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ULAM TYPE STABILITY FOR NONLINEAR HAHN DIFFERENCE EQUATIONS WITH DELAY

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Abstract. In this article, we study the Ulam type stability of nonlinear Hahn difference equations with delay over a finite interval. First, we use the Banach fixed point theorem to prove the existence and uniqueness of a solution. Then we establish the Ulam stability for first and second order nonlinear Hahn difference equations with delay. We also extend our analysis to n-th order nonlinear Hahn difference equations with delay. To illustrate our theoretical findings, we provide three examples.

1. Introduction

Hahn [10] developed a difference operator, by drawing from two well-known difference operators: the forward difference operator [4] and the Jackson q-difference operator [3, 5, 6, 27]. Subsequently, Annaby et al. [2] extended the concept by introducing the q,ω-integral a function, which encompasses both Nörlund sums and Jackson q-integrals. Hamaz et al. [11, 16] explored the existence and uniqueness of solutions to Hahn difference equations using the method of successive approximations and examined the stability of first-order Hahn difference equations. Abdelkhaliq et al. [1] investigated the stability of Hahn difference equations within Banach spaces. Additional results on the Hahn difference operator can be found in references [12, 14, 15, 17, 18, 22, 24].

Ulam stability originated from a query about stability addressed in [29], and was later termed Ulam-Hyers stability by Hyers [19]. Rassias [25] further developed this concept into Ulam-Hyers-Rassias stability by incorporating additional variables in the form of functions. Following this, numerous studies have explored the Ulam stability of various equations [8, 9, 20, 21, 26]. For instance, Rus [28] examined Ulam stability in ordinary differential equations, Otrocol et al. [23] looked into the Ulam stability of delay differential equations, and Hamaz et al. [13] studied the Ulam stability of first-order linear quantum difference equations.

Inspired by [15, 28, 23], we consider the equation

$$\begin{gathered}\mathfrak{D}_{q,\omega}x(s) = F(t, x(s), x(\Theta(s))), \quad s \in I_1 = [\omega_0, b],\\ x(s) = y(s), \quad s \in I_2 = [\omega_0 - h_0, \omega_0],\end{gathered} \tag{1.1}$$

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where $F : I_1 \times \mathbb{R}^2 \to \mathbb{R}$ and $\Theta : I_1 \to I_3$, $I_3 = I_1 \cup I_2$, are continuous at $s = \omega_0$, $\Theta(s) \le s$, $h_0 > 0$ and $y : I_2 \to \mathbb{R}$ is the initial value condition. We demonstrate both the existence and uniqueness of the solution to equation (1.1) on I_3 using the Banach fixed point theorem. Additionally, we explore the Ulam stability of equation (1.1) on I_3. Unlike [11], where the method of successive approximations was used and the function f needed to be continuous on the plane $I_1 \times \mathbb{R}$, our approach requires f to be continuous specifically at $s = \omega_0$.

Secondly, we examine the equation

$$\begin{aligned} \mathfrak{D}^2_{q,\omega} x(s) &= F(s, x(s), \mathfrak{D}_{q,\omega} x(s), x(\Theta(s))), \quad s \in I_1, \\ x(s) &= y(s), \quad \mathfrak{D}_{q,\omega} x(s) = \mathfrak{D}_{q,\omega} y(s), \quad s \in I_2, \end{aligned} \tag{1.2}$$

where $F : I_1 \times \mathbb{R}^3 \to \mathbb{R}$ is continuous at $s = \omega_0$. We analyze the existence and uniqueness of the solution to equation (1.2) on I_3 using the Banach fixed point theorem. Subsequently, we establish the Ulam stability of equation (1.2) on I_3 employing Gronwall's inequality. Finally, we analyze the equation

$$\begin{aligned} \mathfrak{D}^n_{q,\omega} x(s) &= F(s, x(s), \mathfrak{D}_{q,\omega} x(s), \dots, \mathfrak{D}^{n-1}_{q,\omega} x(s), x(\Theta(s))), \quad s \in I_1, \\ x(s) &= y(s), \quad \mathfrak{D}^j_{q,\omega} x(s) = \mathfrak{D}^j_{q,\omega} y(s), \quad s \in I_2, \ i = 0, 1, \dots, n-1, \end{aligned} \tag{1.3}$$

where $F : I_1 \times \mathbb{R}^n \to \mathbb{R}$ is continuous at $s = \omega_0$. We extend the results of the Ulam stability to equation (1.3) on I_3.

The remainder of this article is organized as follows: In Section 2, we present notations and relevant preliminaries for the paper. Section 3 is dedicated to the study of the Ulam stability of equation (1.1) on interval I_3. In Section 4, we establish the Ulam stability of equation (1.2) on interval I_3, and provides direct results on the Ulam stability of equation (1.3). Finally, Section 5 includes examples to illustrate these theoretical findings.

2. Preliminaries

Throughout the article, $\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+$ signifies the set of non-negative real numbers, $\mathbb{N}_+$ refers to the set of positive integers, and I_0 represents any interval of $\mathbb{R}$ that includes ω_0.

We define these function spaces

$$S(I_3, \mathbb{R}) = \{f : I_3 \to \mathbb{R} : f(s) \text{ is continuous at } s = \omega_0 \text{ and bounded}\},$$
$$S(I_3, \mathbb{R}_+) = \{f : I_3 \to \mathbb{R}_+ : f(s) \text{ is continuous at } s = \omega_0 \text{ and bounded}\}.$$

Let $S(I_3, \mathbb{R}_+)$ have a subspace $S_1(I_3, \mathbb{R}_+)$ in which all functions are increasing. Obviously,

$$S(I_3, \mathbb{R}) \supseteq S(I_3, \mathbb{R}_+) \supseteq S_1(I_3, \mathbb{R}_+).$$

For $S(I_3, \mathbb{R})$, $S(I_3, \mathbb{R}_+)$ and $S_1(I_3, \mathbb{R}_+)$, let the metric ρ be defined by

$$\rho(v_1, v_2) = \max_{s \in I_3} |v_1(s) - v_2(s)|.$$

Then it is obvious that $S(I_3, \mathbb{R})$, $S(I_3, \mathbb{R}_+)$ and $S_1(I_3, \mathbb{R}_+)$ are complete metric spaces.

