Electronic Journal of Differential Equations, Vol. 2024 (2024), No. 73, pp. 1–12.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2024.73

EXISTENCE OF WEAK SOLUTIONS FOR NONLOCAL DIRICHLET PROBLEMS VIA YOUNG MEASURE THEORY

MOUAD ALLALOU, MOHAMED EL OUAARABI, ABDERRAHMANE RAJI

Abstract. This article investigates the existence of weak solutions for a class of nonlocal problems with Dirichlet boundary conditions. The proof of the existence result relies on Galerkin's approximation and Young's measure theory.

1. Introduction

Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\mathcal{D}$, and $1 < p < \infty$. We prove the existence of weak solutions for the nonlocal problem with Dirichlet boundary conditions,

$$\begin{gathered} -g\Big(\int_{\mathcal{D}} \mathcal{E}(\upsilon)\,\mathrm{d}z\Big)\operatorname{div}[a(z,\nabla\upsilon) + |\nabla\upsilon|^{p-2}\nabla\upsilon] + |\upsilon|^{p-2}\upsilon = f(z,\upsilon) \quad \text{in } \mathcal{D}, \\ \upsilon = 0 \quad \text{on } \partial\mathcal{D}, \end{gathered} \tag{1.1}$$

where

$$\mathcal{E}(\upsilon) = \int_{\mathcal{D}} \big(\mathcal{A}(z,\nabla\upsilon) + \frac{1}{p}|\nabla\upsilon|^p\big)\,\mathrm{d}z.$$

The functions $f : \mathcal{D}\times\mathbb{R} \to \mathbb{R}$, g, $\mathcal{A} : \mathcal{D}\times\mathbb{R}^m \to \mathbb{R}$, and $a : \mathcal{D}\times\mathbb{R}^m \to \mathbb{R}^m$ are subject to conditions specified below.

The study of nonlinear boundary value problems has garnered significant attention over the past few decades, driven by advancements in fields such as elastic mechanics, electrorheological fluids, and image restoration; see [1, 7, 9, 17, 20, 23].

Transmission problems appear in various applications in physics and biology (see [4, 8, 16]). Recently, in [15], the authors investigated the existence of ground-state solutions for a class of Kirchhoff-type transmission problems.

This work aims to explore the existence of weak solutions to problem (1.1) by employing the principles of Young measures theory. Notably, our problem cannot be addressed with a variational framework because of the specific functions g and f. These functions introduce significant technical challenges, necessitating the use of alternative tools, such as Young measures, which facilitate the identification of weak limits. To the best of our knowledge, this is the first study to approach problem (1.1) using this theoretical framework. For an exploration of closely related topics, readers are encouraged to consult references [10, 12] and additional sources

---

2020 *Mathematics Subject Classification.* 35J60, 35J25, 35D30.
Key words and phrases. Weak solution; Dirichlet boundary conditions; existence;
Young measures; Galerkin's approximation; Sobolev space.

Submitted October 5, 2024. Published November 17, 2024.

cited therein. For a thorough discussion on the steady-state case employing Young measures theory, we refer to [2, 3, 13, 22].

A weak solution for (1.1) is defined as a function $\upsilon \in W_0^{1,p}(\mathcal{D})$ satisfying the following equation for all $\Phi \in W_0^{1,p}(\mathcal{D})$,

$$
\begin{aligned}
&g\Big(\int_{\mathcal{D}} \mathcal{F}(\upsilon)\,\mathrm{d}z\Big)\int_{\mathcal{D}} \big(a(z,\nabla\upsilon)+|\nabla\upsilon|^{p-2}\nabla\upsilon\big)\cdot\nabla\Phi\,\mathrm{d}z+\int_{\mathcal{D}}|\upsilon|^{p-2}\upsilon.\Phi\,\mathrm{d}z\\
&=\int_{\mathcal{D}} f(z,\upsilon)\Phi\,\mathrm{d}z.
\end{aligned}
$$

In this article we use the following assumptions:

(A1) $a:\mathcal{D}\times\mathbb{R}^m\to\mathbb{R}^m$ and $f:\mathcal{D}\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions which implies their measurability with respect to $z\in\mathcal{D}$ and continuity with respect to the other variables. Additionally, the mapping $\xi\mapsto a(z,\xi)$ is both a C^1 function and monotonic, i.e.,

$$(a(z,\xi)-a(z,\xi'))\cdot(\xi-\xi')\geq 0 \quad \forall\xi,\xi'\in\mathbb{R}^m.$$

(A2) We can find elements $\alpha_1,\alpha_2\in L^{p'}(\mathcal{D})$, along with $\alpha_3\in L^1(\mathcal{D})$ and positive constants $c_0,c_1>0$, such that

$$
\begin{gathered}
|a(z,\xi)|\leq c_0(\alpha_1(z)+|\xi|^{p-1}),\\
|f(z,w)|\leq\alpha_2(z)+|w|^q,\\
p\mathcal{A}(z,\xi)\geq a(z,\xi)\cdot\xi\geq c_1|\xi|^p-\alpha_3(z)
\end{gathered}
$$

for all $w\in\mathbb{R}$ and $0\leq q<p-1$.

(A3) $\mathcal{A}:\mathcal{D}\times\mathbb{R}^m\to\mathbb{R}$ is a Carathéodory function within the context of (A1). Furthermore, the mapping $\xi\mapsto\mathcal{A}(z,\xi)$ is both convex and C^1-function and it fulfills the relation $a(z,\xi)=\nabla\xi\mathcal{A}(z,\xi)=(\partial\mathcal{A}/\partial\xi)(z,\xi)$.

(A4) $g:W^{1,p}(\mathcal{D})\to(0,+\infty)$ are continuous and bounded on any bounded subset of $W^{1,p}(\mathcal{D})$ such that there are constants $g_0,g_1>0$ satisfying

$$g_0\leq g(s)\leq g_1.$$

Our main result in this article reads as follows.

Theorem 1.1. *Assume that* (A1)–(A4) *hold. Then, problem* (1.1) *has a weak solution in* $W_0^{1,p}(\mathcal{D})$.

The structure of this paper is as follows: Section 2 offers a concise overview of essential aspects of Young measures. In Section 3, we focus on developing the approximate solutions and establishing preliminary estimates. The final section addresses various convergence outcomes and outlines the demonstration of the primary theorem.

2. Preliminaries

2.1. **Fundamentals of Young measures.** In this section, we provide a succinct summary of the fundamental concepts behind generalized Young measures and revisit relevant findings that will be employed in subsequent discussions. Our approach is influenced by the works of [18, 14], with additional insights available in [19] for a more thorough introduction.

