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OUTPUT TRACKING FOR A 1-D WAVE EQUATIONS WITH SPATIALLY VARYING COEFFICIENTS AND SUBJECT TO UNKNOWN DISTURBANCES

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Abstract. In this article, we study the output tracking problem for a wave equation with variable coefficients, and subject to boundary control matched disturbances. Both the disturbances and the reference signal are unknown harmonic signal. The performance output is non-collocated with the control input. Initially, we establish an undisturbed auxiliary system and devise an appropriate internal model dynamic to reformulate the tracking error. Subsequently, we introduce an error-based feedback controller, leveraging an invertible transformation to achieve output tracking. The well-posedness and stability of the closed-loop system are established by applying semigroup theory approach. Finally, we illustrate the effectiveness of these theoretical results with numerical simulations.

1. Introduction

Output tracking is one of the most fundamental challenges in control theory. In numerous engineering scenarios, the primary focus lies in ensuring that the output signal of the control system asymptotically converges to the desired references, even in the presence of disturbances. Additionally, it is imperative that all internal loop states remain within a suitable range. Over the past few decades, extensive research has been conducted on output tracking within the context of beam equations [11, 14, 19], heat equations [12, 17, 29], wave equations [4, 5, 31], and various other partial differential equations (PDEs) [15, 16]. A highly effective method for addressing the output tracking problem is the internal model principle (IMP), which has been established in the literature since [3, 8, 24]. Through the application of the IMP, the task of achieving robust output tracking is significantly streamlined by constructing a dynamic tracking error feedback control system that incorporates a p-copy of the exosystem, where $p \in \mathbb{N}^+$ represents the dimension of the output [21]. Furthermore, several classic results have been extended to infinite-dimensional systems, as demonstrated in studies such as [1, 4, 5, 18, 24], among many others.

On the other hand, the most challenging aspect of output tracking lies in managing disturbances. Various strategies have been devised to tackle disturbances or uncertain parameters in PDEs control problems. These methods include active

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disturbance rejection control [9, 32], sliding mode control [10, 26], adaptive control [13, 30], and IMP [8, 24]. In [5, 6], the authors address regulation problems by identifying unique solutions to regulator equations, from which compensators can be formulated based on kernel equations. Recently, in [7], a control system based on an observer framework was developed for a one-dimensional wave equation, incorporating non-collocated disturbances through the application of approach. In [28], the issue of output tracking for one-dimensional wave equations, which are subject to unknown harmonic disturbances and reference signals, is tackled through the utilization of an adaptive internal model and an adaptive frequency estimation technique.

In this paper, we investigate the output tracking for a wave equation with variable coefficients subject to boundary control matched harmonic disturbance. The system is governed by the PDEs

$$
\begin{gathered}
\Gamma_{tt}(x,t) = (a(x)\Gamma_x(x,t))_x, \quad x \in (0,1),\ t>0,\\
\Gamma_x(0,t) = m\Gamma_t(0,t), \quad t \ge 0,\\
\Gamma_x(1,t) = U(t) + d(t), \quad t \ge 0,\\
\Gamma(x,0) = \Gamma_0(x), \quad \Gamma_t(x,0) = \Gamma_1(x), \quad x \in [0,1],\\
y_p(t) = \Gamma(0,t), \quad t \ge 0,
\end{gathered} \tag{1.1}
$$

where $\Gamma(\cdot,t)$ represents the state of the entire system, $y_p(t)$ is the performance output, $U(t)$ is the control input. m is a known positive constant. Define $a(\cdot) \in C^1[0,1]$ as follows

$$
a(x) = g_1 x + g_2, \ \ g_1 > 0,\ g_2 > 1. \tag{1.2}
$$

For a given reference signal $r(t)$, we aim at finding a controller $U(t)$ so that

$$
\lim_{t\to\infty} e(t) = \lim_{t\to\infty} (y_p(t) - r(t)) = 0. \tag{1.3}
$$

The disturbance $d(t)$ and reference signal $r(t)$ are generated by the following finite-dimensional exosystem

$$
\begin{gathered}
\dot v(t) = Qv(t), \quad v(0) = v_0, \quad t>0,\\
d(t) = F_1 v(t), \quad r(t) = F_2 v(t), \quad t > 0,
\end{gathered} \tag{1.4}
$$

where $Q \in \mathbb{R}^{2q\times 2q}$ is known but the initial value v_0 and $F_j \in \mathbb{R}^{1\times 2q}$, $j = 1,2$ are unknown, which makes the disturbance $d(t)$ and reference signal $r(t)$ unknown. To achieve output tracking, the following assumptions are required:

Assumption 1.1. The tracking error $e(t)$ and its derivative $\dot e(t)$ are available measurements.

