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DELAY-DEPENDENT STABILITY CONDITIONS FOR DELAY DIFFERENTIAL EQUATIONS WITH UNBOUNDED OPERATORS IN BANACH SPACES

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Abstract. We consider the equation $du(t)/dt = Au(t) + Bu(t-h)$ where $t > 0$, h is a positive constant, and A is a linear unbounded and B is a linear bounded operators. We establish explicit delay-dependent conditions for exponential stability, and present applications to partial integro-differential equations with delay.

1. Introduction and statement of the main result

In this article we suggest delay-dependent stability conditions for delay differential equations with unbound operators in a Banach space.

The basic method for the stability analysis of functional differential equations is the Lyapunov-Krasovskij method [4, 13]. By that method, many results have been obtained. Recently, that method has been extended to functional differential equations in a Hilbert space, see [1, 6, 14, 15] and references given therein. In [8, 10] the delay-dependent stability conditions for equations in a Banach space with *bounded operators* have been derived. To the best of our knowledge, the delay-dependent stability conditions for equations in a Banach space with unbounded operators are not investigated in the available literature.

It should be noted that finding the Lyapunov-Krasovskij type functionals or solving the corresponding operator inequalities are often connected with serious mathematical difficulties,

To the contrary, the stability conditions presented in this paper are explicitly formulated in terms of the coefficients and delays. The literature on the delay-dependent stability criteria is rather rich, but mainly equations in a finite dimensional space are considered, see [2, 3, 13].

Everywhere below, $\mathcal{X}$ is a complex Banach space with a norm $\|\cdot\|_{\mathcal{X}} = \|\cdot\|$ and the unit operator $I_{\mathcal{X}} = I$. By $\mathcal{B}(\mathcal{X})$, we denote the set of all bounded linear operators in $\mathcal{X}$. For a linear operator T, $\sigma(T)$ is the spectrum and $\|T\|_{\mathcal{X}} = \|T\|$ is the operator norm of T if it is bounded.

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Furthermore, $C(J,\mathcal{X})$ is the space of $\mathcal{X}$-valued functions f defined and continuous on a finite or infinite real segment J, and equipped with the finite norm.

$$\|f\|_{C(J)} = \|f\|_{C(J,\mathcal{X})} := \sup_{t\in J} \|f(t)\|_{\mathcal{X}}.$$

In addition, $W(J,\mathcal{X})$ is the space of $\mathcal{X}$-valued functions f defined and strongly continuously differentiable on J, and equipped with the norm

$$\|f\|_{W(J)} = \|f\|_{W(J,\mathcal{X})} := \max\{\sup_{t\in J} \|f'(t)\|_{\mathcal{X}}, \sup_{t\in J} \|f(t)\|_{\mathcal{X}}\}.$$

Denote also $R_+ = [0,\infty)$ and $R_\eta = [-\eta,\infty)$ for a finite $\eta > 0$.

Throughout this article A is a closed linear operator with a dense domain $D(A) \subseteq \mathcal{X}$, generating a strongly continuous semigroup e^{At} on $\mathcal{X}$, and $B \in \mathcal{B}(\mathcal{X})$ maps $\mathcal{X}$ into $D(A)$.

Our main object is to study the equation

$$y'(t) = Ay(t) + By(t-h) \quad (t > 0; 0 < h = \text{const.}\infty) \tag{1.1}$$

with the initial condition

$$y(t) = \phi(t) \quad (-h \le t \le 0), \tag{1.2}$$

where $\phi \in W([-h,0],\mathcal{X}) \cap D(A)$ is given.

Various integro-differential equations with differential operators A and integral operators B are examples of (1.1).

A solution of problem (1.1), (1.2) is defined as a continuous function $y(t)$ defined on R_η with values in $D(A)$, having a continuous derivative for all $t > 0$ and the right derivative at zero, and satisfying (1.1), and (1.2).

