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SOLVABILITY OF AN ATTRACTION-REPULSION CHEMOTAXIS NAVIER-STOKES SYSTEM WITH ARBITRARY POROUS MEDIUM DIFFUSION

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Abstract. In this work, we proposed a model that describes the influence of two chemically opposed stimuli in the movement of species living in a fluid environment. We investigated the well-posedness of a system that models the attraction-repulsion chemotaxis Navier-Stokes system with nonlinear diffusion. We validate the existence of a global three-dimensional weak solution. Furthermore, with some restrictions on the nonlinear exponent and degradation coefficients of the chemical signal, we established the existence of a three-dimensional global bounded weak solutions for the system.

1. Introduction

Chemotaxis is a fascinating natural occurrence that involves organisms moving in response to chemical signals in their surroundings. It is a trait possessed by all motile organisms, ranging from single-celled bacteria to complex multicellular beings. This phenomenon plays significant importance in several biological processes, such as immune responses, wound healing, and the movement of microorganisms in their respective habitats. The chemotaxis system was modeled by Keller and Segel [19], and their model has provided a framework for studying this fascinating phenomenon as

$$\begin{aligned} u_t &= \Delta u - \nabla \cdot (\chi u \nabla v), \\ v_t &= \Delta v - \alpha v + \beta u \end{aligned} \tag{1.1}$$

where u refers to species density, v refers to concentration of signal and χ refers to sensitivity function. The global weak solution for the quasilinear degenerate Keller-Segel system of the parabolic-parabolic type was established in [15]. The global weak solutions and their decay properties were discussed for the degenerate Keller-Segel model in [27]. Different results about the singularity, boundedness, and global existence of solutions were obtained for a simplified form of (1.1) in [12, 13, 26].

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A model that describes the behavior of bacteria living near solid-air-water contact lines was developed in [31] as

$$\begin{gathered} u_t + z \cdot \nabla u = \Delta u - \nabla \cdot (\chi(v) u \nabla v), \\ v_t + z \cdot \nabla v = \Delta v - k(v) u, \\ z_t + \tau (z \cdot \nabla) z + \nabla p = \Delta z - u \nabla \phi, \\ \operatorname{div} z = 0 \end{gathered} \tag{1.2}$$

where u refers to cell density, v refers to the concentration of oxygen and z, p refer to the velocity field and pressure of the fluid, respectively. When $\tau = 1$, the system (1.2) is called Keller-Segel-Navier-Stokes system (KS-NS) whereas $\tau = 0$ is called as Keller-Segel-Stokes (KSS) system. The global existence of classical solutions of KS-NS near-constant states in $\mathbb{R}^3$ and the existence of a global weak solution in $\mathbb{R}^2$ under some assumptions were studied in [8]. A numerical method for solving KS-NS was introduced in [5] with a full exploration of its dynamics. The global-in-time existence of weak solutions of KSS under some assumptions was established in [24]. The local-in-time smooth solution for the Keller-Segel-Navier-Stokes (KS-NS) system under minimal assumptions on the sensitivity function and oxygen consumption rate was established in [1]. Additionally, it was demonstrated that the local and global existence of solutions for the KS-NS system when the oxygen concentration follows either a hyperbolic or parabolic type, as detailed in [3]. The global weak solutions for KSS with regularity in initial data and boundedness of large of large-data solutions were investigated in [32]. Global-in-time classical solution of KSS value was established in [21] for specific initial. Results on weak and bounded weak solutions, mild solutions, and boundedness for KS-NS and KSS under some assumption on χ, k, and diffusion property were established in [4, 14, 17, 18, 20].

Suppose the diffusion of the cell is considered as migration in a porous medium then (1.2) becomes

$$\begin{gathered} u_t + z \cdot \nabla u = \Delta u^{1+\alpha} - \nabla \cdot (\chi u \nabla v), \\ v_t + z \cdot \nabla v = \Delta v - k(v) u, \\ z_t + \tau (z \cdot \nabla) z + \nabla p = \Delta z - u \nabla \phi, \\ \operatorname{div} z = 0. \end{gathered} \tag{1.3}$$

The existence of global-in-time solution for (1.3) with $\tau = 0$ in $\mathbb{R}^2$ established in [11], when $\alpha \in (\frac{1}{2}, 1]$ and the same have been established for $\alpha = \frac{1}{3}$ in [22]. Also, global existence of weak solutions for (1.3) was established in [11] for a bounded domain in $\mathbb{R}^3$ with $\alpha \in [0.8, 1]$ and $\tau = 1$. The global existence of weak solutions of (1.3) with $\alpha > \frac{1}{7}$ and $\tau = 0$ was discussed in [30] under the assumption that initial data are sufficiently regular and positive. The global existence of bounded weak solutions of (1.3) with $\alpha > 0$ and $\tau = 0$ studied in [29] under some assumptions on initial data. The weak and bounded weak solutions were established globally in [7] under some assumptions on χ, k, and diffusive exponent.

The movement of organisms in biological processes is greatly impacted by chemical stimuli, which can either attract or repel them. Chemoattraction represents an organism moving towards an increasing signal concentration, and chemorepulsion means that an organism moves away from an increasing signal concentration, which are both important factors in chemical migration. A recent study in [25] focused on modeling a system that highlights the chemotactic response of microglial cells as

$$\begin{aligned} u_t &= \Delta u - \nabla\cdot(\chi(v)u\nabla v) + \nabla\cdot(\xi(w)u\nabla w), \\ v_t &= \Delta v + \beta u - \gamma v, \\ w_t &= \Delta w + \delta u - \eta w \end{aligned} \tag{1.4}$$

where u denotes species density, v denotes concentration of chemoattractant, and w denotes concentration of chemorepellant. Global solvability, the existence of steady states and blow up of (1.4)were discussed in [28] with the assumption of $\gamma = \eta$. Also, the same generalized for $\gamma, \eta > 0$ in [10]. The classical solutions and steady states of (1.4) were studied globally in [16] for one dimension. The existence of a global bounded classical solution for (1.4) with logistic source under the assumption of limitation on growth in the logistic source was established in [33].

One interesting phenomenon in biology is the migration of species, which is thought to be influenced by attraction-repulsion chemotaxis signals in fluid. As a result, equation (1.4) can be coupled with the Navier-Stokes equation to better understand the dynamics of these phenomena as

$$\begin{aligned} u_t &= \Delta u - \nabla\cdot(\chi(v)u\nabla v) + \nabla\cdot(\xi(w)u\nabla w), \\ v_t &= \Delta v + \beta u - \gamma v, \\ w_t &= \Delta w + \delta u - \eta w, \\ z_t + \tau(z\cdot\nabla)z + \nabla p &= \Delta z - u\nabla\phi \end{aligned} \tag{1.5}$$

where u denotes density of bacteria, v denotes concentration of attraction signal, w refers concentration of repulsive signal, and z refers velocity field of the fluid. The global-in-time classical solution for (1.5) with logistic source under the assumption that initial data are sufficiently regular and non-negative was discussed in [34]. The uniqueness and existence of mild global solutions were discussed in [9] in the Besov-Morrey type of (1.5).

However, there is no paper in the literature to study the attraction-repulsion chemotaxis-Navier-Stokes equation with nonlinear diffusion exponent $1+\alpha$. In this paper, we considered (1.5) with the diffusion of the bacteria assumed as migration in a porous medium, in $\mathbb{R}^3 \times [0,T)$ with $T > 0$, as

$$\begin{aligned} u_t + z\cdot\nabla u &= \Delta u^{1+\alpha} - \nabla\cdot(\chi(v)u\nabla v) + \nabla\cdot(\xi(w)u\nabla w), \\ v_t + z\cdot\nabla v &= \Delta v + \beta u - \gamma v, \\ w_t + z\cdot\nabla w &= \Delta w + \delta u - \eta w, \\ z_t + \tau(z\cdot\nabla)z + \nabla p &= \Delta z - u\nabla\phi, \\ \operatorname{div} z &= 0, \end{aligned} \tag{1.6}$$

where u refers species density, v refers concentration of attraction signal, w refers concentration of repulsive signal, z refers velocity field of the fluid, p refers hydrostatic pressure, ϕ refers gravitational potential, ν refers outward normal vector, χ and ξ are non-negative sensitivity functions and $\alpha, \beta, \gamma, \delta, \eta$ are positive constants. Therefore, in this work, we have attempted to study the existence of weak and bounded weak solutions of the proposed model for $\alpha > 0$ and $\alpha > 1/8$ respectively with some assumption in the chemotactic sensitivity function.

This paper is organized as follows. In section 2, we defined the weak solution and suitable approximation problem of the model (1.6) with $\tau = 1$. Also, a priori estimate is derived using the approximation problem and weak solution established globally using the priori estimate. In section 3, we defined the bounded weak solution and suitable approximation for the proposed model (1.6) with $\tau = 0$. We demonstrated the existence of global bounded weak solution.

2. Existence of weak solutions

We considered the attraction-repulsion chemotaxis Navier-Stokes system, that is, (1.6) with $\tau = 1$ in this secion, as follows:

$$\begin{gathered}
u_t + z\cdot\nabla u = \Delta u^{1+\alpha} - \nabla\cdot(\chi(v)u\nabla v) + \nabla\cdot(\xi(w)u\nabla w),\\
v_t + z\cdot\nabla v = \Delta v + \beta u - \gamma v,\\
w_t + z\cdot\nabla w = \Delta w + \delta u - \eta w,\\
z_t + (z\cdot\nabla)z + \nabla p = \Delta z - u\nabla\phi,\\
\operatorname{div} z = 0.
\end{gathered}\tag{2.1}$$

This section first defines the weak solution for (2.1). Then, we introduce the suitable approximation problem for (2.1). Before demonstrating our main finding, namely weak solutions to (2.1) in $\mathbb{R}^3$, we first establish some key lemma used to substantiate our main findings.

Definition 2.1. For $\alpha > 0$ and $T \in (0,\infty)$, the unknown functions $u, v, w \geq 0$ and z represents a vector function on $\mathbb{R}^3 \times (0,T)$, (u,v,w,z) is termed as a weak solution of (2.1) if

(1) $u(1+|x|+|\log u|) \in L^\infty(0,T;L^1(\mathbb{R}^3))$,
(2) Given $p \in [1, 1+\alpha]$, $u \in L^\infty(0,T;L^p(\mathbb{R}^3))$, and $\nabla u^{\frac{p+\alpha}{2}} \in L^2(0,T;L^2(\mathbb{R}^3))$,
(3) $v, w \in L^\infty(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^2(\mathbb{R}^3))$,
(4) $v,\ w \in L^\infty(\mathbb{R}^3 \times [0,T))$,
(5) $z \in L^\infty(0,T;L^2(\mathbb{R}^3))$,
(6) $\nabla z \in L^2(0,T;L^2(\mathbb{R}^3))$,
(7) Given a test function $\varphi \in C_0^\infty(\mathbb{R}^3 \times [0,T))$, it holds

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3}(-u\varphi_t - zu\cdot\nabla\varphi + \nabla u^{1+\alpha}\cdot\nabla\varphi - u\chi(v)\nabla v\cdot\nabla\varphi + u\xi(w)\nabla w\cdot\nabla\varphi)\,dx\,dt\\
&= \int_{\mathbb{R}^3} u_0\varphi(\cdot,0)dx,
\end{aligned}$$

$$\int_0^t\int_{\mathbb{R}^3}(-v\varphi_t - zv\cdot\nabla\varphi + \nabla v\cdot\nabla\varphi - \beta u\varphi + \gamma v\varphi)\,dx\,dt = \int_{\mathbb{R}^3} v_0\varphi(\cdot,0)dx,$$

$$\int_0^t\int_{\mathbb{R}^3}(-w\varphi_t - zw\cdot\nabla\varphi + \nabla w\cdot\nabla\varphi - \delta u\varphi + \eta w\varphi)\,dx\,dt = \int_{\mathbb{R}^3} w_0\varphi(\cdot,0)dx,$$

(8) Given a test function $\psi \in C_0^\infty(\mathbb{R}^3 \times [0,T), \mathbb{R}^3)$ with $\nabla\cdot\psi = 0$, it holds

$$\int_0^t\int_{\mathbb{R}^3}(-z\cdot\psi_t + \nabla z\cdot\nabla\psi + ((z\cdot\nabla)z)\cdot\psi + u\nabla\phi\cdot\psi)\,dx\,dt = \int_{\mathbb{R}^3} z_0\cdot\psi(\cdot,0)dx.$$

The problem we are dealing with here is quite complex due to the strong presence of degeneracy in the diffusion terms. Hence, we introduced an approximation problem to the proposed model (2.1) which allows us to overcome the degeneracy problem and make progress in our analysis. By considering the regularized problem of (2.1) as below, we have made significant strides in understanding the underlying dynamics and behavior of the system.

$$\begin{gathered} u_{\epsilon_t} + z_\epsilon \cdot \nabla u_\epsilon = \Delta(u_\epsilon + \epsilon)^{1+\alpha} - \nabla \cdot (\chi(v_\epsilon) u_\epsilon \nabla v_\epsilon) + \nabla \cdot (\xi(w_\epsilon) u_\epsilon \nabla w_\epsilon), \\ v_{\epsilon_t} + z_\epsilon \cdot \nabla v_\epsilon = \Delta v_\epsilon + \beta u_\epsilon - \gamma v_\epsilon, \\ w_{\epsilon_t} + z_\epsilon \cdot \nabla w_\epsilon = \Delta w_\epsilon + \delta u_\epsilon - \eta w_\epsilon, \\ z_{\epsilon_t} + (z_\epsilon \cdot \nabla) z_\epsilon + \nabla p_\epsilon = \Delta z_\epsilon - u_\epsilon \nabla \phi, \\ \operatorname{div} z_\epsilon = 0 \end{gathered} \tag{2.2}$$

with initial conditions

$$u_{0_\epsilon} = \phi_\epsilon * u_0, \quad v_{0_\epsilon} = \phi_\epsilon * v_0, \quad w_{0_\epsilon} = \phi_\epsilon * w_0, \quad z_{0_\epsilon} = \phi_\epsilon * z_0,$$

where ϕ_ϵ is a usual mollifier with $\epsilon \in (0,1)$. Hereafter, throughout the paper we use all unknown $(u_\epsilon, v_\epsilon, w_\epsilon, z_\epsilon)$ as (u, v, w, z) for notation simplicity. According to the standard theory of existence and regularity, for every $\epsilon > 0$, equation (2.2) admits a local-in-time classical solution. The proof for this assertion is not provided here because it follows a similar methodology as in [2, 23].