Assumption 1.2. The spectrum of Q is $\{\pm i\omega_j, j = 1,2,\dots,q\}$, where $0 < \omega_1 < \omega_2 < \dots < \omega_q$ are distinct known parameters.

Under Assumption 1.2, the disturbance and reference signal can be rewritten as

$$
d(t) = \sum_{j=1}^{q} \left(A_{1j}\sin\omega_j t + B_{1j}\cos\omega_j t\right), \tag{1.5}
$$

$$
r(t) = \sum_{j=1}^{q} \left(A_{2j}\sin\omega_j t + B_{2j}\cos\omega_j t\right), \tag{1.6}
$$

where $\{\omega_j\}$ represent known frequencies and $\{A_{1j}\}$, $\{B_{1j}\}, \{A_{2j}\}$, $\{B_{2j}\}$ are unknown amplitudes.

The rest of thisarticle is organized as follows. In Section 2, we reconstruct the measurable tracking error using the undisturbed auxiliary system and the proper internal model dynamic. In Section 3, we focus on the construction of error-based feedback controller. In Section 4, we prove the well-posedness and stability of the closed-loop system. Finally, we present some numerical simulations in Section 5, and our conclusions are concluded in Section 6.

2. Estimation

Before we design the controller to reject unknown disturbances and achieve output tracking, it is necessary to estimate these disturbances. Frist we construct an auxiliary system based on the measurable tracking error and its derivative

$$\begin{gathered}\hat{\Gamma}_{tt}(x,t) = (a(x)\hat{\Gamma}_x(x,t))_x,\\ \hat{\Gamma}_x(0,t) = -k_1(e(t) - \hat{\Gamma}(0,t)) + k_2\hat{\Gamma}_t(0,t) + (m-k_2)\dot{e}(t),\\ \hat{\Gamma}_x(1,t) = U(t),\end{gathered} \tag{2.1}$$

where $k_1,\ k_2 > 0$. For simplicity, in system (2.1) and hereafter we omit the initial value when there is no confusion. Let

$$\tilde{\Gamma}(x,t) = \Gamma(x,t) - \hat{\Gamma}(x,t). \tag{2.2}$$

Then

$$\begin{gathered}\tilde{\Gamma}_{tt}(x,t) = (a(x)\tilde{\Gamma}_x(x,t))_x,\\ \tilde{\Gamma}_x(0,t) = k_1\tilde{\Gamma}(0,t) + k_2\tilde{\Gamma}_t(0,t) - (k_1F_2 + k_2F_2Q + mF_2Q)v(t),\\ \tilde{\Gamma}_x(1,t) = F_1v(t).\end{gathered} \tag{2.3}$$

Now we introduce a new system

$$\begin{gathered}w_{tt}(x,t) = (a(x)w_x(x,t))_x,\\ w_x(0,t) = k_1w(0,t) + k_2w_t(0,t),\\ w_x(1,t) = 0,\end{gathered} \tag{2.4}$$

where $k_1,\ k_2 > 0$ and $a(x)$ satisfies (1.2). We consider system (2.4) in the state space $\mathcal{H} = H^1(0,1) \times L^2(0,1)$ equipped with the inner product

$$\langle(\mu_1,\nu_1),(\mu_2,\nu_2)\rangle_{\mathcal{H}} = \int_0^1 [a(x)\mu_1'(x)\mu_2'(x) + \nu_1(x)\nu_2(x)]\,dx + k_1\mu_1(0)\mu_2(0). \tag{2.5}$$

System (2.4) can be easily rewritten as an evolution equation in $\mathcal{H}$:

$$\frac{d}{dt}\left(w(\cdot,t),w_t(\cdot,t)\right) = \mathcal{A}\left(w(\cdot,t),w_t(\cdot,t)\right), \tag{2.6}$$

where operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is defined by

$$\begin{gathered}\mathcal{A}(f,g) = (g,(af')'),\ \ \forall (f,g) \in D(\mathcal{A}),\\ D(\mathcal{A}) = \left\{(f,g) \in H^2(0,1) \times H^1(0,1) \mid f'(1) = 0, f'(0) = k_1f(0) + k_2g(0)\right\}.\end{gathered} \tag{2.7}$$

According to [23], we can see that $\mathcal{A}$ generates an exponentially stable C_0-semigroup on $\mathcal{H}$, that is, there exist two constants $\zeta_0,\ \iota_0 > 0$ such that

$$\|e^{\mathcal{A}t}\|_{\mathcal{H}} \le \zeta_0 e^{-\iota_0 t}, \quad \forall t \ge 0. \tag{2.8}$$

