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WELL-POSEDNESS OF SOLUTIONS FOR THE 2D STOCHASTIC QUASI-GEOSTROPHIC EQUATION IN CRITICAL FOURIER-BESOV-MORREY SPACES

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Abstract. In this article, we apply the Itô integral to obtain the global solutions for stochastic quasi-geostrophic equations in Fourier-Besov-Morrey spaces. For comparison we also give the corresponding results of the deterministic quasi-geostrophic equations. We assume the initial data is F_0 measurable and the right-hand side is a random function in a Morrey space, to obtain the well posedness of stochastic quasi-geostrophic equations.

1. Introduction:

In this article we study the two-dimensional dissipative stochastic quasi-geostrophic (SQG) equation

$$\begin{gathered} \partial_t\theta + V_\theta\cdot\nabla\theta + \mu\Lambda^{2\alpha}\theta = g\dot{W}, \quad x\in\mathbb{R}^2, t>0,\\ V_\theta = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta),\\ \theta(0,x)=\theta_0(x). \end{gathered} \tag{1.1}$$

Where $\mu > 0$ is the dissipative coefficient, $\alpha \geq 1/2$ is a real number, $\theta(t, x)$ is a real-valued function of two space variables t and x. The function θ represents the potential temperature, g $\dot{W}$ is a random external force and W is a standard infinite dimensional Wiener process. The velocity V_θ is incompressible and determined from θ by a stream function ζ,

$$V_\theta = \big(-\frac{\partial\zeta}{\partial x_2}, \frac{\partial\zeta}{\partial x_1}\big), \tag{1.2}$$

where the ζ function is satisfied

$$\Lambda\zeta = -\theta. \tag{1.3}$$

We define the operator Λ by the fractional power of $-\Delta$:

$$\Lambda v = (-\Delta)^{1/2}v, \quad \mathcal{F}(\Lambda v) = \mathcal{F}((-\Delta)^{\frac12}v) = |\xi|\mathcal{F}(v),$$

and more generally

$$\mathcal{F}(\Lambda^{2\alpha}v) = \mathcal{F}((-\Delta)^{\alpha}v) = |\xi|^{2\alpha}\mathcal{F}(v),$$

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where $\mathcal{F}$ is the Fourier transform. The relation between (1.2) and (1.3) can be determined by using the Riesz transform as follows

$$V_\theta = (\partial_{x_2}\Lambda^{-1}\theta, -\partial_{x_1}\Lambda^{-1}\theta) = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta),$$

where $\mathcal{R}_j$, $j = 1, 2$, is the Riesz transform defined by

$$\mathcal{R}_j = \partial_{x_j}(-\Delta)^{-1/2}.$$

The Quasi-geostrophic equations are frequently used to study the large-scale motions of the ocean and atmosphere at mid-latitudes, for the modelling of marine and atmospheric circulation, as well as for stability, frontogenesis and turbulence studies, because they are much simpler than the basic equations These equations have been the subject of several researchers. The solutions depend discontinuously on the initial data, the approximate finite element solutions converge in regions of free flow [12], and the wavenumber energy spectra of the solutions asymptotically approach the statistical equilibrium spectra of the spectrally truncated equations (Kraichnan 1967).

When $\mu = 0$ and $g = 0$, the equation (1.1) will be called the two-dimensional non-dissipative quasigeostrophic equation, it was introduced by Constantin, Majda, and Tabak in 1994 [9].

When $g = 0$. It is clear that the small case is mathematically very interesting to understand, or equivalently when the dissipation tends to 0, the quasi-geostrophic equation converges to the three dimensional Navier-Stokes equations. Although, this model is physically interesting in the case $\alpha = \frac{1}{2}$. Mathematically, the power $\alpha = 1/2$ corresponds to the index for which the nonlinear term and the dissipation are of the same order (in the sense that V_θ and Λ are two operators deriving once). The previous remarks motivate the following definition.

Definition 1.1 ([2]). We distinguish the following 3 cases:

- If $0 \le \alpha < 1/2$, then (1.1) is called *supercritical*.
- If $\alpha = 1/2$, then (1.1) is called *critical*.
- If $1/2 < \alpha \le 1$, then (1.1) is called *subcritical*.

There is a copious literature on well-posedness for fluid dynamics PDEs with singular data in different spaces, where the conditions are taken in norms of critical spaces. For instance, for Navier-Stokes equations and related models, we have well-posedness results in the critical case of the following spaces: Lebesgue space L^p [6], Marcinkiewicz space $L^{p,\infty}$ [10], Morrey spaces $\mathcal{M}_p^\mu$ [7, 13, 14, 17], Besov-Morrey spaces $\mathcal{N}^s_{p,\lambda,q}$ [8, 22], Fourier-Besov spaces $\mathcal{FB}^s_{p,q}$ [5, 9], Fourier-Besov-Morrey spaces $\mathcal{FN}^s_{p,\lambda,q}$ [1, 3].