Consider the functional

$$E(t) := \int_{\mathbb{R}^3} u(\log u + 2\langle x \rangle)\, dx + \|u\|_{1+\alpha}^{1+\alpha} + \|\nabla v\|_2^2 + \|\nabla w\|_2^2 + \|z\|_2^2 \tag{2.3}$$

and

$$D(t) := \|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + \|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + \|\Delta v\|_2^2 + \|\Delta w\|_2^2 + \|\nabla z\|_2^2, \tag{2.4}$$

where $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$. Next, we present two lemmas with some assumptions on sensitivity functions and nonlinear exponent, which play a major role in establishing the main result of the paper.

Lemma 2.2. *Let (u, v, w, z) be a classical solution of (2.2), for all $\epsilon \in (0,1)$ and initial data $(u_{0_\epsilon}, v_{0_\epsilon}, w_{0_\epsilon}, z_{0_\epsilon})$ satisfies the following independent of ϵ:*

(1) $u_{0_\epsilon}(1 + |x| + |\log u_{0_\epsilon}|) \in L^1(\mathbb{R}^3)$,
(2) $u_{0_\epsilon} \in L^{1+\alpha}(\mathbb{R}^3)$,
(3) $v_{0_\epsilon},\ w_{0_\epsilon} \in L^\infty(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$,
(4) $z_{0_\epsilon} \in L^2(\mathbb{R}^3)$.

Assume that

$$\alpha > \frac{1}{6}, \quad \chi', \xi' \in L^\infty_{\rm loc}, \quad \phi \in W^{2,\infty}(\mathbb{R}^3). \tag{2.5}$$

Then, given $t \in (0, T])$,

$$\sup_{0 \le \tau \le t} E(\tau) + \int_0^t D(\tau) d\tau < C, \tag{2.6}$$

where $C > 0$ is a constant solely depends on initial data.

Proof. By integrating $(2.2)_1$, we can obtain $\|u(t)\|_1 \equiv \|u_0\|_1$, which leads us to the conclusion that the total mass of u conserved. Additionally, by applying the maximal principle to $(2.2)_2$ and $(2.2)_3$, we obtain $\|v\|_{L^\infty(\mathbb{R}^3 \times [0,T))} \le \|v_0\|_\infty$ and $\|w\|_{L^\infty(\mathbb{R}^3 \times [0,T))} \le \|w_0\|_\infty$.

Case (i): $1/6 < \alpha \le 1/3$. First, multiply $(2.2)_1$ by $\log u$ and integrating to obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^3} u\log u\,dx + \int_{\mathbb{R}^3} \nabla \log u \cdot \nabla (u+\epsilon)^{1+\alpha}\,dx \\ &= \int_{\mathbb{R}^3} \nabla u \cdot (\chi \nabla v)\,dx - \int_{\mathbb{R}^3} \nabla u \cdot (\xi \nabla w)\,dx. \end{aligned} \tag{2.7}$$

The second term in LHS of above is evaluated as

$$\begin{aligned} \int_{\mathbb{R}^3} \nabla \log u \cdot \nabla (u+\epsilon)^{1+\alpha}\,dx &\ge \int_{\mathbb{R}^3} \nabla \log u \cdot (1+\alpha)u^{\alpha}\nabla u\,dx \\ &= \frac{4}{1+\alpha}\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2. \end{aligned} \tag{2.8}$$

The first term in RHS of (2.7) is evaluated using Young's inequality as

$$\int_{\mathbb{R}^3} \nabla u \cdot (\chi \nabla v)\,dx \le \frac{2\overline{\chi}}{1+\alpha}\Big(\epsilon_1 \|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + C(\epsilon_1)\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\,dx\Big), \tag{2.9}$$

where we used that $|\nabla u| = \frac{2}{1+\alpha}u^{\frac{1-\alpha}{2}}\big|\nabla u^{\frac{1+\alpha}{2}}\big|$ and $\overline{\chi} := \sup_{\mathbb{R}^3\times[0,T)}|\chi(v)|$.

The last term in RHS of (2.7) is evaluated using Young's inequality as

$$\begin{aligned} &\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\,dx \\ &= \int_{\mathbb{R}^3} u^{1-\alpha}\nabla v \cdot \nabla v\,dx \\ &\le C_1\Big(\int_{\mathbb{R}^3} |\nabla u^{1-\alpha}||\nabla v|\,dx + \int_{\mathbb{R}^3} u^{1-\alpha}|\Delta v|\,dx\Big) \\ &\le C_1\Big(\int_{\mathbb{R}^3} C_2 u^{\frac{1-3\alpha}{2}}\big|\nabla u^{\frac{1+\alpha}{2}}\big||\nabla v|\,dx + \int_{\mathbb{R}^3} u^{1-\alpha}|\Delta v|\,dx\Big) \\ &\le C_1C_2\int_{\mathbb{R}^3} \epsilon_2\big|\nabla u^{\frac{1+\alpha}{2}}\big|^2 + C(\epsilon_2)u^{1-3\alpha}|\nabla v|^2\,dx + C_1\int_{\mathbb{R}^3} u^{1-\alpha}|\Delta v|\,dx, \end{aligned}$$

where we used that $\big|\nabla u^{1-\alpha}\big| = \frac{2(1-\alpha)}{(1+\alpha)}u^{\frac{1-3\alpha}{2}}\big|\nabla u^{\frac{1+\alpha}{2}}\big|$. Using the above in (2.9), we obtain

$$\begin{aligned} &\int_{\mathbb{R}^3} \nabla u \cdot (\chi \nabla v)\,dx \\ &\le C_1''\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + C_2''\int_{\mathbb{R}^3} u^{1-3\alpha}|\nabla v|^2\,dx + C_3''\int_{\mathbb{R}^3} u^{1-\alpha}|\Delta v|\,dx, \end{aligned} \tag{2.10}$$

where $C_i'' = \frac{2\overline{\chi}C_i'}{1+\alpha}$ for $i \in \{1,2,3\}$, $C_1' = \epsilon_1 + C_1C_2C(\epsilon_1)\epsilon_2$, $C_2' = C_1C_2C(\epsilon_1)C(\epsilon_2)$ and $C_3' = C_1C(\epsilon_1)$. Following the same procedure as above, we obtain

$$\begin{aligned} &-\int_{\mathbb{R}^3} \nabla u \cdot (\xi \nabla w)\,dx \\ &\le C_4'\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + C_5'\int_{\mathbb{R}^3} u^{1-3\alpha}|\nabla w|^2\,dx + C_6'\int_{\mathbb{R}^3} u^{1-\alpha}|\Delta w|\,dx, \end{aligned} \tag{2.11}$$

where $C_i', i \in \{4,5,6\}$ are positive constants. Using (2.8), (2.10) and (2.11) in (2.7), we obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^3} u\log u\,dx + C'\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 \\ &\le C_2''\int_{\mathbb{R}^3} u^{1-3\alpha}|\nabla v|^2\,dx + C_3''\int_{\mathbb{R}^3} u^{1-\alpha}|\Delta v|\,dx \\ &\quad + C_5'\int_{\mathbb{R}^3} u^{1-3\alpha}|\nabla w|^2\,dx + C_6'\int_{\mathbb{R}^3} u^{1-\alpha}|\Delta w|\,dx, \end{aligned} \tag{2.12}$$

where $C' = \frac{4}{1+\alpha} - C_1'' - C_4'$. Multiply $(2.2)_1$ by $\langle x\rangle$ and integrate over $\mathbb{R}^3$. Then using Young's inequality along with simple algebraic calculations (See [22, (3.6)] and [6, (2.13)]) leads to

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^3}\langle x\rangle u\,dx &= \int_{\mathbb{R}^3} uz\cdot\nabla\langle x\rangle\,dx + \int_{\mathbb{R}^3}(u+\epsilon)^{1+\alpha}\Delta\langle x\rangle\,dx \\ &\quad + \int_{\mathbb{R}^3}\nabla\langle x\rangle\cdot u\chi\nabla v\,dx - \int_{\mathbb{R}^3}\nabla\langle x\rangle\cdot u\xi\nabla w\,dx \\ &\le C_7\big(1+\|z\|_2^2+\|\nabla v\|_2^2+\|\nabla w\|_2^2\big) + \big(C(\epsilon)+\epsilon\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2\big). \end{aligned} \tag{2.13}$$

Multiply $(2.2)_1$ by u^α and integrating to obtain

$$\begin{aligned} &\frac{1}{1+\alpha}\frac{d}{dt}\|u\|_{1+\alpha}^{1+\alpha} + \int_{\mathbb{R}^3}\nabla u^\alpha\cdot\nabla(u+\epsilon)^{1+\alpha}\,dx \\ &= \int_{\mathbb{R}^3}\nabla u^\alpha\cdot u(\chi\nabla v)\,dx - \int_{\mathbb{R}^3}\nabla u^\alpha\cdot u(\xi\nabla w)\,dx. \end{aligned} \tag{2.14}$$

Using $|\nabla u^\alpha| = \alpha u^{\frac{\alpha-1}{2}}|\nabla u|$, $|u^{1+\alpha}| = (1+\alpha)u^{\frac{\alpha}{2}}|\nabla u|$, and

$$|\nabla u^{\frac{1+2\alpha}{2}}|^2 = \frac{(1+2\alpha)^2}{4}u^{\frac{2\alpha-1}{2}}|\nabla u|^2,$$

we obtain

$$\int_{\mathbb{R}^3}\nabla u^\alpha\cdot\nabla(u+\epsilon)^{1+\alpha}\,dx \ge \frac{4\alpha(1+\alpha)}{(1+2\alpha)^2}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2. \tag{2.15}$$

The first term in RHS of (2.14) evaluated using Young's inequality as

$$\begin{aligned} \int_{\mathbb{R}^3}\nabla u^\alpha\cdot u(\chi\nabla v)\,dx &\le C_\chi\int_{\mathbb{R}^3}\big|\nabla u^{\frac{1+2\alpha}{2}}\big|(u^{\frac12}|\nabla v|)\,dx \\ &\le C_\chi\epsilon_3\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + C_\chi C(\epsilon_3)\int_{\mathbb{R}^3} u|\nabla v|^2\,dx. \end{aligned} \tag{2.16}$$

The last term in above evaluated using Young's inequality as

$$\begin{aligned} &\int_{\mathbb{R}^3} u|\nabla v|^2\,dx \\ &\le C_9\int_{\mathbb{R}^3}|\nabla u||\nabla v| + u|\Delta v|\,dx \\ &\le \frac{2C_9}{1+2\alpha}\int_{\mathbb{R}^3} u^{\frac{1-2\alpha}{2}}\big|\nabla u^{\frac{1+2\alpha}{2}}\big||\nabla v|\,dx + C_9\int_{\mathbb{R}^3} u|\Delta v|\,dx \\ &\le \frac{2C_9\epsilon_4}{1+2\alpha}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + \frac{2C_9C(\epsilon_4)}{1+2\alpha}\int_{\mathbb{R}^3} u^{1-2\alpha}|\nabla v|^2\,dx + C_9\int_{\mathbb{R}^3} u|\Delta v|\,dx. \end{aligned}$$

Using the above inequality in (2.16), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^3} \nabla u^\alpha \cdot u(\chi \nabla v)\, dx \\ &\le C_{10}' \|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + C_{11}' \int_{\mathbb{R}^3} u^{1-2\alpha} |\nabla v|^2\, dx + C_{12}' \int_{\mathbb{R}^3} u|\Delta v|\, dx,\end{aligned} \tag{2.17}$$

where $\overline{\chi} := \sup_{\mathbb{R}^3\times[0,T)} |\chi(v)|$, $C_\chi = \frac{2\alpha\overline{\chi}}{1+\alpha}$, $C_{10}' = C_\chi \epsilon_3 + \frac{2C_\chi C_9 C(\epsilon_3)\epsilon_4}{1+2\alpha}$, $C_{11}' = \frac{2C_9 C_\chi C(\epsilon_3) C(\epsilon_4)}{1+2\alpha}$ and $C_{12}' = C_\chi C_9 C(\epsilon_3)$. Following the same procedure as above, we obtain

$$\begin{aligned}&-\int_{\mathbb{R}^3} \nabla u^\alpha \cdot u(\xi \nabla w)\, dx \\ &\le C_{13}' \|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + C_{14}' \int_{\mathbb{R}^3} u^{1-2\alpha} |\nabla w|^2\, dx + C_{15}' \int_{\mathbb{R}^3} u|\Delta w|\, dx,\end{aligned} \tag{2.18}$$

where $C_i', i \in \{13, 14, 15\}$ are positive constants. Using (2.15), (2.17) and (2.18) in (2.14), we have

$$\begin{aligned}&\frac{d}{dt}\|u\|_{1+\alpha}^{1+\alpha} + C_{15}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 \\ &\le C_{11}' \int_{\mathbb{R}^3} u^{1-2\alpha}|\nabla v|^2\, dx + C_{14}' \int_{\mathbb{R}^3} u^{1-2\alpha}|\nabla w|^2\, dx + C_{12}' \int_{\mathbb{R}^3} u|\Delta v| dx \\ &\quad + C_{15}' \int_{\mathbb{R}^3} u|\Delta w| dx,\end{aligned} \tag{2.19}$$

where $C_{15} = \frac{4\alpha(1+\alpha)^2}{(1+2\alpha)^2} - C_{10}' - C_{13}'$.