Theorem 2.1. *Let k_1, $k_2 > 0$. Then for any initial state $(\tilde{\Gamma}(\cdot,0), \tilde{\Gamma}_t(\cdot,0)) \in \mathcal{H}$, system* (2.3) *admits a unique solution $(\tilde{\Gamma}, \tilde{\Gamma}_t) \in C([0,\infty);\mathcal{H})$. Moreover, there exists a vector $\gamma \in \mathbb{R}^{1\times 2q}$ such that*

$$\chi_1(t) = \tilde{\Gamma}(0,t) - \gamma v(t), \tag{2.9}$$

satisfying

$$\lim_{t\to\infty} \chi_1(t) = 0, \tag{2.10}$$

exponentially. Furthermore, the following hidden regularity holds

$$e^{\iota_1 \cdot}\dot{\chi}_1(\cdot) \in L^2([0,\infty);\mathbb{R}), \tag{2.11}$$

where ι_1 is a positive constant.

Proof. A direct computation shows that the adjoint operator $\mathcal{A}^*$ of $\mathcal{A}$ satisfies

$$\begin{gathered} \mathcal{A}^*(f,g) = (-g, -(af')'), \quad \forall (f,g) \in D(\mathcal{A}^*), \\ D(\mathcal{A}^*) = \left\{(f,g) \in H^2(0,1) \times H^1(0,1) : f'(1) = 0, f'(0) = -k_1 f(0) - k_2 g(0)\right\}. \end{gathered} \tag{2.12}$$

System (2.3) can be rewritten as

$$\frac{d}{dt}\left(\tilde{\Gamma}(\cdot,t), \tilde{\Gamma}_t(\cdot,t)\right) = \mathcal{A}\left(\tilde{\Gamma}(\cdot,t), \tilde{\Gamma}_t(\cdot,t)\right) + \mathcal{B}v(t), \tag{2.13}$$

where $\mathcal{B} = (0,\ -(k_1F_2 + k_2F_2Q + mF_2Q)\delta(\cdot) + F_1\delta(\cdot - 1))$ with the Dirac distribution $\delta(\cdot)$. Since $v(t)$ satisfies (1.4), we have $\mathcal{B}v(t) \in H^1_{loc}([0,\infty);[D(\mathcal{A}^*)]')$. Hence, by [25], system (2.3) admits a unique solution $\left(\tilde{\Gamma}, \tilde{\Gamma}_t\right) \in C([0,\infty);\mathcal{H})$. Now we show (2.9)-(2.11). Let

$$\Omega(x) = (\Omega_1(x),\ \Omega_2(x))^T \in \mathbb{R}^{2\times 2q}, \quad x \in [0,1]. \tag{2.14}$$

By recalling (1.4) and (2.13), we have

$$\frac{d}{dt}\begin{bmatrix}\tilde{\Gamma}(\cdot,t) - \Omega_1(\cdot)v(t)\\ \tilde{\Gamma}_t(\cdot,t) - \Omega_2(\cdot)v(t)\end{bmatrix} = \mathcal{A}\begin{bmatrix}\tilde{\Gamma}(\cdot,t) - \Omega_1(\cdot)v(t)\\ \tilde{\Gamma}_t(\cdot,t) - \Omega_2(\cdot)v(t)\end{bmatrix} + (\mathcal{A}\Omega(\cdot) - \Omega(\cdot)Q + \mathcal{B})v(t). \tag{2.15}$$

Since $e^{\mathcal{A}t}$ is exponentially stable, it must have $\operatorname{Re}\lambda < 0$ for any $\lambda \in \sigma(\mathcal{A})$. This together with **Assumption 1.2** implies that $\sigma(\mathcal{A}) \cap \sigma(Q) = \emptyset$. Furthermore, from $\mathcal{A} \in \mathcal{L}(X_1, [D(\mathcal{A}^*)]')$ and $\mathcal{B} \in \mathcal{L}(\mathbb{R}, [D(\mathcal{A}^*)]')$, it follows from [22] that the Sylvester equation $\mathcal{A}\Omega - \Omega Q = -\mathcal{B}$ admits a solution $\Omega \in \mathcal{L}(\mathbb{R}^{2q}, [D(\mathcal{A}^*)]')$. Then, the exponential stability of $e^{\mathcal{A}t}$ implies that $\lim_{t\to\infty}\tilde{\Gamma}(\cdot,t) = \Omega_1(\cdot)v(t)$ exponentially. Letting $\gamma := \Omega_1(0)$, we conclude that (2.9)-(2.11) hold. The proof is complete. □

Combining equations (2.2) and (2.9), we have

$$e(t) = \hat{\Gamma}(0,t) + p(t) + \chi_1(t), \tag{2.16}$$

where $p(t) = (\gamma - F_2)v(t)$. By Assumption 1.2, the term $p(t) = (\gamma - F_2)v(t)$ contains the sinusoids of no more than q distinct frequencies. Then $p(t)$ can be expressed as

$$p(t) = \sum_{j=1}^{q}(A_{3j}\sin\omega_j t + B_{3j}\cos\omega_j t), \tag{2.17}$$

where $\{A_{3j}\}$ and $\{B_{3j}\}$ are uncertain parameters.