Definition 2.1. [10] Assume function $\mathfrak{f} : I_0 \to \mathbb{R}$ is continuous at $s = \omega_0$. Then Hahn difference operator is defined by

$$\mathfrak{D}_{q,\omega} \mathfrak{f}(s) = \begin{cases} \frac{\mathfrak{f}(qs+\omega) - \mathfrak{f}(t)}{s(q-1)+\omega}, & t \ne \omega_0, \\ \mathfrak{f}'(\omega_0), & t = \omega_0, \end{cases}.$$

where $0 < q < 1$ and $\omega > 0$ are constants, $\omega_0 = \frac{\omega}{1-q}$.

Definition 2.2. [2] Assume function $\mathfrak{g} : I_0 \to \mathbb{R}$ is continuous at $s = \omega_0$ and let $[a_1, a_2] \subset I_0$. Then the Hahn integral of $\mathfrak{g}$ from a_1 to a_2 has the form

$$\int_{a_1}^{a_2} \mathfrak{g}(s_1) d_{q,\omega} s_1 = \int_{\omega_0}^{a_2} \mathfrak{g}(s_1) d_{q,\omega} s_1 - \int_{\omega_0}^{a_1} \mathfrak{g}(s_1) d_{q,\omega} s_1,$$

where

$$\int_{\omega_0}^{x} \mathfrak{g}(s_1) d_{q,\omega} s_1 = (x(1-q) - \omega) \sum_{j=0}^{\infty} q^j \mathfrak{g}(\sigma^j(x)) = \sum_{j=0}^{\infty} (\sigma^j(x) - \sigma^{j+1}(x)) \mathfrak{g}(\sigma^k(x))$$

for $x \in I_0$, and

$$\sigma^j(x) = q^j x + \omega [j]_q, \quad x \in I_0, \quad [j]_q = \frac{1-q^j}{1-q},$$

and the series $(x(1-q) - \omega) \sum_{k=0}^{\infty} q^k \mathfrak{g}(\sigma^k(x))$ converges at $x = a_1$ and $x = a_2$.

We can noted that

$$|\int_{a_1}^{a_2} \mathfrak{g}(s_1) d_{q,\omega} s_1| \le \int_{a_1}^{a_2} |\mathfrak{g}(s_1)| d_{q,\omega} s_1, \quad \forall a_1,\ a_2 \in I_0,\ a_1 < a_2,$$

is not necessarily true [2]. However, for $a_1 = \omega_0$, we can obtain

$$|\int_{\omega_0}^{a_2} \mathfrak{g}(s_1) d_{q,\omega} s_1| \le \int_{\omega_0}^{a_2} |\mathfrak{g}(s_1)| d_{q,\omega} s_1, \quad \forall a_2 \in I_0,\ a_2 > \omega_0.$$

Additionally, we can obtain that

$$\int_{\omega_0}^{a_1} |\mathfrak{g}(s_1)| d_{q,\omega} s_1 \le \int_{\omega_0}^{a_2} |\mathfrak{g}(s_1)| d_{q,\omega} s_1, \quad \forall a_1, a_2 \in I_0,\ \omega_0 < a_1 < a_2, \tag{2.1}$$

is not necessarily true. If function $|\mathfrak{g}|$ is increasing on I_0, inequality (2.1) holds.

Definition 2.3. [2] Assume function $\zeta : I_0 \to \mathbb{R}$ is continuous at $s = \omega_0$ and $1 - \zeta(s)(s - \sigma(s)) \neq 0$, $\forall\ s \in I_0$. Then exponential functions $e_\zeta(s)$ and $E_\zeta(s)$ are given by

$$e_\zeta(s) = \frac{1}{\prod_{j=0}^{\infty} (1 - \zeta(\sigma^j(s)) q^j (s - \sigma(s)))}, \tag{2.2}$$

$$E_\zeta(s) = \prod_{j=0}^{\infty} (1 + \zeta(\sigma^j(s)) q^j (s - \sigma(s))). \tag{2.3}$$

It is obvious that (2.2) and (2.3) are convergent since $\sum_{j=0}^{\infty} \mid \zeta(\sigma^j(s)) \mid q^j (s - \sigma(s))$ is convergent. For $\zeta(s) = a_0 \in \mathbb{R}$ for all $s \in I_0$, we have

$$\begin{aligned} e_{a_0}(s) &= \frac{1}{\prod_{j=0}^{\infty} (1 - a_0 q^j (s - \sigma(s)))} \\ &= \sum_{j=0}^{\infty} \frac{(a_0(s - \sigma(s)))^j}{(q:q)_j}, \quad |s - \omega_0| < \frac{1}{|a_0(1-q)|}, \end{aligned} \tag{2.4}$$

and

$$E_{a_0}(s) = \prod_{j=0}^{\infty} (1 + a_0 q^j (s - \sigma(s))) = \sum_{j=0}^{\infty} \frac{q^{\frac{1}{2} j(j-1)} (a_0(s - \sigma(s)))^j}{(q:q)_j}, \quad s \in \mathbb{R}, \tag{2.5}$$

where

$$(a:q)_n=\begin{cases}\prod_{j=1}^{n}(1-aq^{j-1}), & n\in\mathbb{N}_+,\\ 1, & n=0.\end{cases}$$

The proofs of (2.4) and (2.5) can be found in [7].