We denote by $\mathcal{C}_0(\mathbb{R}^m)$ the closure of the space comprising continuous functions on $\mathbb{R}^m$ with compact support concerning the $|\cdot|_\infty$-norm. Its dual space can be identified as $\mathcal{M}(\mathbb{R}^m)$ the space encompassing signed Radon measures with finite mass. The duality pairing for $\sigma : \mathcal{D} \to \mathcal{M}(\mathbb{R}^m)$ is defined as

$$\langle \sigma, \psi \rangle = \int_{\mathbb{R}^m} \psi(\eta)\, \mathrm{d}\sigma(\eta).$$

Lemma 2.1 ([19])**.** *Assume that the sequence $\{y_\mu\}_{\mu \geq 1}$ is bounded in $L^\infty(\mathcal{D}; \mathbb{R}^m)$. Then there exist a subsequence still denoted $\{y_\mu\}_\mu$ and a Borel probability measure σ_z on $\mathbb{R}^m$ for a.e. $z \in \mathcal{D}$, such that for almost each $h \in \mathcal{C}(\mathbb{R}^m)$ we have*

$$\psi(y_\mu) \to^* \bar{\psi} \quad \text{weakly in } L^\infty(\mathcal{D}),$$

where

$$\bar{\psi}(z) = \int_{\mathbb{R}^m} \psi(\eta) d\sigma_z(\eta).$$

Definition 2.2. We call $\sigma = \{\sigma_z\}_{z \in \mathcal{D}}$ the family of Young measures associated with the subsequence $\{y_\mu\}_\mu$. It is shown in [6], that if for all $R > 0$

$$\limsup_{\mu \to \infty} |\{z \in \mathcal{D} \cap B_R(0) : |y_\mu(z)| \geq L\}| = 0,$$

then for any measurable $\mathcal{D}' \subset \mathcal{D}$,

$$\psi(z, y_\mu) \to \langle \sigma_z, \psi(z, .)\rangle = \int_{\mathbb{R}^m} \psi(z, \eta) d\sigma_z(\eta) \quad \text{weakly in } L^1(\mathcal{D})$$

for any Carathéodory function $\psi : \mathcal{D}' \times \mathbb{R}^m \to \mathbb{R}$ such that $\psi(z, y_\mu)$ is equi-integrable.

If we consider $y_\mu = \nabla w_\mu$, where $w_\mu : \mathcal{D} \to \mathbb{R}$, the above properties remain true, and the following lemma can be proved in a similar way as in [5, Lemma 4.1].

Lemma 2.3. *Let (∇w_μ) be a bounded sequence in $L^p(\mathcal{D}; \mathbb{R}^m)$. Then the Young measure σ_z generated by ∇w_μ in $L^p(\mathcal{D}; \mathbb{R}^m)$ satisfies:*

1. *$|\sigma_z|_{\mathcal{M}(\mathbb{R}^m)} = 1$ for a.e. $z \in \mathcal{D}$, i.e., σ_z is a probability measure.*
2. *The weak L^1-limit of ∇w_μ is given by $\langle \sigma_z, id\rangle = \int_{\mathbb{R}^m} \eta \cdot d\sigma_z(\eta)$.*
3. *σ_z satisfies $\langle \sigma_z, id\rangle = \nabla \upsilon(z)$ for a.e. $z \in \mathcal{D}$.*

We will need the following Fatou-type inequality.

Lemma 2.4. *Let $\psi : \mathcal{D} \times \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function and $w_\mu : \mathcal{D} \to \mathbb{R}$ a sequence of measurable functions such that ∇w_μ generates the Young measure σ_z with $|\sigma_z|_{g(\mathbb{R}^m)} = 1$ for almost every $z \in \mathcal{D}$. Then*

$$\liminf_{\mu \to \infty} \int_{\mathcal{D}} \psi(z, \nabla w_\mu)\, \mathrm{d}z \geq \int_{\mathcal{D}} \int_{\mathbb{R}^m} \psi(z, \eta)\, \mathrm{d}\sigma_z(\eta)\, \mathrm{d}z$$

provided that the negative part of $\psi(z, \nabla w_\mu)$ is equi-integrable.

3. Proof of Theorem 1.1

Let us consider the functional $\mathcal{L}(\upsilon) : W_0^{1,p}(\mathcal{D}) \to \mathbb{R}$ given by

$$\begin{aligned} \Phi \mapsto g\Big(\int_{\mathcal{D}} \big(\mathcal{A}(z, \nabla \upsilon) + \frac{1}{p}|\nabla \upsilon|^p\, \mathrm{d}z\big)\Big)\Big[\int_{\mathcal{D}} a(z, \nabla \upsilon) \cdot \nabla \Phi\, \mathrm{d}z \\ + \int_{\mathcal{D}} |\nabla \upsilon|^{p-2} \nabla \upsilon \cdot \nabla \Phi\, \mathrm{d}z\Big] + \int_{\mathcal{D}} |\upsilon|^{p-2}\upsilon.\Phi\, \mathrm{d}z - \int_{\mathcal{D}} f(z, \upsilon)\Phi\, \mathrm{d}z, \end{aligned}$$

for arbitrary $\upsilon \in W_0^{1,p}(\mathcal{D})$ and $\Phi \in W_0^{1,p}(\mathcal{D})$.

Lemma 3.1. *The functional $\mathcal{L}(\upsilon)$ is well defined, linear and bounded.*

Proof. Firstly, utilizing Hölder inequality and conditions (A2)–(A4), we establish

$$\begin{aligned}
|\Lambda_1| &:= \Big|g\Big(\int_{\mathcal{D}}(A(z,\nabla\upsilon)+\frac{1}{p}|\nabla\upsilon|^p)\,\mathrm{d}z\Big)\\
&\quad\times\Big[\int_{\mathcal{D}}a(z,\nabla\upsilon)\nabla\Phi\,\mathrm{d}z+\int_{\mathcal{D}}|\nabla\upsilon|^{p-2}\nabla\upsilon\nabla\Phi\,\mathrm{d}z\Big]\Big|\\
&\le g_1\Big(\int_{\mathcal{D}}|a(z,\nabla\upsilon)|\cdot|\nabla\Phi|\,\mathrm{d}z+\int_{\mathcal{D}}|\nabla u|^{p-1}\cdot|\nabla\Phi|\,\mathrm{d}z\Big)\\
&\le g_1\Big(\int_{\mathcal{D}}c_0(\alpha_1(z)+|\nabla\upsilon|^{p-1})|\nabla\Phi|\,\mathrm{d}z\Big)+\|\nabla\upsilon\|_p^{p-1}\,\|\nabla\Phi\|_p\\
&\le C(\|\alpha_1\|_{p'}+|\nabla\upsilon\|_p^{p-1})\|\nabla\Phi\|_p+\|\nabla\upsilon\|_p^{p-1}\|\nabla\Phi\|_p\\
&\le C\|\nabla\Phi\|_p.
\end{aligned}$$