Lemma 2.2. *The $p(t) = (\gamma - F_2)v(t)$ can be generated by the exosystem*

$$\begin{aligned} \dot{Z}(t) &= \Upsilon Z(t), \quad Z(0) = Z_0, \\ p(t) &= \hat{F} Z(t), \end{aligned} \tag{2.18}$$

where $Z(t) = \left(z_1(t), z_2(t), \dots, z_{2q}(t)\right)^\top \in \mathbb{R}^{2q}$,

$$\begin{gathered} \Upsilon = \begin{pmatrix} 0_{(2q-1)\times 1} & & & & I_{(2q-1)\times(2q-1)} & & \\ -\varpi_1 & 0 & -\varpi_2 & 0 & \dots & -\varpi_q & 0 \end{pmatrix}, \\ \hat{F} = (1, 0, 0, \dots, 0), \end{gathered} \tag{2.19}$$

the $I_{(2q-1)\times(2q-1)}$ denotes $(2q-1)\times(2q-1)$ identity matrix, and $\varpi_1, \dots, \varpi_q$ are chosen so that

$$l^{2q} + \varpi_q l^{2q-2} + \varpi_{q-1} l^{2q-4} + \dots + \varpi_1 = (l^2 + \omega_1^2)\dots(l^2 + \omega_q^2). \tag{2.20}$$

Proof. We choose $\varpi_1, \dots, \varpi_q$ to satisfy (2.20). By (2.17), we obtain

$$\varpi_1 p(t) + \varpi_2 p''(t) + \dots + \varpi_q p^{(2q-2)}(t) + p^{(2q)}(t) = 0.$$

Let $Z(t) = \left(p(t), \dot{p}(t), \cdots, p^{(2q-1)}(t)\right)^\top$, and then we show that $Z(t)$ satisfies (2.18). The proof is complete. □

From (2.16) and (2.18), the output tracking of system (1.1) is converted into a new output tracking problem for system (2.1) and (2.18). However, $p(t)$ is not suitable for the controller design since the initial state $Z(0)$ of the exosystem (2.18) is unknown. To overcome this, we introduce the internal model dynamic

$$\dot{\hat{\Phi}}(t) = \Xi \hat{\Phi}(t) + \hat{K}(e(t) - \hat{\Gamma}(0, t)), \tag{2.21}$$

where $\hat{\Phi}(t) = \left[\hat{\phi}(t), \hat{\phi}'(t), \dots, \hat{\phi}^{(2q-1)}(t)\right]^\top \in \mathbb{R}^{2q\times 1}$,

$$\begin{gathered} \Xi = \begin{pmatrix} 0_{(2q-1)\times 1} & & & I_{(2q-1)\times(2q-1)} & & \\ -\eta_1 & -\eta_2 & -\eta_3 & \dots & -\eta_{2q-1} & -\eta_{2q} \end{pmatrix}, \\ \hat{K} = (0, 0, \dots, \xi)^\top, \end{gathered} \tag{2.22}$$

where the parameter $\eta = (\eta_1, \eta_2, \eta_3, \dots, \eta_{2q})$ is chosen so that Ξ is Hurwitz and $\xi > 0$.

Before we proceed, we need the following Lemma 2.3 from [2].

Lemma 2.3. *Suppose that (S, F) is observable and $(\Xi, \hat{K})$ is controllable, where S, $\Xi \in \mathbb{R}^{2q\times 2q}$ and $F \in \mathbb{R}^{1\times 2q}$, $\hat{K} \in \mathbb{R}^{2q\times 1}$. The Sylvester equation*

$$PS - \Xi P = \hat{K} F, \tag{2.23}$$

admits a unique invertible matrix P.

Lemma 2.4. *Consider system (2.21) with the controller pair $(\Xi, \hat{K})$, where $\Xi \in \mathbb{R}^{2q\times 2q}$ and $\hat{K} \in \mathbb{R}^{2q\times 1}$ are given by (2.22). Then, there exists a unique nonsingular matrix P such that*

$$\chi_2(t) = p(t) - \hat{F} P^{-1} \hat{\Phi}(t) \tag{2.24}$$

tends to zero exponentially as t goes to infinity.