Lemma 2.4 ([2])**.** *Assume $\mathfrak{f},\mathfrak{g}:I_0\to\mathbb{R}$ are continuous at $s=\omega_0$. Then*

$$\int_a^b \mathfrak{g}(s)\mathfrak{D}_{q,\omega}(\mathfrak{f}(s))d_{q,\omega}s+\int_a^b \mathfrak{D}_{q,\omega}(\mathfrak{g}(s))\mathfrak{f}(\sigma(s))d_{q,\omega}s=\mathfrak{f}(s)\mathfrak{g}(s)\big|_a^b,\quad a,\ b\in I_0.$$

Lemma 2.5 (Gronwall's inequality)**.** *Assume $\mathfrak{f}$, $\mathfrak{g}:I_0\to\mathbb{R}$ are continuous at $s=\omega_0$ and $\zeta:I_0\to\mathbb{R}_+$ is continuous at $s=\omega_0$. Let $1-\zeta(s)(s-\sigma(s))>0$ for all $s\in I_0$. If*

$$\mathfrak{f}(s)\le\mathfrak{g}(s)+\int_{\omega_0}^{s}\zeta(s_1)\mathfrak{f}(s_1)d_{q,\omega}s_1,\quad\forall s\in I_0,$$

then

$$\mathfrak{f}(s)\le\mathfrak{g}(s)+e_\zeta(s)\int_{\omega_0}^{s}\zeta(s_1)E_{-s_1}(\sigma(s_1))\mathfrak{g}(s_1)d_{q,\omega}s_1.\tag{2.6}$$

Let $\zeta(s)=a_0\in\mathbb{R}_+$, for all $s\in I_0$. If

$$\mathfrak{f}(s)\le\mathfrak{g}(s)+\int_{\omega_0}^{s}a_0\mathfrak{f}(s_1)d_{q,\omega}s_1,\quad s\in[\omega_0,\omega_0+\frac{1}{a_0(1-q)}],$$

then

$$\mathfrak{f}(s)\le\mathfrak{g}(s)+a_0e_{a_0}(s)\int_{\omega_0}^{s}E_{-a_0}(\sigma(s_1))\mathfrak{g}(s_1)d_{q,\omega}s_1.$$

Lemma 2.6 ([23])**.** *Assume $(Y,d,\le)$ is an ordered metric space. $V:Y\to Y$ is an increasing Picard operator ($F_V=\{y_V^*\}$ denotes the fixed point set of operator V). Then, for $y\in Y$, we have*

- (i) *if $y\le V(y)$, then $y\le y_V^*$;*
- (ii) *if $y\ge V(y)$, then $y\ge y_V^*$.*

3. Ulam stability of equation (1.1)

Definition 3.1 ([28])**.** Assuming there is a real number $c>0$, for for all $\varepsilon>0$ and for all y satisfy

$$|\mathfrak{D}_{q,\omega}y(s)-F(s,y(s),y(\Theta(s)))|\le\varepsilon,\ s\in I_1,\tag{3.1}$$

equation (1.1) has a solution x with

$$|y(s)-x(s)|\le c\varepsilon,\quad\forall s\in I_3.$$

Then (1.1) has Ulam-Hyers stability on I_3.

Definition 3.2 ([28])**.** Assuming there is a function $\theta:\mathbb{R}_+\to\mathbb{R}_+$ and $\theta(0)=0$, for each solution y of inequality (3.1), equation (1.1) has a solution x with

$$|y(s)-x(s)|\le\theta(\varepsilon),\ \forall s\in I_3.$$

Then (1.1) has generalized Ulam-Hyers stability on I_3.

Definition 3.3 ([28])**.** Assuming there is $c > 0$, for all y satisfy

$$|\mathfrak{D}_{q,\omega}y(s) - F(s, y(s), y(\Theta(s)))| \le \varepsilon\varphi(s),\ s \in I_1, \tag{3.2}$$

equation (1.1) has a solution x with

$$|y(s) - x(s)| \le c\varepsilon\varphi(s),\ \forall s \in I_3.$$

Then (1.1) has Ulam-Hyers-Rassias stability with respect to φ on I_3.

Definition 3.4 ([28])**.** Assuming there is $c > 0$, for all y satisfy

$$|\mathfrak{D}_{q,\omega}y(s) - F(s, y(s), y(\Theta(s)))| \le \varphi(s),\ s \in I_1, \tag{3.3}$$

equation (1.1) has a solution x with

$$|y(s) - x(s)| \le c\varphi(s),\ \forall s \in I_3.$$

Then (1.1) has generalized Ulam-Hyers-Rassias stability with respect to φ on I_3.

Remark 3.5. A function y satisfies inequality (3.1) if and only if there is a function $\beta : \mathbb{R} \to \mathbb{R}$ such that

- (i) $|\beta(s)| \le \varepsilon$ for all $s \in I_1$;
- (ii) $\mathfrak{D}_{q,\omega}y(s) = F(s, y(s), y(\Theta(s))) + \beta(s)$ for all $s \in I_1$.

The same statements apply to inequalities (3.2) and (3.3).

In this article, we use the following assumptions:

- (A1) there is a real number $L_F > 0$ such that for all $s \in I_1$, x_j, $y_j \in \mathbb{R}$, $j = 1, 2$,
$$|F(s, x_1, x_2) - F(s, y_1, y_2)| \le L_F \sum_{j=1}^{2} |x_j - y_j|.$$
- (A2) $b - \omega_0 < \frac{1}{2L_F}$.
- (A3) $\varphi : I_1 \to \mathbb{R}$ is increasing and continuous at $s = \omega_0$.

Theorem 3.6. *Under assumptions* (A1), (A2), *Equaton* (1.1) *has (i) a unique solution on I_3, and (ii) Ulam-Hyers stability on I_3.*

Proof. (i) Equation (1.1) is equivalent to the Hahn integral equation

$$x(s) = \begin{cases} y(s), & s \in I_2, \\ y(\omega_0) + \int_{\omega_0}^{s} F(s_1, x(s_1), x(\Theta(s_1)))d_{q,\omega}s_1, & s \in I_1. \end{cases} \tag{3.4}$$

We consider the mapping $G : S(I_3, \mathbb{R}) \to S(I_3, \mathbb{R})$ as

$$(Gx)(s) = \begin{cases} y(s), & s \in I_2, \\ y(\omega_0) + \int_{\omega_0}^{s} F(s_1, x(s_1), x(\Theta(s_1)))d_{q,\omega}s_1, & s \in I_1. \end{cases}$$

For all $v \in S(I_3, \mathbb{R})$, we have

$$\begin{aligned} |(Gu)(s) - (Gv)(s)| &\le \int_{\omega_0}^{s} |F(s_1, u(s_1), u(\Theta(s_1))) - F(s_1, v(s_1), v(\Theta(s_1)))|d_{q,\omega}s_1 \\ &\le \int_{\omega_0}^{s} 2L_F \max_{s\in I_3} |u(s) - v(s)|d_{q,\omega}s_1 \\ &\le 2L_F(b - \omega_0) \max_{s\in I_3} |u(s) - v(s)|. \end{aligned}$$

By (A2) and the Banach fixed point theorem, equation (1.1) has a unique solution on I_3.