Multiplying $(2.2)_2$ by $-\Delta v$ and integrating to obtain

$$\frac{d}{dt}\|\nabla v\|_2^2 + 2\|\Delta v\|_2^2 \le \int_{\mathbb{R}^3} \Delta v \cdot (z \cdot \nabla v) dx - \beta \int_{\mathbb{R}^3} u|\Delta v| dx + \gamma \int_{\mathbb{R}^3} v|\Delta v| dx. \tag{2.20}$$

The first term in RHS of above is evaluated as

$$\int_{\mathbb{R}^3} \Delta v \cdot (z \cdot \nabla v) dx = \int_{\mathbb{R}^3} \sum_{i,j} v\ \partial_i \partial_j v \partial_i z_j\, dx \le C_0 \|\Delta v\|_2^2 \|\nabla z\|_2^2.$$

Using the above in (2.20), we obtain

$$\frac{d}{dt}\|\nabla v\|_2^2 + 2\|\Delta v\|_2^2 \le C_0 \|\Delta v\|_2^2 \|\nabla z\|_2^2 - \beta \int_{\mathbb{R}^3} u|\Delta v|\, dx + \gamma \int_{\mathbb{R}^3} v|\Delta v|\, dx. \tag{2.21}$$

Following similar procedure as above, we obtain

$$\frac{d}{dt}\|\nabla w\|_2^2 + 2\|\Delta w\|_2^2 \le C_1 \|\Delta w\|_2^2 \|\nabla z\|_2^2 - \delta \int_{\mathbb{R}^3} u|\Delta w| dx + \eta \int_{\mathbb{R}^3} w|\Delta w| dx. \tag{2.22}$$

Multiply $(2.2)_4$ by z and integrating, we obtain

$$\frac{d}{dt}\|z\|_2^2 + 2\|\nabla z\|_2^2 \le C_{16}' \int_{\mathbb{R}^3} u|z|\, dx, \tag{2.23}$$

where $C'_{16} = \|\nabla\phi\|_{L^\infty(\mathbb{R}^3)}$. Adding (2.12), (2.13), (2.19) and (2.21)-(2.23), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\Big(\int_{\mathbb{R}^3} u(\log u + 2\langle x\rangle)\,dx + \|u\|_{1+\alpha}^{1+\alpha} + \|\nabla v\|_2^2 + \|\nabla w\|_2^2 + \|z\|_2^2\Big)\\
&+ C_0\Big(\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + \|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + \|\Delta v\|_2^2 + \|\Delta w\|_2^2 + \|\nabla z\|_2^2\Big)\\
&\le C'_0\Big(\int_{\mathbb{R}^3} u^{1-3\alpha}|\nabla v|^2\,dx + \int_{\mathbb{R}^3} u^{1-3\alpha}|\nabla w|^2\,dx + \int_{\mathbb{R}^3} u^{1-2\alpha}|\nabla v|^2\,dx\\
&\quad + \int_{\mathbb{R}^3} u^{1-2\alpha}|\nabla w|^2\,dx + \int_{\mathbb{R}^3} u^{1-\alpha}|\Delta v|dx + \int_{\mathbb{R}^3} u^{1-\alpha}|\Delta w|dx + \int_{\mathbb{R}^3} u|\Delta v|\,dx\\
&\quad + \int_{\mathbb{R}^3} u|\Delta w|\,dx + \int_{\mathbb{R}^3} v|\Delta v|\,dx + \int_{\mathbb{R}^3} w|\Delta w|\,dx + \int_{\mathbb{R}^3} u|z|\,dx\Big).
\end{aligned}
\tag{2.24}
$$

To deduce (2.6) from above, we estimate the integrals in RHS as follows:

As $0 \le 1 - 3\alpha < 2/3$, using the Young's, Hölder and Sobolev inequality, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^3} u^{1-3\alpha}|\nabla v|^2\,dx &\le \begin{cases} \int_{\mathbb{R}^3}\big(C(\epsilon_4) + \epsilon_4 u^{2/3}\big)|\nabla v|^2\,dx & \text{if } 1/6 < \alpha < 1/3,\\ \|\nabla v\|_2^2 & \text{if } \alpha = 1/3. \end{cases}\\
&\le \begin{cases} C(\epsilon_4)\|\nabla v\|_2^2 + \epsilon_4\|u_0\|^{2/3}\|\Delta v\|_2^2 & \text{if } 1/6 < \alpha < 1/3,\\ \|\nabla v\|_2^2 & \text{if } \alpha = 1/3. \end{cases}
\end{aligned}
\tag{2.25}
$$

Following the same procedure as above, we obtain

$$
\int_{\mathbb{R}^3} u^{1-3\alpha}|\nabla w|^2\,dx \le \begin{cases} C(\epsilon_5)\|\nabla w\|_2^2 + \epsilon_5\|u_0\|^{2/3}\|\Delta w\|_2^2 & \text{if } 1/6 < \alpha < 1/3,\\ \|\nabla w\|_2^2 & \text{if } \alpha = 1/3, \end{cases}
\tag{2.26}
$$

$$
\int_{\mathbb{R}^3} u^{1-2\alpha}|\nabla v|^2\,dx \le C(\epsilon_6)\|\nabla v\|_2^2 + \epsilon_6\|u_0\|^{2/3}\|\Delta v\|_2^2,
\tag{2.27}
$$

$$
\int_{\mathbb{R}^3} u^{1-2\alpha}|\nabla w|^2\,dx \le C(\epsilon_7)\|\nabla w\|_2^2 + \epsilon_7\|u_0\|^{2/3}\|\Delta w\|_2^2.
\tag{2.28}
$$

Next, we estimate the integral $\int_{\mathbb{R}^3} u^{1-\alpha}|\Delta v|\,dx$. Using the Hölder, Young and Gagliardo-Nierenberg inequalities in the integral, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^3} u^{1-\alpha}|\Delta v|\,dx &\le C(\epsilon_8)\|u\|_{2-\alpha}^{2-\alpha} + \epsilon_8\|\Delta v\|_2^2\\
&\le C(\epsilon_8)C_{16}\|u_0\|_1^{\frac{1+4\alpha}{2+3\alpha}}\|\nabla u^{\frac{1+\alpha}{2}}\|_2^{\frac{6-6\alpha}{2+3\alpha}} + \epsilon_8\|\Delta v\|_2^2\\
&\le C(\epsilon_8)C_{16}\Big(C(\epsilon_9)\|u_0\|_1^{\frac{1+4\alpha}{2+3\alpha}} + \epsilon_9\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2\Big) + \epsilon_8\|\Delta v\|_2^2\\
&= C_{17} + C_{18}\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + \epsilon_8\|\Delta v\|_2^2,
\end{aligned}
\tag{2.29}
$$

where $C_{17} = C_{16}C(\epsilon_8)C(\epsilon_9)\|u_0\|_1^{\frac{1+4\alpha}{2+3\alpha}}$ and $C_{18} = C_{16}C(\epsilon_8)\epsilon_9$. Here, we used that $\frac{4}{3} \le \frac{6-6\alpha}{2+3\alpha} < 2$. Following the same procedure as above, we obtain

$$
\int_{\mathbb{R}^3} u^{1-\alpha}|\Delta w|\,dx \le C_{19} + C_{20}\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + \epsilon_{10}\|\Delta w\|_2^2.
\tag{2.30}
$$

Next, we estimate the integral $\int_{\mathbb{R}^3} u|\Delta v|\,dx$. Using the Young and Gagliardo-Nierenberg inequalities in the integral, we obtain

$$\begin{aligned}\int_{\mathbb{R}^3} u|\Delta v|\,dx &\le C(\epsilon_{11})\|u\|_2^2 + \epsilon_{11}\|\Delta v\|_2^2\\ &\le C(\epsilon_{11})C_{21}\|u_0\|_1^{\frac{1+6\alpha}{2+6\alpha}}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^{\frac{6}{2+6\alpha}} + \epsilon_{11}\|\Delta v\|_2^2\\ &\le C(\epsilon_{11})C_{21}\Big(C(\epsilon_{12})\|u_0\|_1^{\frac{1+6\alpha}{2+3\alpha}} + \epsilon_{12}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2\Big) + \epsilon_{11}\|\Delta v\|_2^2\\ &= C_{21}' + C_{22}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + \epsilon_{11}\|\Delta v\|_2^2,\end{aligned} \tag{2.31}$$

where $C_{21}' = C_{21}C(\epsilon_{11})C(\epsilon_{12})\|u_0\|_1^{\frac{1+6\alpha}{2+3\alpha}}$ and $C_{22} = C_{21}\epsilon_{12}C(\epsilon_{11})$. Here, we used that $\frac{3}{2} \le \frac{6}{2+6\alpha} < 2$. Following the same procedure as above, we obtain

$$\int_{\mathbb{R}^3} u|\Delta w|\,dx \le C_{23} + C_{24}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + \epsilon_{13}\|\Delta w\|_2^2. \tag{2.32}$$

Next we estimate the integrals $\int_{\mathbb{R}^3} v|\Delta v|\,dx$, $\int_{\mathbb{R}^3} w|\Delta w|\,dx$. Using the Young and Gagliardo-Nierenberg inequalities in the integrals, we obtain

$$\int_{\mathbb{R}^3} v|\Delta v|\,dx \le C(\epsilon_{14}')C_0\|v_0\|_1^{\frac{8}{7}}\|\Delta v\|_2^{\frac{6}{7}} + \epsilon_{14}'\|\Delta v\|_2^2, \tag{2.33}$$

$$\int_{\mathbb{R}^3} w|\Delta w|\,dx \le C(\epsilon_{15}')C_1\|w_0\|_1^{\frac{8}{7}}\|\Delta w\|_2^{\frac{6}{7}} + \epsilon_{15}'\|\Delta w\|_2^2. \tag{2.34}$$

Next, we estimate the integral $\int_{\mathbb{R}^3} u|z|\,dx$. Using the Young, Gagliardo-Nierenberg and Sobolev inequalities in the integral, we obtain

$$\begin{aligned}\int_{\mathbb{R}^3} u|z|\,dx &\le C_{25}\|u\|_{\frac{6}{5}}\|z\|_6 \le \epsilon_{16}\|u\|_{\frac{6}{5}}^2 + C(\epsilon_{16})\|\nabla z\|_2^2\\ &\le C_{25}C_{25}'\|u_0\|_1^{\frac{3+10\alpha}{2+6\alpha}}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^{\frac{2}{2+6\alpha}} + C(\epsilon_{16})\|\nabla z\|_2^2\\ &\le C_{25}C_{25}'\|u_0\|_1^{\frac{3+10\alpha}{2+6\alpha}}\big(\epsilon_{17}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + C(\epsilon_{17})\big) + C(\epsilon_{16})\|\nabla z\|_2^2\\ &= C_{26} + C_{27}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + C(\epsilon_{16})\|\nabla z\|_2^2,\end{aligned} \tag{2.35}$$

where $C_{26} = \epsilon_{16}C_{25}'\|u_0\|_1^{\frac{3+10\alpha}{2+6\alpha}}C(\epsilon_{17})$ and $C_{27} = \epsilon_{16}C_{25}'\|u_0\|_1^{\frac{3+10\alpha}{2+6\alpha}}\epsilon_{17}$. Here we used that $1 < \frac{6}{5} < 3 + 6\alpha$ and $0 < \frac{2}{2+6\alpha} < 2$. Substituting (2.25) – (2.35) in (2.24), we have

$$\begin{aligned}&\frac{d}{dt}\Big(\int_{\mathbb{R}^3} u(\log u + 2\langle x\rangle)\,dx + \|u\|_{1+\alpha}^{1+\alpha} + \|\nabla v\|_2^2 + \|\nabla w\|_2^2 + \|z\|_2^2\Big)\\ &+ \overline{C}_0\Big(\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + \|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + \|\Delta v\|_2^2 + \|\Delta w\|_2^2 + \|\nabla z\|_2^2\Big)\\ &\le \overline{C}_0'\big(1 + \|\nabla v\|_2^2 + \|\nabla w\|_2^2\big).\end{aligned} \tag{2.36}$$

Integrating the above with respect to t, we obtain (2.6).

Case (ii): $1/3 < \alpha \le 1$. First, multiplying $(2.2)_1$ by $\log u$ and integrating we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^3} u\log u\,dx + \frac{4}{1+\alpha}\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 = \int_{\mathbb{R}^3} \nabla u\cdot(\chi\nabla v)\,dx - \int_{\mathbb{R}^3} \nabla u\cdot(\xi\nabla w)\,dx. \tag{2.37}$$

First, we evaluate the RHS of (2.37) as follows:

$$\begin{aligned}\int_{\mathbb{R}^3} \nabla u \cdot (\chi \nabla v)\, dx &\le \frac{2\overline{\chi}}{1+\alpha}\Big(\epsilon_1 \|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + C(\epsilon_1)\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\, dx\Big)\\ &= C_{28}\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + C_{29}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\, dx,\end{aligned} \tag{2.38}$$

where $C_{28} = \frac{2\overline{\chi}\epsilon_1}{1+\alpha}$ and $C_{29} = \frac{2\overline{\chi}C(\epsilon_1)}{1+\alpha}$. Following the same procedure as above, we obtain

$$-\int_{\mathbb{R}^3} \nabla u \cdot (\xi \nabla w)\, dx \le C_{30}\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + C_{31}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla w|^2\, dx. \tag{2.39}$$

Using (2.8), (2.38) and (2.39) in (2.37), we have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^3} u\log u\, dx + C_{32}\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2\\ &\le C_{29}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2 dx + C_{31}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla w|^2 dx.\end{aligned} \tag{2.40}$$

where $C_{32} = \frac{4}{1+\alpha} - C_{28} + C_{30}$. Now, multiply $(2.2)_1$ by u^α and integrating we obtain

$$\begin{aligned}&\frac{1}{1+\alpha}\frac{d}{dt}\|u\|_{1+\alpha}^{1+\alpha} + \frac{4\alpha(1+\alpha)}{(1+2\alpha)^2}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2\\ &= \int_{\mathbb{R}^3} \nabla u^\alpha \cdot u(\chi\nabla v)\, dx - \int_{\mathbb{R}^3} \nabla u^\alpha \cdot u(\xi \nabla w)\, dx.\end{aligned} \tag{2.41}$$

The last term in RHS of (2.16) is evaluated as

$$\begin{aligned}\int_{\mathbb{R}^3} u|\nabla v|^2\, dx &\le C_9\int_{\mathbb{R}^3} |\nabla u||\nabla v| + u|\Delta v|\, dx\\ &\le \frac{2C_9}{1+\alpha}\int_{\mathbb{R}^3} u^{\frac{1-\alpha}{2}}\big|\nabla u^{\frac{1+\alpha}{2}}\big||\nabla v|\, dx + C_9\int_{\mathbb{R}^3} u|\Delta v|\, dx\\ &\le \frac{2C_9}{1+\alpha}\Big(\epsilon_{18}\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + C(\epsilon_{18})\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\, dx\Big)\\ &\quad + C_9\int_{\mathbb{R}^3} u|\Delta v|\, dx.\end{aligned}$$

Using above in (2.16), the first term in RHS of (2.41) is evaluated as

$$\begin{aligned}\int_{\mathbb{R}^3} \nabla u^\alpha \cdot u(\chi\nabla v) dx &\le C_{33}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + C_{34}\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2\\ &\quad + C_{35}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2 dx + C_{36}\int_{\mathbb{R}^3} u|\Delta v|\, dx,\end{aligned} \tag{2.42}$$

where $C_\chi = \frac{2\alpha\overline{\chi}}{1+\alpha}$, $C_{33} = C_\chi C(\epsilon_3)$, $C_{34} = \frac{2C_\chi\epsilon_3 C_9\epsilon_{18}}{1+\alpha}$, $C_{35} = \frac{2C_\chi\epsilon_3 C_9 C(\epsilon_{18})}{1+\alpha}$ and $C_{36} = C_\chi\epsilon_3 C_9$. Following the same procedure as above, we obtain

$$\begin{aligned}-\int_{\mathbb{R}^3} \nabla u^\alpha \cdot u(\xi\nabla w)\, dx &\le \epsilon_{20}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + C_{37}\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2\\ &\quad + C_{38}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla w|^2\, dx + C_{39}\int_{\mathbb{R}^3} u|\Delta w|\, dx.\end{aligned} \tag{2.43}$$