Let

$$\int_0^\infty \|e^{As}\|_{\mathcal{X}} ds < \infty. \tag{1.3}$$

Since AB is defined on the whole $\mathcal{X}$, due to the Banach theorem [12, Section 2] AB is bounded, and consequently,

$$\psi_A := \int_0^\infty \|e^{As}AB\|_{\mathcal{X}} ds < \infty.$$

In addition, put $M = A + B$ and assume that

$$\int_0^\infty \|e^{Ms}\|_{\mathcal{X}} ds < \infty. \tag{1.4}$$

Therefore

$$\psi_M := \int_0^\infty \|e^{Ms}B\|_{\mathcal{X}} ds < \infty.$$

Now we are in a position to formulate the main result of the paper.

Theorem 1.1. *Let conditions* (1.3),(1.4) *and*

$$h\psi_M(\psi_A + \|B\|_{\mathcal{X}}) < 1 \tag{1.5}$$

hold. Then problem (1.1), (1.2) *with* $\phi \in W(-h,0) \cap D(A)$ *has a unique solution* $y(t)$*, which satisfies the inequality* $\|y\|_{C(R_+)} \le c_0\|\phi\|_{W(-h,0)}$*, where the constant* $c_0 \ge 1$ *does not depend on* ϕ*.*

The proof of this theorem is presented in the next section. Theorem 1.1 gives us the conditions for the Lyapunov stability with respect to $W(-h,0)$.

We will say that (1.1) is exponentially stable with respect to $W(-h,0)$, if there are constants $\alpha>0$ and $c_1\ge 1$ independent of $\phi\in W(-h,0)$, such that

$$\|y(t)\|_{\mathcal{X}}\le c_1e^{-\alpha t}\|\phi\|_{W(-h,0)}\quad (t\ge 0)$$

for any solution of (1.1), (1.2).

Assume that the semigroups e^{At} and e^{Mt} are exponentially stable:

$$\|e^{At}\|_{\mathcal{X}}\le c_Ae^{-\alpha_A t}\quad\text{and}\quad \|e^{Mt}\|_{\mathcal{X}}\le c_Me^{-\alpha_M t}, \tag{1.6}$$

where $t\ge 0$, $\alpha_A>0$, $\alpha_M>0$, $c_A\ge 1$, $c_M\ge 1$. Then

$$\psi_A\le \|AB\|_{\mathcal{X}}\int_0^\infty c_Ae^{-\alpha_A t}dt=c_A\|AB\|_{\mathcal{X}}/\alpha_A,\psi_M\le c_M\|B\|_{\mathcal{X}}/\alpha_M.$$

So (1.5) is provided by the inequality

$$\frac{hc_M\|B\|_{\mathcal{X}}}{\alpha_M}\Big(\frac{c_A\|AB\|_{\mathcal{X}}}{\alpha_A}+\|B\|_{\mathcal{X}}\Big)<1. \tag{1.7}$$

Now Theorem 1.1 implies

$$\|y\|_{C(R_+)}\le c_2\|\phi\|_{W(-h,0),} \tag{1.8}$$

where c_2 does not depend on ϕ.

In the next section we also show that Theorem 1.1 implies the following result.

Corollary 1.2. *Let conditions* (1.6) *and* (1.7) *hold. Then* (1.1) *is exponentially stable with respect to* $W(-h,0)$.

This corollary is sharp: if $B=0$, then its conditions are necessary for the exponential stability. Moreover, its conditions are necessary if $h=0$ and $A=0$.

2. Proofs of Theorem 1.1 and Corollary 1.2

Proof of Theorem 1.1. According to [9, Theorem 1], problem (1.1), (1.2) has a unique differentiable solution $y(t)$. Since $y(t)\in D(A)$, by the variation of constants formula [5, Sect. III.1], (1.1) is equivalent to the equation

$$y(t)=e^{At}\phi(0)+\int_0^t e^{A(t-s)}By(s-h)ds.$$

Consequently, in view of (1.1),

$$dy(t)/dt=Ay(t)+By(t-h)=A(e^{At}\phi(0)+\int_0^t e^{A(t-s)}By(s-h)ds)+By(t-h).$$