Conversely, we can also infer, based on the growth condition of f in (A2) and the Hölder inequality, that

$$\begin{aligned}
|\Lambda_2| &:= \Big|\int_{\mathcal{D}}f(z,\upsilon)\Phi\,\mathrm{d}z\Big|\\
&\le\int_{\mathcal{D}}|f(z,\upsilon)\Phi|\,\mathrm{d}z\\
&\le(\|\alpha_2\|_{p'}+\|\upsilon\|_p^{p-1})\|\Phi\|_p\\
&\le\lambda(\|\alpha_2\|_{p'}+\lambda^{p-1}\|\nabla\upsilon\|_p^{p-1})\|\nabla\Phi\|_p,
\end{aligned}$$

with λ denoting the constant in Poincare's inequality, there exists a positive constant λ such that

$$\|\Phi\|_p\le\lambda\|\nabla\Phi\|_p\quad\forall\Phi\in W_0^{1,p}(\mathcal{D}). \tag{3.1}$$

On the other hand,

$$\begin{aligned}
|\Lambda_3| &:= \Big|\int_{\mathcal{D}}|\upsilon|^{p-2}\upsilon\Phi\,\mathrm{d}z\Big|\\
&\le\int_{\mathcal{D}}|\upsilon|^{p-1}|\Phi|\,\mathrm{d}z\\
&\le\Big(\frac{1}{p'}+\frac{1}{p}\Big)\|\upsilon\|_p^{p-1}\|\nabla\Phi\|_p.
\end{aligned}$$

As the estimates of Λ_i for $i=1,2,3$ are finite, $\mathcal{L}(\upsilon)$ is well defined. Moreover, $\mathcal{L}(\upsilon)$ is linear and for all $\Phi\in W_0^{1,p}(\mathcal{D})$, the inequality

$$|\langle\mathcal{L}(\upsilon),\Phi\rangle|\le|\Lambda_1|+|\Lambda_2|+|\Lambda_3|\le C\|\nabla\Phi\|_p$$

holds, indicating that $\mathcal{L}(\upsilon)$ is bounded. □

By Lemma 3.1, we can define the operator $\mathcal{L}: W_0^{1,p}(\mathcal{D})\to W^{-1,p'}(\mathcal{D})$, that satisfies the following result.

Proposition 3.2. *The restriction of $\mathcal{L}$ to a finite dimensional linear subspace $\mathfrak{O}$ of $W_0^{1,p}(\mathcal{D})$ is continuous.*

Proof. Let $\mathfrak{O}$ be a finite linear subspace of $W_0^{1,p}(\mathcal{D})$. Suppose (υ_μ) is a sequence in $\mathfrak{O}$ that converges to υ in $\mathfrak{O}$.

Firstly, $\upsilon_\mu\to\upsilon$ and $\nabla\upsilon_\mu\to\nabla\upsilon$ almost everywhere.

Secondly,

$$\int_{\mathcal{D}} |\upsilon_\mu - \upsilon|^p \,\mathrm{d}z \to 0 \quad \text{and} \quad \int_{\mathcal{D}} |\nabla\upsilon_\mu - \nabla\upsilon|^p \,\mathrm{d}z \to 0,$$

since $\upsilon_\mu \to \upsilon$ strongly in $\mathfrak{D}$. Hence, there exist $Q_1, Q_2 \in L^1(\mathcal{D})$ such that $|\upsilon_\mu - \upsilon|^p \leq Q_1$ and $|\nabla\upsilon_\mu - \nabla\upsilon|^p \leq Q_2$. We know that for $\gamma > 1$

$$|t_1 + t_2|^\gamma \leq 2^{\gamma-1}(|t_1|^\gamma + |t_2|^\gamma).$$

Then

$$|\upsilon_\mu|^p = |\upsilon_\mu - \upsilon + \upsilon|^p \leq 2^{p-1}(|\upsilon_\mu - \upsilon|^p + \|\upsilon\|^p) \leq 2^{p-1}(Q_1 + \|\upsilon\|^p).$$

Like in the demonstration of $|\upsilon_\mu|^p$, it follows that $|\upsilon_\mu|_p$ and $|\nabla\upsilon_\mu|_p$ are bounded by a constant C. Thus, the continuity condition in (A1), (A3) and (A4) permits to deduce that

$$\begin{aligned} &g\Big(\int_\Omega (\mathcal{A}(z,\nabla\upsilon_k) + \frac{1}{p}|\nabla\upsilon_k|^p)dz\Big)\Big(\int_\Omega a(x,\nabla\upsilon_k)\nabla\Phi(z)\,\mathrm{d}z \\ &- \int_\Omega |\nabla\upsilon_k|^{p-2}\nabla\upsilon_k\nabla\Phi(z)\,\mathrm{d}z\Big) + \int_\Omega |\upsilon_k|^{p-2}\upsilon_k.\Phi\,\mathrm{d}z \end{aligned}$$

converges to

$$\begin{aligned} &g\Big(\int_\Omega (\mathcal{A}(z,\nabla\upsilon) + \frac{1}{p}|\nabla\upsilon|^p)dz\Big)\Big(\int_\Omega a(x,\nabla\upsilon)\nabla\Phi(z)\,\mathrm{d}z \\ &- \int_\Omega |\nabla\upsilon|^{p-2}\nabla\upsilon\nabla\Phi(z)\,\mathrm{d}z\Big) + \int_\Omega |\upsilon|^{p-2}\upsilon.\Phi\,\mathrm{d}z \end{aligned}$$

and

$$f(z,\upsilon_\mu)\Phi(z) \to f(z,\upsilon)\Phi(z)$$

almost everywhere as $k \to \infty$. Indeed, if $\mathcal{D}'$ is a measurable subset of $\mathcal{D}$, and $\Phi \in W_0^{1,p}(\mathcal{D})$, then

$$\begin{aligned} &\int_{\Omega'} |a(z,\nabla\upsilon_k)\cdot\nabla\Phi - |\nabla\upsilon_k|^{p-2}\nabla\upsilon_k\cdot\nabla\Phi|\,\mathrm{d}z \\ &\leq \int_{\Omega'} c_0(\alpha_1(z) + |\nabla\upsilon_k|^{p-1})|\nabla\Phi|\,\mathrm{d}z + \int_{\Omega'} |\nabla\upsilon_k|^{p-1}\cdot|\nabla\Phi|\,\mathrm{d}z \\ &\leq \Big(c_0|\alpha_1|_{p'} + (c_0+1)\underbrace{\|\nabla\upsilon_k\|_p^{p-1}}_{\leq C}\Big)\Big(\int_{\Omega'} |\nabla\Phi|^p\,\mathrm{d}z\Big)^{1/p} \end{aligned}$$

and (without loss of generality, we can assume $q = p - 1$)

$$\begin{aligned} \int_{\mathcal{D}'} |f(z,\upsilon_\mu)\Phi(z)|\,\mathrm{d}z &\leq \int_{\mathcal{D}'} (\alpha_2(z) + |\upsilon_\mu|^{p-1})|\Phi|\,\mathrm{d}z \\ &\leq \lambda(|\alpha_2|_{p'} + \underbrace{\|\upsilon_\mu\|_p^{p-1}}_{\leq C})\Big(\int_{\mathcal{D}'} |\nabla\Phi|^p\,\mathrm{d}z\Big)^{1/p}, \end{aligned}$$