Using (2.42) and (2.43) in (2.41), we have

$$\begin{aligned} &\frac{d}{dt}\|u\|_{1+\alpha}^{1+\alpha} + C_{40}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 - C_{41}\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 \\ &\le C_{35}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2 dx + C_{36}\int_{\mathbb{R}^3} u|\Delta v| dx \\ &\quad + C_{38}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla w|^2 dx + C_{39}\int_{\mathbb{R}^3} u|\Delta w| dx, \end{aligned} \tag{2.44}$$

where $C_{40} = \frac{4\alpha(1+\alpha)^2}{(1+2\alpha)^2} - C_{33} - \epsilon_{20}$ and $C_{41} = C_{37} + C_{34}$. It is clear that $(2.2)_2, (2.2)_3$ and $(2.2)_4$ are independent of α and therefore, adding (2.21)-(2.23) with (2.40), (2.13), (2.44), we obtain

$$\begin{aligned} &\frac{d}{dt}\Big(\int_{\mathbb{R}^3} u(\log u + 2\langle x\rangle)\, dx + \|u\|_{1+\alpha}^{1+\alpha} + \|\nabla v\|_2^2 + \|\nabla w\|_2^2 + \|z\|_2^2\Big) \\ &+ C_{42}\Big(\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + \|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + \|\Delta v\|_2^2 + \|\Delta w\|_2^2 + \|\nabla z\|_2^2\Big) \\ &\le C_{43}\Big(\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\, dx + \int_{\mathbb{R}^3} u^{1-\alpha}|\nabla w|^2\, dx + \int_{\mathbb{R}^3} u|\Delta v|\, dx \\ &\quad + \int_{\mathbb{R}^3} u|\Delta w|\, dx + \int_{\mathbb{R}^3} v|\Delta v|\, dx + \int_{\mathbb{R}^3} w|\Delta w|\, dx + \int_{\mathbb{R}^3} u|z|\, dx\Big). \end{aligned} \tag{2.45}$$

As $0 \le 1 - \alpha < \frac{2}{3}$, using the Young's, Hölder and Sobolev inequality, we obtain

$$\begin{aligned} \int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\, dx &\le \begin{cases} \int_{\mathbb{R}^3} \big(C(\epsilon_{21}) + \epsilon_{21}u^{2/3}\big)|\nabla v|^2\, dx & \text{if } 1/3 < \alpha < 1, \\ \|\nabla v\|_2^2 & \text{if } \alpha = 1. \end{cases} \\ &\le \begin{cases} C(\epsilon_{21})\|\nabla v\|_2^2 + \epsilon_{21}\|u_0\|^{2/3}\|\Delta v\|_2^2 & \text{if } \frac{1}{3} < \alpha < 1, \\ \|\nabla v\|_2^2 & \text{if } \alpha = 1. \end{cases} \end{aligned} \tag{2.46}$$

Following the same procedure as above, we obtain

$$\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla w|^2\, dx \le \begin{cases} C(\epsilon_{22})\|\nabla w\|_2^2 + \epsilon_{22}\|u_0\|^{2/3}\|\Delta w\|_2^2 & \text{if } 1/3 < \alpha < 1, \\ \|\nabla w\|_2^2 & \text{if } \alpha = 1. \end{cases} \tag{2.47}$$

As $0 < \frac{6}{2+6\alpha} < 2$ and $1 < 6/5 < 3 + 6\alpha$, following the same procedure as (2.31), (2.35) obtained and from previous case, we have

$$\begin{aligned} \int_{\mathbb{R}^3} u|\Delta v|\, dx &\le C_{44} + C_{45}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + \epsilon_{23}\|\Delta v\|_2^2, \\ \int_{\mathbb{R}^3} u|\Delta w|\, dx &\le C_{46} + C_{47}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + \epsilon_{24}\|\Delta w\|_2^2, \\ \int_{\mathbb{R}^3} v|\Delta v|\, dx &\le C(\epsilon'_{14})C_0\|v_0\|_1^{\frac{8}{7}}\|\Delta v\|_2^{\frac{6}{7}} + \epsilon'_{14}\|\Delta v\|_2^2, \\ \int_{\mathbb{R}^3} w|\Delta w|\, dx &\le C(\epsilon'_{15})C_1\|w_0\|_1^{\frac{8}{7}}\|\Delta w\|_2^{\frac{6}{7}} + \epsilon'_{15}\|\Delta w\|_2^2, \\ \int_{\mathbb{R}^3} u|z|\, dx &\le C_{26} + C_{27}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + C(\epsilon_{16})\|\nabla z\|_2^2. \end{aligned}$$

Substituting (2.46), (2.47) and the above estimates in (2.45), we obtain

$$\begin{aligned} &\frac{d}{dt}\Big(\int_{\mathbb{R}^3} u(\log u + 2\langle x\rangle)\,dx + \|u\|_{1+\alpha}^{1+\alpha} + \|\nabla v\|_2^2 + \|\nabla w\|_2^2 + \|z\|_2^2\Big)\\ &\quad + \overline{C}_{42}\Big(\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + \|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + \|\Delta v\|_2^2 + \|\Delta w\|_2^2 + \|\nabla z\|_2^2\Big)\\ &\le \overline{C}_{43}\big(1 + \|\nabla v\|_2^2 + \|\nabla w\|_2^2\big). \end{aligned} \tag{2.48}$$

Integrating above with respect to t, we obtain (2.6).

Case (iii): $\alpha > 1$. First, multiplying $(2.2)_1$ by $\log u$ and integrating we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^3} u\log u\,dx + \frac{4}{1+\alpha}\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 = \int_{\mathbb{R}^3}\nabla u\cdot(\chi\nabla v)\,dx - \int_{\mathbb{R}^3}\nabla u\cdot(\xi\nabla w)\,dx. \tag{2.49}$$

Using the assumption $\chi', \xi' \in L^\infty_{\rm loc}$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^3}\nabla u\cdot(\chi\nabla v)\,dx &\le C_{44}\int_{\mathbb{R}^3} u|\nabla v|^2\,dx + C_{44}\int_{\mathbb{R}^3} u|\Delta v|\,dx\\ -\int_{\mathbb{R}^3}\nabla u\cdot(\xi\nabla w)\,dx &\le C_{45}\int_{\mathbb{R}^3} u|\nabla w|^2\,dx + C_{45}\int_{\mathbb{R}^3} u|\Delta w|\,dx. \end{aligned}$$

Using the above in (2.49), we obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^3} u\log u\,dx + \frac{4}{1+\alpha}\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2\\ &\le C_{44}\int_{\mathbb{R}^3} u|\nabla v|^2\,dx + C_{45}\int_{\mathbb{R}^3} u|\nabla w|^2\,dx\\ &\quad + C_{44}\int_{\mathbb{R}^3} u|\Delta v|dx + C_{45}\int_{\mathbb{R}^3} u|\Delta w|dx. \end{aligned} \tag{2.50}$$

Now, multiplying $(2.2)_1$ by u^α and integrating we obtain

$$\begin{aligned} &\frac{1}{1+\alpha}\frac{d}{dt}\|u\|_{1+\alpha}^{1+\alpha} + \frac{4\alpha(1+\alpha)}{(1+2\alpha)^2}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2\\ &\le \int_{\mathbb{R}^3}\nabla u^\alpha\cdot u(\chi\nabla v)\,dx - \int_{\mathbb{R}^3}\nabla u^\alpha\cdot u(\xi\nabla w)\,dx. \end{aligned} \tag{2.51}$$

From (2.16), we obtain

$$\int_{\mathbb{R}^3}\nabla u^\alpha\cdot u(\chi\nabla v)\,dx \le C_{50}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + C_{51}\int_{\mathbb{R}^3} u|\nabla v|^2\,dx, \tag{2.52}$$

where $C_{50} = C_\chi\epsilon_3$ and $C_{51} = C_\chi C(\epsilon_3)$. Following the same procedure as previous, we obtain

$$-\int_{\mathbb{R}^3}\nabla u^\alpha\cdot u(\xi\nabla w)\,dx \le C_{52}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + C_{53}\int_{\mathbb{R}^3} u|\nabla w|^2\,dx. \tag{2.53}$$

Substituting (2.52) and (2.53) in (2.51), we obtain

$$\frac{d}{dt}\|u\|_{1+\alpha}^{1+\alpha} + C_{49}'\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 \le C_{51}\int_{\mathbb{R}^3} u|\nabla v|^2\,dx + C_{53}\int_{\mathbb{R}^3} u|\nabla w|^2\,dx, \tag{2.54}$$

where $C'_{49} = \frac{4\alpha(1+\alpha)^2}{(1+2\alpha)^2} - C_{50} + C_{52}$. Adding (2.50), (2.13), (2.54) and (2.21)-(2.23), we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_{\mathbb{R}^3} u(\log u + 2\langle x\rangle)\,dx + \|u\|_{1+\alpha}^{1+\alpha} + \|\nabla v\|_2^2 + \|\nabla w\|_2^2 + \|z\|_2^2\Big)\\
&+ C_{54}\Big(\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + \|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + \|\Delta v\|_2^2 + \|\Delta w\|_2^2 + \|\nabla z\|_2^2\Big)\\
&\le C_{55}\ \Big(\int_{\mathbb{R}^3} u|\nabla v|^2\,dx + \int_{\mathbb{R}^3} u|\nabla w|^2\,dx + \int_{\mathbb{R}^3} u|\Delta v|\,dx + \int_{\mathbb{R}^3} u|\Delta w|\,dx\\
&\quad + \int_{\mathbb{R}^3} v|\Delta v|\,dx + \int_{\mathbb{R}^3} w|\Delta w|\,dx + \int_{\mathbb{R}^3} u|z|\,dx\Big).
\end{aligned}\tag{2.55}$$

Next, we estimate the first integral $\int_{\mathbb{R}^3} u|\nabla v|^2\,dx$ of RHS. As $\frac{1+\alpha}{2} > 1$, using Young's inequality, we obtain $u \le \epsilon_{26} + C(\epsilon_{26})u^{\frac{1+\alpha}{2}}$. Using the previous equation and Young's inequality, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} u|\nabla v|^2 dx\\
&\le \epsilon_{26}\|\nabla v\|_2^2 + C(\epsilon_{26})\int_{\mathbb{R}^3} u^{\frac{1+\alpha}{2}}|\nabla v|^2\,dx\\
&= C(\epsilon_{26})\int_{\mathbb{R}^3} u^{\frac{1+\alpha}{2}}\nabla v\cdot\nabla v\,dx + \epsilon_{26}\|\nabla v\|_2^2\\
&\le C(\epsilon_{26})C_{56}\Big(\int_{\mathbb{R}^3} \nabla u^{\frac{1+\alpha}{2}}\cdot\nabla v + u^{\frac{1+\alpha}{2}}\Delta v\,dx\Big) + \epsilon_{26}\|\nabla v\|_2^2\\
&\le C(\epsilon_{26})C_{56}\Big(\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + \|\nabla v\|_2^2 + \|u\|_{1+\alpha}^{1+\alpha} + \|\Delta v\|_2^2\Big) + \epsilon_{26}\|\nabla v\|_2^2.
\end{aligned}\tag{2.56}$$

Now, we estimate the third term of RHS of above. Using the Gagliardo-Nierenberg and Young inequality, we obtain

$$\|u\|_{1+\alpha}^{1+\alpha} \le C_{57}\|u_0\|_1^{\frac{2+2\alpha}{2+3\alpha}}\|\nabla u^{\frac{1+\alpha}{2}}\|_2^{\frac{6\alpha}{2+3\alpha}} \le C_{57}C(\epsilon_{27}) + \epsilon_{27}\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2.$$

Here, we have used that $\frac{6\alpha}{2+3\alpha} < 2$. Substituting the above in (2.56), we obtain

$$\int_{\mathbb{R}^3} u|\nabla v|^2\,dx \le C'_{54} + C'_{55}\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + C'_{56}\|\nabla v\|_2^2 + C'_{57}\|\Delta v\|_2^2,\tag{2.57}$$

where $C'_{54} = C(\epsilon_{26})C(\epsilon_{27})C_{56}C_{57}$, $C'_{55} = C(\epsilon_{26})C_{56}+C(\epsilon_{26})C_{56}\epsilon_{27}$, $C'_{56} = C(\epsilon_{26})C_{56}+\epsilon_{26}$ and $C'_{57} = C(\epsilon_{26})C_{56}$. Following the same procedure as in the previous estimate, and from previous case, we obtain

$$\begin{gathered}
\int_{\mathbb{R}^3} u|\nabla w|^2\,dx \le C'_{58} + C'_{59}\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + C'_{60}\|\nabla w\|_2^2 + C'_{61}\|\Delta w\|_2^2,\\
\int_{\mathbb{R}^3} u|\Delta v|\,dx \le C_{44} + C_{45}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + \epsilon_{23}\|\Delta v\|_2^2,\\
\int_{\mathbb{R}^3} u|\Delta w|\,dx \le C_{46} + C_{47}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + \epsilon_{24}\|\Delta w\|_2^2,\\
\int_{\mathbb{R}^3} v|\Delta v|\,dx \le C(\epsilon'_{14})C_0\|v_0\|_1^{\frac{8}{7}}\|\Delta v\|_2^{\frac{6}{7}} + \epsilon'_{14}\|\Delta v\|_2^2,\\
\int_{\mathbb{R}^3} w|\Delta w|\,dx \le C(\epsilon'_{15})C_1\|w_0\|_1^{\frac{8}{7}}\|\Delta w\|_2^{\frac{6}{7}} + \epsilon'_{15}\|\Delta w\|_2^2,
\end{gathered}$$

$$\int_{\mathbb{R}^3} u|z|\,dx \le C_{26} + C_{27}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + C(\epsilon_{16})\|\nabla z\|_2^2.$$

Substituting the above estimates and (2.57) in (2.55), we have

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_{\mathbb{R}^3} u(\log u + 2\langle x\rangle)\,dx + \|u\|_{1+\alpha}^{1+\alpha} + \|\nabla v\|_2^2 + \|\nabla w\|_2^2 + \|z\|_2^2\Big)\\
&+ \overline{C}_{54}\Big(\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + \|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + \|\Delta v\|_2^2 + \|\Delta w\|_2^2 + \|\nabla z\|_2^2\Big)\\
&\le \overline{C}_{55}\big(1 + \|\nabla v\|_2^2 + \|\nabla w\|_2^2\big).
\end{aligned} \tag{2.58}$$

Integrating above with respect to t, we obtain (2.6). □

Lemma 2.3. *Let (u,v,w,z) be a classical solution of (2.2), for $\epsilon \in (0,1)$ and initial data $(u_{0_\epsilon}, v_{0_\epsilon}, w_{0_\epsilon}, z_{0_\epsilon})$ satisfies the following conditions, independent of ϵ,*

(1) $u_0(1+|x|+|\log u_0|) \in L^1(\mathbb{R}^3)$,
(2) $u_0 \in L^{1+\alpha}(\mathbb{R}^3)$,
(3) $v_0,\ w_0 \in L^\infty(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$,
(4) $z_0 \in L^2(\mathbb{R}^3)$.