Proof. Since $(\Upsilon, \hat{F})$ is observable, we can replace (S, F) of Lemma 2.3 by the $(\Upsilon, \hat{F})$. Then, the Sylvester equation

$$P\Upsilon - \Xi P = \hat{K}\hat{F}, \tag{2.25}$$

has a unique invertible matrix P by Lemma 2.3. We define the error

$$\tilde{Z}(t) = PZ(t) - \hat{\Phi}(t). \tag{2.26}$$

From (2.16), (2.18) and (2.21), we have

$$\dot{\tilde{Z}}(t) = \Xi\tilde{Z}(t) - \hat{K}\chi_1(t). \tag{2.27}$$

Since P has its invertible matrix, Ξ is a Hurwitz, and $\chi_1(t)$ satisfies Theorem 2.1, we have that $\tilde{Z}(t)$ tends exponentially to zero and so does for $\chi_2(t) = \hat{F}P^{-1}\tilde{Z}(t)$. Furthermore, by differentiating (2.27) with respect to t, we obtain $\tilde{Z}''(t) = \Xi\dot{\tilde{Z}}(t) - \hat{K}\dot{\chi}_1(t)$. Owing to Ξ is Hurwitz and $\chi_1(t)$ satisfies (2.11), there exists $\iota_2 > 0$ such that

$$e^{\iota_2 \cdot}\dot{\tilde{Z}}(\cdot) \in L^2([0,\infty);\mathbb{R}), \quad e^{\iota_2 \cdot}\dot{\chi}_2(\cdot) \in L^2([0,\infty);\mathbb{R}). \tag{2.28}$$

The proof is complete. □

Substituting (2.24) into (2.16), we obtain

$$e(t) = \hat{\Gamma}(0,t) + \hat{F}P^{-1}\hat{\Phi}(t) + \chi(t), \tag{2.29}$$

where $\chi(t) = \chi_1(t) + \chi_2(t)$ converges exponentially to zero as $t \to \infty$. In particular, from (2.11) and (2.28), we deduce that

$$e^{\iota \cdot}\dot{\chi}(\cdot) \in L^2([0,\infty);\mathbb{R}), \quad \iota > 0. \tag{2.30}$$

Next, we determine the matrix P of (2.25). Let P_j, $j = 1, 2, \dots, 2q$ denote the jth row of P. Then expanding $P\Upsilon - \Xi P = \hat{K}\hat{F}$ gives

$$\begin{pmatrix} P_1\Upsilon - P_2 \\ P_2\Upsilon - P_3 \\ \vdots \\ P_{2q-1}\Upsilon - P_{2q} \\ P_{2q}\Upsilon + \sum_{j=1}^{2q}\eta_j P_j \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ \xi & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{2.31}$$

By a straightforward calculation, one gets

$$P = \xi\left(\eta_1 I + \eta_2\Upsilon + \dots + \eta_{2q}\Upsilon^{2q-1} + \Upsilon^{2q}\right)^{-1}. \tag{2.32}$$

Substituting (2.29) into (2.21) and using (2.25), we obtain

$$\dot{\hat{\Phi}}(t) = P\Upsilon P^{-1}\hat{\Phi}(t) + \hat{K}\chi(t). \tag{2.33}$$

Noting that $\Upsilon P^{-1} = P^{-1}\Upsilon$, we rewrite (2.33) as

$$\dot{\hat{\Phi}}(t) = \Upsilon\hat{\Phi}(t) + \hat{K}\chi(t). \tag{2.34}$$

Remark 2.5. Solving (2.34) gives

$$\hat{\Phi}(t) = e^{\Upsilon t}\hat{\Phi}(0) + \int_0^t e^{\Upsilon(t-\tau)}\hat{K}\chi(\tau)d\tau.$$

Since $\chi(t)$ converges exponentially to zero and Υ has eigenvalues $\pm i\omega_1$, $\pm i\omega_2, \dots$, $\pm i\omega_q$, we can conclude that $\hat{\Phi}(t)$ is bounded, i.e., $\|\hat{\Phi}(t)\| < \infty$.