Since AB is bounded the integral $\int_0^t e^{A(t-s)}ABy(s-h)ds$ $(0<t<\infty)$ converges and

$$A\int_0^t e^{A(t-s)}By(s-h)ds=\int_0^t e^{A(t-s)}ABy(s-h)ds.$$

Thus, (1.1) can be written as

$$\frac{dy}{dt}=Ae^{At}\phi(0)+\int_0^t e^{A(t-s)}ABy(s-h)ds+By(t-h).$$

Hence, with the notation

$$|y|_t:=\sup_{0\le s\le t}\|y(s)\|_{\mathcal{X}}\ \ (0<t<\infty)\quad\text{and}\quad a_0:=\sup_{t\ge 0}\|e^{At}\|_{\mathcal{X}},$$

we can write

$$|y'|_t \le a_0\|A\phi(0)\|_{\mathcal{X}} + \int_0^t \|e^{A(t-s)}ABy(s-h)\|_{\mathcal{X}}ds + \|B\|_{\mathcal{X}} \sup_{0\le s\le t}\|y(s-h)\|_{\mathcal{X}},$$

and therefore

$$|y'|_t \le a_0\|A\phi(0)\|_{\mathcal{X}} + (\psi_A + \|B\|_{\mathcal{X}}) \sup_{0\le s\le t}\|y(s-h)\|_{\mathcal{X}},$$

i. e.

$$|y'|_t \le a_0\|A\phi(0)\|_{\mathcal{X}} + (\psi_A + \|B\|_{\mathcal{X}})(\|\phi\|_{C(-h,0)} + |y|_t). \tag{2.1}$$

From (1.1) and (1.2) it follows that

$$\phi'(0) = A\phi(0) + B\phi(-h).$$

Hence,

$$\|A\phi(0)\|_{\mathcal{X}} \le (1 + \|B\|_{\mathcal{X}})\|\phi\|_{W(-h,0).}$$

Now (2.1) yields

$$|y'|_t \le a_0(1 + \|B\|_{\mathcal{X}})\|\phi\|_{W(-h,0)} + (\psi_A + \|B\|_{\mathcal{X}})\|\phi\|_{C(-h,0)} + (\psi_A + \|B\|_{\mathcal{X}})|y|_t$$

and thus

$$|y'|_t \le \hat{c}\|\phi\|_{W(-h,0)} + (\psi_A + \|B\|_{\mathcal{X}})|y|_t, \tag{2.2}$$

where

$$\hat{c} = a_0(1 + \|B\|_{\mathcal{X}}) + \psi_A + \|B\|_{\mathcal{X}}.$$

Furthermore, we rewrite (1.1) as

$$y'(t) = My(t) + B(y(t-h) - y(t)) \quad (t > 0). \tag{2.3}$$

Recall that $M = A + B$. from the above mentioned variation of constants formula,

$$y(t) = e^{Mt}\phi(0) + \int_0^t e^{M(t-s)}B(y(s-h) - y(s))ds.$$

Hence,

$$|y|_t \le m_0\|\phi(0)\|_{\mathcal{X}} + \int_0^t \|e^{M(t-s)}B\|_{\mathcal{X}}ds \ \sup_{s\le t}\|y(s-h) - y(s)\|_{\mathcal{X}}, \tag{2.4}$$

where $m_0 := \sup_{t\ge 0}\|e^{Mt}\|_{\mathcal{X}}$, and therefore

$$|y|_t \le m_0\|\phi(0)\|_{\mathcal{X}} + \psi_M \sup_{0\le s\le t}\|y(s-h) - y(s)\|_{\mathcal{X}}. \tag{2.5}$$

Note that

$$\begin{aligned}\|y(s-h) - y(s)\|_{\mathcal{X}} &= \|\int_{s-h}^{s} y'(s_1)ds_1\|_{\mathcal{X}} \\ &\le h\|y'\|_{C(-h,t)} \\ &\le h\|\phi'\|_{C(-h,0)} + h|y'|_t \ \ (s \le t).\end{aligned}$$

Using (2.5), we arrive at the inequality

$$|y|_t \le m_0\|\phi(0)\|_{\mathcal{X}} + \psi_M h(\|\phi'\|_{C(-h,0)} + |y'|_t).$$