(ii) Let y satisfy (3.1) and x represent the unique solution of (1.1). Then we can get (3.4). Consequently, by Remark 3.5, we obtain

$$\begin{aligned} &|y(s)-x(s)| \\ &\leq \int_{\omega_0}^{s} |\beta(s_1)| d_{q,\omega} s_1 \\ &\quad + \Big| \int_{\omega_0}^{s} F(s_1, y(s_1), y(\Theta(s_1))) - F(s_1, x(s_1), x(\Theta(s_1))) d_{q,\omega} s_1 \Big| \\ &\leq \varepsilon(s-\omega_0) + \int_{\omega_0}^{s} L_F(|y(s_1)-x(s_1)| + |y(\Theta(s_1)) - x(\Theta(s_1))|) d_{q,\omega} s_1. \end{aligned} \tag{3.5}$$

Let us define $V : S(I_3, \mathbb{R}_+) \to S(I_3, \mathbb{R}_+)$ by

$$(Vu)(s) = \begin{cases} 0, & s \in I_2, \\ \varepsilon(s-\omega_0) + \int_{\omega_0}^{s} L_F(u(s_1) + u(\Theta(s_1))) d_{q,\omega} s_1, & s \in I_1. \end{cases}$$

For all $u, v \in S(I_3, \mathbb{R}_+)$, we have

$$\begin{aligned} |(Vu)(s) - (Vv)(s)| &\leq \int_{\omega_0}^{s} L_F(|u(s_1) - v(s_1)| + |u(\Theta(s_1)) - v(\Theta(s_1))|) d_{q,\omega} s_1 \\ &\leq 2L_F(b-\omega_0) \max_{s \in I_3} |u(s) - v(s)|. \end{aligned}$$

Then V is a contraction mapping in $S(I_3, \mathbb{R}_+)$. For all $u_1, v_1 \in S_1(I_3, \mathbb{R}_+)$, we can also obtain

$$|(Vu_1)(s) - (Vv_1)(s)| \leq 2L_F(b-\omega_0) \max_{s \in I_3} |u_1(s) - v_1(s)|.$$

Then V is also a contraction mapping in $S_1(I_3, \mathbb{R}_+)$. Thus, according to Banach fixed theorem, V has the unique fixed point $u^* \in S_1(I_3, \mathbb{R}_+)$ in $S(I_3, \mathbb{R}_+)$. We obtain

$$u^*(s) = \varepsilon(s-\omega_0) + \int_{\omega_0}^{s} L_F(u^*(s_1) + u^*(\Theta(s_1))) d_{q,\omega} s_1, \quad s \in I_1.$$

Since $u^* \in S_1(I_3, \mathbb{R}_+)$ is increasing, we have

$$u^*(s) \leq \varepsilon(s-\omega_0) + \int_{\omega_0}^{s} 2L_F u^*(s_1) d_{q,\omega} s_1, \quad s \in I_1. \tag{3.6}$$

By using Lemma 2.5 (Gronwall's inequality), from (3.6) it follows that

$$u^*(s) \leq \varepsilon(s-\omega_0) + e_{2L_F}(s) 2L_F \int_{\omega_0}^{s} E_{-2L_F}(\sigma(s_1)) \varepsilon(s_1 - \omega_0) d_{q,\omega} s_1.$$

Therefore, based on Lemma 2.4, we get

$$\begin{aligned} u^*(s) &\leq \varepsilon(s-\omega_0) - e_{2L_F}(s) \int_{\omega_0}^{s} \mathfrak{D}_{q,\omega}(E_{-2L_F}(s_1)) \varepsilon(s_1 - \omega_0) d_{q,\omega} s_1 \\ &\leq \frac{\varepsilon(e_{2L_F}(b) - 1)}{2L_F}. \end{aligned}$$

Let $u = |y - x|$. According to (3.5), we have $u \leq V(u)$. By using Lemma 2.6, we obtain

$$u(s) \leq u^*(s), \; s \in I_1.$$

Then

$$|y(s) - x(s)| \le \frac{\varepsilon(e_{2L_F}(b) - 1)}{2L_F}, \ s \in I_1.$$

Thus equation (1.1) has Ulam-Hyers stability on I_3. □

Corollary 3.7. *Under assumptions* (A1)–(A3)*, Equation* (1.1) *has generalized Ulam-Hyers stability on* I_3*.*

Theorem 3.8. *Under assumptions* (A1)–(A3)*, Equation* (1.1) *has Ulam-Hyers-Rassias stability with respect to* φ *on* I_3*.*

Proof. By Theorem 3.6 (i), equation (1.1) has the unique solution on I_3. Let y satisfy (3.2). We can obtain (3.4). Thus, by using Remark 3.5, we have

$$\begin{aligned} &|y(s) - x(s)| \\ &\le \varepsilon(b-\omega_0)\varphi(s) + \Big|\int_{\omega_0}^{s} F(s_1, y(s_1), y(\Theta(s_1))) - F(s_1, x(s_1), x(\Theta(s_1)))d_{q,\omega}s_1\Big| \\ &\le \varepsilon(b-\omega_0)\varphi(s) + \int_{\omega_0}^{s} L_F(|y(s_1) - x(s_1)| + |y(\Theta(s_1 u)) - x(\Theta(s_1))|)d_{q,\omega}s_1. \end{aligned} \tag{3.7}$$

As in the proof of Theorem 3.6 (ii), let the operator $V : S(I_3, \mathbb{R}_+) \to S(I_3, \mathbb{R}_+)$ be defined by

$$(Vu)(s) = \begin{cases} 0, & s \in I_2, \\ \varepsilon(b-\omega_0)\varphi(s) + \int_{\omega_0}^{s} L_F((Vu)(s_1) + (Vu)(\Theta(s_1)))d_{q,\omega}s_1, & s \in I_1. \end{cases}$$

Then, V has a unique fixed point $u^* \in S(I_3, \mathbb{R}_+)$ such that u^* is increasing and

$$u^*(s) \le \varepsilon(b-\omega_0)\varphi(s) + \int_{\omega_0}^{s} 2L_F u^*(s_1)d_{q,\omega}s_1. \tag{3.8}$$