3. Controller design

From Section 2, the tracking error $e(t)$ can be represented by the state $\hat{\Phi}$, $\hat{\Gamma}$ and $\hat{\Gamma}_t$. Since the state $\hat{\Phi}$, $\hat{\Gamma}$ and $\hat{\Gamma}_t$ are available for the controller design and $\chi(t)$ converges exponentially to zero, we can design a feedback control for system (2.1) and (2.21) such that

$$\lim_{t\to\infty} \hat{e}(t) = \lim_{t\to\infty} \Big(\hat{\Gamma}(0,t) + \hat{F}P^{-1}\hat{\Phi}(t)\Big) = 0. \tag{3.1}$$

Then, this controller has the same control effect on system (1.1). Motivated by [20], we introduce the transformation

$$y(x,t) = \hat{\Gamma}(x,t) - \Sigma(x)\hat{\Phi}(t), \tag{3.2}$$

where $\Sigma(x)$ satisfies

$$\begin{gathered} (a(x)\Sigma'(x))' = \Sigma(x)\Upsilon^2, \\ \Sigma'(0) = -\hat{F}P^{-1}(k_1 I + (k_2 - m)\Upsilon), \\ \Sigma(0) = -\hat{F}P^{-1}. \end{gathered} \tag{3.3}$$

Substituting (3.2) into (2.1) and using (2.29) and (2.34), we obtain

$$\begin{gathered} y_{tt}(x,t) = (a(x)y_x(x,t))_x - \Sigma(x)\Upsilon\hat{K}\chi(t) - \Sigma(x)\hat{K}\dot{\chi}(t), \\ y_x(0,t) = my_t(0,t) - k_1\chi(t) - k_2(\hat{F}P^{-1}\hat{K}\chi(t) + \dot{\chi}(t)) + m\dot{\chi}(t), \\ y_x(1,t) = U(t) - \Sigma'(1)\hat{\Phi}(t), \\ y(0,t) = \hat{e}(t). \end{gathered} \tag{3.4}$$

Evidently, the output regulation problem of $\hat{\Gamma}(0,t) - p(t) \to 0$ has been converted into a stabilization problem $y(\cdot,t) \to 0$ when t goes to infinity. Then, we propose the output feedback control as follows:

$$\begin{aligned} U(t) &= -ky_t(1,t) - y(1,t) + \Sigma'(1)\hat{\Phi}(t) \\ &= -k\hat{\Gamma}_t(1,t) - \Gamma(1,t) + k\Sigma(1)\Upsilon\hat{\Phi}(t) - \Sigma(1)\hat{\Phi}(t) + \Sigma'(1)\hat{\Phi}(t), \end{aligned} \tag{3.5}$$

where k is a positive constant. Under the controller (3.5), system (3.4) becomes

$$\begin{gathered} y_{tt}(x,t) = (a(x)y_x(x,t))_x - \Sigma(x)\Upsilon\hat{K}\chi(t) - \Sigma(x)\hat{K}\dot{\chi}(t), \\ y_x(0,t) = my_t(0,t) - k_1\chi(t) - k_2(\hat{F}P^{-1}\hat{K}\chi(t) + \dot{\chi}(t)) + m\dot{\chi}(t), \\ y_x(1,t) = -ky_t(1,t) - y(1,t), \\ y(0,t) = \hat{e}(t). \end{gathered} \tag{3.6}$$

Lemma 3.1. *Suppose that m, $k > 0$. When $\chi(t) \equiv 0$, system (3.6) is exponentially stable in $\mathcal{H}$.*

Proof. When $\chi(t) \equiv 0$, system (3.6) reads

$$\begin{gathered} y_{tt}(x,t) = (a(x)y_x(x,t))_x, \\ y_x(0,t) = my_t(0,t), \\ y_x(1,t) = -ky_t(1,t) - y(1,t). \end{gathered} \tag{3.7}$$

Let $\tilde{y}(x,t) = y(1-x,t)$. Then, $\tilde{y}(x,t)$ satisfies

$$\begin{gathered} \tilde{y}_{tt}(x,t) = (a(x)\tilde{y}_x(x,t))_x, \\ \tilde{y}_x(0,t) = k\tilde{y}_t(0,t) + \tilde{y}(0,t), \\ \tilde{y}_x(1,t) = -m\tilde{y}_t(1,t). \end{gathered} \tag{3.8}$$

We define the energy of system (3.8) as

$$E(t) = \frac{1}{2}\int_0^1 a(x)\tilde{y}_x^2(x,t) + \tilde{y}_t^2(x,t)dx + \frac{a(0)}{2}\tilde{y}^2(0,t). \tag{3.9}$$