Using Hölder's and Poincaré inequalities, along with (3.1). Moreover, we have

$$g(\int_{\mathcal{D}'} \Big(\mathcal{A}(z,\nabla\upsilon_\mu) + \frac{1}{p}|\nabla\upsilon_\mu|^p\Big)\,\mathrm{d}z) \leq g_1 < \infty,$$

by (A4) and the boundedness of $|\nabla\upsilon_\mu|_p$. By utilizing the Vitali Theorem, we can establish the continuity of $\mathcal{L}$. □

Remark 3.3. In this section, we have used only the condition $q \le p-1$. Thus Lemma 3.1 and Proposition 3.2 are still valid as $q = p-1$.

Now, the problem (1.1) is equivalent to find a solution $\upsilon \in W_0^{1,p}(\mathcal{D})$ such that

$$\langle \mathcal{L}(\upsilon), \Phi\rangle = 0 \quad \text{for all } \Phi \in W_0^{1,p}(\mathcal{D}).$$

To find such a solution we apply a Galerkin scheme. Since $W_0^{1,p}(\mathcal{D})$ is separable there exists a sequence $(\mathfrak{O}_\mu)$ of finite dimensional subspaces such that $\cup_{\mu\ge1}\mathfrak{O}_\mu$ is dense in $W_0^{1,p}(\mathcal{D})$. Let $\{x_1,\dots,x_r\}$ be a basis of $\mathfrak{O}_\mu$ where $\dim \mathfrak{O}_\mu = r$. Next, Let us define

$$\begin{gathered}\mathcal{G} : \mathbb{R}^r \to \mathbb{R}^r,\\ (d_i)_{i=1,\dots,r} \to (\langle \mathcal{L}(d_i x_i), x_j\rangle)_{j=1,\dots,r}.\end{gathered}$$

Proposition 3.4. *$\mathcal{L}$ is continuous and $\mathcal{G}(d)\cdot d \to \infty$ as $|d|_{\mathbb{R}^r} \to \infty$.*

Proof. $\mathcal{G}$ is trivially continuous, by the continuity of $\mathcal{L}$ restricted to $\mathfrak{O}_\mu$ (see Proposition 3.2 if necessary). Consider $d \in \mathbb{R}^r$ and $\upsilon = d_i x_i \in \mathfrak{O}_\mu$ (with conventional summation). The condition $|d|_{\mathbb{R}^r} \to \infty$ is equivalent to $\|\upsilon\|_{1,p} \to \infty$, and we have

$$\mathcal{G}(d)\cdot d = \langle \mathcal{L}(\upsilon), \upsilon\rangle.$$

Note that

$$\begin{aligned}\Lambda_4 &:= g\Big(\int_{\mathcal{D}} (A(z,\nabla\upsilon) + \frac{1}{p}|\nabla\upsilon|^p)\,\mathrm{d}z\Big)\Big[\int_{\mathcal{D}} a(x,\nabla\upsilon)\cdot\nabla\upsilon\,\mathrm{d}z + \int_{\mathcal{D}} |\nabla\upsilon|^p\,\mathrm{d}z\Big]\\ &\ge g_0\Big[\int_{\mathcal{D}} a(x,\nabla\upsilon)\cdot\nabla\upsilon\,\mathrm{d}z + \int_{\mathcal{D}} |\nabla\upsilon|^p\,\mathrm{d}z\Big] \quad \text{(by (A4))}\\ &\ge g_0\Big(\int_{\mathcal{D}} c_1|\nabla\upsilon|^p\,\mathrm{d}z - \int_{\mathcal{D}} \alpha_3(z)\,\mathrm{d}z\Big) + g_0\int_{\mathcal{D}} |\nabla\upsilon|^p\,\mathrm{d}z\\ &\ge C_{\min}\int_{\mathcal{D}} |\nabla\upsilon|^p\,\mathrm{d}z - C'\int_{\mathcal{D}} \alpha_3(z)\,\mathrm{d}z,\end{aligned}$$

since $\beta \ge 1$. Finally, from the growth condition (A2) and (3.1) we have

$$\begin{aligned}|\Lambda_5| &:= |\int_{\mathcal{D}} f(z,\upsilon)\upsilon\,\mathrm{d}z|\\ &\le \int_{\mathcal{D}} |f(z,\upsilon)\upsilon|\,\mathrm{d}z\\ &\le \int_{\mathcal{D}} (\alpha_2(z) + \|\upsilon\|^{p-1})\|\upsilon\|\,\mathrm{d}z\\ &\le \lambda\|\alpha_2\|_{p'}\|\nabla\upsilon\|_p + \lambda^{p+1}\|\nabla\upsilon\|_p^{p+1}.\end{aligned}$$

Hence

$$\begin{aligned}\langle \mathcal{L}(\upsilon), \upsilon\rangle &\ge \Lambda_4 - \Lambda_5\\ &\ge C_{min}\|\nabla\upsilon\|_p^p - C'\|\alpha_3\|_{p'} - \lambda\|\alpha_2|\|_{p'}\|\nabla u\|_p - \lambda^{p+1}\|\nabla\upsilon\|_p^{p+1} \to \infty\end{aligned}$$

as $\|\upsilon\|_{1,p} \to \infty$, since $C_{min}, C' > 0$ and $(p > \max(1, q+1))$. □

Proposition 3.5. *For all $k \in \mathbb{N}$ there exists $\upsilon_\mu \in \mathfrak{O}_\mu$ such that*

$$\langle \mathcal{L}(\upsilon_\mu), \Phi\rangle = 0 \quad \textit{for all } \Phi \in \mathfrak{O}_\mu.$$

Proof. By Proposition 3.4, there exists $R > 0$ such that for all $d \in \partial B_R(0) \subset \mathbb{R}^r$ we have $\mathcal{G}(d) \cdot d > 0$, and the usual topological argument [24, Proposition 2.8], there exists $z \in B_R(0)$ such that $\mathcal{G}(z) = 0$. Hence, for all $k \in \mathbb{N}$ there exists $\upsilon_\mu \in \mathfrak{O}_\mu$ such that $\langle \mathcal{L}(\upsilon_\mu), \varphi \rangle = 0$ for all $\varphi \in \mathfrak{O}_\mu$. $\square$

Proposition 3.6. *The constructed sequence (υ_μ) in Proposition 3.5 is uniformly bounded, i.e., there is a constant $R > 0$ such that $\|\upsilon_\mu\|_{1,p} \leq R$ for all $k \in \mathbb{N}$.*

