p$ denotes the fractional p-Laplacian defined by

$$(-\Delta)^s_p u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\, dy, \tag{1.2}$$

for $x \in \Omega$, with $s \in (0,1)$, $1 < p$, $sp < N$, and $\lambda > 0$.

As pointed out by Lindgren and Lidqvist [21, page 801], it is not sufficient to prescribe the boundary values only on $\partial\Omega$, but instead, we have to assume that $u = 0$ in the whole complement $\mathbb{R}^N \backslash \Omega$ because a change in u done outside Ω can impact the fractional p-Laplacian operator $(-\Delta)^s_p$. For more details, see Nezza et al. [12], Lindgren et al. [21], and references therein.

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In this work, the functions f and V satisfy the following hypotheses:

(H1) Assume that $p-1 < q < p_s^* - 1$, where $p_s^* := \frac{Np}{N-sp}$ is the fractional critical Sobolev exponent, and there exist $A, B > 0$ such that

$$A(s^q - 1) \leqslant f(s) \leqslant B(s^q + 1), \quad \text{for } s > 0, \tag{1.3}$$

$$f(s) = 0, \quad \text{for } s \leqslant -1. \tag{1.4}$$

(H2) There exist $\theta > p$ and $K \in \mathbb{R}$ such that f satisfies the Ambrosetti-Rabinowitz type condition

$$sf(s) \geqslant \theta F(s) + K, \quad \text{for all } s \in \mathbb{R}, \tag{1.5}$$

where $F(s) = \int_0^s f(\xi)\, d\xi$, for $s \in \mathbb{R}$, is the primitive of f.

(H3) $V \in L^\infty(\mathbb{R}^N)$ and $V(x) \geqslant -c_V$, for a.e. $x \in \mathbb{R}^N$, where $0 < c_V < \lambda_1$ and λ_1 is the first eigenvalue of $((-\Delta)_p^s, W_0^{s,p}(\Omega))$.

The first two results establish the existence and multiplicity of solutions for problem (1.1) when $f(0) \neq 0$.

Theorem 1.1. *Assume that Ω is a bounded domain with a Lipschitz boundary $\partial\Omega$ and* (H1)–(H3) *are satisfied with $f(0) \neq 0$. Then there exists $\lambda_0 > 0$ such tha, for all $\lambda \in (0, \lambda_0)$, problem* (1.1) *has at least two solutions.*

To obtain a positive solution, we need to assume that $p \geqslant 2$ to get enough regularity of solutions up to the boundary of Ω and $V(x) \geqslant 0$, for a.e $x \in \Omega$. In this case, we obtain the following multiplicity result.

Theorem 1.2. *In addition to the hypotheses of Theorem 1.1, assume that $V(x) \geqslant 0$ for a.e. $x \in \Omega$, $p \geqslant 2$, Ω is bounded and satisfies the interior ball condition at any $x \in \partial\Omega$, and*

$$p - 1 < q < \min\big\{\frac{sp}{N} p_s^*, p_s^* - 1\big\}.$$

Then, there exists $\lambda^* > 0$ such that, for all $0 < \lambda < \lambda^*$, problem (1.1) *has at least two solutions. Moreover:*

(a) *If $f(0) > 0$, then both solutions are positive.*
(b) *If $f(0) < 0$, then at least one of the solutions is positive.*

Remark 1.3. Observe that statement (b) encompasses the semipositone case. See, for example, Castro et al. [8] and references therein.

When $u \equiv 0$ is a solution of problem (1.1), called the trivial solution, to obtain a multiplicity result, we need an additional condition on the primitive of f.

Theorem 1.4. *Assume that Ω is a bounded domain with a Lipschitz boundary $\partial\Omega$ and* (H1)–(H3) *are satisfied. Moreover, assume that $f(0) = 0$ and*

$$\limsup_{s \to 0} \frac{F(s)}{|s|^p} = 0.$$

Then there exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ problem (1.1) *has at least two nontrivial solutions.*

Problems involving the fractional p-Laplacian have been an object of intensive research in the last years in many branches of science such as in phase transition phenomena, population dynamics, and game theory (see [2, 7, 11, 12, 16, 17, 18, 19, 21, 22, 25, 29]). Valdinoci [30] presents a self-contained exposition on how a simple random walk with possibly long jumps is related to the fractional p-Laplacian operator. For more insights on the applications, we refer to Iannizzotto et al. [16] and Caffarelli [7] where the authors provide a detailed review of current applications and challenges faced when dealing with these nonlocal operators.

This article was motivated by the results obtained by Castro et al. [8] for the case of the p-Laplacian operator and by Lopera et al. [22] for the fractional p-Laplacian. In those articles, the authors proved the existence of a positive solution for problem (1.1) when the potential $V \equiv 0$. The existence result was obtained by showing that the associated energy functional for problem (1.1) had the geometry of the mountain-pass theorem of Ambrosetti-Rabinowitz [1]. They also proved that the solution was positive by using some new regularity results and Hopf's Lemma.

The main goal of this work is to extend the results of Lopera et al. [22] by proving the existence of at least two solutions for problem (1.1). We will use a variant of the mountain-pass theorem due to Hofer [15] and infinite-dimensional Morse theory to obtain the existence of a second solution for both cases where $f(0) \neq 0$ and $f(0) = 0$, respectively.

This article is organized as follows: In Section 2 we present some preliminary results that will be used throughout this work. In Section 3, we prove that the associated energy functional to problem (1.1) has a critical point u_λ of mountain-pass type. In Section 4, we apply infinite-dimensional Morse theory to compute the critical groups of the associated energy functional at infinity. In Section 5, we compute the critical groups of the associated energy functional for problem (1.1) at the origin. Finally, we prove the existence and multiplicity results in Section 6.

2. Preliminaries

In this work, we will use a variational approach to study the existence and multiplicity of solutions for problem (1.1). We start with some notation and preliminary results that will be used throughout this article.

Let Ω be an open bounded subset of in $\mathbb{R}^N$, $N \geqslant 2$, with boundary $\partial\Omega$. Denote by $C(\overline{\Omega})$ the set of continuous functions on $\overline{\Omega}$. The space of γ-Hölder continuous functions is defined by

$$C^\gamma(\overline{\Omega}) = \{u \in C(\overline{\Omega}) : [u]_{C^\gamma(\overline{\Omega})} < \infty\},$$

where $0 < \gamma \leqslant 1$ and

$$[u]_{C^\gamma(\overline{\Omega})} = \sup_{x,y\in\overline{\Omega},x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\gamma}.$$

The space $C^\gamma(\overline{\Omega})$ is a Banach space endowed with the norm

$$\|u\|_{C^\gamma(\overline{\Omega})} = \|u\|_{L^\infty(\Omega)} + [u]_{C^\gamma(\overline{\Omega})}.$$

In some of the regularity results that will be used in this article, it will be required that the domain $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, be a Lipschitz domain. This is the content of the next definition.

Definition 2.1. We will say that $\Omega \subset \mathbb{R}^N$ has a Lipschitz boundary, and call it a Lipschitz domain, if, for every $x_0 \in \partial\Omega$, there exists $r > 0$ and a map $h : B_r(x_0) \to B_1(0)$ such that

(i) h is a bijection,
(ii) h and h^{-1} are both Lipschitz continuous functions,

(iii) $h(\partial\Omega \cap B_r(x_0)) = Q_0$,
(iv) $h(\Omega \cap B_r(x_0)) = Q_+$,

where $B_r(x_0)$ denotes the $n-$dimensional open ball of radius r and center at $x_0 \in \partial\Omega$, and

$$Q_0 := \{(x_1, \dots, x_n) \in B_1(0) \; | x_n = 0\}, \quad Q_+ := \{(x_1, \dots, x_n) \in B_1(0) \; | x_n > 0\}.$$

Next, we introduce the space of functions where the energy functional associated with problem (1.1) will be defined. Let $s \in (0,1)$ and $1 \leqslant p < \infty$, and denote by

$$W_0^{s,p}(\Omega) = \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e in } \mathbb{R}^N \backslash \Omega\} \tag{2.1}$$

the subset of the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$,

$$W^{s,p}(\mathbb{R}^N) = \big\{u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy < \infty \big\},$$

endowed with the norm

$$\|u\|_{s,p} := \left(\|u\|_p^p + [u]_{s,p}^p\right)^{1/p}, \tag{2.2}$$

where $\|\cdot\|_p$ denotes the norm in $L^p(\Omega)$ for $1 \leqslant p < \infty$ and

$$[u]_{s,p} := \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy, \tag{2.3}$$

is the Gagliardo seminorm. It can be shown that $W^{s,p}(\mathbb{R}^N)$, endowed with the norm $\|\cdot\|_{s,p}$ defined in (2.2) and (2.3), is a Banach space, and $W_0^{s,p}(\Omega) \subset W^{s,p}(\mathbb{R}^N)$ is a closed subspace. In the case $1 < p < \infty$, $W^{s,p}(\Omega)$ is a reflexive Banach space (see Asso et al. [2, Section 2.1]).

By a Sobolev-type inequality (see [12, Theorem 6.7]), it can be shown that the space $W_0^{s,p}(\Omega)$ can also be endowed with the norm

$$\|u\| := [u]_{s,p}, \tag{2.4}$$

for $s \in (0,1)$ and $1 \leqslant p < \infty$.

We will denote by $\widetilde{W}^{s,p}(\Omega)$ the Sobolev space

$$\big\{u \in L^p_{loc}(\mathbb{R}^N) : \exists U \supset\supset \Omega \text{ s.t } \|u\|_{W^{s,p}(U)} + \int_{\mathbb{R}^N} \frac{|u(x)|^{p-1}}{(1+|x|)^{N+ps}} \, dx < \infty \big\},$$

where $\Omega \subset \mathbb{R}^N$ is a bounded set (see [19, Definition 2.1] for more details). Since Ω is a bounded set, it follows from [11, Remark 1.1] that $W_0^{s,p}(\Omega) \subset \widetilde{W}^{s,p}(\Omega)$. We will refer to the space $\widetilde{W}^{s,p}(\Omega)$ during the proof of a comparison principle for problem (1.1). For more details on fractional Sobolev spaces, see [12, Section 2], [6], and references therein.