The derivative of $E(t)$ along (3.8) satisfies

$$\dot{E}(t) = -ma(1)\tilde{y}_t^2(1,t) - ka(0)\tilde{y}_t^2(0,t). \tag{3.10}$$

We establish the energy multiplier as follows:

$$\varphi(t) = \int_0^1 (x-1)\tilde{y}_x(x,t)\tilde{y}_t(x,t)dx + 2s\tilde{y}(0,t)\int_0^1 (1-x)\tilde{y}_t(x,t)dx + l\tilde{y}^2(0,t), \tag{3.11}$$

where s, $l > 0$ to be determined later. Obviously, $|\varphi(t)| \le ME(t)$, $M > 0$. A direct computation shows that

$$\begin{aligned} \dot{\varphi}(t) &= \frac{1}{2}\tilde{y}_t^2(0,t) + \frac{1}{2}a^2(0)\tilde{y}_x^2(0,t) - \frac{1}{2}\int_0^1 \left[a(x)\tilde{y}_x^2(x,t) + \tilde{y}_t^2(x,t)\right]dx \\ &\quad + \frac{1}{2}\int_0^2 a'(2)(x-1)\tilde{y}_x^2(x,t)dx + 2s\tilde{y}_t(0,t)\int_0^1 (1-x)\tilde{y}_t(x,t)dx \\ &\quad + 2l\tilde{y}(0,t)\tilde{y}_t(0,t) - 2sa(0)\tilde{y}(0,t)\tilde{y}_x(0,t) + 2s\tilde{y}(0,t)\int_0^1 a(x)\tilde{y}_x(x,t)dx \\ &\le \big(\frac{1}{2} - 2sa(0)k + lr\big)\tilde{y}_t^2(0,t) + \Big(a^2(0) + s - 2sa(0) + \frac{l}{r}\Big)\tilde{y}^2(0,t) \\ &\quad - \frac{1}{2}\int_0^1 \left[a(x)\tilde{y}_x^2(x,t) + \tilde{y}_t^2(x,t)\right]dx, \end{aligned}$$

where $r > 0$ is chosen so that $\frac{1}{2} - 2sa(0)k + lr > 0$. Since $a(0) > 1/2$, we can choose sufficiently large s such that $a^2(0) + s - 2sa(0) + \frac{l}{r} < 0$. Therefore, there exists $M_0 > 0$ such that $\dot{\varphi}(t) \le (\frac{1}{2} - 2sa(0)k + lr)\tilde{y}_t^2(0,t) - M_0E(t)$. Let

$$\Pi(t) = E(t) + \frac{\mu}{M}\varphi(t), \quad \mu > 0. \tag{3.12}$$

Then

$$(1-\mu)E(t) \le \Pi(t) \le (1+\mu)E(t), \quad \dot{\Pi}(t) \le -\frac{M_0\mu}{M(1+\mu)}\Pi(t), \tag{3.13}$$

for all sufficiently small $\mu > 0$. This shows that

$$E(t) \le \frac{1+\mu}{1-\mu}e^{-\frac{M_0\mu}{M(1+\mu)}t}E(0). \tag{3.14}$$

The proof is complete. □

Lemma 3.2. *Suppose that $k > 0$. For any initial state $(y(\cdot,0), y_t(\cdot,0)) \in \mathcal{H}$, system (3.6) admits a unique solution $(y, y_t) \in C([0,\infty);\mathcal{H})$, which is exponentially stable. Moreover,* $\lim_{t\to\infty} |\hat{e}(t)| = 0$.

Proof. System (3.6) can be written abstractly as

$$\frac{d}{dt}(y(\cdot,t), y_t(\cdot,t)) = \mathcal{A}_1(y(\cdot,t), y_t(\cdot,t)) + B_1\chi(t) + B_2\dot{\chi}(t), \tag{3.15}$$

where

$$B_1 = \Big(0, -\Sigma(\cdot)\Upsilon\hat{K} - k_1\delta(\cdot) - k_2\hat{F}P^{-1}\hat{K}\delta(\cdot)\Big), \quad B_2 = \Big(0, -\Sigma(\cdot)\hat{K} - (k_2 - m)\delta(\cdot)\Big)$$

with the Dirac distribution $\delta(\cdot)$. Thanks to the result of Lemma 3.1, $\mathcal{A}_1$ generates an exponentially stable C_0-semigroup on $\mathcal{H}$, and B_1, B_2 are admissible for $e^{\mathcal{A}_1 t}$ (see [25]). Therefore, by [27] and Lemma 2.4, system (3.15) is exponentially stable, which admits a unique solution $(y, y_t) \in C([0,\infty);\mathcal{H})$. Besides, we obtain $\lim_{t\to\infty} |\hat{e}(t)| = 0$. □

Lemma 3.3. *Equation* (3.3) *admits a unique solution* $\Sigma(\cdot) \in C^\infty([0,1];\mathbb{R}^{2q})$.