Now (2.2) implies

$$|y|_t \le \|\phi\|_{W(-h,0)}(m_0 + \psi_M h + h\psi_M\hat{c}) + h\psi_M(\psi_A + \|B\|_{\mathcal{X}})|y|_t,$$

or

$$|y|_t \le \hat{c}_2\|\phi\|_{W(-h,0)} + h\psi_M(\psi_A + \|B\|_{\mathcal{X}})|y|_t,$$

where $\hat{c}_2 = m_0 + \psi_M h + h\psi_M \hat{c}$. According (1.5) we obtain

$$|y|_t \le \hat{c}_2 \|\phi\|_{W(-h,0)} (1 - h\psi_M(\psi_A + \|B\|_{\mathcal{X}}))^{-1}.$$

Hence, letting $t \to \infty$, we obtain

$$\|y\|_{C(R_+)} \le (1 - h\psi_M(\psi_A + \|B\|_{\mathcal{X}}))^{-1} \hat{c}_2 \|\phi\|_{W(-h,0)}.$$

This proves the required result. □

Proof of Corollary 1.2. Substitute

$$y(t) = e^{-\epsilon t} y_\epsilon(t) \tag{2.6}$$

with $\epsilon > 0$ into (1.1). We obtain the equation

$$y_\epsilon'(t) = (A + \epsilon I) y_\epsilon(t) + B e^{\epsilon h} y_\epsilon(t-h) \quad (t > 0). \tag{2.7}$$

Put $M(\epsilon) = A + \epsilon I + B e^{\epsilon h}$. We have

$$\begin{aligned} e^{M(\epsilon)t} - e^{Mt} &= \int_0^t e^{M(t-s)} (M(\epsilon) - M) e^{M(\epsilon)s} ds \\ &= -\int_0^t e^{M(t-s)} (\epsilon I + B(e^{\epsilon h} - 1)) e^{M(\epsilon)s} ds. \end{aligned}$$

Hence,

$$\begin{aligned} \|e^{M(\epsilon)t}\| &\le \|e^{Mt}\| + \int_0^t \|e^{M(t-s)}\| \delta(\epsilon) \|e^{M(\epsilon)s}\| ds \\ &\le e^{-\alpha_M t} + \delta(\epsilon) \int_0^t e^{-\alpha_M (t-s)} \|e^{M(\epsilon)s}\| ds, \end{aligned}$$

where $\delta(\epsilon) = \|\epsilon I + B(e^{\epsilon h} - 1)\| \to 0$ as $\epsilon \to 0$. Thus we obtain

$$\|e^{(M(\epsilon) + \alpha_M I)t}\| \le 1 + \delta(\epsilon) \int_0^t \|e^{(M(\epsilon) + \alpha_M I)s}\| ds.$$

Now the Gronwall lemma yields

$$\|e^{M(\epsilon)t}\| \le e^{-\alpha_M(\epsilon) t},$$

where $\alpha_M(\epsilon) = \alpha_M - \delta(\epsilon)$. So $\alpha_M(0) = \alpha_M$. If (1.6), (1.7) hold, then for small enough $\epsilon > 0$,

$$\frac{h c_M e^{\epsilon h} \|B\|}{\alpha_M(\epsilon)} \Big(\frac{c_A e^{\epsilon h} \|AB + \epsilon B\|}{\alpha_A - \epsilon} + e^{\epsilon h} \|B\| \Big) < 1.$$

From inequality (1.8), which follows from Theorem 1.1, a solution of (2.7) with the initial function $\phi \in W(-h, 0)$ satisfies the inequality $\|y_\epsilon\|_{C(R_+)} \le c_\epsilon \|\phi\|_{W(-h,0)}$, where c_ϵ does not depend on ϕ. Hence, (2.6) yields

$$\|y(t)\|_{C(R_+)} \le c_\epsilon e^{-\epsilon t} \|\phi\|_{W(-h,0)} \quad (t \ge 0).$$

This proves the exponential stability. □

3. Example

In this section $\mathcal{X} = L^2(0,1)$, where $L^2(0,1) = L^2$ is the traditional Hilbert space of complex-valued functions defined on $[0,1]$ with the scalar product

$$(f, f_1) = \int_0^1 f(x)\overline{f}_1(x)dx \quad (f, f_1 \in L^2).$$

We consider the equation

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + b(x)u(t,x) + \int_0^1 K(x,s)u(t-h,s)ds \tag{3.1}$$

for $0 \le x \le 1$ and $t \ge 0$, where $b(x)$ is a complex valued function defined and bounded on $[0,1]$; $K(x,s)$ is defined on $[0,1]\times[0,1]$, twice continuously differentiable in $x \in [0,1]$, and bounded and measurable in s, and $K(0,s) = K(1,s) = 0$ ($s \in [0,1]$).