Proof. By Proposition 3.4 there exists $R > 0$ with the property that $\langle \mathcal{L}(\upsilon), \upsilon \rangle > 1$ whenever $\|\upsilon\|_{1,p} > R$. Hence, for the sequence of Galerkin approximations $\upsilon_\mu \in \mathfrak{O}_\mu$ which satisfy $\langle \mathcal{L}(\upsilon_\mu), \upsilon_\mu \rangle = 0$ by the Proposition 3.5, we get the uniform boundedness of (υ_μ) in $W_0^{1,p}(\mathcal{D})$. $\square$

4. Proofs and properties for the convergence

In this section, we present general convergence findings pertaining to the functions denoted as $a(\cdot)$, $\mathcal{A}(\cdot)$, and $f(\cdot)$. Given that the sequence (υ_μ) remains within bounded limits in the space $W_0^{1,p}(\mathcal{D})$, as established in Propositions 3.4, 3.5, and 3.6, we can infer, based on the assertions of Lemma 2.4, the existence of a Young measure denoted as σ_x. This measure is generated by the gradients of υ_μ within the space $L^p(\mathcal{D}; \mathbb{R}^m)$.

We define

$$\widetilde{a}(z, \nabla\upsilon) = a(z, \nabla\upsilon) + |\nabla\upsilon|^{p-2}\nabla\upsilon,$$

where $\widetilde{a}$ adheres to conditions (A1)-(A3) with both coercivity and growth rate set to p, specifically,

$$\begin{gathered} \widetilde{a}(z,\xi).\xi \geq |\xi|^p, \\ |\widetilde{a}(z,\xi)| \leq |\xi|^{p-1} + S(c_3, p, q). \end{gathered} \tag{4.1}$$

Lemma 4.1. *The Young measure σ_z generated by $\nabla\upsilon_\mu$ satisfies*

$$(\widetilde{a}(z,\eta) - \widetilde{a}(z,\nabla\upsilon)) \cdot (\eta - \nabla\upsilon) = 0 \quad \text{on } \operatorname{supp}\sigma_z,$$

where $\operatorname{supp}\sigma_z$ *is the support of σ_z for a.e. $z \in \mathcal{D}$.*

Proof. We consider the sequence

$$\begin{aligned} e_\mu &:= (\widetilde{a}(z,\nabla\upsilon_\mu) - \widetilde{a}(z,\nabla\upsilon)) \cdot (\nabla\upsilon_\mu - \nabla\upsilon) \\ &= \widetilde{a}(z,\nabla\upsilon_\mu) \cdot (\nabla\upsilon_\mu - \nabla\upsilon) - \widetilde{a}(z,\nabla\upsilon) \cdot (\nabla\upsilon_\mu - \nabla\upsilon) \\ &= e_{\mu,1} + e_{\mu,2}. \end{aligned}$$

Given the growth condition of $\widetilde{a}$ in (4.1) and the weak convergence described in Lemma 2.3, it follows that

$$\liminf_{\mu\to\infty} \int_{\mathcal{D}} e_{\mu,2}\,\mathrm{d}z = \int_{\mathcal{D}} \widetilde{a}(z,\nabla\upsilon) \cdot \Big(\underbrace{\int_{\mathbb{R}^m} \eta d\sigma_z(\eta)}_{=:\nabla\upsilon(z)} - \nabla\upsilon \Big)\,\mathrm{d}z = 0.$$

Thus

$$e := \liminf_{\mu\to\infty} \int_{\mathcal{D}} e_\mu dz = \liminf_{\mu\to\infty} \int_{\mathcal{D}} e_{\mu,1} dz.$$

By the growth condition on $\widetilde{a}$ in (4.1), $(\widetilde{a}(z,\nabla\upsilon_\mu)\cdot\nabla\upsilon)$ is equi-integrable. Let us fix an arbitrary measurable subset $\mathcal{D}'\subset\mathcal{D}$. Then, the coercivity condition in (4.1) implies

$$\int_{\mathcal{D}'}|\min(\widetilde{a}(z,\nabla\upsilon_\mu)\cdot\nabla\upsilon_\mu,0)|\,\mathrm{d}z\le\int_{\mathcal{D}'}|\nabla\upsilon_\mu|^p\,\mathrm{d}z<\infty, \tag{4.2}$$

This shows the equi-integrability of $(\widetilde{a}(z,\nabla\upsilon_\mu)\cdot\nabla\upsilon_\mu)$. Therefore $(\widetilde{a}(z,\nabla\upsilon_\mu)\cdot(\nabla\upsilon_\mu-\nabla\upsilon))$ is also equi-integrable, and by Lemma 2.4, this yields

$$\int_{\mathcal{D}}\int_{\mathbb{R}^m}\widetilde{a}(z,\eta)\cdot(\eta-\nabla\upsilon)d\sigma_z(\eta)\,\mathrm{d}z\le\liminf_{\mu\to\infty}\int_{\mathcal{D}}\widetilde{a}(z,\nabla\upsilon_\mu)\cdot(\nabla\upsilon_\mu-\nabla\upsilon)\,\mathrm{d}z=e.$$

Now, let us show that $e\le 0$. By Propositions 3.5, we can write

$$\begin{aligned}
&g\Big(\int_\Omega(\mathcal{A}(z,\nabla\upsilon_k)+\frac{1}{p}|\nabla\upsilon_k|^p)dz\Big)\\
&\times\Big(\int_\Omega a(x,\nabla\upsilon_k)\cdot(\nabla\upsilon_k-\nabla\upsilon_k)\,\mathrm{d}z-\int_\Omega|\nabla\upsilon_k|^{p-2}\nabla\upsilon_k\cdot(\nabla\upsilon_k-\nabla\upsilon)dz\Big)\\
&=\int_\Omega f(z,\upsilon_k)(\upsilon_k-\upsilon)\,\mathrm{d}z-\int_\Omega|\upsilon_k|^{p-2}\upsilon_k.(\upsilon_k-\upsilon)\,\mathrm{d}z.
\end{aligned}$$