Assume that

$$\alpha > 0, \quad \phi \in W^{2,\infty}(\mathbb{R}^3), \quad \chi', \xi' \in L^\infty_{\rm loc} \quad \textit{with } \chi'(\cdot) \ge \chi_0,\ \xi'(\cdot) \ge \xi_0 \tag{2.59}$$

for some constant $\chi_0 > 0, \xi_0 > 0$. Then, given $t \in (0,T]$, we have

$$\sup_{0\le\tau\le t} E(\tau) + \int_0^t D(\tau)d\tau < C, \tag{2.60}$$

where $C > 0$ is a constant solely depends on initial data.

Proof. It is sufficient to prove only for $0 < \alpha \le 1/6$. Further, for $\alpha > 1/6$ is already established in the previous lemma. Multiplying $(2.2)_1$ by $\log u$ and integrating we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^3} u\log u\,dx + \frac{4}{1+\alpha}\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 \le \int_{\mathbb{R}^3} \nabla u\cdot(\chi\nabla v)\,dx - \int_{\mathbb{R}^3} \nabla u\cdot(\xi\nabla w)\,dx. \tag{2.61}$$

The first term in RHS of the above is evaluated using our assumption and Young's inequality as

$$\begin{aligned}
&\int_{\mathbb{R}^3} \nabla u\cdot(\chi\nabla v)\,dx\\
&= -\int_{\mathbb{R}^3} (\chi'|\nabla v|^2 + \chi\Delta v)u\,dx\\
&\le -\hat{\chi}_1 \int_{\mathbb{R}^3} u|\nabla v|^2\,dx + C_{58}\int_{\mathbb{R}^3} |\nabla u||\nabla v|\,dx\\
&\le -\chi_0 \int_{\mathbb{R}^3} u|\nabla v|^2\,dx + C_{58}C_{59}\int_{\mathbb{R}^3} u^{\frac{1-\alpha}{2}}\big|\nabla u^{\frac{1+\alpha}{2}}\big||\nabla v|\,dx\\
&\le -\chi_0 \int_{\mathbb{R}^3} u|\nabla v|^2\,dx + C_{60}\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + C_{61}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\,dx,
\end{aligned} \tag{2.62}$$

where $C_{60} = C_{58}C_{59}\epsilon_{26}$ and $C_{61} = C_{58}C_{59}C(\epsilon_{26})$. Here we used that $|\nabla u| = \frac{2u^{\frac{1-\alpha}{2}}}{1+\alpha}\big|\nabla u^{\frac{1+\alpha}{2}}\big|$. Similarly, we obtain

$$
\begin{aligned}
&-\int_{\mathbb{R}^3} \nabla u\cdot(\xi\nabla w)dx \\
&\le -\xi_0\int_{\mathbb{R}^3} u|\nabla w|^2\,dx + C_{60}'\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + C_{61}'\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla w|^2\,dx.
\end{aligned}
\tag{2.63}
$$

Using (2.62) and (2.63) in (2.61), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3} u\log u\,dx + C_{62}\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + \chi_0\int_{\mathbb{R}^3} u|\nabla v|^2dx + \xi_0\int_{\mathbb{R}^3} u|\nabla w|^2\,dx \\
&\le C_{61}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\,dx + C_{61}'\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla w|^2dx,
\end{aligned}
\tag{2.64}
$$

where $C_{62} = \frac{4}{1+\alpha} - C_{60} - C_{60}'$. Now, multiplying $(2.2)_1$ by u^α and integrating we obtain

$$
\begin{aligned}
&\frac{1}{1+\alpha}\frac{d}{dt}\|u\|_{1+\alpha}^{1+\alpha} + \frac{4\alpha(1+\alpha)}{(1+2\alpha)^2}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 \\
&= \int_{\mathbb{R}^3} \nabla u^\alpha\cdot u(\chi\nabla v)\,dx - \int_{\mathbb{R}^3} \nabla u^\alpha\cdot u(\xi\nabla w)\,dx.
\end{aligned}
\tag{2.65}
$$

Now, we estimate the first integral of RHS. Using Young's inequality, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^3} \nabla u^\alpha\cdot u(\chi\nabla v)\,dx &\le C_\chi\int_{\mathbb{R}^3} \big|\nabla u^{\frac{1+2\alpha}{2}}\big|(u^{\frac12}|\nabla v|)\,dx \\
&\le C_{63}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + C_{63}'\int_{\mathbb{R}^3} u|\nabla v|^2\,dx,
\end{aligned}
\tag{2.66}
$$

where $\overline{\chi} := \sup_{\mathbb{R}^3}|\chi(c)|$, $C_\chi = \frac{2\alpha\overline{\chi}}{1+\alpha}$, $C_{63} = C_\chi\epsilon_{27}$, and $C_{63}' = C_\chi C(\epsilon_{27})$. Following the procedure above, we obtain

$$
-\int_{\mathbb{R}^3} \nabla u^\alpha\cdot u(\xi\nabla w)\,dx \le C_{64}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + C_{64}'\int_{\mathbb{R}^3} u|\nabla w|^2\,dx. \tag{2.67}
$$

Using (2.66) and (2.67) in (2.65), we obtain

$$
\frac{d}{dt}\|u\|_{1+\alpha}^{1+\alpha} + C_{65}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 \le C_{63}'\int_{\mathbb{R}^3} u|\nabla v|^2\,dx + C_{64}'\int_{\mathbb{R}^3} u|\nabla w|^2\,dx, \tag{2.68}
$$

where $C_{67} = \frac{4\alpha(1+\alpha)^2}{(1+2\alpha)^2} - C_{63} - C_{64}$. As α does not affect $(2.2)_2, (2.2)_3$ and $(2.2)_4$, (2.23) in previous Lemma 2.2 holds. Adding (2.64), (2.13), (2.68) and (2.21)-(2.23), we have

$$
\begin{aligned}
&\frac{d}{dt}\Big(\int_{\mathbb{R}^3} u(\log u + 2\langle x\rangle)\,dx + \|u\|_{1+\alpha}^{1+\alpha} + \|\nabla v\|_2^2 + \|\nabla w\|_2^2 + \|z\|_2^2\Big) \\
&+ C_{67}\Big(\int_{\mathbb{R}^3} u|\nabla v|^2 + u|\nabla w|^2dx + \|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + \|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 \\
&+ \|\Delta v\|_2^2 + \|\Delta w\|_2^2 + \|\nabla z\|_2^2\Big) \\
&\le C_{68}\Big(\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2dx + \int_{\mathbb{R}^3} u^{1-\alpha}|\nabla w|^2dx - \int_{\mathbb{R}^3} u|\Delta v|dx \\
&\quad - \int_{\mathbb{R}^3} u|\Delta w|dx + \int_{\mathbb{R}^3} v|\Delta v|dx + \int_{\mathbb{R}^3} w|\Delta w|dx + \int_{\mathbb{R}^3} u|z|\,dx\Big).
\end{aligned}
\tag{2.69}
$$

Now, we estimate the first integral of the of RHs. Using $u^{1-\alpha} \le C(\epsilon_{28}) + \epsilon_{28} u$ in the integral and choosing sufficiently small ϵ_{28}, we obtain

$$\int_{\mathbb{R}^3} u^{1-\alpha} |\nabla v|^2 \, dx \le C_{69} \|\nabla v\|_2^2. \tag{2.70}$$

Similarly, we obtain

$$\int_{\mathbb{R}^3} u^{1-\alpha} |\nabla w|^2 \, dx \le C'_{69} \|\nabla w\|_2^2. \tag{2.71}$$

From the previous Lemma, we have

$$\begin{gathered}
\int_{\mathbb{R}^3} v|\Delta v| \, dx \le C(\epsilon'_{14}) C_0 \|v_0\|_1^{\frac{8}{7}} \|\Delta v\|_2^{\frac{6}{7}} + \epsilon'_{14} \|\Delta v\|_2^2, \\
\int_{\mathbb{R}^3} w|\Delta w| \, dx \le C(\epsilon'_{15}) C_1 \|w_0\|_1^{\frac{8}{7}} \|\Delta w\|_2^{\frac{6}{7}} + \epsilon'_{15} \|\Delta w\|_2^2, \\
\int_{\mathbb{R}^3} u|z| \, dx \le C_{26} + C_{27} \|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + C(\epsilon_{16}) \|\nabla z\|_2^2.
\end{gathered}$$

Using (2.70), (2.71) and the above estimates in (2.69), we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_{\mathbb{R}^3} u(\log u + 2\langle x\rangle) \, dx + \|u\|_{1+\alpha}^{1+\alpha} + \|\nabla v\|_2^2 + \|\nabla w\|_2^2 + \|z\|_2^2 \Big) \\
&+ \overline{C}_{67} \Big(\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + \|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + \|\Delta v\|_2^2 + \|\Delta w\|_2^2 + \|\nabla z\|_2^2 \Big) \\
&\le \overline{C}_{68} \big(1 + \|\nabla v\|_2^2 + \|\nabla w\|_2^2 \big).
\end{aligned} \tag{2.72}$$

Integrating the above with respect to t, we obtain (2.60).. □

Now, we are ready to state the primary finding of this paper.

Theorem 2.4. *Suppose that the initial data (u_0, v_0, w_0, z_0) satisfies the following:*

(1) $u_0(1 + |x| + |\log u_0|) \in L^1(\mathbb{R}^3)$,
(2) $u_0 \in L^{1+\alpha}(\mathbb{R}^3)$,
(3) $v_0, w_0 \in L^\infty(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$,
(4) $z_0 \in L^2(\mathbb{R}^3)$,

and either one of the assumptions (2.5) *or* (2.59) *holds. Then for any $T > 0$,* (2.2) *possesses a weak soluion (u, v, w, z) that satisfies*

$$\begin{aligned}
&\sup_{0 \le t \le T} \Big(\int_{\mathbb{R}^3} u(|\log u| + 2\langle x\rangle) \, dx + \|u\|_{1+\alpha}^{1+\alpha} + \|\nabla v\|_2^2 + \|\nabla w\|_2^2 + \|z\|_2^2 \Big) \\
&+ \int_0^T \Big(\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + \|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + \|\Delta v\|_2^2 + \|\Delta w\|_2^2 + \|\nabla z\|_2^2 \Big) dt < C,
\end{aligned} \tag{2.73}$$

where C is a constant that solely depends on initial data.

Proof. Recall that, the solutions of (2.2) in $\mathbb{R}^3 \times [0, T)$ with initial conditions $(u_{0_\epsilon}, v_{0_\epsilon}, w_{0_\epsilon}, z_{0_\epsilon})$ is given by

$$u_{0_\epsilon} = \phi_\epsilon * u_0, \quad v_{0_\epsilon} = \phi_\epsilon * v_0, \quad w_{0_\epsilon} = \phi_\epsilon * w_0, \quad z_{0_\epsilon} = \phi_\epsilon * z_0,$$

where ϕ_ϵ is a usual mollifier with $\epsilon \in (0, 1)$. The uniformity of the estimates obtained in Lemma 2.2, regardless of the value of ϵ, is ensured by the convergence of $(u_{0_\epsilon}, v_{0_\epsilon}, w_{0_\epsilon}, z_{0_\epsilon})$. In other words, it means that the constant C in (2.6) can be selected without dependence on ϵ. In a similar way, we obtain constants such that for $q < \infty$,

(1) u_ϵ bounded in $L^\infty((0,T)\times\mathbb{R}^3)$,
(2) $\nabla u_\epsilon^{\frac{q+\alpha}{2}}$ bounded in $L^2((0,T)\times\mathbb{R}^3)$,
(3) $v_\epsilon, w_\epsilon, z_\epsilon$ bounded in $L^\infty(0,T;W^{1,q}(\mathbb{R}^3))$,
(4) $v_\epsilon, w_\epsilon, z_\epsilon$ bounded in $L^q(0,T;W^{2,q}(\mathbb{R}^3))$,
(5) $v_{\epsilon_t}, w_{\epsilon_t}, z_{\epsilon_t}$ bounded in $L^q(0,T;L^q(\mathbb{R}^3))$.

Our estimate allowed the local solution to be extended to arbitrary $(0,T)$ (as in [22, 27, 11, 7]). Let $k\ge 2+\alpha$ be chosen. Then u_{ϵ_t} and $u^k_{\epsilon_t}$ belong to $L^1(0,T;W^{-2,2}(\mathbb{R}^3))$ (as in [22]), where the dual space of $W^{2,2}(\mathbb{R}^3)$ is denoted by $W^{-2,2}(\mathbb{R}^3)$. Using the Aubin-Lions compactness lemma, we have a weak limit (u,v,w,z) as $\epsilon\to 0$ which is a weak solution. □

The above theorem can indeed be proven for a bounded domain with Neumann boundary conditions for u, v, and w, as well as no-slip boundary conditions for z. To be more specific, suppose we have a smooth boundary for the bounded domain Ω, and we are considering the system (2.2) within $\Omega\times[0,T)$, with the aforementioned boundary conditions as

$$\frac{\partial u}{\partial\nu}=\frac{\partial v}{\partial\nu}=\frac{\partial w}{\partial\nu}=0,\quad z=0\quad\text{on } \partial\Omega. \tag{2.74}$$

Theorem 2.5. *Suppose that the initial data (u_0,v_0,w_0,z_0) satisfies the following:*

(1) $u_{0_\epsilon}\in L^1(\Omega)\cap L^{1+\alpha}(\Omega)$,
(2) v_{0_ϵ} *and* $w_{0_\epsilon}\in L^\infty(\Omega)\cap H^1(\Omega)$,
(3) $z_{0_\epsilon}\in L^2(\Omega)$.

and either one of the assumptions (2.5) *or* (2.59) *holds by replacing $\mathbb{R}^3$ by Ω. Then for each $T>0$, system* (2.2) *with boundary conditions* (2.74) *possesses a weak soluion (u,v,w,z) that satisfies*

$$\begin{aligned}&\sup_{0\le t\le T}\Big(\int_\Omega u|\log u|\,dx+\|u\|_{1+\alpha}^{1+\alpha}+\|\nabla v\|_2^2+\|\nabla w\|_2^2+\|z\|_2^2\Big)\\&+\int_0^T\Big(\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2+\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2+\|\Delta v\|_2^2+\|\Delta w\|_2^2+\|\nabla z\|_2^2\Big)dt<C,\end{aligned}\tag{2.75}$$

where C is a constant that solely depends on initial data.