The small perturbations (numerical, empirical, and physical errors) or thermodynamic fluctuations present in fluid flows are frequently modeled using stochastic components in the equations of motion. Additionally, they are employed in order to comprehend turbulence better. Consequently, stochastic partial differential equations (SDE) such as quasi-geostrophic equations (QG), stochastic Navier-Stokes equations are gaining more and more interest in fluid mechanics research.

The well-posedness of partial differential equations (PDEs) in Fourier-Besov-Morrey spaces is crucial for several reasons related to both the theoretical and practical aspects of solving PDEs. These spaces allow for the description of functions that may not be smooth but still satisfy certain integrability conditions. They are typically used when analyzing the regularity of solutions to PDEs with less regular data or in the context of singular integrals. They are useful for dealing with solutions that are not necessarily in classical Sobolev spaces but still exhibit sufficient regularity for solving certain types of PDEs. In this research Fourier-Besov-Morrey spaces are particularly suited for handling the scaling properties of nonlinear operators that arise in stochastic quasi-geostrophic equations.

Biswas [5] established the global-in-time well-posedness of (1.1) in the space $\mathcal{FB}^s_{p,q}$ (wich is a particular case of Fourier-Besov-Morrey space $\mathcal{FN}^s_{p,\lambda,q}$ by taking $\lambda = 0$). In 2003, Chae and Lee [8] obtained the global well-posedness in the super-critical dissipative quasigeostrophic equations in $\mathcal{N}^s_{p,\lambda,q}$. Recently, Azanzal et al. [2] obtained the existence in Fourier-Besov-Morrey spaces $\mathcal{FN}^s_{p,\lambda,q}$ but in the deterministic case. Consequently, our research extends the previous works.

Definition 1.2 ([23])**.** Let $(\Omega, F, P, \{F_t\}_{t\in[0,T]})$ be a filtered probability space with the expectation $\mathbb{E}$ and $T > 0$, we designate by $\mathcal{M}_T$ the smallest σ-algebra of F_t adapted distribution processes f defined on $\Omega \times [0,T] \times \mathbb{R}^3$, which are progressively measurable, more precisely $f(\omega, t, \cdot) \in F_t \times \mathcal{B}[0,T]$ for all $t \in [0,T]$.

Definition 1.3. Let $\left(\Omega, F, P, \{F_t\}_{t\geq 0}, W\right)$ be a fixed probability basis, the divergence free process θ is a mild-solution of (1.1), if $\theta(\omega,\cdot) \in \tilde{L}^4(0,t;\mathcal{F}\dot{\mathcal{N}}^{3-2\alpha-\frac{2}{P}}_{p,\lambda,q}) \cap \mathcal{M}_t$ for all $t \geq 0$ and

$$\theta(t) = T_\alpha(t)\theta_0 - \int_0^t T_\alpha(t-s)(V_\theta \cdot \nabla\theta)(s)\,ds + \int_0^t T_\alpha(t-s) g\,dW. \tag{1.4}$$

To examine how stochastic forces affect quasi-geostrophic equations, we first present the outcome of the deterministic quasi-geostrophic equations, or the case $g = 0$ in (1.1). So to speak

$$\begin{gathered}\partial_t\theta + V_\theta \cdot \nabla\theta + \mu\Lambda^{2\alpha}\theta = 0, \quad x \in \mathbb{R}^2, t > 0,\\ V_\theta = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta),\\ \theta(0,x) = \theta_0(x).\end{gathered} \tag{1.5}$$

Theorem 1.4 ([2] Deterministic case)**.** *Let $g = 0$, $1 \leq p < \infty$, $1 \leq \lambda < 2$, $1 \leq q \leq \infty$, $\frac{1}{2} \leq \alpha \leq 2 + \frac{\lambda-2}{2p}$ and $\theta_0 \in \mathcal{F}\dot{\mathcal{N}}^{3-2\alpha-\frac{2-\lambda}{P}}_{p,\lambda q}$. Then, there exists a constant $\beta > 0$ such that if θ_0 satisfies $\|\theta_0\|_{\mathcal{F}\dot{\mathcal{N}}^{3-2\alpha-\frac{2-\lambda}{P}}_{p,\lambda,q}} \leq \beta$, then (1.5) admits a unique global solution $\theta \in \mathcal{C}(\mathbb{R}^+, \mathcal{F}\dot{\mathcal{N}}^{3-2\alpha-\frac{2-\lambda}{P}}_{p,\lambda,q}) \cap \mathcal{L}^1(\mathbb{R}^+, \mathcal{F}\dot{\mathcal{N}}^{3-\frac{2-\lambda}{P}}_{p,\lambda,q})$, such that $\|\theta\|_X \lesssim \|\theta_0\|_{\mathcal{F}\dot{\mathcal{N}}^{3-2\alpha-\frac{2-\lambda}{P}}_{p,\lambda,q}}$, where*

$$X = \mathcal{L}^\infty(\mathbb{R}^+, \mathcal{F}\dot{\mathcal{N}}^{3-2\alpha-\frac{2-\lambda}{P}}_{p,\lambda,q}) \cap \mathcal{L}^1(\mathbb{R}^+, \mathcal{F}\dot{\mathcal{N}}^{3-\frac{2-\lambda}{P}}_{p,\lambda,q}).$$

Furthermore, the solution θ depends continuously on the initial data θ_0.