In this article, we shall denote by X the fractional Sobolev space $W_0^{s,p}(\Omega)$. We define $J_\lambda : X \to \mathbb{R}$, the energy functional associated with problem (1.1), by

$$J_\lambda(u) = \frac{1}{p}\|u\|^p + \frac{1}{p}\int_\Omega V(x)|u|^p \, dx - \lambda \int_\Omega F(u) \, dx, \quad \text{for } u \in X, \tag{2.5}$$

and $\lambda > 0$ with $\|\cdot\|$ defined in (2.4).

The functional J_λ is well-defined and $J_\lambda \in C^1(X,\mathbb{R})$. It can be shown that the Fréchet derivative of J_λ is given by

$$\begin{aligned}\langle J_\lambda'(u),\varphi\rangle = \int_{\mathbb{R}^{2N}} \frac{\Phi_p(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy \\ + \int_\Omega V(x)|u|^{p-2}u\varphi\,dx - \lambda\int_\Omega f(u)\varphi\,dx,\end{aligned} \tag{2.6}$$

for all $\varphi \in X$, where $\Phi_p : \mathbb{R}\to\mathbb{R}$ is given by $\Phi_p(s) = |s|^{p-2}s$, for $s \in \mathbb{R}$.

We will say that u is a weak solution of problem (1.1) if u is a critical point of J_λ; namely,

$$\langle J_\lambda'(u),\varphi\rangle = 0, \quad \text{for all } \varphi \in X. \tag{2.7}$$

For every $1 < p_1 < p_s^*$, we shall denote by C_{p_1} the optimal constant in the Sobolev embedding theorem; namely,

$$\|u\|_{p_1} \leqslant C_{p_1}\|u\|, \quad \text{for all } u \in X, \tag{2.8}$$

see [12, Theorem 6.7].

In the proof of the existence of a solution of mountain-pass type, we will need the following result due to Lindgren and Lindqvist [21].

Theorem 2.2 ([21, Thm. 5])**.** *There exists a non-negative minimizer u in$W_0^{s,p}(\Omega)$, $u \not\equiv 0$, and $u = 0$ in $\mathbb{R}^N\backslash\Omega$ of the fractional Rayleigh quotient:*

$$\lambda_1 = \inf_{u\in W_0^{s,p}(\Omega)\backslash\{0\}} \frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(u)-u(x)|^p}{|y-x|^{\alpha p}}\,dx\,dy}{\int_{\mathbb{R}^N}|u(x)|^p\,dx}. \tag{2.9}$$

It satisfies the Euler-Lagrange equation

$$\begin{aligned}&\int_{\mathbb{R}^{2N}} \frac{|u(y)-u(x)|^{p-2}(u(y)-u(x))(\varphi(y)-\varphi(x))}{|y-x|^{\alpha p}}\,dx\,dy \\ &= \lambda\int_{\mathbb{R}^N}|u|^{p-2}u\varphi\,dx,\end{aligned} \tag{2.10}$$

with $\lambda = \lambda_1$ whenever $\varphi \in C_c^\infty(\Omega)$. If $\alpha p > 2N$, the minimizer is in $C^{0,\beta}(\mathbb{R}^N)$ with $\beta = \alpha - 2N/p$.

Theorem 2.2 motivated the following definition.

Definition 2.3 ([21, Definition 6])**.** *We say that $u \not\equiv 0$, $u \in W_0^{s,p}(\Omega)$, $s = \alpha - n/p$, is an* eigenfunction *of Ω, if the Euler-Lagrange equation* (2.10) *holds for all test functions $\varphi \in C_c^\infty(\Omega)$. The corresponding λ is called an* eigenvalue.

Remark 2.4. The minimizer found in Theorem 2.2 is called the *first eigenfunction* of $((-\Delta)_p^s, W_0^{s,p}(\Omega))$.

To use some of the minimax theorems in the literature, we have to check that the associated energy functional also satisfies some kind of compactness condition.

Definition 2.5. We will say that $(u_n) \subset X$ is a PS-sequence for J if

$$|J(u_n)| \leqslant C \quad \text{for all } n, \text{ and } J'(u_n) \to 0 \text{ as } n\to\infty,$$

where C is a positive constant. We say that a functional $J \in C^1(X,\mathbb{R})$ satisfies the Palais-Smale condition (PS-condition) if any PS-sequence $(u_n) \subset X$ possesses a convergent subsequence.

To prove the existence of a second solution for (1.1) in Theorem 1.1, we will need the concept of critical groups from infinite-dimensional Morse Theory.

Define $J_\lambda^c = \{w \in X \,|\, J_\lambda(w) \leqslant c\}$, the sub-level set of J_λ at c, and set

$$\mathcal{K} = \{u \in X \,|\, J_\lambda'(u) = 0\},$$

the critical set of J_λ. For an isolated critical point u_0 of J_λ, the q–critical groups of J_λ at u_0, with coefficients in a field $\mathbb{F}$ of characteristic 0, are defined by

$$C_k(J_\lambda, u_0) = H_k(J_\lambda^{c_0} \cap U, J_\lambda^{c_0} \cap U \backslash \{u_0\}), \quad \text{for all } k \in \mathbb{Z},$$

where $c_0 = J_\lambda(u_0)$, U is a neighborhood of u_0 that contains no critical points of J_λ other than u_0, and H_* denotes the singular homology groups. The critical groups are independent of the choice of U by the excision property of homology (see Hatcher [14]). For more information on the definition of critical groups, we refer the reader to [9, 26, 24, 23].

Next, we present the concept of the critical groups at infinity introduced by Bartsch and Li in [4]. Assume that $J_\lambda \in C^1(X, \mathbb{R})$ satisfies the Palais-Smale condition. Let $\mathcal{K} = \{u \in X : J_\lambda'(u) = 0\}$ be the set of critical points of J_λ and assume that under these assumptions the critical value set is bounded from below; that is,

$$a_o < \inf J_\lambda(\mathcal{K}),$$

for some $a_o \in \mathbb{R}$. The critical groups at infinity are defined by

$$C_k(J_\lambda, \infty) = H_k(X, J_\lambda^{a_o}), \quad \text{for all } k \in \mathbb{Z}, \tag{2.11}$$

(see [4]). These critical groups are well-defined as a consequence of the Second Deformation Theorem (see Perera and Schechter [26, Lemma 1.3.7]).

In this work, we use the concept of a critical point of a functional being of a mountain-pass type. We use the definition found in Hofer [15] and Montreanu et al. [24].

Definition 2.6 ([24, Definition 6.98])**.** Let X be a Banach space, $J \in C^1(X, \mathbb{R})$, and $u_0 \in \mathcal{K}$. We say that u_0 is of *mountain-pass type* if, for any open neighborhood U of u_0, the set $\{w \in U \,|\, J(w) < J(u_0)\}$ is nonempty and not path-connected.

The critical groups of mountain-pass type can be described by the following proposition found in Montreanu et al. [24]:

Proposition 2.7 ([24, Proposition 6.100])**.** *Let X be a reflexive Banach space, $J \in C^1(X, \mathbb{R})$, and $u_0 \in \mathcal{K}$ be isolated in $J(\mathcal{K})$. If u_0 is of mountain-pass type, then $C_1(J, u_0) \ncong 0$.*

Put $\mathcal{K}_d = \{u \in X | J(u) = d, J'(u) = 0\}$, the critical set at level d. One of the critical points that will be obtained in the proof of Theorem 1.1 satisfies a variant of the mountain-pass theorem due to Hofer, which we present next for the reader's convenience.

Theorem 2.8 ([15])**.** *Assume that X is a real Banach space. Let $J \in C^1(X, \mathbb{R})$ satisfy the Palais-Smale condition and assume that e_0 and e_1 are distinct points in X. Define*

$$A = \{a \in C([0, 1], X)| : a(i) = e_i, \text{ for } i = 0, 1\}, \tag{2.12}$$

$$d = \inf_{a \in A} \sup J(|a|), \quad |a| = a([0, 1]), \quad c = \max\{J(e_0), J(e_1)\}. \tag{2.13}$$

If $d > c$, the set $\mathcal{K}_d$ is non-empty. Moreover, there exists at least one critical point u_0 in $\mathcal{K}_d$ that is either a local minimum or of mountain-pass type. If all the critical points in $\mathcal{K}_d$ are isolated in X the set $\mathcal{K}_d$ contains a critical point of mountain-pass type.

Remark 2.9. Once we prove that the functional J_λ defined in (2.5) satisfies the conditions of Theorem 2.8 in Section 4, for the case $f(0) \neq 0$, assuming that J_λ has only one critical point u_λ, it will follow from Proposition 2.7 that

$$C_1(J_\lambda, u_\lambda) \ncong 0. \tag{2.14}$$

We do not have information about the other critical groups $C_k(J_\lambda, u_\lambda)$ when $k \neq 1$. But, the fact that $C_1(J_\lambda, u_\lambda)$ is nontrivial will be enough to prove the existence of a second critical point for the functional J_λ. A similar argument will be used in Section 6 for the case $f(0) = 0$.

Finally, the last result we will need to prove multiplicity results for problem (1.1) for the case $f(0) \neq 0$ is found in Bartsch and Li [4].

Proposition 2.10 ([4, Proposition 3.6])**.** *Suppose that $J \in C^1(X, \mathbb{R})$ satisfies the Palais-Smale condition at level c for every $c \in \mathbb{R}$. If $\mathcal{K} = \emptyset$, then $C_k(J, \infty) \cong 0$ for all $k \in \mathbb{Z}$. If $\mathcal{K} = \{u_\lambda\}$, then $C_k(J, \infty) \cong C_k(J, u_\lambda)$, for all $k \in \mathbb{Z}$.*

We shall prove in Section 4 that $C_k(J_\lambda, \infty) \cong 0$ for all $k > 0$; that is, the critical groups of J_λ at infinity are all trivial for $k \neq 0$. In particular, we will have $C_1(J_\lambda, \infty) \cong 0$. Hence, assuming, by a way of contradiction, that J_λ has only the critical point u_λ found in Section 3, we will then obtain a contradiction based on the result of Proposition 2.10 and the assertion in (2.14).

In the next section, we will prove the existence of a mountain-pass type solution for problem (1.1).

3. Existence and a priori estimates

3.1. **Existence of a mountain-pass type solution.** In this section, we show that the functional J_λ defined in (2.5) satisfies the conditions of the variant of the mountain-pass theorem due to Hofer [15] as presented in Theorem 2.8.