Proof. We address only the modification to be done in above proof, as the proof is similar. As $\|u\|_{L^1(\Omega)}$ takes care the negative part of $\int_\Omega u\log u$, L^1 estimate of $u\langle x\rangle$ (2.13) is not needed. Also, (2.31) can be replaced by

$$\|u\|_2^2\le C_0\|u_0\|_1^{\frac{1+6\alpha}{2+6\alpha}}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^{\frac{6}{2+6\alpha}}+C_0\|u\|_1^2. \tag{2.76}$$

The rest of the proof is completed by employing similar ideas as those in Theorem 2.4, and hence omitted. □

3. Existence of bounded weak solutions

We consider the model the attraction-repulsion chemotaxis Stokes system, that is, (1.6) with $\tau = 0$, in this section as

$$\begin{aligned} u_t + z \cdot \nabla u &= \Delta u^{1+\alpha} - \nabla \cdot (\chi(v) u \nabla v) + \nabla \cdot (\xi(w) u \nabla w), \\ v_t + z \cdot \nabla v &= \Delta v + \beta u - \gamma v, \\ w_t + z \cdot \nabla w &= \Delta w + \delta u - \eta w, \\ z_t + \nabla p &= \Delta z - u \nabla \phi, \\ \operatorname{div} z &= 0. \end{aligned} \tag{3.1}$$

This section first defines the bounded weak solution for (3.1). For the rationale outlined in previous section, we introduce the suitable approximation problem for (3.1). Before demonstrating our main finding, namely bounded weak solutions to (3.1) in bounded domain with smooth boundary, we first establish a key lemma used to substantiate our main findings.

Definition 3.1. For $\alpha > 0$ and $T \in (0, \infty)$, a weak solution (u, v, w, z) as introduced in Definition 2.1 is termed as a bounded weak solution of (3.1) if

(1) Given $p \in [1, \infty)$, we have $u \in L^\infty((0,T); \mathbb{R}^3)$ and $\nabla u^{\frac{p+\alpha}{2}} \in L^2(0, T; L^2(\mathbb{R}^3))$,
(2) Given $q \in [2, \infty)$, we have $v, w, z \in L^q(0, T; W^{2,q}(\mathbb{R}^3))$ and $v_t, w_t, z_t \in L^q(0, T; L^q(\mathbb{R}^3))$.

We define an approximation problem for the above system as

$$\begin{aligned} u_{\epsilon_t} + z_\epsilon \cdot \nabla u_\epsilon &= \Delta (u_\epsilon + \epsilon)^{1+\alpha} - \nabla \cdot (\chi(v_\epsilon) u_\epsilon \nabla v_\epsilon) + \nabla \cdot (\xi(w_\epsilon) u_\epsilon \nabla w_\epsilon), \\ v_{\epsilon_t} + z_\epsilon \cdot \nabla v_\epsilon &= \Delta v_\epsilon + \beta u_\epsilon - \gamma v_\epsilon, \\ w_{\epsilon_t} + z_\epsilon \cdot \nabla w_\epsilon &= \Delta w_\epsilon + \delta u_\epsilon - \eta w_\epsilon, \\ z_{\epsilon_t} + \nabla p_\epsilon &= \Delta z_\epsilon - u_\epsilon \nabla \phi, \\ \operatorname{div} z_\epsilon &= 0 \end{aligned} \tag{3.2}$$

with initial conditions

$$u_{0_\epsilon} = \phi_\epsilon * u_0, \quad v_{0_\epsilon} = \phi_\epsilon * v_0, \quad w_{0_\epsilon} = \phi_\epsilon * w_0, \quad z_{0_\epsilon} = \phi_\epsilon * z_0,$$

where ϕ_ϵ is a usual mollifier with $\epsilon \in (0, 1)$. According to the standard theory of existence and regularity, for every $\epsilon > 0$, equation (2.2) admits a local-in-time classical solution. The proof for this assertion is not provided here because it follows a similar methodology as in [2, 23]. Hereafter, we use the unknowns (u_ϵ, v_ϵ, w_ϵ, and z_ϵ) as (u, v, w, z) for simplicity of notation. First, we deduce some estimates, independent of ϵ, of the solution to (3.2) which are uniform in nature. Using those estimates, local bounded weak solution is extended to arbitrary $(0, T)$ and we construct bounded weak solution of (3.1).

Lemma 3.2. *Suppose that (u, v, w, z) is a classical solution of (3.2), for all $\epsilon \in (0, 1)$ and the initial data $(u_{0_\epsilon}, v_{0_\epsilon}, w_{0_\epsilon}, z_{0_\epsilon})$ satisfies (1) – (4) of Lemma 2.2 along with $u_{0_\epsilon} \in L^\infty(\mathbb{R}^3)$ and $v_{0_\epsilon}, w_{0_\epsilon}, z_{0_\epsilon} \in W^{1,q}(\mathbb{R}^3)$ for any $q \in [2, \infty)$. Furthermore, assume that*

$$\alpha > \frac{1}{8}, \quad \gamma = \eta = 0, \quad \chi', \xi' \in L^\infty_{\rm loc} \quad \text{with } \chi'(\cdot) \geq \chi_0,\ \xi'(\cdot) \geq \xi_0, \tag{3.3}$$

for some positive constants χ_0, ξ_0. Then, for each $t \in (0, T]$, we have:

(1) *For* $1 \le p \le \infty$,

$$u \in L^\infty(0,T;L^p(\mathbb{R}^3)) \text{ and } \nabla u^{\frac{p+\alpha}{2}} \in L^2(0,T;L^2(\mathbb{R}^3)), \tag{3.4}$$

(2) *For* $2 \le q < \infty$,

$$v,w,z \in L^\infty(0,T;W^{1,q}(\mathbb{R}^3)) \cap L^q(0,T;W^{2,q}(\mathbb{R}^3)), \tag{3.5}$$

(3) *For* $2 \le q < \infty$,

$$v_t, w_t, z_t \in L^q(0,T;L^q(\mathbb{R}^3)). \tag{3.6}$$

Proof. For $1 \le p \le 1+\alpha$, by Lemma 2.2, (3.4) holds. It is sufficient to show that for $\alpha > \frac{1}{8}$, u satisfies (3.4), as $(3.4)-(3.6)$ follows from that. Multiplying $(3.2)_1$ by u^{p-1} and integrating it by parts, we have

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|u\|_p^p + \int_{\mathbb{R}^3} \nabla u^{p-1} \cdot \nabla (u+\epsilon)^{1+\alpha}\,dx \\ &= -\int_{\mathbb{R}^3} u^{p-1}\nabla\cdot(\chi u \nabla v)\,dx + \int_{\mathbb{R}^3} u^{p-1}\nabla\cdot(\xi u\nabla w)\,dx. \end{aligned} \tag{3.7}$$

Also, we have

$$\int_{\mathbb{R}^3} \nabla u^{p-1}\cdot\nabla(u+\epsilon)^{1+\alpha}\,dx \ge \frac{4(p-1)(1+\alpha)}{(p+\alpha)^2}\|\nabla u^{\frac{p+\alpha}{2}}\|_2^2, \tag{3.8}$$

by using that $\nabla u^{p-1}\cdot\nabla u^{1+\alpha} = (p-1)(1+\alpha)u^{p-2+\alpha}|\nabla u|^2$ and $|\nabla u^{\frac{p+\alpha}{2}}|_2^2 = \frac{(p+\alpha)^2}{4}u^{p+\alpha-2}|\nabla u|^2$. The first term in RHS of (3.7) is evaluated using Young's inequality as

$$\begin{aligned} &-\int_{\mathbb{R}^3} u^{p-1}\nabla\cdot(\chi u\nabla v)\,dx \\ &\le \frac{2\overline{\chi}}{(p+\alpha)}\int_{\mathbb{R}^3} |\nabla u^{\frac{p+\alpha}{2}}| u^{\frac{p-\alpha}{2}}|\nabla v|\,dx \\ &\le \frac{2\overline{\chi}}{(p+\alpha)}\Big(\epsilon_{29}\|\nabla u^{\frac{p+\alpha}{2}}\|_2^2 + C(\epsilon_{29})\int_{\mathbb{R}^3} u^{p-\alpha}|\nabla v|^2 dx\Big) \\ &\le C_{70}\|\nabla u^{\frac{p+\alpha}{2}}\|_2^2 + C_{71}\int_{\mathbb{R}^3} u^{p-\alpha}|\nabla v|^2\,dx, \end{aligned} \tag{3.9}$$

by using that $|\nabla u| = \frac{2}{p+\alpha}u^{\frac{p-\alpha}{2}}|\nabla u^{\frac{p+\alpha}{2}}|$ where $\overline{\chi} = \sup_{\mathbb{R}^3\times[0,T)}\chi(v)$, $C_{70} = \frac{2\overline{\chi}\epsilon_{29}}{(p+\alpha)}$ and $C_{71} = \frac{2\overline{\chi}C(\epsilon_{29})}{(p+\alpha)}$. Similarly, we have

$$\int_{\mathbb{R}^3} u^{p-1}\nabla\cdot(\xi u\nabla w)\,dx \le C_{72}\|\nabla u^{\frac{p+\alpha}{2}}\|_2^2 + C_{73}\int_{\mathbb{R}^3} u^{p-\alpha}|\nabla w|^2\,dx. \tag{3.10}$$

Using (3.8)-(3.10) in (3.7), we have

$$\frac{1}{p}\frac{d}{dt}\|u\|_p^p + C_{74}\|\nabla u^{\frac{p+\alpha}{2}}\|_2^2 \le C_{71}\int_{\mathbb{R}^3} u^{p-\alpha}|\nabla v|^2\,dx + C_{73}\int_{\mathbb{R}^3} u^{p-\alpha}|\nabla w|^2\,dx \tag{3.11}$$

where $C_{74} = \frac{4(p-1)(1+\alpha)}{(p+\alpha)^2} - C_{70} - C_{72}$. The first term in RHS of the above is evaluated using the Hölder, Sobolev and Young inequalities as

$$\begin{aligned}
&\int_{\mathbb{R}^3} u^{p-\alpha}|\nabla v|^2\,dx \\
&\le \|u^{p-\alpha}\|^2_{\frac{p}{p-\alpha}} \||\nabla v|^2\|_{\frac{p}{\alpha}} \\
&\le C_{p_1}\|u\|_p^{p-\alpha}\|\Delta v\|^2_{\frac{6p}{2p+3\alpha}} \le C_{p_1}\Big(\frac{\alpha}{p} + \frac{p-\alpha}{p}\|u\|_p^p\Big)\|\Delta v\|^2_{\frac{6p}{2p+3\alpha}}.
\end{aligned} \tag{3.12}$$

Similarly, we have

$$\int_{\mathbb{R}^3} u^{p-\alpha}|\nabla w|^2\,dx \le C_{p_2}\Big(\frac{\alpha}{p} + \frac{p-\alpha}{p}\|u\|_p^p\Big)\|\Delta w\|^2_{\frac{6p}{2p+3\alpha}}. \tag{3.13}$$

Using (3.12) and (3.13) in (3.11), we have

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\|u\|_p^p + C_{74}\|\nabla u^{\frac{p+\alpha}{2}}\|_2^2 &\le C'_{71}\Big(\|\Delta v\|^2_{\frac{6p}{2p+3\alpha}} + \|\Delta w\|^2_{\frac{6p}{2p+3\alpha}}\Big)\|u\|_p^p \\
&+ C'_{72}\Big(\|\Delta v\|^2_{\frac{6p}{2p+3\alpha}} + \|\Delta w\|^2_{\frac{6p}{2p+3\alpha}}\Big).
\end{aligned}$$

Using the Gronwall inequality above, we have

$$\begin{aligned}
\|u\|_p^p \le \exp\Big(&\int_0^t \|\Delta v(s)\|^2_{\frac{6p}{2p+3\alpha}} + \|\Delta w(s)\|^2_{\frac{6p}{2p+3\alpha}}\,ds\Big) \\
&\times \int_0^t \|\Delta v(s)\|^2_{\frac{6p}{2p+3\alpha}} + \|\Delta w(s)\|^2_{\frac{6p}{2p+3\alpha}}\,ds + \|u_0\|_p^p.
\end{aligned} \tag{3.14}$$

Hence, (3.4) holds, whenever the following holds

$$\int_0^T \|\Delta v(s)\|^2_{\frac{6p}{2p+3\alpha}} + \|\Delta w(s)\|^2_{\frac{6p}{2p+3\alpha}}\,ds < \infty, \quad 1+\alpha < p < \infty. \tag{3.15}$$

We prove the above statement in two case: (i) $\alpha > 1/3$ and (ii) $1/8 < \alpha \le 1/3$.