By (A4) we have

$$\begin{aligned}
&g_0\int_{\mathcal{D}}a(z,\nabla\upsilon_\mu)\cdot(\nabla\upsilon_\mu-\nabla\upsilon)\,\mathrm{d}z\\
&\le\int_{\mathcal{D}}f(z,\upsilon_\mu)(\upsilon_\mu-\upsilon)\,\mathrm{d}z-\int_{\mathcal{D}}|\upsilon_\mu|^{p-2}\upsilon_\mu(\upsilon_\mu-\upsilon)\,\mathrm{d}z\\
&\quad+g_0\int_{\mathcal{D}}|\nabla\upsilon_\mu|^{p-2}\nabla\upsilon_\mu\cdot(\nabla\upsilon_\mu-\nabla\upsilon)\,\mathrm{d}z.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_{\mathcal{D}}a(z,\nabla\upsilon_\mu)\cdot(\nabla\upsilon_\mu-\nabla\upsilon)\,\mathrm{d}z\\
&\le\frac{1}{g_0}\Big(\int_{\mathcal{D}}f(z,\upsilon_\mu)(\upsilon_\mu-\upsilon)\,\mathrm{d}z-\int_{\mathcal{D}}|\upsilon_\mu|^{p-2}\upsilon_\mu.(\upsilon_\mu-\upsilon)\,\mathrm{d}z\Big)\\
&\quad-\int_{\mathcal{D}}|\nabla\upsilon_\mu|^{p-2}\nabla\upsilon_\mu\cdot(\nabla\upsilon_\mu-\nabla\upsilon)dz\\
&\le C_m\int_{\mathcal{D}}f(z,\upsilon_\mu)(\upsilon_\mu-\upsilon)\,\mathrm{d}z\\
&\le C_m(\|d_2\|_{p'}+\underbrace{\|\upsilon_\mu\|_p^{p-1}}_{\le C})\|\upsilon_\mu-\upsilon\|_p\to 0\quad\text{as }k\to\infty.
\end{aligned}$$

This is achieved by Hölder's inequality and the fact that $\upsilon_\mu\to\upsilon$ in $W_0^{1,p}(\mathcal{D})$. Hence, we have

$$\int_{\mathcal{D}}\int_{\mathbb{R}^m}\widetilde{a}(z,\eta)\cdot(\eta-\nabla\upsilon)d\sigma_z(\eta)dz\le 0.$$

In conclusion, we can infer from this and Equation (4.2) that

$$\int_{\mathcal{D}}\int_{\mathbb{R}^m}\Big(\widetilde{a}(z,\eta)-\widetilde{a}(z,\nabla\upsilon)\Big)\cdot(\eta-\nabla\upsilon)\,\mathrm{d}\sigma_z(\eta)\,\mathrm{d}z\le 0.$$

The function $\widetilde{a}$ being monotonic, the integral above evaluates to zero with respect to the product measure $\mathrm{d}\sigma_z(\eta) \otimes \mathrm{d}z$, meaning that

$$\int_{\mathcal{D}} \int_{\mathbb{R}^m} \Big(\widetilde{a}(z,\eta) - \widetilde{a}(z,\nabla \upsilon)\Big) \cdot (\eta - \nabla \upsilon)\, \mathrm{d}\sigma_z(\eta) \otimes \mathrm{d}z = 0.$$

Consequently, we obtain

$$\Big(\widetilde{a}(z,\eta) - \widetilde{a}(z,\nabla \upsilon)\Big) \cdot (\eta - \nabla \upsilon) = 0 \quad \text{on supp } \sigma_z. \qquad \square$$

Proposition 4.2. *For almost every $z \in \mathcal{D}$, the support of σ_z is contained within the set where $\widetilde{\mathcal{A}}$ coincides with the supporting hyper-plane L defined as*

$$L := \{(\eta, \widetilde{\mathcal{A}}(z,\nabla \upsilon) + \widetilde{a}(z,\nabla \upsilon) \cdot (\eta - \nabla \upsilon))\},$$

that is

$$\operatorname{supp} \sigma_z \subset K_z = \{\eta \in \mathbb{R}^m : \widetilde{\mathcal{A}}(z,\eta) = \widetilde{\mathcal{A}}(z,\nabla \upsilon) + \widetilde{a}(z,\nabla \upsilon) \cdot (\eta - \nabla \upsilon)\}.$$

Proof. Let $\eta \in \operatorname{supp} \sigma_z$. By Lemma 4.1 implies for all $t \in [0,1]$, we have

$$(1-t)\big(\widetilde{a}(z,\eta) - \widetilde{a}(z,\nabla \upsilon)\big) \cdot (\eta - \nabla \upsilon) = 0. \tag{4.3}$$

Therefore, by the monotonicity condition and (4.3), we obtain

$$\begin{aligned} 0 &\le (1-t)\Big(\widetilde{a}(z,\eta) - \widetilde{a}\big(z,\nabla \upsilon + t(\eta - \nabla \upsilon)\big)\Big) \cdot (\eta - \nabla \upsilon) \\ &= (1-t)\Big(\widetilde{a}(z,\nabla \upsilon) - \widetilde{a}\big(z,\nabla \upsilon + t(\eta - \nabla \upsilon)\big)\Big) \cdot (\eta - \nabla \upsilon). \end{aligned} \tag{4.4}$$

Using the monotonicity condition, we have

$$\Big(\widetilde{a}(z,\nabla \upsilon) - \widetilde{a}\big(z,\nabla \upsilon + t(\eta - \nabla \upsilon)\big)\Big) \cdot t(\nabla \upsilon - \eta) \ge 0,$$

and since $t \in [0,1]$, we deduce that

$$\Big(\widetilde{a}(z,\nabla \upsilon) - \widetilde{a}\big(z,\nabla \upsilon + t(\eta - \nabla \upsilon)\big)\Big) \cdot (1-t)(\nabla \upsilon - \eta) \ge 0. \tag{4.5}$$

Combining (4.4) and (4.5) we find that

$$\Big(\widetilde{a}(z,\nabla \upsilon) - \widetilde{a}\big(z,\nabla \upsilon + t(\eta - \nabla \upsilon)\big)\Big) \cdot (\eta - \nabla \upsilon) = 0.$$

It follows from (A3) that

$$\begin{aligned} \widetilde{\mathcal{A}}(z,\eta) &= \widetilde{\mathcal{A}}(z,\nabla \upsilon) + \int_0^1 \widetilde{a}\big(z,\nabla \upsilon + t(\eta - \nabla \upsilon)\big) \cdot (\eta - \nabla \upsilon)\, \mathrm{d}z \\ &= \widetilde{\mathcal{A}}(z,\nabla \upsilon) + \widetilde{a}(z,\nabla \upsilon) \cdot (\eta - \nabla \upsilon). \end{aligned}$$

Hence $\eta \in K_z$, i.e., supp $\sigma_z \subset K_z$ for almost every $z \in \mathcal{D}$. $\square$

Now, we establish the proof of our main result.