Our main result reads as follows.

Theorem 1.5 (Stochastic ase)**.** *Assume $1 \leq q \leq \infty$ and $\frac{1}{2} \leq \alpha \leq \frac{3}{2} + \frac{\lambda}{4}$. Let $(\Omega, F, P, \{F_t\}_{t\geq 0}, W)$ be a probability basis and θ_0 be F_0 measurable, $g \in \mathcal{M}_T$. Assume that for any positive T,*

$$(1+T)\|g\|_{\tilde{L}^4_\Omega \tilde{L}^4_T(\mathcal{F}\dot{\mathcal{N}}^{2-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q})} + \|u_0\|_{\tilde{L}^4_\Omega(\mathcal{F}\dot{\mathcal{N}}^{2-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q})} < +\infty.$$

Then, there exists a unique global mild solution of (1.1) *in* $\tilde{L}^4(0,\ T;\mathcal{F}\dot{\mathcal{N}}^{2-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q})$, *for all* ω *in* $\widetilde{\Omega}$, *with* $\widetilde{\Omega}$ *is a random set with positive probability.*

The structure of this article is as follows. In the second section, we review Bernstein's lemma, the fixed point lemma, the definition of the Fourier-Besov-Morrey space, and the Littlewood-Paley theory. In the third section, we present the proof of the Theorem 1.5 .

2. Preliminaries

Here we provide notation and review the fundamental characteristics of Fourier-Besov-Morrey spaces, which will be utilized throughout the article. First, we should review the Fourier transform, Littlewood-Paley theory, and the definitions of our spaces. For more details we refer to [15, 4]. The Fourier transform is described as

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx.$$

And its inverse Fourier transform as

$$\check{f}(x) = \mathcal{F}^{-1}f(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\,d\xi.$$

Let $\varphi \geq 0$ be a C_0^∞ function with $\operatorname{supp}\varphi \subset \{3/4 \leq |\xi| \leq 8/3\}$ and nonnegative $\chi \in C_0^\infty(B(0,\frac{4}{3}))$ such that

$$\chi(\xi) + \sum_{j\geq 0}\varphi_j(\xi) = 1, \quad \xi \in \mathbb{R}^n,$$
$$\sum_{j\in\mathbb{Z}}\varphi_j(\xi) = 1, \quad \xi \in \mathbb{R}^n\backslash\{0\},$$

where $\varphi_j(\xi) = \varphi(2^{-j}\xi)$.

Let $\mathcal{S}'_h = \mathcal{S}'/\mathcal{P}$, where $\mathcal{P}$ is the set of polynomials,

$$h_j = \mathcal{F}^{-1}\varphi_j, \quad \tilde{h}_j = \mathcal{F}^{-1}\chi_j = \mathcal{F}^{-1}\chi(2^{-j}.),$$
$$\dot{\Delta}_j = \mathcal{F}^{-1}\varphi_j *,\ \dot{S}_j u = \mathcal{F}^{-1}\chi_j * u,$$

where

$$\chi_j = \chi(2^{-j}\cdot).$$

Now we give some definitions concerning Fourrier-Besov-Morrey space.

Definition 2.1. (i) Let $1 \leq p \leq \infty$ and $0 \leq \lambda < d$. The homogeneous Morrey space M_p^λ is the set of all functions $f \in L^p(B(x,r))$ such that

$$\|f\|_{M_p^\lambda} = \sup_{x\in\mathbb{R}^d}\sup_{r>0} r^{-\frac{\lambda}{p}}\|f\|_{L^p(B(x,r))} < \infty, \tag{2.1}$$

where $B(x,r)$ is the open ball in $\mathbb{R}^d$ centered at x and with radius $r > 0$.

(ii) Let $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$. We can define the homogeneous Fourier-Besov space $FB^s_{p,q}$ as the set of all distributions $f \in \mathcal{S}'\backslash\mathcal{P}$ is the set of all polynomials such that the norm $\|f\|_{FB^s_{p,q}}$ is finite, where

$$\|f\|_{FB^s_{p,q}} := \begin{cases} \Big(\sum_{j\in\mathbb{Z}} 2^{jsq}\|\varphi_j\hat{f}\|^q_{L^p}\Big)^{1/q} & \text{for } q < \infty \\ \sup_{j\in\mathbb{Z}} 2^{js}\|\varphi_j\hat{f}\|_{L^p} & \text{for } q = \infty. \end{cases} \tag{2.2}$$