First, by conditions (1.3) and (1.4), it can be shown that there exists $B_1 > 0$ such that

$$F(s) \leqslant B_1(|s|^{q+1} + 1), \quad \text{for all } s \in \mathbb{R}. \tag{3.1}$$

It also follows from (1.3) and (1.4) that, for all $s \geqslant 0$, there exist $A_1, C_1 > 0$ such that

$$F(s) \geqslant A_1(s^{q+1} - C_1), \quad \text{for all } s \geqslant 0. \tag{3.2}$$

In what follows, let $r > 0$ be the positive number

$$r = \frac{1}{q+1-p}, \tag{3.3}$$

where p, q satisfy the conditions in hypothesis (H1).

In the next two lemmas, we prove the geometric conditions in Theorem 2.8.

Lemma 3.1. *There exist $\tau > 0$, $c_1 > 0$ and $\hat{\lambda}_2 \in (0,1)$ such that if $\|u\| = \tau\lambda^{-r}$ then $J_\lambda(u) \geqslant c_1(\tau\lambda^{-r})^p$ for all $\lambda \in (0, \hat{\lambda}_2)$, where r is given in* (3.3).

Proof. By the Sobolev embedding theorem and hypothesis (H3), it follows from the definition of J_λ in (2.5) that

$$\begin{aligned} J_\lambda(u) &= \frac{1}{p}\|u\|^p + \frac{1}{p}\int_\Omega V(x)|u|^p\,dx - \lambda\int_\Omega F(u)dx \\ &\geqslant \frac{1}{p}\|u\|^p - \frac{c_V}{\lambda_1 p}\|u\|^p - \lambda B_1 C_{q+1}^{q+1}\|u\|^{q+1} - \lambda B_1|\Omega|, \end{aligned} \tag{3.4}$$

for all $u \in X$. Let $\tau > 0$ be a small enough constant such that the following identity is satisfied:

$$1 - \frac{c_V}{\lambda_1} = \frac{3}{2}pC_{q+1}^{q+1}B_1\tau^{q+1-p}. \tag{3.5}$$

Next, setting $\|u\| = \lambda^{-r}\tau$ in (3.4) and using that $r(q+1)+1 = -rp$, we obtain

$$J_\lambda(u) \geqslant \lambda^{-rp}\Big[\frac{\tau^p}{p}\Big(1 - \frac{c_V}{\lambda_1}\Big) - B_1C_{q+1}^{q+1}\tau^{q+1} - \lambda^{1+rp}B_1|\Omega|\Big], \tag{3.6}$$

for all $u \in X$.

Then, by (3.5), it follows from (3.6) that

$$J_\lambda(u) \geqslant \lambda^{-rp}\Big(\frac{1}{2}pC_{q+1}^{q+1}B_1\tau^{q+1-p} - \lambda^{1+rp}|\Omega|B_1\Big), \tag{3.7}$$

for all $u \in X$.

Finally, choose $\lambda \in (0, \hat{\lambda}_2)$ with $\hat{\lambda}_2 := \tau^{p/(1+rp)}(4pB_1|\Omega|)^{-1/(1+rp)}$. Then, for this choice of λ, we obtain from (3.7) that

$$J_\lambda(u) \geqslant c_1(\tau\lambda^{-r})^p; \quad \text{for } u \in X,$$

where $c_1 = \frac{1}{4p}$. This completes the proof. □

Lemma 3.2. *Let $\varphi_o \in X$ be such that $\varphi_o > 0$ and $\|\varphi_o\| = 1$. There exists $\hat{\lambda}_1 > 0$ such that if $\lambda \in (0, \hat{\lambda}_1)$ then $J_\lambda(c\lambda^{-r}\varphi_o) \leqslant 0$, where r is given by* (3.3).

Proof. Set $\ell = c\lambda^{-r}$, where $c, \lambda > 0$ are positive constants to be chosen shortly. Then, by hypothesis (H1), the estimate (3.2), and the characterization of the first eigenvalue of the fractional p-Laplacian from Theorem 2.2, we obtain

$$\begin{aligned} J_\lambda(\ell\varphi_o) &= \frac{1}{p}\|\ell\varphi_o\|^p + \frac{1}{p}\int_\Omega V(x)|\ell\varphi_o|^p\,dx - \lambda\int_\Omega F(\ell\varphi_o)\,dx \\ &\leqslant \frac{\ell^p}{p}\|\varphi_o\|^p + \frac{\ell^p}{p}\|V\|_\infty\|\varphi_o\|_p^p - \lambda A_1\ell^{q+1}\int_\Omega \varphi_o^{q+1}dx + \lambda A_1C_1|\Omega| \\ &\leqslant \frac{\ell^p}{p}\Big(1 + \frac{1}{\lambda_1}\|V\|_\infty\|\varphi_o\|^p - \lambda pA_1\ell^{q+1-p}\|\varphi_o\|_{q+1}^{q+1}\Big) + \lambda A_1C_1|\Omega|. \end{aligned} \tag{3.8}$$

Next, we define $c > 0$ such that

$$c^{q+1} = \frac{2c^p}{pA_1\|\varphi_o\|_{q+1}^{q+1}}\Big(1 + \frac{1}{\lambda_1}\|V\|_\infty\Big). \tag{3.9}$$

Then, by (3.9) and the definition of ℓ, it follows from (3.8) that

$$J_\lambda(\ell\varphi_o) \leqslant \lambda^{-rp}\frac{c^p}{p}\Big[-\Big(1 + \frac{1}{\lambda_1}\|V\|_\infty\Big) + \lambda^{1+rp}A_1C_2|\Omega|\Big]. \tag{3.10}$$

We set

$$\hat{\lambda}_1 = \Big[\frac{1 + \frac{1}{\lambda_1}\|V\|_\infty}{2pA_1C_2|\Omega|}\Big]^{\frac{1}{1+rp}}.$$

Then, it follows from (3.10) that

$$J_\lambda(\ell\varphi_o) \leqslant -\frac{c^p}{2p}\lambda^{-rp} \leqslant 0,$$

for all $\lambda \in (0, \hat{\lambda}_1)$, which establishes the lemma. □

In the next lemma, we will show that the functional J_λ satisfies the Palais-Smale condition.

Lemma 3.3. *Assume that* (H1)–(H3) *are satisfied and* $\lambda \in (0, \lambda_3)$ *with* $\lambda_3 := \min\{\hat{\lambda}_1, \hat{\lambda}_2\}$*, where* $\hat{\lambda}_1$ *is given by Lemma 3.2 and* $\hat{\lambda}_2$ *is given by Lemma 3.1. Then,* J_λ *satisfies the Palais-Smale condition.*

Proof. Let (u_n) be a Palais-Smale sequence for J_λ in X; that is,

$$|J_\lambda(u_n)| \leqslant C, \quad \text{for all } n; \tag{3.11}$$

where $C > 0$ is a constant and there exists a sequence of positive numbers (ε_n) such that

$$\langle J_\lambda'(u_n), \varphi\rangle \leqslant \varepsilon_n\|\varphi\|, \quad \text{for all } n, \tag{3.12}$$

and all $\varphi \in X$ and $\varepsilon_n \to 0$ as $n \to \infty$. In particular, setting $\varphi = u_n$ in (3.12), we obtain that there exists $N_1 > 0$ such that

$$|\langle J_\lambda'(u_n), u_n\rangle| \leqslant \|u_n\|, \quad \text{for all } n \geqslant N_1.$$

Hence, we can write

$$-\|u_n\|^p - \|u_n\| \leqslant -\|u_n\|^p + \langle J'(u_n), u_n\rangle, \quad \text{for } n \geqslant N_1. \tag{3.13}$$

Thus, using the definition of the Fréchet derivative of J_λ given in (2.6), and the definition of the norm $\|\cdot\|$ given in (2.4) and (2.3), we obtain from (3.13) that

$$-\|u_n\|^p - \|u_n\| \leqslant \int_\Omega V(x)|u_n|^p dx - \lambda\int_\Omega f(u_n)u_n\, dx, \tag{3.14}$$

for $n \geqslant N_1$.

On the other hand, using the estimate in (1.5) in hypothesis (H2), we have that

$$\frac{1}{p}\|u_n\|^p - \frac{\lambda}{\theta}\int_\Omega f(u_n)u_n\, dx + \frac{\lambda}{\theta}K|\Omega| \leqslant \frac{1}{p}\|u_n\|^p - \lambda\int_\Omega F(u_n)\, dx$$

for $n \in \mathbb{N}$; so that, using the definition of J_λ in (2.5),

$$\frac{1}{p}\|u_n\|^p - \frac{\lambda}{\theta}\int_\Omega f(u_n)u_n\, dx + \frac{\lambda}{\theta}K|\Omega| \leqslant J_\lambda(u_n) - \frac{1}{p}\int_\Omega V(x)|u_n|^p, dx, \tag{3.15}$$

for all $n \in \mathbb{N}$.

Now, it follows from (3.15) and the hypothesis in (3.11) that

$$\frac{1}{p}\|u_n\|^p - \frac{\lambda}{\theta}\int_\Omega f(u_n)u_n\, dx + \frac{\lambda}{\theta}K|\Omega| \leqslant C - \frac{1}{p}\int_\Omega V(x)|u_n|^p\, dx, \tag{3.16}$$

for $n \in \mathbb{N}$.

Next, we multiply on both sides of the estimate in (3.14) by $\frac{1}{\theta}$ and add $\frac{1}{p}\|u_n\|^p$ on both sides of the inequality to obtain

$$\begin{aligned} &\big(\frac{1}{p} - \frac{1}{\theta}\big)\|u_n\|^p - \frac{1}{\theta}\|u_n\| \\ &\leqslant \frac{1}{p}\|u_n\|^p + \frac{1}{\theta}\Big(\int_\Omega V(x)|u_n|^p\, dx - \lambda\int_\Omega f(u_n)u_n\, dx\Big), \end{aligned} \tag{3.17}$$

for $n \geqslant N_1$. It follows from the estimate (3.16) that

$$\begin{aligned} &\frac{1}{p}\|u_n\|^p + \frac{1}{\theta}\Big(\int_\Omega V(x)|u_n|^p\,dx - \lambda\int_\Omega f(u_n)u_n\,dx\Big) \\ &\leqslant C - \big(\frac{1}{p}-\frac{1}{\theta}\big)\int_\Omega V(x)|u_n|^p\,dx, \end{aligned} \tag{3.18}$$

for $n \in \mathbb{N}$. Consequently, combining the estimates in (3.17) and (3.18),

$$\big(\frac{1}{p}-\frac{1}{\theta}\big)\|u_n\|^p - \frac{1}{\theta}\|u_n\| \leqslant C - \big(\frac{1}{p}-\frac{1}{\theta}\big)\int_\Omega V(x)|u_n|^p\,dx, \tag{3.19}$$

for $n \geqslant N_1$.