Proof of Theorem 1.1. Since $\xi \mapsto \widetilde{\mathcal{A}}(z,\xi)$ is convex, we can represent it as

$$\widetilde{\mathcal{A}}(z,\eta) =: \mathcal{H}(\eta) \ge \widetilde{\mathcal{A}}(z,\nabla \upsilon) + \widetilde{a}(z,\nabla \upsilon) \cdot (\eta - \nabla \upsilon) =: R(\eta)$$

for all $\eta \in \mathbb{R}^m$. Assuming that $\eta \mapsto \mathcal{H}(\eta)$ is a C^1-function, as specified in the hypothesis, we obtain the following relationships for any $\xi \in \mathbb{R}^m, t \in \mathbb{R}$

$$\frac{\mathcal{H}(\eta + t\xi) - \mathcal{H}(\eta)}{t} \ge \frac{R(\eta + t\xi) - R(\eta)}{t} \quad \text{for } t > 0,$$

$$\frac{\mathcal{H}(\eta+t\xi)-\mathcal{H}(\eta)}{t} \le \frac{R(\eta+\tau\xi)-R(\eta)}{t} \quad \text{for } t<0.$$

Consequently, we can deduce that $\nabla_\eta \mathcal{H} = \nabla_\eta R$, which implies

$$\widetilde{a}(z,\eta) = \widetilde{a}(z,\nabla\upsilon) \quad \text{for all } \eta \in K_z \supset \operatorname{supp}\sigma_z. \tag{4.6}$$

Since $\widetilde{a}(z,\nabla\upsilon_\mu)$ is equi-integrable, by (4.6) and Lemma 2.3, its weak L^1-limit satisfies

$$\begin{aligned}\bar{a}(z) &= \int_{\mathbb{R}^m} \widetilde{a}(z,\eta)\,\mathrm{d}\sigma_z(\eta)\\ &= \int_{\operatorname{supp}\sigma_z} \widetilde{a}(z,\eta)\,\mathrm{d}\sigma_z(\eta)\\ &= \int_{\operatorname{supp}\sigma_z} \widetilde{a}(z,\nabla\upsilon)\,\mathrm{d}\sigma_z(\eta)\\ &= \widetilde{a}(z,\nabla\upsilon).\end{aligned} \tag{4.7}$$

Next, if we consider the following Carathéodory function

$$\mathcal{B}(z,\eta) = |\widetilde{a}(z,\eta) - \bar{a}(z)|, \quad \eta \in \mathbb{R}^m,$$

then, since $a(z,\nabla\upsilon_\mu)$ is equi-integrable, we conclude that $\mathcal{B}_\mu(z) := \mathcal{B}(z,\nabla\upsilon_\mu)$ is also equi-integrable, and its weak L^1-limit is

$$\mathcal{B}_\mu \to \bar{\mathcal{B}} \qquad \text{in } L^1(\mathcal{D}),$$

where

$$\bar{\mathcal{B}}(z) = \int_{\mathbb{R}^m} |\widetilde{a}(z,\eta) - \bar{a}(z)|\,\mathrm{d}\sigma_z(\eta) = \int_{\operatorname{supp}\sigma_z} |\widetilde{a}(z,\eta) - \bar{a}(z)|\,\mathrm{d}\sigma_z(\eta) = 0,$$

by (4.5) and (4.6). Notably, the convergence of $\mathcal{B}_\mu$ is strong since $\mathcal{B}_\mu \ge 0$.

Applying (4.1), we derive theinequalities

$$\widetilde{\mathcal{A}}(z,\nabla\upsilon_\mu) \ge \frac{1}{p}\widetilde{a}(z,\nabla\upsilon_\mu)\cdot\nabla\upsilon_\mu \ge \frac{1}{p}|\nabla\upsilon_\mu|^p;$$

thus

$$\int_{\mathcal{D}'} |\min(\widetilde{\mathcal{A}}(z,\nabla\upsilon_\mu),0)|\,\mathrm{d}z < \infty.$$

Consequently, $\widetilde{\mathcal{A}}(z,\nabla\upsilon_\mu)$ is both bounded and equi-integrable. As a result, its weak L^1-limit is $\int_{\mathbb{R}^m} \widetilde{\mathcal{A}}(z,\eta)\,\mathrm{d}\sigma_z(\eta)$. Utilizing Lemma 4.1, we obtain

$$\int_{\mathbb{R}^m} \widetilde{\mathcal{A}}(z,\eta)\,\mathrm{d}\sigma_z(\eta) = \int_{\operatorname{supp}\sigma_z} \widetilde{\mathcal{A}}(z,\eta)\,\mathrm{d}\sigma_z(\eta) = \widetilde{\mathcal{A}}(z,\nabla\upsilon),$$

by Equation (4.2). The continuity of the function g in (A4) and $\mathcal{B}_\mu \to 0$ in $L^1(\mathcal{D})$, imply that

$$\begin{aligned}&g\Big(\int_\Omega (\mathcal{A}(z,\nabla\upsilon_k) + \frac{1}{p}|\nabla\upsilon_k|^p)\,\mathrm{d}z\Big)\\ &\times\Big(\int_\Omega a(z,\nabla\upsilon_k)\nabla\Phi(z)\,\mathrm{d}z - \int_\Omega |\nabla\upsilon_k|^{p-2}\nabla\upsilon_k\nabla\Phi(z)\,\mathrm{d}z\Big) + \int_\Omega |\upsilon_k|^{p-2}\upsilon_k\,\mathrm{d}z\end{aligned}$$

converges to

$$g\Big(\int_\Omega (A(z,\nabla\upsilon) + \frac{1}{p}|\nabla\upsilon|^p)\,\mathrm{d}z\Big)$$

$$\times \Big(\int_\Omega a(z,\nabla \upsilon)\nabla\Phi(z)\,\mathrm{d}z - \int_\Omega |\nabla \upsilon|^{p-2}\nabla \upsilon \nabla \Phi(z) dz \Big) + \int_\Omega \|\upsilon\|^{p-2}\upsilon\,\mathrm{d}z.$$

To complete the proof, we need to consider the term $\int_{\mathcal{D}} f(z,\upsilon_\mu)\Phi(z)\,\mathrm{d}z$. We know that (υ_μ) is bounded in $W_0^{1,p}(\mathcal{D})$ according to Propositions 3.4, 3.5, and 3.6, up to a subsequence, $\upsilon_\mu \to \upsilon$ in $L^p(\mathcal{D})$. For some ϵ positive, we have

$$\int_{\mathcal{D}} |\upsilon_\mu - \upsilon|^p\,\mathrm{d}z \geq \int_{\{z\in\mathcal{D}:|\upsilon_\mu-\upsilon|\geq\epsilon\}} |\upsilon_\mu - \upsilon|^p\,\mathrm{d}z \geq \epsilon^p |\{z\in\mathcal{D}: |\upsilon_\mu - \upsilon| \geq \epsilon\}|,$$

which implies

$$|\{z\in\mathcal{D}: |\upsilon_\mu - \upsilon| \geq \epsilon\}| \leq \frac{1}{\epsilon^p}\int_{\mathcal{D}} |\upsilon_\mu - \upsilon|^p\,\mathrm{d}z \to 0 \quad \text{as } k\to\infty,$$

thus $\upsilon_\mu \to \upsilon$ in measure and almost everywhere.
The continuity of the function f in (A1) implies

$$f(z,\upsilon_\mu)\Phi(z) \to f(z,\upsilon)\Phi(z)$$

almost everywhere. From the growth condition in $\mathcal{F}$ and the uniform bound in Propositions 3.4, 3.5, and 3.6, it follows that $f(z,\upsilon_\mu)\Phi(z)$ is equi-integrable. Consequently,

$$f(z,\upsilon_\mu)\Phi(z) \to f(z,\upsilon)\Phi(z) \quad \text{in } L^1(\mathcal{D}),$$

and then, by the Vitali convergence theorem, we establish that

$$\lim_{\mu\to\infty}\int_{\mathcal{D}} f(z,\upsilon_\mu)\Phi(z)\,\mathrm{d}z = \int_{\mathcal{D}} f(z,\upsilon)\Phi(z)\,\mathrm{d}z \quad \forall \Phi \in \cup_{\mu\geq 1}\mathfrak{O}_\mu.$$