(iii) Let $1 \le p, q \le \infty$, $0 \le \lambda < n$ and $s \in \mathbb{R}$. The homogeneous Fourier-Besov-Morrey space $\mathcal{FN}^s_{p,\lambda,q}$ is defined as the set of all distributions $f \in \mathcal{S}'\backslash\mathcal{P}$, such that the norm $\|f\|_{\mathcal{FN}^s_{p,\lambda,q}}$ is finite, where

$$\|f\|_{\mathcal{FN}^s_{p,\lambda,q}} := \begin{cases} (\sum_{j\in\mathbb{Z}} 2^{jsq}\|\varphi_j \hat{f}\|^q_{\mathrm{M}^\lambda_p})^{1/q} & \text{for } q < \infty \\ \sup_{j\in\mathbb{Z}} 2^{js}\|\varphi_j \hat{f}\|_{\mathrm{M}^\lambda_p} & \text{for } q = \infty. \end{cases} \tag{2.3}$$

Note that the space $\mathcal{FN}^s_{p,\lambda,q}(\mathbb{R}^n)$ equipped with the norm (2.2) is a Banach space. Since $\mathrm{M}^0_p = L^p$, we have $\mathcal{FN}^s_{p,0,q} = F\dot{B}^s_{p,q}$, and $\mathcal{FN}^s_{1,0,1} = F\dot{B}^s_{1,1} = \mathcal{X}^s$ where $\mathcal{X}^s$ is the Lei-Lin space [4].

Definition 2.2. *Let $1 \le \rho \le +\infty$ and $T \in (0, +\infty]$, the space $\tilde{L}^\rho_T(\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}(\mathbb{R}^d))$ called Chemin–Lerner type space is defined as the set of tempered distributions in $\mathcal{S}'\Big(\mathbb{R}\times\mathbb{R}^d\Big)/\mathcal{P}$ with respect to the norm*

$$\|f\|_{\tilde{L}^\rho_T(\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}(\mathbb{R}^d))} := \|\Big\{2^{js}\|\varphi_j\hat{f}\|_{L^\rho([0,T];M^\lambda_p(\mathbb{R}^d))}\Big\}\|_{l^q(j\in\mathbb{Z})} < +\infty.$$

Definition 2.3. *Let $p, r, \sigma \in (1, +\infty]$, $1 \le q < +\infty$, $s \in \mathbb{R}$ and $T > 0$. We define the space $(\tilde{L}^\sigma_\Omega \tilde{L}^r_T \mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}(\mathbb{R}^d))$ called Chemin–Lerner type space of Bochner (CLBFB) as the space of distribution process $g \in \mathcal{M}_T$ such that $g(\omega, t) \in \mathcal{S}'_h(\mathbb{R}^d)$ and the quasi-norm*

$$\|g\|_{\tilde{L}^\sigma_\Omega \tilde{L}^r_T \mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}} = \|\Big\{2^{js}\big[\mathbb{E}(\|\varphi_j\hat{g}(t)\|_{L^r_T M^\lambda_p})^\sigma\big]^{1/\sigma}\Big\}\|_{l^q(j\in\mathbb{Z})} < +\infty.$$

Lemma 2.4 ([16])**.** *Let $1 \le p_1, p_2, p_3 < \infty$ and $0 \le \lambda_1,\ \lambda_2,\ \lambda_3 < d$.*

(i) (Hölder's inequality) If $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{\lambda_3}{p_3} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}$, then

$$\|fg\|_{\mathrm{M}^{\lambda_3}_{p_3}} \le \|f\|_{\mathrm{M}^{\lambda_1}_{p_1}}\|g\|_{\mathrm{M}^{\lambda_2}_{p_2}}. \tag{2.4}$$

(ii) (Young's inequality) If $\phi \in L^1$ and $h \in \mathrm{M}^{\lambda_1}_{p_1}$, then

$$\|\phi * h\|_{\mathrm{M}^{\lambda_1}_{p_1}} \le \|\phi\|_{L^1}\|h\|_{\mathrm{M}^{\lambda_1}_{p_1}}, \tag{2.5}$$

where $*$ denotes the standard convolution operator.

Now, we recall the Bernstein-type lemma in Fourier variables.

Lemma 2.5. *Let $1 \le p_2 \le p_1 < \infty$, $0 \le \lambda_1, \lambda_2 < d$, $\frac{d-\lambda_1}{p_1} \le \frac{d-\lambda_2}{p_2}$ and let β be a multi-index. If $\mathrm{supp}(\widehat{\varphi}) \subset \{|\xi| \le A2^j\}$, then there is a constant $C > 0$ independent of φ and j such that*

$$\|(i\xi)^\beta\widehat{\varphi}\|_{\mathrm{M}^{\lambda_2}_{p_2}} \le C2^{j|\beta|+j(\frac{d-\lambda_2}{p_2}-\frac{d-\lambda_1}{p_1})}\|\widehat{\varphi}\|_{\mathrm{M}^{\lambda_1}_{p_1}}. \tag{2.6}$$

Definition 2.6 ([18])**.** Let V and W be finite-dimensional inner product spaces, and let $m = (m(t))_{t\ge0}$ be a continuous V-valued semimartingale. Itô's formula says that if $F \in C^2(V;W)$, then

$$dF(m(t)) = DF(m(t))[dm(t)] + \frac{1}{2}D^2F(m(t))[dm(t), dm(t)],$$

where D^kF is the k^{th} derivative of F.