By Lemma 2.5, from (3.8) it follows that

$$\begin{aligned} u^*(s) &\le \varepsilon(b-\omega_0)\varphi(s) + e_{2L_F}(s)2L_F(b-\omega_0)\int_{\omega_0}^{s} E_{-2L_F}(\sigma(s_1))\varepsilon\varphi(s_1)d_{q,\omega}s_1 \\ &\le (b-\omega_0)e_{2L_F}(b)\varepsilon\varphi(s). \end{aligned}$$

Then according to Lemma 2.6, we obtain

$$|y(s) - x(s)| \le (b-\omega_0)e_{2L_F}(b)\varepsilon\varphi(s).$$

Thus, equation (1.1) has Ulam-Hyers-Rassias stability with respect to φ on I_3. □

Corollary 3.9. *Under assumptions* (A1)–(A3)*, Equation* (1.1) *has generalized Ulam-Hyers-Rassias stability with respect to* φ *on* I_3*.*

4. Ulam stability of equation (1.2) and (1.3)

4.1. **Ulam stability of equation** (1.2)**.** In this section, let $S(I_3, \mathbb{R})$ be a Banach space in which all $u \in S(I_3, \mathbb{R})$ with the norm

$$\|u\|_{q,\omega} = \max\big\{\max_{s\in I_3}|u(s)|, \ \max_{s\in I_3}|\mathfrak{D}_{q,\omega}u(s)|\big\}.$$

Lemma 4.1. *Equation* (1.2) *has a solution* $x : I_3 \to \mathbb{R}$ *in the form*

$$x(s) = \begin{cases} y(s), & s \in I_2, \\ y(\omega_0) + (s-\omega_0)\mathfrak{D}_{q,\omega} y(\omega_0) & \\ + \int_{\omega_0}^{s} (s-\sigma(s_1)) F(s_1, x(s_1), \mathfrak{D}_{q,\omega} x(s_1), x(\Theta(s_1))) d_{q,\omega} s_1, & s \in I_1. \end{cases} \tag{4.1}$$

Proof. Equation (1.2) is equivalent to the integral equation

$$x(s) = \begin{cases} y(s), & s \in I_2, \\ y(\omega_0) + (s-\omega_0)\mathfrak{D}_{q,\omega} y(\omega_0) & \\ + \int_{\omega_0}^{s} \int_{\omega_0}^{s_2} F(s_1, x(s_1), \mathfrak{D}_{q,\omega} x(s_1), x(\Theta(s_1))) d_{q,\omega} s_1 d_{q,\omega} s_2, & s \in I_1. \end{cases} \tag{4.2}$$

Then, we have

$$\begin{aligned}
&\int_{\omega_0}^{s} \int_{\omega_0}^{s_2} F(s_1, x(s_1), \mathfrak{D}_{q,\omega} x(s_1), x(\Theta(s_1))) d_{q,\omega} s_1 d_{q,\omega} s_2 \\
&= \int_{\omega_0}^{s} \sum_{j=0}^{\infty} (\sigma^j(s_2) - \sigma^{j+1}(s_2)) \\
&\quad \times F(\sigma^j(s_2), x(\sigma^j(s_2)), \mathfrak{D}_{q,\omega} x(\sigma^j(s_2)), x(\Theta(\sigma^j(s_2)))) d_{q,\omega} s_2 \\
&= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} (\sigma^i(s) - \sigma^{i+1}(s))(\sigma^{j+i}(s) - \sigma^{j+1+i}(s)) \\
&\quad \times F(\sigma^{j+i}(s), x(\sigma^{j+i}(s)), \mathfrak{D}_{q,\omega} x(\sigma^{j+i}(s)), x(\Theta(\sigma^{j+i}(s)))) \\
&= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} q^i (s-\sigma(s)) q^{j+i} (s-\sigma(s)) \\
&\quad \times F(\sigma^{j+i}(s), x(\sigma^{j+i}(s)), \mathfrak{D}_{q,\omega} x(\sigma^{j+i}(s)), x(\Theta(\sigma^{j+i}(s)))) \\
&= (s-\sigma(s))^2 \Big[F(s, x(s), \mathfrak{D}_{q,\omega} x(s), x(\Theta(s))) \\
&\quad + q(1+q) F(\sigma(s), x(\sigma(s)), \mathfrak{D}_{q,\omega} x(\sigma(s)), x(\Theta(\sigma(s)))) \\
&\quad + q^2(1+q+q^2) F(\sigma^2(s), x(\sigma^2(s)), \mathfrak{D}_{q,\omega} x(\sigma^2(s)), x(\Theta(\sigma^2(s)))) \\
&\quad + q^3(1+q+q^2+q^3) F(\sigma^3(s), \mathfrak{D}_{q,\omega} x(\sigma^3(s)), x(\sigma^3(s)), x(\Theta(\sigma^3(s)))) + \dots \Big] \\
&= (s-\sigma(s))^2 \sum_{j=0}^{\infty} \frac{q^j(1-q^{j+1}) F(\sigma^j(s), x(\sigma^j(s)), \mathfrak{D}_{q,\omega} x(\sigma^j(s)), x(\Theta(\sigma^j(s))))}{1-q} \\
&= \sum_{j=0}^{\infty} (\sigma^j(s) - \sigma^{j+1}(s))(s - \sigma^{j+1}(s)) F(\sigma^j(s), x(\sigma^j(s)), \mathfrak{D}_{q,\omega} x(\sigma^j(s)), x(\Theta(\sigma^j(s)))) \\
&= \int_{\omega_0}^{s} (s-\sigma(s_1)) F(\tau, x(s_1), \mathfrak{D}_{q,\omega} x(s_1), x(\Theta(s_1))) d_{q,\omega} s_1.
\end{aligned}$$

Thus, we obtain (4.1). □

Now we introduce two more assumptions:

(A4) there is a real number $L_F > 0$ such that for all $s \in I_1$, x_j, $y_j \in \mathbb{R}$, $j = 1, 2, 3$,

$$|F(s, x_1, x_2, x_3) - F(s, y_1, y_2, y_3)| \le L_F \sum_{j=1}^{3} |x_j - y_j|.$$

(A5) $(b-\omega_0)^2 < \frac{1+q}{3L_F}$.