Next, use the estimate for the potential V in hypothesis (H3) to obtain from (3.19) that

$$\big(\frac{1}{p}-\frac{1}{\theta}\big)\|u_n\|^p - \frac{1}{\theta}\|u_n\| \leqslant C + \big(\frac{1}{p}-\frac{1}{\theta}\big)\frac{c_V}{\lambda_1}\|u_n\|^p, \quad \text{for } n \geqslant N_1, \tag{3.20}$$

where we have used the definition of λ_1 in (2.9).

Rearranging (3.20) we obtain

$$\big(\frac{1}{p}-\frac{1}{\theta}\big)\big(1-\frac{c_V}{\lambda_1}\big)\|u_n\|^p - \frac{1}{\theta}\|u_n\| \leqslant C, \quad \text{for } n \geqslant N_1,$$

from which we obtain that (u_n) is bounded in $W_0^{s,p}(\Omega)$.

Hence, since (u_n) is bounded in X, we may invoke the Banach-Alaoglu theorem (see [20, Theorem 2.18]) to deduce, passing to a subsequence if necessary, that there exists $u \in X$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } X \quad \text{as } n \to \infty.$$

Furthermore, since $1 < q+1 < p^*$, by the Sobolev embedding theorem, we can also assume that

$$\begin{gathered} u_n \to u \quad \text{in } L^{q+1}(\Omega) \quad \text{as } n \to \infty \\ u_n(x) \to u(x) \quad \text{a.e. in } \Omega \quad \text{as } n \to \infty. \end{gathered} \tag{3.21}$$

Next, we put $q_1' = \frac{q+1}{q}$; so that, $q' > 1$, and $q'q = q+1$. Hence, by (3.1) we obtain

$$|f(u_n)|^{q_1'} \leqslant B_1^{q_1'}(|u_n|^q + 1)^{q_1'} \leqslant C_1(|u_n|^{qq_1'} + 1) \leqslant C_1(|u_n|^{q+1} + 1), \tag{3.22}$$

for all $n \in \mathbb{N}$, where C_1 is a positive constant. Thus, applying Hölder's inequality with exponent q_1' in (3.22) and its conjugate, we obtain

$$\lambda\int_\Omega f(u_n)(u_n-u)\,dx \leqslant C(\|u_n\|_{q+1}+1)\|u_n-u\|_{q+1} \leqslant C\|u_n-u\|_{q+1},$$

where C is a positive constant.

Consequently, letting $n \to \infty$ in the previous estimate and applying (3.21) with the Lebesgue dominated convergence theorem, we obtain

$$\lambda\int_\Omega f(u_n)(u_n-u)dx \to 0, \quad \text{as } n \to \infty. \tag{3.23}$$

Next, we put $p' = \frac{p}{p-1}$ (recall that we are assuming $p > 1$); so that $p' > 1$ and $p'(p-1) = p$. Then, by Hölder's inequality we have

$$\int_\Omega |V(x)||u_n|^{p-1}|u_n-u|\,dx \leqslant \|V\|_\infty\|u_n\|_p^{p-1}\|u_n-u\|_p \leqslant C\|u_n-u\|_p \leqslant C\|u_n-u\|_{q+1},$$

for all $n \in \mathbb{N}$, where C is a positive constant. Hence, letting $n \to \infty$ in the previous estimate and applying eqrefpss13 with the Lebesgue dominated convergence theorem, we obtain

$$\int_\Omega |V(x)||u_n|^{p-1}|u_n - u|\,dx \to 0, \quad \text{as } n \to \infty\ . \tag{3.24}$$

Next, since (u_n) is a Palais-Smale sequence in X, it follows from (3.12), (3.23), and (3.24) that

$$\lim_{n\to\infty} \int_{\mathbb{R}^{2N}} \frac{\Phi_p(u_n(x) - u_n(y))((u_n - u)(x) - (u_n - u)(y))}{|x - y|^{N+sp}}\,dx = 0. \tag{3.25}$$

Once again, using the fact that u is the weak limit of u_n we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^{2N}} \frac{\Phi_p(u(x) - u(y))((u_n - u)(x) - (u_n - u)(y))}{|x - y|^{N+sp}}\,dx = 0. \tag{3.26}$$

On the other hand, it follows from appling Hölder's inequality as in [22, Lemma 3] that

$$\begin{aligned}
&\int_\Omega \frac{\Phi_p(u_n(x) - u_n(y)) - \Phi_p(u(x) - u(y))}{|x - y|^{N+sp}}((u_n - u)(x) - (u_n - u)(y))\,dx\,dy \\
&= \int_\Omega \Big[\frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N+sp}} - \frac{\Phi_p(u_n(x) - u_n(y))(u(x) - u(y))}{|x - y|^{N+sp}} \\
&\quad - \frac{\Phi_p(u(x) - u(y))(u_n(x) - u_n(y))}{|x - y|^{N+sp}} + \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}}\Big]\,dx\,dy \\
&\geqslant \|u_n\|^p - \|u_n\|^{p-1}\|u\| - \|u_n\|\|u\|^{p-1} + \|u\|^p \\
&= (\|u_n\|^{p-1} - \|u\|^{p-1})(\|u_n\| - \|u\|).
\end{aligned} \tag{3.27}$$

Then, in view of

$$\left(\|u_n\|^{p-1} - \|u\|^{p-1}\right)(\|u_n\| - \|u\| \geqslant 0, \quad \text{for all } n,$$

it follows from (3.25),(3.26), and (3.27) that

$$\lim_{n\to\infty} ([\|u_n\|^{p-1} - \|u\|^{p-1})(\|u_n\| - \|u\|) = 0,$$

from which we obtain

$$\lim_{n\to\infty} \|u_n\| = \|u\|. \tag{3.28}$$

Finally, by (3.28) and that $u_n \rightharpoonup u$ weakly in X, we conclude that $u_n \to u$ strongly in X. Hence, J_λ satisfies the Palais-Smale condition. □

Next, we present the main result of this section.

Theorem 3.4. *Assume that* (H1)–(H3) *are satisfied. Then, for λ sufficiently small, the functional J_λ has a critical point $u_\lambda \in X$ of mountain-pass type. Moreover,*

$$c_1\lambda^{-rp} \leqslant J_\lambda(u_\lambda) \leqslant c_2\lambda^{-rp}, \tag{3.29}$$

where c_1 and c_2 are positive constants independent of λ, and r is given in (3.3).

Proof. It follows from Lemmas 3.8, 3.2, 3.3, that, for each $\lambda \in (0, \lambda_3)$, the functional J_λ defined in (2.5) satisfies the conditions of Theorem 2.8. Therefore, J_λ possesses a critical point, u_λ, with critical value characterized by

$$J_\lambda(u_\lambda) = \inf_{a\in A} \max J_\lambda(|a|),$$

with

$$A = \{a \in C([0,1], X) : a(0) = 0,\ a(1) = c\lambda^{-r}\varphi_o\},$$

where $a(1)$ is obtained in Lemma 3.2 and $|a| = a([0,1])$.

Furthermore, by Lemma 3.2, we observe that

$$J_\lambda(sc\lambda^{-r}\varphi_o) \leqslant c_2\lambda^{-rp}, \quad \text{for } 0 \leqslant s \leqslant 1,$$

where c_0 is a positive constant independent of λ. Hence, we conclude that

$$J_\lambda(u_\lambda) \leqslant c_2\lambda^{-rp}.$$

Finally, it follows from Lemma 3.1 that there exists a positive constant c_1 independent of λ such that

$$c_1\lambda^{-rp} \leqslant J_\lambda(u), \quad \text{for all } \|u\| = \tau\lambda^{-r}.$$

Then, it follows from the characterization of the critical value that

$$c_1\lambda^{-rp} \leqslant J_\lambda(u_\lambda).$$

This completes the proof. □

The next two results will be used in the proof of a comparison principle for problem (1.1).

Lemma 3.5. *Assume that* (H1)–(H3) *are satisfied and let* u_λ *be the mountain-pass critical point of* J_λ *given in Theorem 3.4. There exists a constant* c *such that*

$$\|u_\lambda\| \leqslant c\lambda^{-r}. \tag{3.30}$$

and r *is given in* (3.3).

Proof. Let u_λ be a critical point of J_λ given by Theorem 3.4. Then, it follows from (2.7) that

$$\langle J'_\lambda(u), \varphi\rangle = 0, \quad \text{for all } \varphi \in X. \tag{3.31}$$

Then, setting $\varphi = u_\lambda$ in (3.31) and using (2.6), we obtain

$$\|u_\lambda\|^p + \int_{\mathbb{R}^N} V(x)|u_\lambda|^p\,dx = \lambda\int_\Omega f(u_\lambda)u_\lambda\,dx.$$

It then follows from the Ambrosetti-Rabinowitz type condition in (1.5) that

$$\begin{aligned}
\Big(\frac{1}{p} - \frac{1}{\theta}\Big)\|u_\lambda\|^p &= \frac{1}{p}\|u_\lambda\|^p - \frac{1}{\theta}\Big(\lambda\int_\Omega f(u_\lambda)u_\lambda\,dx - \int_{\mathbb{R}^N} V(x)|u_\lambda|^p\,dx\Big)\\
&\leqslant \frac{1}{p}\|u_\lambda\|^p - \frac{\lambda}{\theta}\Big(\int_\Omega \theta F(u_\lambda)\,dx + K|\Omega|\Big) + \frac{1}{\theta}\int_{\mathbb{R}^N} V(x)|u_\lambda|^p\,dx\\
&\leqslant \frac{1}{p}\|u_\lambda\|^p - \lambda\int_\Omega F(u_\lambda)\,dx + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u_\lambda|^p\,dx - \frac{\lambda K}{\theta}|\Omega|\\
&\leqslant J_\lambda(u_\lambda) + C\lambda^{-rp};
\end{aligned}$$

so that, using (3.29) in Theorem 3.4, (3.30) follows. □

Finally, we present lower and upper estimates for $\|u_\lambda\|_\infty$, where u_λ is the critical point obtained in Theorem 3.4. These results will be used in the proof of comparison principle for problem (1.1).