Since $\cup_{\mu\geq1}\mathfrak{O}_\mu$ is dense in $W_0^{1,p}(\mathcal{D})$, it follows that υ is a weak solution of (1.1) as desired. □

References

[1] Acerbi, E.; Mingione, G.; *Regularity results for stationary electro-rheological fluids.* Arch. Ration. Mech. Anal. **164** (2002), 213–259.

[2] Allalou, M.; El Ouaarabi, M.; El Hammar, H.; Raji, A.; *On a class of obstacle problem via Young measure in generalized Sobolev space.* Advances in Operator Theory. **9**(3) (2024), 1–23.

[3] Allalou, M.; El Ouaarabi, M.; Raji, A.; *Existence and uniqueness results for a class of obstacle problem via Young's measure theory.* Analysis and Mathematical Physics, **14**(6) (2024), 123.

[4] Ayoujil, A.; Moussaoui, M.; *Multiplicity results for nonlocal elliptic transmission problem with variable exponent.* Bol. Soc. Parana. Mat. **33**(2) (2015), 185–199.

[5] Azroul, E.; Balaadich, F.; *Existence of solutions for a class of Kirchhoff-type equations via Young measures.* Numerical Functional Analysis and Oplimization, **42** (2021) No. 4, 460-473.

[6] Ball, J. M.; *A version of the fundamental theorem for young measures.* In: PDEs and Continuum Models of Phase Transitions, pp. 207–215. Springer, Berlin, 1989.

[7] Ben Omrane, I.; Ben Slimane, M.; Gala, S.; Ragusa, M. A.; *A new regularity criterion for the 3D nematic liquid crystal flows.* Analysis and Applications.(2024), 1–20.

[8] Cekic, B.; Mashiyev, R. A.; *Nontrivial solution for a nonlocal elliptic transmission problem in variable exponent Sobolev spaces.* Abstr. Appl. Anal. **2010** (2010), 385048.

[9] Chen, Y.; Levine, S.; Rao, M.; *Variable exponent, linear growth functionals in image restoration.* SIAM J. Appl. Math. **66**(4) (2006), 1383–1406.

[10] Chung, N. T.; Ngo, Q. A.; *A multiplicity result for a class of equations of p-Laplacian type with sign-changing nonlinearities.* Glasgow Math. J. **51**(3) (2009), 513–524.

[11] Dai, G., Hao, R.; *Existence of solutions for a $p(x)$-Kirchhoff-type equation*, J. Math. Anal. Appl. **359** (2009), 275–284.

[12] De Napoli, P.; Mariani, M. C.; *Mountain pass solutions to equations of p-Laplacian type.* Nonlinear Anal. Theory Methods Appl. **54**(7) (2003), 1205–1219.

[13] Hungerbuhler, N.; *Quasi-linear parabolic systems in divergence form with weak monotonicity.* Duke Math. J. **107**(3) (2001), 497–520.

[14] Kristensen, J.; Rindler, F.; *Characterization of generalized gradient Young measures generated by sequences in $W^{1,1}$ and BV*, Arch. Ration. Mech. Anal. **167** (2010), 539–598.

[15] Li, F.; Zhang, Y.; Zhu, X.; Liang, Z.; *Ground-state solutions to Kirchhoff-type transmission problems with critical perturbation.* J. Math. Anal. Appl. **482**(2) (2020), 123568.

[16] Ma, T. F.; Rivera, J. M.; *Positive solutions for a nonlinear nonlocal elliptic transmission problem.* Applied Mathematics Letters. **16**(2) (2003), 243–248.

[17] Mohiuddine, S. A.; Das, A.; Alotaibi, A.; *Existence of solutions for infinite system of nonlinear q-fractional boundary value problem in Banach spaces.* Filomat. **37**(30) (2023), 10171–10180.

[18] Rindler, F.; *Lower semicontinuity for integral functionals in the space of functions of bounded deformation via rigidity and Young measures*, Arch. Ration. Mech. Anal., **202** (2013), 63–113.

[19] Rindler, F.; *Lower Semicontinuity and Young Measures for Integral Functionals with Linear Growth*, Ph.D. thesis, University of Oxford, 2011.

[20] Su, H.; *Nonlinear n-Order m-Point Semipositive Boundary Value Problems and Applications.* Journal of Function Spaces. **2023**(1) (2023), 8868353.

[21] Zhang, W.; Zuo, J.; Zhao, P.; *Multiplicity and Concentration of Positive Solutions for (p,q)-Kirchhoff Type Problems.* J Geom Anal. **33** (2023), 159.

[22] Zhou, S.; *A note on nonlinear elliptic systems involving measures*, Electron J. Differential Eq., 2000 (2000), no. 08, 1–6.

[23] Zhikov, V. V. E.; *Averaging of functionals of the calculus of variations and elasticity theory.* Izv. Akad. Nauk SSSR Ser. Mat. 50(4) (1986), 675–710.

[24] Zeidler, E.; *Nonlinear functional analysis and its applications: II/B: nonlinear monotone operators.* Springer Science and Business Media, 2013.

Mouad Allalou
Applied Mathematics and Scientific Computing Laboratory, Faculty of Science and Technics, Sultan Moulay Slimane University, Beni Mellal, BP 523, 23000, Morocco
Email address: mouadallalou@gmail.com

Mohamed El Ouaarabi
Fundamental and Applied Mathematics Laboratory, Faculty of Sciences Aïn Chock, Hassan II University, Casablanca, BP 5366, 20100, Morocco.
Applied Mathematics and Scientific Computing Laboratory, Faculty of Science and Technics, Sultan Moulay Slimane University, Beni Mellal, BP 523, 23000, Morocco
Email address: mohamed.elouaarabi@etu.univh2c.ma, mohamedelouaarabi93@gmail.com

Abderrahmane Raji
Applied Mathematics and Scientific Computing Laboratory, Faculty of Science and Technics, Sultan Moulay Slimane University, Beni Mellal, BP 523, 23000, Morocco
Email address: rajiabd2@gmail.com