Theorem 4.2. *Assume* (A4) *and* (A5) *hold. Then* (1.2) *has the unique solution on* I_3.

Proof. By Lemma 4.1, equation (1.2) has a solution $x : I_3 \to \mathbb{R}$ in the form of (4.1). Let the mapping $V : S(I_3, \mathbb{R}) \to S(I_3, \mathbb{R})$ be define by

$$V(x)(s) = \begin{cases} y(s), & s \in I_2, \\ y(\omega_0) + (s-\omega_0)\mathfrak{D}_{q,\omega} y(\omega_0) \\ + \int_{\omega_0}^{s} (s-\sigma(s_1)) F(s_1, x(s_1), \mathfrak{D}_{q,\omega} x(s_1), x(\Theta(s_1))) d_{q,\omega} s_1, & s \in I_1. \end{cases}$$

For all $x, y \in S(I_3, \mathbb{R})$, we obtain

$$\begin{aligned} &|V(x)(s) - V(y)(s)| \\ &\le \int_{\omega_0}^{s} (s-\sigma(s_1)) L_F(|x(s_1) - y(s_1)| + |\mathfrak{D}_{q,\omega} x(s_1) - \mathfrak{D}_{q,\omega} y(s_1)| \\ &\quad + |x(\Theta(s_1)) - y(\Theta(s_1))|) d_{q,\omega} s_1 \\ &\le \int_{\omega_0}^{s} 3L_F (s-\sigma(s_1)) \|x-y\|_{q,\omega} d_{q,\omega} s_1 \\ &\le \frac{3L_F (s-\omega_0)^2}{(1+q)} \|x-y\|_{q,\omega}. \end{aligned}$$

Then, we can get

$$\|V(x) - V(y)\|_{q,\omega} \le \frac{3L_F (b-\omega_0)^2}{(1+q)} \|x-y\|_{q,\omega}.$$

By (A5) and the Banach fixed theorem, (1.2) has the unique solution on I_3. ☐

Lemma 4.3. *Assume* (A5) *holds and* $\eta(s) = 3L_F(s-\omega_0)$, $s \in \mathbb{R}$. *Then* $e_\eta(s) > 0$ *is increasing on* I_1 *and* $1 - \eta(s)(s-\sigma(s)) > 0$, *for all* $s \in I_1$.

Proof. According to the definition of Hahn integral, by calculation, we obtain

$$\int_{\omega_0}^{s} \eta(s) d_{q,\omega} s = \frac{3L_F (s-\omega_0)^2}{(1+q)}.$$

From condition (A5), for $s \in I_1$, we have

$$\sum_{k=0}^{\infty} \eta(\sigma^k(s))(\sigma^k(s) - \sigma^{k+1}(s)) < 1.$$

Then, $1 - \eta(\sigma^k(s))(\sigma^k(s) - \sigma^{k+1}(s)) \in (0,1)$ for all $k \in \mathbb{N}_0$, $s \in I_1$. Thus, $e_\eta(s) > 0$ is increasing on I_1 and $1 - \eta(s)(s-\sigma(s)) > 0$ for all $s \in I_1$. ☐

Theorem 4.4. *Assume* (A4) *and* (A5) *hold. Then equation* (1.2) *has Ulam-Hyers stability on* I_3.

Proof. According to Theorem 4.2, we can know equation (1.2) has the unique solution on I_3. Let y satisfy the inequality

$$|\mathfrak{D}_{q,\omega}^2 y(s) - F(s, y(s), \mathfrak{D}_{q,\omega} y(s), y(\Theta(s)))| \le \varepsilon, \ s \in I_1.$$

Let x represent the unique solution to (1.2). Then, we obtain (4.1).

For $s \in I_1$, according to Remark 3.5, we obtain

$$
\begin{aligned}
&|y(s)-x(s)| \\
&\le \int_{\omega_0}^{s}\int_{\omega_0}^{s_2} \varepsilon + L_F(|y(s_1)-x(s_1)| \\
&\quad + |\mathfrak{D}_{q,\omega}y(s_1)-\mathfrak{D}_{q,\omega}x(s_1)| + |y(\Theta(s_1))-x(\Theta(s_1))|) d_{q,\omega}s_1 d_{q,\omega}s_2.
\end{aligned} \tag{4.3}
$$

Let

$$
\begin{aligned}
\phi(s) = \max\Big\{ &\max_{s_1\in[\omega_0-h_0,s]} |y(s_1)-x(s_1)|, \\
&\max_{s_1\in[\omega_0-h_0,s]} |\mathfrak{D}_{q,\omega}y(s_1)-\mathfrak{D}_{q,\omega}x(s_1)|\Big\}.
\end{aligned}
$$

Then ϕ is increasing and

$$
\begin{gathered}
|y(s_1)-x(s_1)| \le \phi(s_1), \quad |y(\Theta(s_1))-x(\Theta(s_1))| \le \phi(s_1), \\
|\mathfrak{D}_{q,\omega}y(s_1)-\mathfrak{D}_{q,\omega}x(s_1)| \le \phi(s_1).
\end{gathered}
$$

Consequently,

$$
\begin{aligned}
|y(s)-x(s)| &\le \int_{\omega_0}^{s}\int_{\omega_0}^{s_2} \varepsilon + 3L_F\phi(s_1) d_{q,\omega}s_1 d_{q,\omega}s_2 \\
&\le \int_{\omega_0}^{s} (\varepsilon + 3L_F\phi(s_2))(s_2-\omega_0) d_{q,\omega}s_2 \\
&= \frac{\varepsilon(s-\omega_0)^2}{1+q} + \int_{\omega_0}^{s} 3L_F\phi(s_2)(s_2-\omega_0) d_{q,\omega}s_2.
\end{aligned}
$$

For all $s_1 \in [\omega_0, s]$, we have

$$
|y(s_1)-x(s_1)| \le \frac{\varepsilon(s_1-\omega_0)^2}{1+q} + \int_{\omega_0}^{s_1} 3L_F\phi(s_2)(s_2-\omega_0) d_{q,\omega}s_2.
$$