Lemma 3.6. *Assume that* (H1)–(H3) *are satisfied. Let* u_λ *be a weak solution of problem* (1.1) *obtained via Theorem 3.4 and* λ_3 *be as in Lemma 3.3. Then, there exists a constant* C *such that, for all* $0 < \lambda < \lambda_3$,

$$C\lambda^{-r} \leqslant \|u_\lambda\|_\infty, \tag{3.32}$$

and r *is given in* (3.3).

Proof. By estimate in (3.29) for $J_\lambda(u_\lambda)$ in Theorem 3.4, and that $\min F > -\infty$, we obtain

$$\begin{aligned} \lambda \int_\Omega f(u_\lambda)u_\lambda \, dx &= \|u_\lambda\|^p + \int_{\mathbb{R}^N} V(x)|u|^p \, dx \\ &= pJ_\lambda(u_\lambda)\, dx + p\lambda \int_\Omega F(u_\lambda)\, dx \\ &\geqslant pC\lambda^{-rp} + p\lambda|\Omega| \min F \\ &\geqslant C\lambda^{-rp}. \end{aligned} \tag{3.33}$$

On the other hand, by the growth of f in (1.3), we obtain

$$\lambda \int_\Omega f(u_\lambda)u_\lambda \, dx \leqslant B\lambda \|u_\lambda\|_\infty^{q+1}.f \tag{3.34}$$

Combining the estimates (3.33) and (3.34), we obtain (3.32). □

In the proof of the comparison principle, we will need the following regularity result found in Mosconi et al. [25].

Lemma 3.7 ([25, Lemma 2.3])**.** *Let* $g \in L^t(\Omega)$, $N/(sp) < t \leqslant \infty$ *and* $u \in W_0^{1,p}(\Omega)$ *be a weak solution of* $(-\Delta)_s^p = g$ *in* Ω. *Then*

$$\|u\|_\infty \leqslant C\|g\|_t^{1/(p-1)}.$$

The following theorem due to Ianizotto et al. [17] establishes a sharp boundary regularity result for the fractional p-Laplacian, for $p \geqslant 2$. The assumption of $p \geqslant 2$ will allow us to obtain enough regularity up to the boundary of Ω to obtain a positive solution for (1.1).

Theorem 3.8 ([17, Theorem 1.1])**.** *Let* $p \geqslant 2$, Ω *be a bounded domain with* $C^{1,1}$ *boundary and* $d(x) = \operatorname{dist}(x, \partial\Omega)$. *There exist* $\alpha \in (0, s)$ *and* $C > 0$ *depending on* N, Ω, p *and* s, *such that, for all* $g \in L^\infty(\Omega)$, *a weak solution* $u \in W_0^{s,p}(\Omega)$ *of the problem*

$$\begin{gathered} (-\Delta)_p^s(u) = g; \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^N \backslash \Omega, \end{gathered}$$

satisfies $u/d^s \in C^\alpha(\overline{\Omega})$ *and*

$$\|\frac{u}{d^s}\|_{C^\alpha(\overline{\Omega})} \leqslant C\|g\|_\infty^{\frac{1}{p-1}}.$$

Finally, we present the last result of this section that will be used to prove the existence of a positive solution for problem (1.1).

Lemma 3.9. *Assume that* (H1)–(H3) *are satisfied. Let* $\lambda_3 > 0$ *be as in Lemma 3.3. Then, there exist* $\alpha \in (0, s]$ *and a constant* $C > 0$ *such that, for all* $0 < \lambda < \lambda_3$, *the solution u_λ given in Theorem 3.4 of the problem* (1.1) *satisfies $u_\lambda/d^s \in C^\alpha(\overline{\Omega})$. Furthermore*

$$\|u_\lambda\|_\infty \leqslant C\lambda^{-r},$$
$$\|\frac{u_\lambda}{d^s}\|_{C^\alpha(\overline{\Omega})} \leqslant C\lambda^{-r}.$$

and r is given in (3.3).

Proof. It follows from the assumption $Nq/(sp) < p_s^*$ that there exists $t > 1$ such that $\frac{N}{sp} < t$ and $tq < p_s^*$, which implies $t(p-1) < p_s^*$. Set $g := \lambda f \circ u_\lambda + V\Phi_p(u_\lambda)$. Since $W_0^{s,p}(\Omega) \hookrightarrow L^{tq}(\Omega)$ is a continuous embedding and $|g| \leqslant A_1\lambda(|u_\lambda|^q + 1) + \|V\|_\infty |u_\lambda|^{p-1}$ we obtain

$$\begin{aligned}\int_\Omega |\lambda f(u_\lambda)(x) + V\Phi_p(u_\lambda)|^t\,dx &\leqslant \lambda^t \int_\Omega |A_1(u_\lambda^q + 1)|^t\,dx + \|V\|_\infty^t \int_\Omega |u_\lambda|^{t(p-1)}\,dx \\ &\leqslant \lambda^t C \int_\Omega (|u_\lambda|^{qt} + 1)\,dx + \|V\|_\infty^t \int_\Omega |u_\lambda|^{t(p-1)}\,dx.\end{aligned}$$

Hence, $g \in L^t(\Omega)$ and it follows from Lemma 3.7 that

$$\|u_\lambda\|_\infty \leqslant \|g\|_t^{\frac{1}{p-1}}. \tag{3.35}$$

On the other hand, by Lemma 3.5, we have

$$\begin{aligned}\|g\|_t &\leqslant C_1\lambda\|u_\lambda\|_{tq}^q + C_2\|u_\lambda\|_{t(p-1)}^{p-1} \\ &\leqslant C_1\lambda\|u_\lambda\|^q + C_2\|u_\lambda\|^{p-1} \\ &\leqslant C(\lambda^{1-rq} + \lambda^{-r(p-1)}).\end{aligned}$$

Therefore, from (3.35) and the fact that $-r = (1-rq)/(p-1)$ we obtain that

$$\|u_\lambda\|_\infty \leq \|g\|_t^{1/(p-1)} \leq C\lambda^{-r}. \tag{3.36}$$

Thus, $u_\lambda \in L^\infty(\Omega)$ and then $g \in L^\infty(\Omega)$. Hence, by Theorem 3.8, there exists $\alpha \in (0,s]$ and $C > 0$, depending only on N, p, s and Ω, such that the solution u_λ satisfies $u_\lambda/d^s \in C^\alpha(\overline{\Omega})$ and

$$\|\frac{u_\lambda}{d^s}\|_{C^\alpha(\overline{\Omega})} \leqslant C\|g\|_\infty^{\frac{1}{p-1}} \leqslant \lambda^{-r}.$$ □

3.2. **Existence of a positive solution.** To prove that the solution u_λ found in Subsection 3.1 is positive, we will list two results found in Del Pezzo et al. [11] and one theorem due to Ianizzoto et al. [19], which will lead us to a comparison principle for the fractional p-Laplacian problem in (1.1).

First, we recall two basic definitions that will be used in this section for the reader's convenience.

Definition 3.10. Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, be an open set. We say that $x_o \in \partial\Omega$ satisfies the interior ball condition if there is $x \in \Omega$ and $r > 0$ such that

$$B_r(x) \subset \Omega, \quad \text{and } x_o \in \partial B_r(x),$$

where $B_r(x) = \{z \in \mathbb{R}^N : |z - x| < r\}$.

Next, we recall the concept of a function $u \in \widetilde{W}^{s,p}(\Omega)$ being a super-solution of the fractional p-Laplacian problem (1.1).

Definition 3.11. Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with $N \geqslant 1$. We say that $u \in \widetilde{W}^{s,p}(\Omega)$ is a super-solution of (1.1) if

$$\int_{\mathbb{R}^{2N}} \frac{\Phi_p(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy + \int_\Omega V(x)|u|^{p-2}u\varphi\,dx \geqslant \lambda \int_\Omega f(u)\varphi\,dx,$$

for each $\varphi \in \widetilde{W}^{s,p}(\Omega)$.

The next two theorems due to Del Pezzo et al. [11] will play a special role in the main result to be discussed in this section.

Theorem 3.12 ([11, Theorem 1.4])**.** *Let $c \in C(\overline{\Omega})$ be a non-positive function and $u \in \widetilde{W}^{s,p}(\Omega) \cap C(\overline{\Omega})$ be a weak super-solution of*

$$(-\Delta)_p^s u(x) = c(x)|u(x)|^{p-2}u(x), \quad \text{for } x \in \Omega. \tag{3.37}$$

If Ω is bounded and $u \geqslant 0$ a.e. in $\mathbb{R}^N \backslash \Omega$, then either $u > 0$ in Ω or $u = 0$ a.e. in $\mathbb{R}^N$.

Theorem 3.13 ([11, Theorem 1.5])**.** *Let Ω satisfy the interior ball condition at $x_0 \in \partial\Omega$, $c \in C(\overline{\Omega})$, and $u \in \widetilde{W}^{s,p}(\Omega) \cap C(\overline{\Omega})$ be a weak super-solution of (3.37). Suppose that Ω is bounded, $c(x) \leqslant 0$ in Ω and $u \geqslant 0$ a.e. in $\mathbb{R}^N \backslash \Omega$. Then, either $u = 0$ a.e. in $\mathbb{R}^N$, or*

$$\liminf_{x \to x_0,\ x \in B} \frac{u(x)}{(d(x))^s} > 0, \tag{3.38}$$

where $B \subseteq \Omega$ is an open ball in Ω, such that $x_0 \in \partial B$, and d is the distance from x to $\mathbb{R}^N \backslash B$.

Next, we present a version of the comparison principle for problem (1.1) motivated by a result due to Lindgren et al. [21, Lemma 9] (see also Ianizzotto et al. [19, Proposition 2.10]).