Case (i): $\alpha > \frac{1}{3}$. The first term in (3.15) is evaluated using the standard maximal regularity estimate of heat equation. In $(2.2)_2$, we have

$$\begin{aligned}
&\int_0^T \|\Delta v(s)\|^2_{\frac{6p}{2p+3\alpha}}\,ds \\
&\le C_{75}\Big(\|\nabla v_0\|^2_{\frac{6p}{2p+3\alpha}} + \int_0^T \|u(s)\|^2_{\frac{6p}{2p+3\alpha}}\,ds + \int_0^T \|z\cdot\nabla v\|^2_{\frac{6p}{2p+3\alpha}}\,ds\Big).
\end{aligned} \tag{3.16}$$

For $p > 1+\alpha$, using an interpolation inequality, we have

$$\begin{aligned}
\int_0^T \|u(s)\|^2_{\frac{6p}{2p+3\alpha}}\,ds &\le C_{76}\int_0^t \|u(s)\|_1^{2-\frac{(1+2\alpha)(4p-3\alpha)}{2p(1+3\alpha)}}\|u(s)\|_{3+6\alpha}^{\frac{(1+2\alpha)(4p-3\alpha)}{2p(1+3\alpha)}}\,ds \\
&\le C_{76}\int_0^T \|\nabla u^{\frac{1+2\alpha}{2}}(s)\|_2^{\frac{4}{1+3\alpha}-\frac{3\alpha}{p(1+3\alpha)}}\,ds.
\end{aligned} \tag{3.17}$$

The last term in RHS of (3.16) is evaluated using $z \in L^\infty(0,T;L^6(\mathbb{R}^3))$ and the maximal regularity estimate for the heat equation in $(3.2)_2$ as

$$
\begin{aligned}
&\int_0^T \|z\cdot\nabla v\|^2_{\frac{6p}{2p+3\alpha}}\,ds\\
&\le \int_0^T \|z(s)\|_6^2\|\nabla v(s)\|^2_{\frac{6p}{p+3\alpha}}\,ds\\
&\le C\int_0^T \|\Delta v(s)\|^2_{\frac{2p}{p+\alpha}}\,ds\\
&\le CC_1\Big(\|\nabla v_0\|^2_{\frac{2p}{p+\alpha}}+\int_0^T\|w(s)\|^2_{\frac{2p}{p+\alpha}}\,ds+\int_0^T\|z\cdot\nabla v\|^2_{\frac{2p}{p+\alpha}}\,ds\Big)\\
&\le C_2+C_3\int_0^T\|\nabla v(s)\|^2_{\frac{6p}{2p+3\alpha}}\,ds+C_4\int_0^T\|w(s)\|^2_{\frac{2p}{p+\alpha}}\,ds\\
&\le C_2+C_3\int_0^T\|\nabla v(s)\|^2_{\frac{6p}{2p+3\alpha}}\,ds\\
&\quad +C_4C_5\int_0^T\|w(s)\|_1^{2-\frac{3(1+2\alpha)(p-\alpha)}{2(1+3\alpha)p}}\|w(s)\|_{3+6\alpha}^{\frac{3(1+2\alpha)(p-\alpha)}{2(1+3\alpha)p}}\,ds\\
&\le C_2+C_3\int_0^T\|\nabla v(s)\|^2_{\frac{6p}{2p+3\alpha}}\,ds+C_6\int_0^T\|\nabla w^{\frac{1+2\alpha}{2}}(s)\|_2^{\frac{3(p-\alpha)}{p(1+3\alpha)}}\,ds.
\end{aligned}
\tag{3.18}
$$

The above holds because $\alpha > 1/3$. Similarly we obtain the same estimate for $\int_0^T \|\Delta w(s)\|^2_{\frac{6p}{2p+3\alpha}}\,ds$. Therefore (3.4) holds for $p\in(\max\{1+\alpha,3\alpha\},\infty)$ and it can be extended for $p\in(1+\alpha,\infty)$. Also it implies the following holds for every $1\le p<\infty$: Given $q<\infty$, we have

$$
v_t,\nabla^2 v, w_t,\nabla^2 w, z_t,\nabla^2 z\in L^q((0,T)\times\mathbb{R}^3),\quad \nabla v,\nabla w\in L^\infty((0,T)\times\mathbb{R}^3).
$$

Using the above in (3.11) and an interpolation inequality, we have

$$
\frac{d}{dt}\|u\|_p^p \le C_{77}\|u\|_{p-\alpha}^{p-\alpha}\le C_{77}\|u\|_1^{\frac{\alpha}{p-1}}\|u\|_p^{\frac{p(p-\alpha-1)}{p-1}}\le C_{77}\|u\|_p^{\frac{p(p-\alpha-1)}{p-1}}.
$$

Using Gronwall's inequality above, we have

$$
\|u(t)\|_p \le (C_{77}p^2t)^{1/p}+\|u_0\|_p,\quad t\le T.
$$

Letting $p\to\infty$, we have $u\in L^\infty(0,T;L^p(\mathbb{R}^3))$ for $p\in(1+\alpha,\infty)$.

Case (ii): $\frac18<\alpha\le\frac13$. We prove (3.15) by showing that (3.4) holds, by deriving for $p\in[1,1+4\alpha)$. We estimate the first term in RHS of (3.11) using $\left|\nabla u^{p-\alpha}\right| = C_0 u^{\frac{p-3\alpha}{2}}\left|\nabla u^{\frac{p+\alpha}{2}}\right|$ and Young's inequality as

$$
\begin{aligned}
\int_{\mathbb{R}^3} u^{p-\alpha}|\nabla v|^2dx &= \int_{\mathbb{R}^3} u^{p-\alpha}\nabla v\cdot\nabla v\,dx\\
&\le C_0'\Big(\int_{\mathbb{R}^3}|\nabla u^{p-\alpha}||\nabla v|\,dx+\int_{\mathbb{R}^3}u^{p-\alpha}|\Delta v|\,dx\Big)\\
&\le C_0'\Big(\int_{\mathbb{R}^3}C_0u^{\frac{p-3\alpha}{2}}\left|\nabla u^{\frac{p+\alpha}{2}}\right||\nabla v|\,dx+\int_{\mathbb{R}^3}u^{p-\alpha}|\Delta v|\,dx\Big)\\
&\le C_{78}\|\nabla u^{\frac{p+\alpha}{2}}\|^2+C_{79}\int_{\mathbb{R}^3}u^{p-3\alpha}|\nabla v|^2dx+C_0'\int_{\mathbb{R}^3}u^{p-\alpha}|\Delta v|dx.
\end{aligned}
$$

Similarly, we have

$$\begin{aligned}&\int_{\mathbb{R}^3} u^{p-\alpha}|\nabla w|^2\,dx\\&\leq C'_{78}\|\nabla u^{\frac{p+\alpha}{2}}\|^2 + C'_{79}\int_{\mathbb{R}^3} u^{p-3\alpha}|\nabla w|^2\,dx + C'_1\int_{\mathbb{R}^3} u^{p-\alpha}|\Delta w|\,dx.\end{aligned}$$

Using the above estimates in (3.11), we have

$$\begin{aligned}\frac{d}{dt}\|u\|_p^p + C'\|\nabla u^{\frac{p+\alpha}{2}}\|_2^2 &\leq C_{80}\int_{\mathbb{R}^3} u^{p-3\alpha}|\nabla v|^2\,dx + C_{81}\int_{\mathbb{R}^3} u^{p-\alpha}|\Delta v|\,dx\\&\quad + C_{82}\int_{\mathbb{R}^3} u^{p-3\alpha}|\nabla w|^2\,dx + C_{83}\int_{\mathbb{R}^3} u^{p-\alpha}|\Delta w|\,dx.\end{aligned}\tag{3.19}$$

Integrating the above with respect to t, we obtain

$$\begin{aligned}&\|u\|_p^p + C'\int_0^t \|\nabla u^{\frac{p+\alpha}{2}}\|_2^2 ds\\&\leq C_{80}\int_0^t\int_{\mathbb{R}^3} u^{p-3\alpha}|\nabla v|^2\,dx\,ds + C_{82}\int_0^t\int_{\mathbb{R}^3} u^{p-3\alpha}|\nabla w|^2\,dx\,ds\\&\quad + C'_{80}\int_0^t \|u\|_{p-\alpha+1}^{p-\alpha+1}ds + C'_{81}\int_0^t \|\Delta v\|_{p-\alpha+1}^{p-\alpha+1} + \|\Delta w\|_{p-\alpha+1}^{p-\alpha+1}ds + \|u_0\|_p^p.\end{aligned}\tag{3.20}$$

Evaluating first term of RHS using the Hölder inequality and a maximal regularity estimate, we obtain

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^3} u^{p-3\alpha}|\nabla v|^2 dx ds\\&\leq C_{84}\int_0^t \|u^{p-3\alpha}(s)\|_{\frac{1+\alpha}{p-3\alpha}}\|\nabla v(s)^2\|_{\frac{1+\alpha}{1+4\alpha-p}}\,ds\\&\leq C_{84}C_{85}\int_0^t \|\Delta v(s)\|^2_{\frac{6+6\alpha}{5+14\alpha-3p}}\\&\leq C_{84}C_{85}\Big(\|\nabla v_0\|^2_{\frac{6+6\alpha}{5+14\alpha-3p}} + \int_0^t \|u\|^2_{\frac{6+6\alpha}{5+14\alpha-3p}}ds + \int_0^t \|z\cdot\nabla v\|^2_{\frac{6+6\alpha}{5+14\alpha-3p}}ds\Big)\end{aligned}\tag{3.21}$$

for all p satisfying $1+\alpha < p < 1+4\alpha$. Above, we used that $1/8 < \alpha \leq 1/3$ and $p \in (1, 1+4\alpha)$. As $1+\alpha < \frac{6+6\alpha}{5+14\alpha-3p} < 3p+3\alpha$ and using interpolation inequality, the second term in RHS of (3.21) is evaluated as

$$\int_0^t \|u(s)\|^2_{\frac{6+6\alpha}{5+14\alpha-3p}}ds \leq \int_0^t \|u\|_{1+\alpha}^{2(1-t_1)}\|u\|_{3p+3\alpha}^{2t_1}\,ds \leq \int_0^t \|\nabla u^{\frac{p+\alpha}{2}}\|_2^{\frac{4t_1}{p+\alpha}}\,ds \tag{3.22}$$

where $t_1 = \frac{(3p-14\alpha+1)(p+\alpha)}{2(3p+2\alpha-1)}$. The third term in RHS of (3.21) is estimated using the Gagliardo-Nierenberg inequality and maximal regularity estimate as

$$\begin{aligned}
&\int_0^t \|z\cdot\nabla v\|^2_{\frac{6+6\alpha}{5+14\alpha-3p}}ds \\
&\le C'_{84}\int_0^t \|\nabla v\|_{\frac{6+6\alpha}{4+13\alpha-3p}} \\
&\le C'_{85}\int_0^t \|\Delta v\|_{\frac{2+2\alpha}{2+5\alpha-p}} \\
&\le C'_{85}\Big(\|\nabla v_0\|^2_{\frac{2+2\alpha}{2+5\alpha-p}} + \int_0^t \|u\|^2_{\frac{2+2\alpha}{2+5\alpha-p}}ds + \int_0^t \|z\cdot\nabla v\|^2_{\frac{2+2\alpha}{2+5\alpha-p}}ds\Big) \\
&\le C'_{85}\|\nabla v_0\|^2_{\frac{2+2\alpha}{2+5\alpha-p}} + C'_{85}\int_0^t \|\nabla u^{\frac{p+\alpha}{2}}\|_2^{\frac{8(1+\alpha)}{(p+\alpha)(2+5\alpha-p)}}ds \\
&+ C_{86}\int_0^t \|\nabla v\|^2_{\frac{6+6\alpha}{5+14\alpha-3p}}ds.
\end{aligned} \tag{3.23}$$

Substituting (3.22) amd (3.23) in (3.21), we have

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3} u^{p-3\alpha}|\nabla v|^2dxds \\
&\le C_{87}\|\nabla v_0\|^2_{\frac{6+6\alpha}{5+14\alpha-3p}} + C_{88}\|\nabla v_0\|^2_{\frac{2+2\alpha}{2+5\alpha-p}} + C_{89}\int_0^t \|\nabla u^{\frac{p+\alpha}{2}}\|_2^{\frac{4t_1}{p+\alpha}}ds \\
&\quad + C_{90}\int_0^t \|\nabla u^{\frac{p+\alpha}{2}}\|_2^{\frac{8(1+\alpha)}{(p+\alpha)(2+5\alpha-p)}}ds + C_{91}\int_0^t \|\nabla v\|^2_{\frac{6+6\alpha}{5+14\alpha-3p}}ds.
\end{aligned} \tag{3.24}$$

Proceeding as above, we obtain

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3} u^{p-3\alpha}|\nabla w|^2dxds \\
&\le C'_{87}\|\nabla w_0\|^2_{\frac{6+6\alpha}{5+14\alpha-3p}} + C'_{88}\|\nabla w_0\|^2_{\frac{2+2\alpha}{2+5\alpha-p}} + C'_{89}\int_0^t \|\nabla u^{\frac{p+\alpha}{2}}\|_2^{\frac{4t_1}{p+\alpha}}ds \\
&\quad + C'_{90}\int_0^t \|\nabla u^{\frac{p+\alpha}{2}}\|_2^{\frac{8(1+\alpha)}{(p+\alpha)(2+5\alpha-p)}}ds + C'_{91}\int_0^t \|\nabla w\|^2_{\frac{6+6\alpha}{5+14\alpha-3p}}ds.
\end{aligned} \tag{3.25}$$

Similarly, we deduce

$$\begin{aligned}
&C'_{80}\int_0^t \|u\|^{p-\alpha+1}_{p-\alpha+1}ds + C'_{81}\int_0^t \|\Delta v\|^{p-\alpha+1}_{p-\alpha+1} + \|\Delta w\|^{p-\alpha+1}_{p-\alpha+1}ds \\
&\le C_{91}\|\nabla v_0\|^{p-\alpha+1}_{p-\alpha+1} + C_{92}\|\nabla w_0\|^{p-\alpha+1}_{p-\alpha+1} + C_{93}\|\nabla v_0\|^2_{\frac{6(1+p-\alpha)}{7+p-\alpha}} \\
&\quad + C_{94}\|\nabla w_0\|^2_{\frac{6(1+p-\alpha)}{7+p-\alpha}} + C_{95}\|\nabla u^{\frac{p+\alpha}{2}}\|_2^{\frac{6(p-2\alpha)}{2\alpha+3p-1}} \\
&\quad + C_{96}\|\nabla u^{\frac{p+\alpha}{2}}\|_2^{\frac{1+5\alpha-5p}{1-3\alpha-3p}} + C_{97}\|\Delta v_0\|_2^{\frac{3(p-\alpha-1)}{2(p-\alpha+1)}} + C_{98}\|\Delta w_0\|_2^{\frac{3(p-\alpha-1)}{2(p-\alpha+1)}}.
\end{aligned} \tag{3.26}$$

Using the above estimates in (3.19) and Young's inequality, for $p \in (1+\alpha, 1+4\alpha)$ we have

$$\|u\|_p^p + C'_{91}\int_0^t \|\nabla u^{\frac{p+\alpha}{2}}\|_2^2ds \le C'_{92}, \quad 0<t<T. \tag{3.27}$$