We consider the boundary conditions

$$u(0,t) = u(1,t) = 0 \quad (t \ge 0). \tag{3.2}$$

We will consider problem (3.1), (3.2) in $L^2(0,1)$ with

$$D(A) = \{f \in L^2(0,1) : f'' \in L^2(0,1), f(0) = f(1) = 0\},$$

A and B are defined by

$$(Af)(x) = \frac{d^2 f(x)}{dx^2} + b(x)f(x) \ \ (f \in D(A)),$$
$$(Bf)(x) = \int_0^1 K(x,s)f(s)ds \ \ (f \in L^2).$$

Thus B maps $L^2(0,1)$ into $D(A)$. Also

$$(ABf)(x) = \int_0^1 [K''(x,s) + b(x)K(x,s)]f(s)ds \quad (f \in L^2(0,1)).$$

Simple calculations show that the largest eigenvalue of the self-adjoint operator $\frac{d^2}{dx^2}$ on $D(A)$ is $-\pi^2$ and

$$\sup_{f\in D(A)} \operatorname{Re}(Af,f)/(f,f) \le \hat{\nu}_A := -\pi^2 + \sup_x \operatorname{Re} b(x).$$

Note that the function $w(t) = e^{At}w(0)$ with $w(0) \in D(A)$ satisfies

$$\begin{aligned}\frac{d}{dt}(w(t),w(t)) &= (w'(t),w(t)) + (w(t),w'(t)) \\ &= (Aw(t),w(t)) + (w(t),Aw(t)) \\ &\le 2\hat{\nu}_A(w(t),w(t)).\end{aligned}$$

Hence

$$\frac{d}{dt}\|w(t)\| \le \hat{\nu}_A\|w(t)\|$$

and therefore

$$\|e^{At}\| \le e^{\hat{\nu}_A t} \quad (t \ge 0). \tag{3.3}$$

With

$$\hat{\nu}_A = -\pi^2 + \sup_x \operatorname{Re} b(x) < 0$$

we have

$$\psi_A = \int_0^\infty \|e^{At}AB\| dt \le \|AB\|/|\hat\nu_A|. \tag{3.4}$$

Furthermore, with $M = A + B$, we obtain

$$\sup_{f\in D(A)} \operatorname{Re}(Mf,f)/(f,f) \le \hat\nu_M := \hat\nu_A + \hat\nu_B$$

where

$$\hat\nu_B := \frac{1}{2} \sup_{f\in L^2(0,1)} ((B+B^*)f,f)/(f,f) < \infty,$$

where B^* is the adjoint of B, i.e. $\hat\nu_B$ is the largest eigenvalue of the self-adjoint operator $(B+B^*)/2$. With $\hat\nu_M < 0$ similarly to (3.3) and (3.4) we have

$$\begin{gathered} \|e^{Mt}\|_{L^2} \le e^{\hat\nu_M t} \quad (t \ge 0), \\ \psi_M = \int_0^\infty \|e^{Mt}B\| dt \le \|B\|_{L^2}/|\hat\nu_M|. \end{gathered} \tag{3.5}$$

According to (3.3) and (3.5) $c_A = c_M = 1$. Using Corollary 1.2, we arrive at the following result.

Theorem 3.1. *Let $\hat\nu_A < 0$, $\hat\nu_M < 0$ and*

$$\frac{h\|B\|_{L^2}}{\hat\nu_M}\Big(\frac{\|AB\|_{L^2}}{\hat\nu_A} + \|B\|_{L^2}\Big) < 1\,.$$

Then (3.1), (3.2) *is exponentially stable with respect to $W(-h,0)$.*

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