Theorem 3.14. *Let Ω be a bounded subset of $\mathbb{R}^N$, $N \geqslant 2$, and $u, v \in \widetilde{W}^{s,p}(\Omega)$ satisfy $u \leqslant v$ in $\mathbb{R}^N \backslash \Omega$. Moreover, assume that*

$$\begin{aligned} &\int_{\mathbb{R}^{2N}} \frac{\Phi_p(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy + \int_{\mathbb{R}^N} V(x)\Phi_p(u)\varphi\,dx \\ &\leqslant \int_{\mathbb{R}^{2N}} \frac{\Phi_p(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy + \int_{\mathbb{R}^N} V(x)\Phi_p(v)\varphi\,dx, \end{aligned} \tag{3.39}$$

for all $\varphi \in W_0^{s,p}(\Omega)$, $\varphi \geqslant 0$ a.e. in Ω. If $V(x) \geqslant 0$ for a.e. $x \in \mathbb{R}^N$, then $u \leqslant v$ in Ω.

Proof. We set $\varphi = (u-v)^+$, where $(u-v)^+ = \max\{u-v, 0\}$ denotes the positive part of the function $u-v$, in (3.39) to obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} V(x)(\Phi_p(u)-\Phi_p(v))(v-u)^+(x)\,dx \\ &\leqslant \int_{\mathbb{R}^{2N}} \frac{(\Phi_p(v(x)-v(y))-\Phi_p(u(x)-u(y)))((v-u)^+(x)-(v-u)^+(y))}{|x-y|^{N+sp}}\,dx\,dy. \end{aligned} \tag{3.40}$$

Using an identity found in [21, page 809],

$$\Phi_p(b)-\Phi_p(a) = (p-1)(b-a)\int_0^1 |a+t(b-a)|^{p-2}\,dt,$$

with $b = u(x)$ and $a = v(x)$, we obtain the estimate

$$\begin{aligned}0 &\leqslant (p-1)(u(x)-v(x))(u-v)^+(x)\int_0^1 |v(x)+t(u(x)-v(x))|^{p-2}\,dt \\ &= (\Phi_p(u(x))-\Phi_p(v(x)))(u-v)^+(x).\end{aligned} \tag{3.41}$$

for a.e. $x \in \mathbb{R}^N$. Hence, we conclude that the left-hand side of (3.40) is nonnegative. The remainder of the proof of the theorem follows the same line of reasoning as in [21, Lemma 9] and we omit the arguments here. □

Next, we show that the solution u_λ found through Theorem 3.4 is positive in Ω.

Theorem 3.15. *Assume that $p \geqslant 2$ and $V(x) \geqslant 0$ for a.e $x \in \Omega$. If $p-1 < q < \min\{\frac{sp}{N}p_s^*, p_s^*-1\}$, then there exists $\lambda^* > 0$ such that, for all $0 < \lambda < \lambda^*$, problem* (1.1) *has at least one positive solution $u_\lambda \in C_o^\alpha(\overline{\Omega})$ for some $0 < \alpha < 1$.*

Proof. From Lemma 3.4 we know that, for any $\lambda \in (0, \lambda_3)$, there exists a solution $u_\lambda \in X$. Assume, by a way of contradiction, that there exists a sequence $(\lambda_j)_{j\in\mathbb{N}}$ with $0 < \lambda_j < 1$ such that $\lambda_j \to 0$ as $j \to \infty$ and, for all $j \in \mathbb{N}$, we have

$$|\Omega_j| > 0, \tag{3.42}$$

where $\Omega_j = \{x \in \Omega | u_{\lambda_j}(x) \leqslant 0\}$, for all $j \in \mathbb{N}$, and $|\Omega_j|$ denotes the Lebesgue measure of the set Ω_j.

We set $w_j = \frac{u_{\lambda_j}}{\|u_{\lambda_j}\|_\infty}$. Notice that $w_j(x) \leqslant 0$ for all $x \in \Omega_j$. Thus, by the regularity result in [19, Theorem 1.1], we obtain

$$(-\Delta)_p^s(w_j) = h_j(x, w_j),$$

where $h_j(x,s) := -V(x)\Phi_p(s) + \lambda_j\|u_{\lambda_j}\|_\infty^{1-p} f(\|u_{\lambda_j}\|_\infty s)$. Using that $\lambda_j\|u_{\lambda_j}\|_\infty^{1-p} < 1$ and, by Lemma 3.9, $\lambda_j\|u_{\lambda_j}\|_\infty^{q+1-p} < C$ for j large, and $1 - r(1-p+q) = 0$, we obtain

$$\begin{aligned}|h_j(x,s)| &\leqslant |V(x)||s|^{p-1} + \lambda_j\|u_{\lambda_j}\|_\infty^{1-p} B((\|u_{\lambda_j}\|_\infty|s|)^q + 1) \\ &\leqslant \|V\|_\infty|s|^{p-1} + B\lambda_j\|u_{\lambda_j}\|_\infty^{1-p+q}|s|^q + B\lambda_j\|u_{\lambda_j}\|_\infty^{1-p} \\ &\leqslant \|V\|_\infty|s|^{p-1} + B\lambda_j^{1-r(1-p+q)}|s|^q + B \\ &\leqslant C_1|s|^{p^*-1} + C_2.\end{aligned}$$

Using the result of Theorem 3.8, there exists $\alpha \in (0, s]$ such that

$$\begin{aligned}\Big\|\frac{w_j}{d_\Omega^s}\Big\|_{C^\alpha(\overline{\Omega})} &\leqslant \|h_j(x,w_j)\|_\infty^{1/(p-1)} \\ &\leqslant (C_1\|w_j\|_\infty^{p^*-1} + C_2)^{1/(p-1)} = C_3,\end{aligned} \tag{3.43}$$

where C_3 is a positive constant which does not depend on λ_j.

Next, choose β such that $0 < \beta < \alpha$. By Arzelà-Ascoli Theorem (see [28, Theorem 40 on pg. 169]), up to a subsequence, it follows from (3.43) that

$$\lim_{j\to\infty}\frac{w_j}{d_\Omega^s} = \frac{w}{d_\Omega^s}, \quad \text{in } C^\beta(\overline{\Omega}).$$

The next step consists of using the comparison principle to prove that $w(x) \geqslant 0$. Indeed, let $v_0 \in W_0^{s,p}(\Omega)$ be a solution of

$$(-\Delta)_p^s u + V(x)\Phi_p(u) = 1, \quad \text{in } \Omega;$$

$$u = 0, \quad \text{in } \mathbb{R}^N \backslash \Omega,$$

obtained in Appendix 7.

Let $K_j = \frac{\lambda_j}{\|u_{\lambda_j}\|_\infty^{p-1}} \min_{t \in \mathbb{R}} f(t)$, and note that $K_j < 0$. Let $v_j = -(-K_j)^{1/(p-1)} v_0$. Then v_j solves

$$\begin{gathered}(-\Delta)_p^s u + V(x)\Phi_p(u) = K_j, \quad \text{in } \Omega; \\ u = 0, \quad \text{in } \mathbb{R}^N \backslash \Omega.\end{gathered}$$

Observe that, for all $\varphi \in W_0^{s,p}(\Omega)$ with $\varphi \geqslant 0$, we have the estimate

$$\begin{aligned}&\int_{\mathbb{R}^{2N}} \frac{\Phi_p(w_j(x) - w_j(y))}{|x-y|^{N+sp}} (\varphi(x) - \varphi(y))\, dx\, dy + \int_\Omega V(x)\Phi_p(w_j)\varphi\, dx \\ &= \int_\Omega \lambda_j f(u_{\lambda_j}) \|u_{\lambda_j}\|_\infty^{1-p} \varphi\, dx \\ &\geqslant \int_\Omega K_j \varphi\, dx \\ &= \int_{\mathbb{R}^{2N}} \frac{\Phi_p(v_j(x) - v_j(y))}{|x-y|^{N+sp}} (\varphi(x) - \varphi(y))\, dx\, dy + \int_\Omega V(x)\Phi_p(v_j)\varphi\, dx.\end{aligned} \tag{3.44}$$

The above estimate implies that $(-\Delta)_p^s(w_j) \geqslant (-\Delta)_p^s(v_j)$. By the comparison principle stated in Theorem 3.14, we conclude that $w_j \geqslant v_j$. Since $v_j \to 0$, as $j \to \infty$, we obtain $w(x) \geqslant 0$, for $x \in \Omega$.

Next, let $t := Npr/(N-sp) > 1$. By Lemmas 3.6) and 3.9, we have that

$$C_1 \lambda^{-r} \leqslant \|u_\lambda\|_\infty \leqslant C_2 \lambda^{-r}.$$

Then, we obtain

$$\begin{aligned}\lambda_j |f(u_{\lambda_j}(x))| \|u_{\lambda_j}\|_\infty^{1-p} &\leqslant C\lambda_j (|u_{\lambda_j}(x)|^q + 1) \|u_{\lambda_j}\|_\infty^{1-p} \\ &\leqslant C\lambda_j (\|u_{\lambda_j}\|_\infty^q + 1) \|u_{\lambda_j}\|_\infty^{1-p} \\ &\leqslant C\lambda_j (\lambda_j^{-rq} + 1) \lambda_j^{r(p-1)} \\ &\leqslant C\lambda_j \lambda_j^{-rq} \lambda_j^{r(p-1)} \\ &= C\lambda_j^{1-rq+r(p-1)} = C,\end{aligned}$$

where C is a positive constant and $q < p_s^* - 1$. It follows from the previous estimate that

$$\int_\Omega (\lambda_j f(u_j) \|u_{\lambda_j}\|_\infty^{1-p})^t \, dx \leqslant C_\Omega |\Omega|.$$

Thus, $\{\lambda_j f(u_j) \|u_{\lambda_j}\|_\infty^{1-p}\}_j$ is bounded in $L^t(\Omega)$ and we may assume that it converges weakly in $L^t(\Omega)$. Let $z := \lim_{j \rightharpoonup 0} \lambda_j f(u_j) \|u_{\lambda_j}\|_\infty^{1-p}$ be its weak limit. Since f is bounded from below and $\lim_{j \to \infty} \lambda_j \|u_{\lambda_j}\|_\infty^{1-p} = 0$, it follows that $z \geqslant 0$. We claim that $(-\Delta)_p^s(w) = z$. In fact, by Lemmas 3.5 and 3.6, we can follow the same line of reasoning as in the proof of [22, Theorem 1.1] to obtain

$$\begin{aligned}&\lim_{j \to \infty} \int_{\mathbb{R}^{2N}} \frac{|w_j(x) - w_j(y)|^{p-2}(w_j(x) - w_j(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}}\, dx\, dy \\ &= \int_{\mathbb{R}^{2N}} \frac{|w(x) - w(y)|^{p-2}(w(x) - w(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}}\, dx\, dy.\end{aligned} \tag{3.45}$$