Theorem 2.7 (Burkholder-Davis-Gundy inequalities [19]). *For each $p > 0$ there exist two constants c_p and C_p such that for any continuous local martingale Λ vanishing at 0 we have*

$$c_p\mathbb{E}\big[\big\langle m,m\big\rangle_\infty^{p/2}\big] \le \mathbb{E}\big(m_\infty^*\big)^p \le C_p\mathbb{E}\big[\big\langle m,m\big\rangle_\infty^{p/2}\big].$$

Stopping at a time T, Theorem 2.7 leads to the following result, which is nevertheless very important in applications.

Corollary 2.8 ([19]). *For a stopping time T one has*

$$c_p\mathbb{E}\big[\langle m,m\rangle_T^{p/2}\big] \le \mathbb{E}\big(m_T^*\big)^p \le C_p\mathbb{E}\big[\big\langle m,m\big\rangle_T^{p/2}\big].$$

In general for a bounded predictable process H we have

$$\begin{aligned} c_p\mathbb{E}\Big[\Big(\int_0^T H_s^2 d\langle m,\Lambda\rangle_s\Big)^{p/2}\Big] &\le \mathbb{E}\Big[\sup_{t\le T}\Big|\int_0^t H_s dm_s\Big|^p\Big] \\ &\le C_p\mathbb{E}\Big[\Big(\int_0^T H_s^2 d\langle m,m\rangle_s\Big)^{p/2}\Big]. \end{aligned}$$

We shall review briefly the existence and uniqueness of an abstract operator equation in a Banach space at the end of this section. The purpose of this is to illustrate Theorem 1.5.

Lemma 2.9. . *Let $\mathbb{B}$ be a Banach space with norm $\|\cdot\|_{\mathbb{B}}$ and $L:\mathbb{B}\times\mathbb{B}\to\mathbb{B}$ be a bounded bilinear operator satisfying*

$$\|L(\theta_1,\theta_2)\|_{\mathbb{B}} \le \beta\|\theta_1\|_{\mathbb{B}}\|\theta_2\|_{\mathbb{B}},$$

for all $\theta_1,\theta_2\in\mathbb{B}$ and a constant $\beta>0$. Then, if $0<\varepsilon<\frac{1}{4\beta}$ and if $y\in\mathbb{B}$ such that $\|y\|_{\mathbb{B}}\le\varepsilon$, the equation $x:=y+L(x,x)$ has a solution $\overline{x}$ in $\mathbb{B}$ such that $\|\overline{x}\|_{\mathbb{B}}\le 2\varepsilon$. This solution is the only one in the ball $\overline{B}(0,2\varepsilon)$. Moreover, the solution depends continuously on y in the sense: if $\|y'\|_{\mathbb{B}}<\varepsilon$, $x'=y'+L(x',x')$, and $\|x'\|_{\mathbb{B}}\le 2\varepsilon$, then

$$\|\overline{x}-x'\|_{\mathbb{B}} \le \frac{1}{1-4\varepsilon\beta}\|y-y'\|_{\mathbb{B}}\,.$$

3. Proof of Theorem 1.5

The random term in equation (1.4) must also be addressed in order to find a solution, so using the superposition principle, we take the following auxiliary Cauchy problem

$$\begin{gathered} du+\mu\Lambda^{2\alpha}u\,dt = g\,dW \quad \text{in } \Omega\times(0,+\infty)\times\mathbb{R}^2, \\ u_{t=0}=u_0 \quad \text{on } \Omega\times\mathbb{R}^2. \end{gathered} \tag{3.1}$$

Applying the Fourier transform of (3.1) with regard to the spatial variable produces, we obtain

$$\begin{gathered} d\hat{u}+|\xi|^{2\alpha}\hat{u}\,dt = \hat{g}\,dW \quad \text{in } \Omega\times(0,+\infty)\times\mathbb{R}^2, \\ \hat{u}_{t=0}=\hat{u}_0 \quad \text{on } \Omega\times\mathbb{R}^2. \end{gathered} \tag{3.2}$$

we can conclude that this linear stochastic ODE has a unique solution, So by the Fourier transformation we can also obtain the solution of (3.1).We must estimate the solution of (3.1) in order to use the fixed point theory to find the solution of (1.1).

Lemma 3.1. *Let θ_0 be F_0 measurable and f progressively measurable on $\Omega \times [0,T] \times \mathbb{R}^2$, and for any $q \in [2,+\infty), \theta_0 \in \tilde{L}^4_\Omega \mathcal{F}\dot{\mathcal{N}}^{2-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}, f \in \tilde{L}^4_\Omega \tilde{L}^4_T \mathcal{F}\dot{\mathcal{N}}^{2-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}$, the solution u of (3.1) is in the space $\tilde{L}^4_\Omega \tilde{L}^4_T \mathcal{F}\dot{\mathcal{N}}^{2-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}$, and*

$$\|u\|_{\tilde{L}^4_\Omega \tilde{L}^4_T \mathcal{F}\dot{\mathcal{N}}^{2-2\alpha}_{2,\lambda,q}} \le C(1+T)\|f\|_{\tilde{L}^4_\Omega \tilde{L}^4_T \mathcal{F}\dot{\mathcal{N}}^{2-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}} + \|\theta_0\|_{\tilde{L}^4_\Omega \mathcal{F}\dot{\mathcal{N}}^{2-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}}. \tag{3.3}$$