Then

$$
\phi(s) \le \frac{\varepsilon(s-\omega_0)^2}{1+q} + \int_{\omega_0}^{s} 3L_F\phi(s_2)(s_2-\omega_0) d_{q,\omega}s_2. \tag{4.4}
$$

Let $\eta(s) = 3L_F(s-\omega_0)$, $s \in \mathbb{R}$. According to Lemma 2.5 and Lemma 4.3, from (4.4) it follows that

$$
\begin{aligned}
\phi(s) &\le \frac{\varepsilon(s-\omega_0)^2}{1+q} + e_\eta(s)\int_{\omega_0}^{s} \eta(s_2)E_{-\eta}(\sigma(s_2))\frac{\varepsilon(s_2-\omega_0)^2}{1+q} d_{q,\omega}s_2 \\
&\le \frac{\varepsilon(b-\omega_0)^2}{1+q} e_\eta(b).
\end{aligned}
$$

Thus,

$$
|y(s)-x(s)| \le \frac{\varepsilon(b-\omega_0)^2}{1+q} e_\eta(b).
$$

Then (1.2) has Ulam-Hyers stability on I_3. □

Corollary 4.5. *Under assumptions* (A4), (A5), *Equation* (1.2) *has generalized Ulam-Hyers stability on I_3.*

Theorem 4.6. *Assume* (A3)–(A5) *hold. Then* (1.2) *has Ulam-Hyers-Rassias stability with respect to φ on I_3.*

Proof. According to Theorem 4.2, equation (1.2) has the unique solution on I_3. Let y satisfy the inequality

$$|\mathfrak{D}^2_{q,\omega}y(s) - F(s, y(s), \mathfrak{D}_{q,\omega}y(s), y(\Theta(s)))| \le \varepsilon\varphi(s), \quad s \in I_1.$$

Let x represent unique solution to (1.2). According to Remark 3.5, we have

$$\begin{aligned}|y(s) - x(s)| \le \int_{\omega_0}^{s}\int_{\omega_0}^{s_2} \varepsilon\varphi(s_1) + L_F(|y(s_1) - x(s_1)| + |\mathfrak{D}_{q,\omega}y(s_1) - \mathfrak{D}_{q,\omega}x(s_1)|\\ + |y(\Theta(s_1)) - x(\Theta(s_1))|)d_{q,\omega}s_1 d_{q,\omega}s_2.\end{aligned}$$

Let

$$\begin{aligned}\phi(s) = \max\Big\{ &\max_{s_1\in[\omega_0-h_0,s]} |y(s_1) - x(s_1)|,\\ &\max_{s_1\in[\omega_0-h_0,s]} |\mathfrak{D}_{q,\omega}y(s_1) - \mathfrak{D}_{q,\omega}x(s_1)|\Big\}.\end{aligned}$$

As in the proof of Theorem 4.4, we have

$$\begin{aligned}|y(s) - x(s)| &\le \int_{\omega_0}^{s}\int_{\omega_0}^{s_2} \varepsilon\varphi(s_1) + 3L_F\phi(s_1)d_{q,\omega}s_1 d_{q,\omega}s_2\\ &\le \int_{\omega_0}^{s} (\varepsilon\varphi(s_2) + 3L_F\phi(s_2))(s_2 - \omega_0)d_{q,\omega}s_2\\ &\le \frac{\varepsilon\varphi(s)(s-\omega_0)^2}{1+q} + \int_{\omega_0}^{s} 3L_F\phi(s_2)(s_2 - \omega_0)d_{q,\omega}s_2.\end{aligned}$$

Then one has

$$\phi(s) \le \frac{\varepsilon\varphi(s)(s-\omega_0)^2}{1+q} + \int_{\omega_0}^{s} 3L_F\phi(s_2)(s_2 - \omega_0)d_{q,\omega}s_2. \tag{4.5}$$

By using Lemma 2.5, from (4.5) it follows that

$$\begin{aligned}\phi(s) &\le \frac{\varepsilon\varphi(s)(s-\omega_0)^2}{1+q} + e_p(s)\int_{\omega_0}^{s} p(s_2)E_{-\eta}(\sigma(s_2))\frac{\varepsilon\varphi(s_2)(s_2-\omega_0)^2}{1+q}d_{q,\omega}s_2\\ &\le \frac{(b-\omega_0)^2 e_\eta(b)\varepsilon\varphi(s)}{1+q}.\end{aligned}$$

Then

$$|y(s) - x(s)| \le \frac{(b-\omega_0)^2 e_\eta(b)\varepsilon\varphi(s)}{1+q}.$$

Thus, (1.2) has Ulam-Hyers-Rassias stability with respect to φ on I_3. □

Corollary 4.7. *Assume* (A3)–(A5) *hold. Then* (1.2) *has generalized Ulam-Hyers-Rassias stability with respect to φ on I_3.*

4.2. **Ulam stability of equation** (1.3)**.** Based on the definitions of the Hahn difference and q,ω-integral, it is clear that $x : I_3 \to \mathbb{R}$ satisfies (1.3) if and only if x satisfies the corresponding integral equation

$$x(s) = \begin{cases} y(s), & s \in I_2,\\ \sum_{k=0}^{n-1} \frac{(1-q)^k(s-\omega_0)^k}{(q:q)_k}\mathfrak{D}^k_{q,\omega}y(\omega_0) + \int_{\omega_0}^{s}\int_{\omega_0}^{s_n}\dots\int_{\omega_0}^{s_2} F(s_1, x(s_1), & \\ \mathfrak{D}_{q,\omega}x(s_1), \dots, \mathfrak{D}^{n-1}_{q,\omega}x(s_1), x(\Theta(s_1)))d_{q,\omega}s_1 \dots d_{q,\omega}s_{n-1}d_{q,\omega}s_n, & s \in I_1.\end{cases}$$

Now we introduce the next assumptions

(A6) there is a real number $L > 0$ such that for all $s \in I_1$, x_j, $y_j \in \mathbb{R}$, $j = 1, 2, \dots, n$,
$$|F(s, x_1, x_2, \dots, x_n) - F(s, y_1, y_2, \dots, y_n)| \le L \sum_{j=1}^{n} |x_j - y_j|.$$

(A7) $(b - \omega_0)^n < \frac{(q:q)_n}{(n+1)L(1-q)^n}$.