Here we used the facts $\frac{4t_1}{p+\alpha} \in (0,2)$ and $\frac{6+6\alpha}{5+14\alpha-3p} \in [2,6]$ for $1/8 < \alpha \le 1/3$ and $1+\alpha < p < 1+4\alpha$. Hence, we have (3.4) holds for $p \in \left(\frac{2+11\alpha}{3}, 1+4\alpha\right)$, Also, the above can be extended for $p \in (1+\alpha, 1+4\alpha)$. Choose $s_0 = \frac{3}{2} - \frac{3\alpha}{4}$. Then from above, $u \in L^\infty(0,T; L^{s_0}(\mathbb{R}^3))$ as $1 \le s_0 < 1+4\alpha$. To prove (3.15) holds for $\alpha \in [1/8, 1/3)$, we choose $s_1 = \frac{6p}{2p+3\alpha}$ such that $s_1 \in (s_0, 3s_0+3\alpha)$ for $p > 1+\alpha$. Then by the standard maximal regularity estimate and using Hölder inequality, we have

$$\int_0^T \|\Delta v\|_{s_1} \le C_{99} \int_0^T \|u\|_{s_1}^2\,ds + \int_0^T \|z\cdot\nabla v\|_{s_1}^2\,ds. \tag{3.28}$$

The first term in RHS of above evaluated using interpolation inequality as

$$\begin{aligned}
\int_0^T \|u\|_{s_1}^2\,ds &\le C_{91} \int_0^T \|u\|_{s_0}^{2-\frac{(s_0+\alpha)(6p-2ps_0-3\alpha s_0)}{2(2s_0+3\alpha)}} \|u\|_{3s_0+3\alpha}^{\frac{(s_0+\alpha)(6p-2ps_0-3\alpha s_0)}{2(2s_0+3\alpha)}} ds \\
&\le C_{91} \int_0^T \|\nabla u^{\frac{s_0+\alpha}{2}}\|_2^{\frac{p(12-4s_0)-6\alpha s_0}{p(2s_0+3\alpha)}} ds \\
&= C_{91} \int_0^T \|\nabla u^{\frac{s_0+\alpha}{2}}\|_2^{2-\frac{2p(3\alpha+4s-6)+6\alpha s}{p(3\alpha+2s)}} ds.
\end{aligned} \tag{3.29}$$

The second term in RHS of (3.28) evaluated using maximal regularity estimate as

$$\begin{aligned}
&\int_0^T \|z\cdot\nabla v\|_{s_1}^2 ds \\
&\le C'_{91} \int_0^T \|\Delta v\|_{\frac{2p}{p+\alpha}}^2 ds \\
&\le C'_{91} \Big(\|\nabla v_0\|_{\frac{2p}{p+\alpha}}^2 + \int_0^T \|u\|_{\frac{2p}{p+\alpha}}^2 ds + \int_0^T \|z\cdot\nabla v\|_{\frac{2p}{p+\alpha}}^2 ds \Big) \\
&\le C'_{92} + C'_{93} \int_0^T \|u\|_{1+\alpha}^{\frac{(\alpha+1)(3\alpha(2\alpha+1)+6\alpha p+p)}{(5\alpha+2)p}} \|u\|_{3+6\alpha}^{\frac{3(2\alpha+1)(\alpha(\alpha+1)+(\alpha-1)p)}{(5\alpha+2)p}} ds \\
&\le C'_{92} + C'_{94} \int_0^T \|u^{\frac{1+2\alpha}{2}}\|_{1+\alpha}^{\frac{6(\alpha-1)(\alpha-p)}{(5\alpha+2)p}}.
\end{aligned} \tag{3.30}$$

As (3.15) holds for $\alpha > \frac{1}{8}$, we have $u \in L^\infty(0,T; L^p(\mathbb{R}^3))$ for $p \in (1+\alpha, \infty)$. As similar to the above, we have u is bounded in L^∞-norm. □

Theorem 3.3. *Suppose that the initial data (u_0, v_0, w_0, z_0) satisfies (1)-(4) of Theorem 2.4 along with $u_0 \in L^\infty(\mathbb{R}^3)$ and $v_0, w_0, z_0 \in W^{1,q}(\mathbb{R}^3)$ for all $q \in [2,\infty)$. Furthermore, assume that (3.3) holds. Then for each $T > 0$, system (3.2) possesses a bounded weak solution (u, v, w, z) that satisfies*

$$\begin{aligned}
&\|u\|_{L^\infty((0,T)\times\mathbb{R}^3)} + \|\nabla u^{\frac{p+\alpha}{2}}\|_{L^2((0,T)\times\mathbb{R}^3)} \\
&+ \|v\|_{L^q(0,T;W^{2,q}(\mathbb{R}^3))} + \|w\|_{L^q(0,T;W^{2,q}(\mathbb{R}^3))} + \|z\|_{L^q(0,T;W^{2,q}(\mathbb{R}^3))} \\
&+ \|\partial_t v\|_{L^q(0,T;L^q(\mathbb{R}^3))} + \|\partial_t w\|_{L^q(0,T;L^q(\mathbb{R}^3))} + \|\partial_t z\|_{L^q(0,T;L^q(\mathbb{R}^3))} < C,
\end{aligned}$$

where C is a constant depending on the initial data.

Proof. The existence of local weak solutions for model (3.1) can be obtained as for model (2.1). Therefore, we omit the proof here for the sake simplicity. Now, it is sufficient to prove that the system admits bounded weak solution. From Lemma

3.2 and using the Aubin-Lions compactness lemma we have a weak solution, which indeed is a bounded weak solution for (3.1). □

The above theorem can be proven for a bounded domain Ω with Neumann boundary conditions for u, v, and w, as well as no-slip boundary conditions for z as specified in (2.74). Further, we have the result for bounded domain as corollary.

Corollary 3.4. *Suppose that the initial data* (u_0, v_0, w_0, z_0) *satisfies (1)–(4) of Theorem 2.4 when replacing* $\mathbb{R}^3$ *by* Ω*, along with* $u_0 \in L^\infty(\Omega)$ *and* $v_0, w_0, z_0 \in W^{1,q}(\Omega)$ *for any* $q \in [2, \infty)$*. Furthermore, assume that* (3.3) *holds when replacing* $\mathbb{R}^3$ *by* Ω*. Then for each* $T > 0$*, system* (3.2) *with boundary conditions* (2.74) *possesses a bounded weak solution* (u, v, w, z) *that satisfies*

$$\begin{aligned}
&\|u\|_{L^\infty((0,T)\times\Omega)} + \|\nabla u^{\frac{p+\alpha}{2}}\|_{L^2((0,T)\times\Omega)} + \|v\|_{L^q(0,T;W^{2,q}(\Omega))} \\
&+ \|w\|_{L^q(0,T;W^{2,q}(\Omega))} + \|z\|_{L^q(0,T;W^{2,q}(\Omega))} + \|\partial_t v\|_{L^q(0,T;L^q(\Omega))} \\
&+ \|\partial_t w\|_{L^q(0,T;L^q(\Omega))} + \|\partial_t z\|_{L^q(0,T;L^q(\Omega))} < C,
\end{aligned}$$

where C *is a constant depending on the initial data.*

References

[1] Myeongju Chae, Kyungkeun Kang, Jihoon Lee; *Existence of smooth solutions to coupled chemotaxis-fluid equations*, Discret. Contin. Dyn. Syst. **33** (2013), no. 6, 2271–2297.

[2] Myeongju Chae, Kyungkeun Kang, Jihoon Lee; *Existence of smooth solutions to coupled chemotaxis-fluid equations*, Discret. Contin. Dyn. Syst. **33** (2013), no. 6, 2271–2297.

[3] Myeongju Chae, Kyungkeun Kang, Jihoon Lee; *Global Existence and Temporal Decay in Keller-Segel Models Coupled to Fluid Equations*, Commun. Partial Differ. Equations **39** (2014), no. 7, 1205–1235.

[4] Xiaoyu Chen, Jijie Zhao, Qian Zhang; *Global existence of weak solutions for the 3D axisymmetric chemotaxis-Navier-Stokes equations with nonlinear diffusion*, Discret. Contin. Dyn. Syst., **42** (2022), no. 9, 4489–4522.

[5] A. Chertock, K. Fellner, A. Kurganov, A. Lorz, P. A. Markowich; *Sinking, merging and stationary plumes in a coupled chemotaxis-fluid model: a high-resolution numerical approach*, J. Fluid Mech. **694** (2012), 155–190.

[6] Yun-Sung Chung, Kyungkeun Kang; *Existence of global solutions for a chemotaxis-fluid system with nonlinear diffusion*, J. Math. Phys. **57** (2016), no. 4, 041503.

[7] Yun-Sung Chung, Kyungkeun Kang, Jaewoo Kim; *Global Existence of Weak Solutions for a Keller-Segel-Fluid Model with Nonlinear Diffusion*, J. Korean Soc. Math. **51** (2014), no. Issue 3, 635–654.

[8] Renjun Duan, Alexander Lorz, Peter Markowich; *Global Solutions to the Coupled Chemotaxis-Fluid Equations*, Commun. Partial Differ. Equations **35** (2010), no. 9, 1635–1673.

[9] Abelardo Duarte-Rodriguez, Lucas C. F. Ferreira, Elder J. Villamizar-Roa; *Global Existence for an Attraction–Repulsion Chemotaxis-Fluid System in a Framework of Besov–Morrey type*, J. Math. Fluid Mech. **22** (2020).

[10] Elio Espejo, Takashi Suzuki; *Global existence and blow-up for a system describing the aggregation of microglia*, Appl. Math. Lett. **35** (2014), 29–34.

[11] Marco Di Francesco, Alexander Lorz, Peter A. Markowich; *Chemotaxis-fluid coupled model for swimming bacteria with nonlinear diffusion: Global existence and asymptotic behavior*, Discret. Contin. Dyn. Syst. **28** (2010), no. 4, 1437–1453.

[12] Kentarou Fujie, Takasi Senba; *Global existence and boundedness of radial solutions to a two dimensional fully parabolic chemotaxis system with general sensitivity*, Nonlinearity **29** (2016), no. 8, 2417.

[13] Miguel A Herrero, Juan J.L. Velazquez; *Singularity patterns in a chemotaxis model.*, Math. Ann. **306** (1996), no. 3, 583–624.

[14] S. Ishida; *Global existence and boundedness for chemotaxis-Navier-Stokes systems with position-dependent sensitivity in 2D bounded domains*, Discret. Contin. Dyn. Syst. **35** (2015), no. 8, 3463–3482.

[15] Sachiko Ishida, Tomomi Yokota; *Global existence of weak solutions to quasilinear degenerate Keller–Segel systems of parabolic–parabolic type with small data*, J. Differ. Equ. **252** (2012), no. 3, 2469–2491.

[16] Liu Jia, Wang Zhi-An; *Classical solutions and steady states of an attraction–repulsion chemotaxis in one dimension*, J. Biol. Dyn. **6** (2012), no. sup1, 31–41, PMID: 22873673.

[17] Kyungkeun Kang, Hwa Kil Kim; *Existence of Weak Solutions in Wasserstein Space for a Chemotaxis Model Coupled to Fluid Equations*, SIAM J. Math. Anal. **49** (2017), no. 4, 2965–3004.

[18] Kyungkeun Kang, Kyunghwa Kim, Changwook Yoon; *Existence of weak and regular solutions for Keller–Segel system with degradation coupled to fluid equations*, J. Math. Anal. **485** (2020), no. 1, 123750.

[19] Evelyn F. Keller, Lee A. Segel; *Model for chemotaxis*, J. Theor. Biol. **30** (1971), no. 2, 225–234.

[20] Hideo Kozono, Masanari Miura, Yoshie Sugiyama; *Existence and uniqueness theorem on mild solutions to the Keller–Segel system coupled with the Navier–Stokes fluid*, J. Funct. Anal. **270** (2016), no. 5, 1663–1683.

[21] Xie Li, Youjun Xiao; *Global existence and boundedness in a 2D Keller-Segel–Stokes system*, Nonlinear Anal. Real World Appl. **37** (2017), 14–30.

[22] Jian-Guo Liu, Alexander Lorz; *A coupled chemotaxis-fluid model: Global existence*, Ann. l'Institut Henri Poincare Anal. Non Lineaire **28** (2011), no. 5, 643–652.

[23] Jian-Guo Liu, Alexander Lorz; *A coupled chemotaxis-fluid model: Global existence*, Ann. Inst. Henri Poincare (C) Anal. Non Lineaire **28** (2011), no. 5, 643–652.

[24] Alexander Lorz; *A coupled Keller–Segel–Stokes model: global existence for small initial data and blow-up delay*, Commun. Math. Sci. **10** (2012), no. 2, 555–574.

[25] Magdalena Luca, Alexandra Chavez-Ross, Leah Edelstein-Keshet, Alex Mogilner; *Chemotactic signaling, microglia, and Alzheimer's disease senile plaques: Is there a connection?*, Bull. Math. Biol. **65** (2003), no. 4, 693–730.

[26] Toshitaka Nagai; *Global existence of solutions to a parabolic system for chemotaxis in two space dimensions*, Nonlinear Anal. Theory. Methods Appl. **30** (1997), no. 8, 5381–5388, Proceedings of the Second World Congress of Nonlinear Analysts.

[27] Yoshie Sugiyama, Hiroko Kunii; *Global existence and decay properties for a degenerate Keller–Segel model with a power factor in drift term*, J. Differ. Equ. **227** (2006), no. 1, 333–364.

[28] Youshan Tao, Zhi-an Wang; *Competing effects of attraction vs. repulsion in chemotaxis*, Math. Models Methods Appl. Sci. **23** (2013), no. 01, 1–36.

[29] Youshan Tao, Michael Winkler; *Global existence and boundedness in a Keller-Segel-Stokes model with arbitrary porous medium diffusion*, Discret. Contin. Dyn. Syst. **32** (2012), no. 5, 1901–1914.

[30] Youshan Tao, Michael Winkler; *Locally bounded global solutions in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion*, Ann. l'Institut Henri Poincare Anal. Non Lineaire **30** (2013), no. 1, 157–178.

[31] Idan Tuval, Luis Cisneros, Christopher Dombrowski, Charles W. Wolgemuth, John O. Kessler, Raymond E. Goldstein; *Bacterial swimming and oxygen transport near contact lines*, Proc. Natl. Acad. Sci. USA. **102** (2005), no. 7, 2277–2282.

[32] Michael Winkler; *Boundedness and large time behavior in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion and general sensitivity*, J. Math. Fluid Mech. **54** (2015), no. 4, 3789–3828.

[33] Qingshan Zhang, Yuxiang Li; *An attraction-repulsion chemotaxis system with logistic source*, Z Angew. Math. Mech. **96** (2016), no. 5, 570–584.

[34] Pan Zheng, Chunlai Mu, Xuegang Hu; *Global dynamics for an attraction–repulsion chemotaxis–(Navier)–Stokes system with logistic source*, Nonlinear Anal. Real World Appl. **45** (2019), 557–580.

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