On the other hand, since $w_j \to w$ uniformly in $\overline{\Omega}$ and $w \in L^p(\Omega)$, we also have that

$$\lim_{j\to\infty} \int_\Omega V(x)\Phi_p(w_j)\varphi(x)\,dx = \int_\Omega V(x)\Phi_p(w)\varphi(x)\,dx. \tag{3.46}$$

Notice that $w_j \to w$ in $W_0^{s,p}(\Omega)$, which implies that $w \in W_0^{s,p}(\Omega)$. Consequently, by (3.45), (3.46), and the fact that z is the weak limit of $\{\lambda_j f(u_{\lambda_j}\|u_{\lambda_j}\|_\infty^{1-p}\}$, we have that

$$(-\Delta)_p^s w + V\Phi_p(w) = z.$$

That is, w is a weak supersolution of $(-\Delta)_p^s w + V\Phi_p(w) = 0$. Hence, by Theorems 3.12 and 3.13, we have two alternatives: First, $w = 0$ cannot hold since $w_j \to w$ in $C^\beta(\overline{\Omega})$ and $\|w_j\|_\infty = 1$ for all $j \in \mathbb{N}$. Second, $w > 0$ in Ω and, for all $x_0 \in \partial\Omega$,

$$\liminf_{x\to x_0,\, x\in B} \frac{w(x)}{(d(x))^s} > 0,$$

where $B \subseteq \Omega$ is an open ball in Ω, such that $x_0 \in \partial B$, and $d(x)$ is the distance from x to $\mathbb{R}^N \backslash B$ (see (3.38)). Therefore, there exist j_o sufficiently large such that, for all $j \geqslant j_o$, we have $w_j > 0$. But this contradicts that $w_j(x) = \frac{u_{\lambda_j}(x)}{\|u_{\lambda_j}\|_\infty} \leqslant 0$, for $x \in \Omega_j$.

Hence, $|\Omega_j| = 0$ for all $j \in \mathbb{N}$ and we conclude that problem (1.1) has at least one positive solution $u_\lambda \in C_0^\alpha(\overline{\Omega})$, for some $0 < \alpha < 1$. □

4. Computation of critical groups at infinity

In this section, we will obtain the first multiplicity result for problem (1.1). The first step will consist of computing the critical groups of J_λ at infinity as defined in (2.11). This will require to use the concept of two topological spaces being homotopically equivalent.

To show that two topological spaces A and B are homotopically equivalent, denoted by $A \cong B$, one needs to show that there exist functions $\eta : A \to B$ and $i : B \to A$ such that $\eta \circ i \approx id_B$ and $i \circ \eta \approx id_A$, where id denotes the identity function and the symbol $\approx$ denotes the existence of a homotopy.

In particular, if $B \subset A$ and $i : B \to A$ denotes the inclusion function and $\eta : A \to B$ is a deformation retraction from A onto B, then we have that $\eta \circ i \approx id_B$ and $i \circ \eta = id_A$. Hence, to obtain the critical groups of J_λ at infinity, we will prove the existence of a deformation retract from J_λ^{-M} onto S^∞, for some M to be chosen soon, where S^∞ denotes the unit sphere in X. Finally, the result will follow by using an argument with the long exact sequence of the topological pair (X, J^{-M}) and the fact that S^∞ is contractible in X.

Let $S^\infty = \{u \in X : \|u\| = 1\}$ be the unit sphere in X. Notice that, for $u \in S^\infty$, we have that

$$\lim_{t\to\infty} J_\lambda(tu) = -\infty. \tag{4.1}$$

In fact, substituting (3.2) into (2.5) and applying (H3), we obtain

$$J_\lambda(tu) \leqslant \frac{t^p}{p}\left(1 + \|V\|_{L^\infty}\|u\|_p^p\right) - \lambda A_1 t^{q+1}\|u\|_{q+1}^{q+1} + \lambda A_1 C_1|\Omega|, \tag{4.2}$$

for all $u \in S^\infty$. Then, since $p < q + 1$, the result (4.1) follows by letting $t \to \infty$ in (4.2).

Lemma 4.1. *Assume that* (H1), (H2) *are satisfied. Then, there exists* $\widetilde{M} > 0$ *such that, for all* $M \geqslant \widetilde{M}$*,* J_λ^{-M} *is homotopically equivalent to* S^∞*.*

Proof. We will follow a line of reasoning similar to one in [31, Section 3] to show the existence of a deformation retract from J_λ^{-M} to S^∞.

First, notice that the critical value set $J_\lambda(\mathcal{K})$ is bounded from below. In fact, if $u_0 \in \mathcal{K}$, then setting $\varphi = u_0$ in (2.6), we obtain

$$\|u_0\|^p + \int_\Omega V(x)|u_0|^p \, dx = \lambda \int_\Omega f(u_0)u_0 \, dx. \tag{4.3}$$

Next, we substitute (4.3) into (2.5) and use (H2) to obtain

$$\begin{aligned} J_\lambda(u_0) &= \frac{\lambda}{p} \int_\Omega [f(u_0)u_0 - pF(u_0)] \, dx \\ &\geqslant \frac{\lambda}{p} \int_\Omega [(\theta - p)F(u_0) + K] \, dx \\ &\geqslant \frac{\lambda}{p} |\Omega|((\theta - p) \min F + K) =: -a_0, \end{aligned}$$

for all $u_0 \in \mathcal{K}$, and therefore

$$-a_o \leqslant \inf J_\lambda(\mathcal{K}). \tag{4.4}$$

By (4.1), given $u \in S^\infty$ and $M_1 > 0$, there exists $t_0 = t_0(u) \geqslant 1$ such that

$$J_\lambda(tu) < -M_1, \quad \text{for } t_0 \geqslant 1, u \in S^\infty.$$

We define $\widetilde{M} = \min\{-a_o, -M_1\}$. Choosing $M_2 > \widetilde{M}$ such that, for $tu \in J_\lambda^{-M_2}$, we have

$$J_\lambda(tu) = \frac{t^p}{p}\Big(1 + \int_\Omega V(x)|u|^p \, dx\Big) - \lambda \int_\Omega F(tu) \, dx, \tag{4.5}$$

for $t \geqslant 1$.

Using the chain rule, and taking into account that $f(s)s$ is bounded from below, and $\frac{p}{\theta} < 1$, it follows from (4.5) and (H2) that

$$\begin{aligned} \frac{d}{dt} J_\lambda(tu) &= \frac{1}{t}\Big[pJ(tu) + \lambda \int_\Omega (pF(tu) - f(tu)tu) \, dx\Big] \\ &\leqslant \frac{1}{t}\Big[-pM_2 + \lambda \int_\Omega \Big(\frac{p}{\theta} f(tu)(tu) - f(tu)tu - \frac{Kp}{\theta}\Big) \, dx\Big] \\ &\leqslant \frac{1}{t}\Big[-pM_2 + \lambda\big(\frac{p}{\theta} - 1\big) \int_\Omega f(tu)(tu)) \, dx - \lambda \frac{Kp}{\theta}|\Omega|\Big] \\ &\leqslant \frac{1}{t}[-M_o + \hat{K}\lambda], \end{aligned} \tag{4.6}$$

where M_o and $\hat{K}$ are positive constants, for all $tu \in J_\lambda^{-M_2}$.

Choosing λ small enough in (4.6), we obtain

$$\frac{d}{dt} J_\lambda(tu) < 0, \tag{4.7}$$

for $tu \in J_\lambda^{-M_2}$, and $t \geqslant 1$.

Let us take $M \geqslant \widetilde{M}$. Then, combining (4.1) and (4.7), we can invoke the intermediate value theorem to conclude that there exists $T(u) \geqslant 1$ such that

$$J_\lambda(T(u)u) = -M, \quad \text{for } u \in S^\infty.$$

It follows from the implicit function theorem [10, Theorem 15.1] that $T \in C(S^\infty, \mathbb{R})$.

Finally, let $B^\infty = \{u \in X : \|u\| \leqslant 1\}$ be the unit ball in X. We define $\eta : [0,1] \times (X \backslash B^\infty) \to X \backslash B^\infty$ by

$$\eta(t,u) = (1-t)u + tT(u)u,$$

for $t \in [0,1]$ and $u \in X \backslash B^\infty$. Observe that $\eta(0,u) = u$ and $\eta(1,u) \in J_\lambda^{-M}$. Thus, η is a deformation retract from $X \backslash B^\infty$ onto J_λ^{-M}. Since $X \backslash B^\infty \cong S^\infty$, we conclude that

$$J_\lambda^{-M} \cong X \backslash B^\infty \cong S^\infty;$$

that is, J_λ^{-M} is homotopically equivalent to S^∞. □

Since J_λ^{-M} and S^∞ are homotopically equivalent, as shown in the previous lemma, we conclude that the homology groups $H_k(J_\lambda^{-M})$ and $H_k(S^\infty)$ are isomorphic, for all $k \in \mathbb{Z}$ (see [14, Corollary 2.11]). Since S^∞ is also contractible in X (see Benyamini-Sternfeld [5]), we obtain that the singular homology groups $H_k(J_\lambda^{-M})$ have the homology type of a point for all $k \in \mathbb{Z}$; namely,

$$H_k(J_\lambda^{-M}) \cong \delta_{k,0}\mathbb{F}, \quad \text{for all } k \in \mathbb{Z}.$$

Using an argument similar to that in [27, Section 3] with the long exact sequence of reduced homology groups of the topological pair (X, J_λ^{-M}) and the fact that J_λ satisfies the Palais-Smale condition shown in Lemma 3.3, we conclude that the critical groups of J_λ at infinity are given by

$$C_k(J_\lambda, \infty) = H_k(X, J_\lambda^{-M}) \cong \delta_{k,0}\mathbb{F}, \quad \text{for all } k \in \mathbb{Z}. \tag{4.8}$$

5. Computation of critical groups at the origin

In this section, we study the questions of existence and multiplicity for the case $f(0) = 0$. In this case, the function $u \equiv 0$ is also a critical point of J_λ and we need to obtain some information about the critical groups of J_λ at the origin. To obtain another solution, we need to make an additional assumption about the behavior of F at the origin. This is the content of the next lemma.