Consequently, analogous approaches can be used to establish Ulam stability for equation (1.3) on I_3. We now easily present these results without proofs.

Theorem 4.8. *Assume* (A6) *and* (A7) *hold. Then* (1.3) *has the unique solution on* I_3.

Theorem 4.9. *Assume* (A6) *and* (A7) *hold. Then* (1.3) *has Ulam-Hyers stability on* I_3 *with*
$$c = \frac{(1-q)^n (b - \omega_0)^n e_\eta(b)}{(q:q)_n},$$
where $\eta(s) = \frac{(n+1)L(1-q)^{n-1}(s-\omega_0)^{n-1}}{(q:q)_{n-1}}$, *and* $s \in \mathbb{R}$.

Theorem 4.10. *Assume (A6) and (A7) hold. Then equation* (1.3) *has generalized Ulam-Hyers stability on* I_3.

Theorem 4.11. *Assume* (A3), (A6), (A7) *hold. Then* (1.3) *has Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability with respect to* φ *on* I_3.

5. Examples

Example 5.1. We consider the equation
$$\begin{gathered} \mathfrak{D}_{\frac{1}{3},6} x(s) = \frac{2e^{-|x(s)|} + \sin(x(s-10))}{264}, \quad s \in [9, b], \\ x(s) = s^2, \quad s \in [-1, 9], \end{gathered} \tag{5.1}$$
and inequalities
$$\begin{gathered} |\mathfrak{D}_{\frac{1}{3},6} y(s) - \frac{2e^{-|y(s)|} + \sin(y(s-10))}{264}| \le \varepsilon, \quad s \in [9, b], \\ |\mathfrak{D}_{\frac{1}{3},6} y(s) - \frac{2e^{-|y(s)|} + \sin(y(s-10))}{264}| \le \varepsilon e_{\frac{1}{45}}(s), \quad s \in [9, b]. \end{gathered}$$
When $9 < b < 75$, equation (5.1) has the unique solution on $[-1, b]$. Obviously, equation (5.1) has Ulam-Hyers stability on $[8, 75)$ with
$$c = 66(e_{1/66}(75) - 1).$$
Equation (5.1) has Ulam-Hyers-Rassias stability with respect to $e_{\frac{1}{45}}(s)$ on $[-1, 75)$ with
$$c = 66 e_{1/66}(75).$$

Example 5.2. We consider the equation
$$\begin{gathered} \mathfrak{D}^2_{\frac{1}{2},\frac{1}{2}} x(s) = \frac{5\cos(x(s)) + 8\mathfrak{D}_{\frac{1}{2},\frac{1}{2}} x(s) + 8\sin(x(s-5))}{10240} + \frac{\sin(x(s))}{16806}, \quad s \in [1, b], \\ x(s) = s, \quad \mathfrak{D}_{\frac{1}{2},\frac{1}{2}} x(s) = 1, \quad s \in [-4, 1], \end{gathered} \tag{5.2}$$

and inequalities

$$\Big|\mathfrak{D}^2_{\frac12,\frac12}y(s)-\frac{5\cos(y(s))+8\mathfrak{D}_{\frac12,\frac12}y(s)+8\sin(y(s-5))}{10240}-\frac{\sin(y(s))}{16806}\Big|\le\varepsilon,$$

for $s\in[1,b]$, and

$$\Big|\mathfrak{D}^2_{\frac12,\frac12}y(s)-\frac{5\cos(y(s))+8\mathfrak{D}_{\frac12,\frac12}y(s)+8\sin(y(s-5))}{10240}-\frac{\sin(y(s))}{16806}\Big|\le\varepsilon e_{\frac{1}{16}}(s),$$

for $s\in[1,b]$. When $1<b<1+8\sqrt{10}$, equation (5.2) has the unique solution on $[-4,b]$. Then equation (5.2) has Ulam-Hyers stability on $[-4,1+8\sqrt{10})$ and Ulam-Hyers-Rassias stability with respect to $e_{\frac{1}{16}}(s)$ on $[-4,1+8\sqrt{10})$ with

$$c=\frac{1280}{3}e_{p(1+8\sqrt{10})}(1+8\sqrt{10}),$$

where $p(s)=\frac{3(s-1)}{1280}$, $s\in\mathbb{R}$.

Example 5.3. We consider the equation

$$\begin{gathered}\mathfrak{D}^n_{\frac15,8}x(s)=\frac{3\mathfrak{D}^{n-1}x(s)+\sin(\sum_{i=1}^{n-2}\mathfrak{D}^i_{q,\omega}x(s))+2e^{-|x(s)|}+\cos(x(s-1))}{885},\\ s\in[10,b],\\ x(t)=s,\quad \mathfrak{D}_{\frac15,8}x(s)=1,\quad \mathfrak{D}^i_{\frac15,8}x(s)=0,\quad i=2,3,\dots,n-1,\ s\in[9,10],\end{gathered}\tag{5.3}$$

and inequalities

$$\Big|\mathfrak{D}^n_{\frac15,8}y(s)-\frac{3\mathfrak{D}^{n-1}y(s)+\sin(\sum_{i=1}^{n-2}\mathfrak{D}^i_{q,\omega}y(s))+2e^{-|y(s)|}+\cos(y(s-1))}{885}\Big|\le\varepsilon,$$

for $s\in[10,b]$, and

$$\Big|\mathfrak{D}^n_{\frac15,8}y(s)-\frac{3\mathfrak{D}^{n-1}y(s)+\sin(\sum_{i=1}^{n-2}\mathfrak{D}^i_{q,\omega}y(s))+2e^{-|y(s)|}+\cos(y(s-1))}{885}\Big|\le\varepsilon s,$$

for $s\in[10,b]$. When $10<b<\sqrt[n]{\frac{295[n]_q!}{n+1}}+10$, equation (5.3) has the unique solution on $[9,b]$. Therefore, (5.3) has Ulam-Hyers stability on $[9,b]$ and Ulam-Hyers-Rassias stability with respect to $\varphi(s)=s$ on $[9,b]$.

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