Proof. By multiplying the first equation in (3.2) by φ_j on both sides, we obtain

$$d\varphi_j \hat{u} = -|\xi|^{2\alpha}\varphi_j \hat{u} + \varphi_j \hat{g}\, dW.$$

We apply Itô formula to $\|\varphi_j \hat{u}\|_{M_2^\lambda}$ to obtain

$$\begin{aligned}
& d\|\varphi_j \hat{u}\|^2_{M_2^\lambda} \\
&= d\big(\sup_{x\in\mathbb{R}^2}\sup_{r>0} r^{-\frac{\lambda}{2}} \|\varphi_j \hat{u}\|_{L^2(B(x,r))}\big)^2 \\
&= \sup_{x\in\mathbb{R}^2}\sup_{r>0} r^{-\lambda} d\|\varphi_j \hat{u}\|^2_{L^2(B(x,r))} \\
&= \sup_{x\in\mathbb{R}^2}\sup_{r>0} r^{-\lambda}\big[2\langle \varphi_j \hat{u}, -|\xi|^{2\alpha}(\varphi_j \hat{u}) + (\varphi_j \hat{g})\, dW\rangle + \|\varphi_j \hat{g}\|^2_{L^2(B(x,r))}\, dt\big] \\
&= \sup_{x\in\mathbb{R}^2}\sup_{r>0} r^{-\lambda}\big[(-2\||\xi|^{2\alpha}\varphi_j \hat{u}\|^2_{L^2(B(x,r))} + \|\varphi_j \hat{g}\|^2_{L^2(B(x,r))})\, dt + 2\langle \varphi_j \hat{u}, \varphi_j \hat{g}\rangle\, dW\big] \\
&\lesssim \Big(-2\||\xi|^{2\alpha}\varphi_j \hat{u}\|^2_{M_2^\lambda} + \|\varphi_j \hat{g}\|^2_{M_2^\lambda}\Big) + 2\sup_{x\in\mathbb{R}^2}\sup_{r>0} r^{-\lambda}\langle \varphi_j \hat{u}, \varphi_j \hat{g}\rangle\, dW.
\end{aligned} \tag{3.4}$$

where $\langle \cdot,\cdot\rangle$, denotes the inner product in $L^2(\mathbb{R}^2)$.

Again applying Itô formula to $(\|\varphi_j \hat{u}\|^2_{M_2^\lambda} + \varepsilon)^2$ for $\varepsilon > 0$. As a result of (3.3), we have

$$\begin{aligned}
& d(\|\varphi_j \hat{u}\|^2_{M_2^\lambda} + \varepsilon)^2 \\
&= 2(\|\varphi_j \hat{u}\|^2_{M_2^\lambda} + \varepsilon)\Big[(-2\||\xi|^{2\alpha}\varphi_j \hat{u}\|^2_{M_2^\lambda} + \|\varphi_j \hat{g}\|^2_{M_2^\lambda})\, dt \\
&\quad + 2\sup_{x\in\mathbb{R}^2}\sup_{r>0} r^{-\lambda}\langle \varphi_j \hat{u}, \varphi_j \hat{g}\rangle\, dW\Big] + 4\sup_{x\in\mathbb{R}^2}\sup_{r>0} r^{-\lambda}\langle \varphi_j \hat{g}, \varphi_j \hat{u}\rangle^2\, dt.
\end{aligned} \tag{3.5}$$

Considering the sequence of stopping times

$$\tau_N = \begin{cases} \inf\{t \ge 0 : \|\varphi_j \hat{u}\| > N\}, & \text{if } \{t : \|\varphi_j \hat{u}\| > N\} \ne \emptyset, \\ T, & \text{if } \{t : \|\varphi_j \hat{u}\| > N\} = \emptyset. \end{cases}$$

for $N = 1, 2, \ldots$. Integrating (3.5) $[0,t]$ for $t \le \min\{T, \tau_N\}$ over interval and taking the expectation of the resulting term, we obtain

$$\begin{aligned}
& \mathbb{E}(\|\varphi_j \hat{u}\|^2_{M_2^\lambda} + \varepsilon)^2 - \mathbb{E}(\|\varphi_j \hat{\theta}_0\|^2_{M_2^\lambda} + \varepsilon)^2 \\
&= 4\mathbb{E}\int_0^t \sup_{x\in\mathbb{R}^2}\sup_{r>0} r^{-\lambda}\langle \varphi_j \hat{g}, \varphi_j \hat{u}\rangle^2\, ds - 4\mathbb{E}\int_0^t \big(\|\varphi_j \hat{u}\|^2_{M_2^\lambda} + \varepsilon\big)\||\xi|^{2\alpha}\varphi_j \hat{u}\|^2_{M_2^\lambda}\, ds \\
&\quad + 2\mathbb{E}\int_0^t \big(\|\varphi_j \hat{u}\|^2_{M_2^\lambda} + \varepsilon\big)\|\varphi_j \hat{g}\|^2_{M_2^\lambda}\, ds \\
&\quad + 4\mathbb{E}\int_0^t (\|\varphi_j \hat{u}\|^2_{M_2^\lambda} + \varepsilon)\sup_{x\in\mathbb{R}^2}\sup_{r>0} r^{-\lambda}\langle \varphi_j \hat{u}, \varphi_j \hat{g}\rangle\, dW \\
&:= A_1 + A_2 + A_3 + A_4.
\end{aligned}$$