Lemma 5.1. *Assume that the nonlinearity f satisfies* (H1) *and its primitive F satisfies*

$$\limsup_{s \to 0} \frac{F(s)}{|s|^p} = 0. \tag{5.1}$$

Then, the origin is a local minimizer of the functional J_λ and its critical groups are

$$C_k(J_\lambda, 0) \cong \delta_{k,0}\mathbb{F}, \quad \textit{for all } k \in \mathbb{Z}. \tag{5.2}$$

Proof. By condition (5.1), for each given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|s| < \delta \Rightarrow F(s) < \varepsilon |s|^p. \tag{5.3}$$

It follows from (3.1) that there exists a constant $K_1 = K_1(\delta)$ such that

$$|F(s)| \leqslant K_1 |s|^{q+1}, \quad \text{for all } |s| \geqslant \delta. \tag{5.4}$$

In fact, assuming $s \geqslant \delta$ and using (H1) we obtain

$$|F(s)| \leqslant \int_0^s |f(\xi)|\, d\xi \leqslant Bs + \frac{B}{q+1} s^{q+1};$$

so that

$$|F(s)| \leqslant B\big[\delta\big(\frac{s}{\delta}\big) + \frac{\delta^{q+1}}{q+1}\big(\frac{s}{\delta}\big)^{q+1}\big]. \tag{5.5}$$

Since we are assuming that $s \geqslant \delta$; so that $\frac{s}{\delta} \geqslant 1$, it follows from (5.5) that

$$|F(s)| \leqslant B\big[\delta\left(\frac{s}{\delta}\right)^{q+1} + \frac{\delta^{q+1}}{q+1}\left(\frac{s}{\delta}\right)^{q+1}\big], \quad \text{for } s \geqslant \delta, \tag{5.6}$$

from which we obtain that

$$|F(s)| \leqslant \frac{B}{\delta^{q+1}}\left[\delta + \delta^{q+1}\right] s^{q+1}, \quad \text{for } s \geqslant \delta, \tag{5.7}$$

where we have used that $q + 1 > p > 1$, in view of hypothesis (H1). Setting $K_1 = K_1(\delta) = \frac{B}{\delta^{q+1}}[\delta + \delta^{q+1}]$, we see that (5.4) follows from (5.7). The case for $s \leqslant -\delta$ is analogous. Hence, estimate (5.4) is valid for all $|s| \geqslant \delta$.

Next, combine the estimates (5.3) and (5.4) to obtain

$$F(s) \leqslant \varepsilon|s|^p + K_1|s|^{q+1}, \quad \text{for } s \in \mathbb{R}. \tag{5.8}$$

Then, it follows from (5.8) that

$$\int_\Omega F(u)\,dx \leqslant \varepsilon \int_\Omega |u|^p\,dx + K_1 \int_\Omega |u|^{q+1}\,dx;$$

so that, using the Sobolev inequality [12, Theorem 6.7], it follows from the previous estimate that

$$\int_\Omega F(u)\,dx \leqslant C_3\left(\varepsilon + K_1\|u\|^{q+1-p}\right)\|u\|^p, \tag{5.9}$$

for some positive constant C_3.

Setting $\rho = (\frac{\varepsilon}{2K_1})^{1/(q+1-p)}$, we obtain from (5.9) that

$$\|u\| < \rho \Rightarrow \int_\Omega F(u)\,dx \leqslant C_3\varepsilon\|u\|^p. \tag{5.10}$$

It follows from the definition of J_λ in (2.5), (5.10), and (H3) that

$$J_\lambda(u) \geqslant \big(\frac{1}{p} - C_3\lambda\varepsilon\big)\|u\|^p - \frac{c_V}{p}\|u\|_p^p. \tag{5.11}$$

On the other hand, it follows from [21, Section 3] that the first eigenvalue λ_1 of $(-\Delta)_p^s$ is characterized by the minimization of the Rayleigh quotient,

$$\lambda_1 = \inf_{u\in X\setminus\{0\}} \frac{\|u\|^p}{\|u\|_p^p}, \tag{5.12}$$

with $\lambda_1 \in (0,\infty)$; see [21, Theorem 5].

Hence, applying (5.12) in (5.11), we obtain

$$J_\lambda(u) \geqslant \big[\frac{1}{p}\big(1 - \frac{c_V}{\lambda_1}\big) - C_3\lambda\varepsilon\big]\|u\|^p. \tag{5.13}$$

By (H3), $c_V < \lambda_1$, thus we can choose $\varepsilon > 0$ such that

$$\varepsilon < \frac{1}{2pC_3\lambda}\big(1 - \frac{c_V}{\lambda_1}\big). \tag{5.14}$$

Then, by (5.14), we obtain from (5.13) that

$$J_\lambda(u) \geqslant \frac{1}{2pC_3\lambda}\big(1 - \frac{c_V}{\lambda_1}\big)\|u\|^p > J(0), \quad \text{for } 0 < \|u\| < \rho,$$

where $\rho > 0$ is sufficiently small. Consequently, $u = 0$ is a local minimum of J_λ in $B_\rho(0)$. It follows from ([9, Example 1, page 33] that

$$C_k(J_\lambda, 0) \cong \delta_{k,0}\mathbb{F}, \quad \text{for } k \in \mathbb{Z}.$$

□

6. Proofs of main results

Proof of Theorem 1.1. Assume, by a way of contradiction, that $\mathcal{K} = \{u_\lambda\}$ where u_λ is the mountain-pass type solution found in Theorem 3.4. Then, it follows from Proposition 2.7 that

$$C_1(J_\lambda, u_\lambda) \ncong 0. \tag{6.1}$$

Since we are assuming that $\mathcal{K} = \{u_\lambda\}$, we can invoke Proposition 2.10 to obtain

$$C_k(J_\lambda, \infty) \cong C_k(J_\lambda, u_\lambda), \quad \text{for all } k \in \mathbb{Z}. \tag{6.2}$$

In particular, if $k = 1$ in (6.2), we obtain from (4.8) and (6.1) that

$$0 \cong C_1(J_\lambda, \infty) \cong C_1(J_\lambda, u_\lambda) \ncong 0,$$

which is a contradiction. Therefore, J_λ must have at least two critical points and this completes the proof. □

Proof of Theorem 1.2. By Theorem 1.1, we obtain the existence of two solutions for problem (1.1). Furthermore, one of them is of mountain pass type. Next, assume that $p \geqslant 2$ and $V(x) \geqslant 0$ for a.e. $x \in \Omega$. For the case of $f(0) > 0$, it follows from the Comparison Theorem 3.15 that both solutions are positive. For the case of $f(0) < 0$, Theorem 3.15 leads us to the positivity of the mountain-pass type solution. □

Proof of Theorem 1.4. Assume, by a way of contradiction, that $\mathcal{K} = \{0, u_\lambda\}$, where u_λ is the mountain-pass type solution found in Theorem 3.4. Then, it follows from [9, Theorem 4.2, page 35] that

$$H_k(X, J_\lambda^{-M}) \cong C_k(J_\lambda, 0) \oplus C_k(J_\lambda, u_\lambda), \quad \text{for all } k \in \mathbb{Z}. \tag{6.3}$$

In particular, setting $k = 1$ in (6.3) and using (2.14), (4.8), and (5.2), we obtain

$$0 \cong C_1(J_\lambda, 0) \oplus C_1(J_\lambda, u_\lambda) \cong 0 \oplus C_1(J_\lambda, u_\lambda) \ncong 0,$$

which is a contradiction. Therefore, the critical set $\mathcal{K}$ must have at least three critical points. This completes the proof. □

7. Appendix

In this section, we prove that the problem

$$\begin{gathered} (-\Delta)_p^s u(x) + V(x)\Phi_p(u(x)) = 1, \quad \text{for } x \in \Omega; \\ u = 0, \quad \text{in } \mathbb{R}^N \backslash \Omega, \end{gathered} \tag{7.1}$$

has a positive weak solution. We will show that the associated energy functional with problem (7.1) is coercive and weakly lower semi-continuous. Then, the existence result follows by a result found in Evans [13, Theorem 2, Chapter 8].

In fact, the associated functional with problem (7.1) is

$$E(u) := \frac{1}{p}\|u\|_{s,p}^p + \frac{1}{p}\int_\Omega V(x)|u|^p\,dx - \int_\Omega u\,dx, \quad u \in X. \tag{7.2}$$

To prove the coercivity of E, let $(u_n)_n$ be a sequence in X such that $\|u_n\|_{s,p} \to \infty$ as $n \to \infty$. From (2.8) we have that $\|u_n\|_1 \leqslant C_1\|u_n\|_{s,p}$, for all n. Moreover,

$\|u_n\|_p^p \leqslant \frac{1}{\lambda_1}\|u_n\|_{s,p}^p$, for all n. Therefore, applying these estimates and (H3) to (7.2) we obtain

$$\begin{aligned} E(u_n) &\geqslant \frac{1}{p}\|u_n\|_{s,p}^p - \frac{c_V}{p}\|u_n\|_p^p - C_1\|u_n\|_{s,p} \\ &\geqslant \frac{1}{p}\big(1-\frac{c_V}{\lambda_1}\big)\|u_n\|_{s,p}^p - C_1\|u_n\|_{s,p}, \end{aligned} \tag{7.3}$$

for all $n \in \mathbb{N}$.

Since $1-\frac{c_V}{\lambda_1} > 0$ and $p > 1$, we obtain from (7.3) that $E(u_n) \to \infty$ as $n \to \infty$. Now, E is continuous because of its differentiability. Moreover, a simple computation shows that the functional E is convex. Therefore, E is weakly lower semicontinuous (see for example [3, Theorem 1.5.3]). This proves that problem (7.1) has at least one solution $u \in X$, which is nontrivial.

Finally, notice that u is a weak supersolution of the problem

$$(-\Delta)_p^s u(x) + V(x)\Phi_p(u(x)) = 0, \quad \text{for } x \in \Omega,$$

with$u = 0$, in $\mathbb{R}^N \backslash \Omega$. Thus, by Theorem (3.12), it follows that $u > 0$.

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Emer Lopera
Universidad Nacional de Colombia, Manizales, Colombia
Email address: edloperar@unal.edu.co

Leandro Recôva
California State Polytechnic University, 3801 West Temple Avenue, Pomona, CA 91768, USA
Email address: llrecova@cpp.edu

Adolfo Rumbos
Pomona College, Mathematics and Statistics Department. 610 N. College Avenue, Claremont, CA 91711, USA
Email address: arumbos@pomona.edu