Next we estimate A_1, A_3 and A_4 since A_2 is already the form needed. For that we apply Hölder's and Young's inequalities.

$$\begin{aligned} A_1 &\lesssim \mathbb{E}\int_0^t \|\varphi_j\hat{g}\|^2_{M_2^\lambda}\|\varphi_j\hat{u}\|^2_{M_2^\lambda}\,ds \\ &\lesssim \mathbb{E}\sup_{s\in[0,t]}\|\varphi_j\hat{u}\|^2_{M_2^\lambda}\int_0^t \|\varphi_j\hat{g}\|^2_{M_2^\lambda}\,ds \\ &\lesssim \varepsilon\mathbb{E}\sup_{s\in[0,t]}\|\varphi_j\hat{u}\|^4_{M_2^\lambda} + C_\varepsilon tE\int_0^t \|\varphi_j\hat{g}\|^4_{M_2^\lambda}\,ds. \end{aligned}$$

$$A_3 \lesssim \varepsilon\mathbb{E}\sup_{s\in[0,t]}(\|\varphi_j\hat{u}\|^2_{M_2^\lambda}+\varepsilon)^2 + C_\varepsilon tE\int_0^t \|\varphi_j\hat{g}\|^4_{M_2^\lambda}\,ds.$$

Fore estimating the random integral A_4, we apply the Burkholder-Davis-Gundy inequality and Young's inequality.

$$\begin{aligned} A_4 &\lesssim \mathbb{E}\sup_{s'\in[0,t]}\Big|\int_0^{s'}(\|\varphi_j\hat{u}\|^2_{M_2^\lambda}+\varepsilon)\langle\varphi_j\hat{u},\ \varphi_j\hat{g}\rangle dW\Big| \\ &\lesssim \mathbb{E}\sup_{s\in[0,t]}(\|\varphi_j\hat{u}\|^2_{M_2^\lambda}+\varepsilon)\|\varphi_j\hat{u}\|_{M_2^\lambda}\Big(\int_0^t \|\varphi_j\hat{g}\|^2_{M_2^\lambda}ds\Big)^{1/2} \\ &\lesssim \varepsilon\mathbb{E}\sup_{s\in[0,t]}[(\|\varphi_j\hat{u}\|^2_{M_2^\lambda}+\varepsilon)\|\varphi_j\hat{u}\|_{M_2^\lambda}]^{\frac{4}{3}} + C_\varepsilon\mathbb{E}t\int_0^t \|\varphi_j\hat{g}\|^4_{M_2^\lambda}ds. \end{aligned}$$

Combining the estimates of A_1, A_2, A_3 and A_4, assuming $\varepsilon > 0$ to be sufficiently small and passing to the limit as $\varepsilon \to 0$, we obtain

$$\begin{aligned} &\mathbb{E}\sup_{t\in[0,T\wedge\tau_N]}\|\varphi_j\hat{u}\|^4_{M_2^\lambda} + \mathbb{E}\int_0^{T\wedge\tau_N}\|\varphi_j\hat{u}\|^2_{M_2^\lambda}\||\xi|^{2\alpha}\varphi_j\hat{u}\|^2_{M_2^\lambda}ds \\ &\lesssim \mathbb{E}\|\varphi_j\hat{\theta}_0\|^4_{M_2^\lambda} + [1+(T\wedge\tau_N)]\mathbb{E}\int_0^{T\wedge\tau_N}\|\varphi_j\hat{g}\|^4_{M_2^\lambda}ds. \end{aligned} \tag{3.6}$$

Considering the conditions on θ_0 and g, thus $\mathbb{E}\Big(\sup_{t\in[0,T\wedge\tau]}\|\varphi_j\hat{v}\|^4\Big)$ is bounded by a constant independent of N by (3.6). Hence, let $N \to \infty$ and consider $\lim_{N\to\infty}\tau_N = T$, almost certainly. Applying (2.5), we have

$$\begin{aligned} &\mathbb{E}\sup_{t\in[0,T]}\|\varphi_j\hat{u}\|^4_{M_2^\lambda} + 2^{2\alpha j}\mathbb{E}\int_0^T \|\varphi_j\hat{u}\|^4_{M_2^\lambda}ds \\ &\lesssim \mathbb{E}\|\varphi_j\hat{\theta}_0\|^4_{M_2^\lambda} + (1+T)\mathbb{E}\int_0^T \|\varphi_j\hat{g}\|^4_{M_2^\lambda}ds. \end{aligned} \tag{3.7}$$

Hence,

$$2^{2\alpha j}\mathbb{E}\int_0^T \|\varphi_j\hat{u}\|^4_{M_2^\lambda}\,ds \lesssim \mathbb{E}\|\varphi_j\hat{\theta}_0\|^4_{M_2^\lambda} + (1+T)\mathbb{E}\int_0^T \|\varphi_j\hat{g}\|^4_{M_2^\lambda}\,ds.$$

Multiplying the above estimate by $2^{(\frac{3}{2}-2\alpha+\frac{\lambda}{2})j}$ and taking l^q norm, we obtain

$$\|u\|_{\tilde{L}^4_\Omega\tilde{L}^4_T\mathcal{F}\dot{\mathcal{N}}^{2-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}} \le C(1+T)\|g\|_{\tilde{L}^4_\Omega\tilde{L}^4_T\mathcal{F}\dot{\mathcal{N}}^{2-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}} + \|\theta_0\|_{\tilde{L}^4_\Omega\mathcal{F}\dot{\mathcal{N}}^{2-2\alpha+\frac{\lambda}{2}}_{2,q}},$$

where the condition $\alpha \le \frac{3}{2}+\frac{\lambda}{4}$ (i.e, $\frac{3}{2}-2\alpha+\frac{\lambda}{2} \ge 0$). Which yields the conclusion. □

Lemma 3.2 ([20])**.** *. Under condition 3.1, for the solution u of* (3.1)*, there is a set $\widetilde{\Omega}$ with positive probability such that $u(\omega,\cdot,\cdot) \in \tilde{L}^4_T\mathcal{F}\dot{\mathcal{N}}^{2-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}$ with*

$$\|u(\omega,\cdot,\cdot)\|_{\tilde{L}^4_T\mathcal{F}\dot{\mathcal{N}}^{2-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}} \le C_1\big[\|\theta_0\|_{\tilde{L}^4_\Omega\mathcal{F}\dot{\mathcal{N}}^{2-2\alpha}_{2,\lambda q}+\frac{\lambda}{2}} + (1+T)\|g\|_{\tilde{L}^4_\Omega\tilde{L}^4_T\mathcal{F}\dot{\mathcal{N}}^{2-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}}\big],$$

for all $\omega \in \widetilde{\Omega}$, where C_1 is a constant.

Proof of Theorem 1.5. We take the working space $Z := \tilde{L}^4_T\mathcal{F}\dot{\mathcal{N}}^{2-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}$ and define mappings

$$\Psi(\theta) = T_\alpha(t)\theta_0 + \int_0^t T_\alpha(t-s)g\,dW - \int_0^t T_\alpha(t-s)(V_\theta\cdot\nabla\theta)(s)\,ds,$$

$$e_0 = T_\alpha(t)\theta_0 + \int_0^t T_\alpha(t-s)g\,dW,$$

$$B(\theta,\xi) = -\int_0^t T_\alpha(t-s)(V_\theta\cdot\nabla\xi)(s)\,ds.$$

Thus, to solve (1.1) it is sufficient to search for the fixed point of the mapping $\Psi(\theta)$ by (2.9). By the bilinear estimate in Theorem 1.4 [2], we have

$$\|\int_0^t T_\alpha(t-s)(V_\theta\cdot\nabla\theta)(s)\,ds\|_Z \le C_0\|\theta\|_Z\|\theta\|_Z.$$

Thus, Lemma 3.2 leads to

$$\begin{aligned}\|\Psi(\theta)\|_Z &\le \|T_\alpha(t)\theta_0 + \int_0^t T_\alpha(t-s)g\,dW\|_Z + \|\int_0^t T_\alpha(t-s)(V_\theta\cdot\nabla\theta)(s)\,ds\|_Z\\ &\le \|u(\omega,\cdot,\cdot)\|_{\tilde{L}^4_T\mathcal{F}\dot{\mathcal{N}}^{2-2\alpha+\frac{\lambda}{2}}_{2,q}} + C_0\|\theta\|_Z^2.\end{aligned}$$

By (2.9), All that remains is for us to prove that

$$\|u(\omega,\ \cdot,\ \cdot)\|_{\tilde{L}^4_T\mathcal{F}\dot{\mathcal{N}}^{2-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}} \le \frac{1}{4C_1C_0},$$

with positive probability. Indeed, by Lemma 3.2 we have

$$\begin{aligned}\|u(\omega,\cdot,\cdot)\|_{\tilde{L}^4_T\mathcal{F}\dot{\mathcal{N}}^{2-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}} &\le C_1\big[\|\theta_0\|_{\tilde{L}^4_\Omega\mathcal{F}\dot{\mathcal{N}}^{2-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}} + (1+T)\|g\|_{\tilde{L}^4_\Omega\tilde{L}^4_T\mathcal{F}\dot{\mathcal{N}}^{2-2\alpha+\frac{\lambda}{2}}_{2,\lambda,q}}\big]\\ &\le \frac{1}{4C_0},\end{aligned}$$

for all $\omega \in \widetilde{\Omega}$. This completes the proof. □

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