Proof. It is clear that (3.3) is an initial value problem, so we prove the existence of its solution. □

4. Closed-loop system

In this section, we consider the following closed-loop system which is composed of (1.1), (2.1) and (2.21):

$$\begin{gathered}
\Gamma_{tt}(x,t) - (a(x)\Gamma_x(x,t))_x = 0,\\
\Gamma_x(0,t) = m\Gamma_t(0,t),\\
\Gamma_x(1,t) = U(t) + d(t),\\
\hat{\Gamma}_{tt}(x,t) = (a(x)\hat{\Gamma}_x(x,t))_x,\\
\hat{\Gamma}_x(0,t) = -k_1(e(t) - \hat{\Gamma}(0,t)) + k_2\hat{\Gamma}_t(0,t) + (m - k_2)\dot{e}(t),\\
\hat{\Gamma}_x(1,t) = U(t),\\
\dot{\hat{\Phi}}(t) = \Xi\hat{\Phi}(t) + \hat{K}(e(t) - \hat{\Gamma}(0,t)),\\
e(t) = y_p(t) - r(t),\\
U(t) = -k\hat{\Gamma}_t(1,t) - \hat{\Gamma}(1,t) + k\Sigma(1)\Upsilon\hat{\Phi}(t) - \Sigma(1)\hat{\Phi}(t) + \Sigma'(1)\hat{\Phi}(t).
\end{gathered} \tag{4.1}$$

Now we study system (4.1) in the Hilbert space $\mathcal{X} = \mathcal{H} \times \mathcal{H} \times \mathbb{R}^{2q}$.

Theorem 4.1. *Let* k, k_1, $k_2 > 0$. *Suppose that* $d(t)$ *and* $r(t)$ *satisfy* (1.5) *and* (1.6)*, respectively. Then, for any initial state*

$$\Big(\Gamma(\cdot,0), \Gamma_t(\cdot,0), \hat{\Gamma}(\cdot,0), \hat{\Gamma}_t(\cdot,0), \hat{\Phi}(0)\Big) \in \mathcal{X}, \tag{4.2}$$

the closed-loop system (4.1) *admits a unique solution*

$$\Big(\Gamma, \Gamma_t, \hat{\Gamma}, \hat{\Gamma}_t, \hat{\Phi}\Big) \in C([0,\infty);\mathcal{X}), \tag{4.3}$$

satisfying

$$|e(t)| < Le^{-\omega t}, \tag{4.4}$$

for some constants $L > 0$ *and* $\omega > 0$. *Moreover,*

(i) *the state of the closed-loop* (4.1) *is uniformly bounded*

$$\sup_{t\in[0,\infty)} \big\|\big(\Gamma(\cdot,t), \Gamma_t(\cdot,t), \hat{\Gamma}(\cdot,t), \hat{\Gamma}_t(\cdot,t), \hat{\Phi}(t)\big)\big\| < \infty; \tag{4.5}$$

(ii) *when* $d(t) \equiv 0$ *and* $r(t) \equiv 0$*, there exist positive constants* ζ_1 *and* ι_1 *such that*

$$\big\|\big(\Gamma(\cdot,t),\Gamma_t(\cdot,t),\hat{\Gamma}(\cdot,t),\hat{\Gamma}_t(\cdot,t),\hat{\Phi}(t)\big)\big\|_{\mathcal{H}\times\mathcal{H}\times\mathbb{R}^{2q}} \le \zeta_1 e^{-\iota_1 t}, \quad \forall t>0. \tag{4.6}$$

Proof. According to (2.2) and (2.16), we just need to consider that

$$\begin{gathered}\hat{\Gamma}_{tt}(x,t) = (a(x)\hat{\Gamma}_x(x,t))_x,\\ \hat{\Gamma}_x(0,t) = -k_1(p(t)+\chi_1(t)) - k_2(\dot{p}(t)+\dot{\chi}_1(t)) + m\dot{e}(t),\\ \hat{\Gamma}_x(1,t) = -k\hat{\Gamma}_t(1,t) - \Gamma(1,t) + k\Sigma(1)\Upsilon\hat{\Phi}(t) - \Sigma(1)\hat{\Phi}(t) + \Sigma'(1)\hat{\Phi}(t),\end{gathered} \tag{4.7}$$

and

$$\dot{\hat{\Phi}}(t) = \Xi\hat{\Phi}(t) + \hat{K}(p(t)+\chi_1(t)), \tag{4.8}$$

where $\chi_1(t)$ is given by (2.9) and $p(t) = (\gamma - F_2)v(t)$. By Remark 2.5 and the transformation (3.2), we obtain that

$$\begin{gathered}\dot{\hat{\Phi}}(t) = \Upsilon\hat{\Phi}(t) + \hat{K}\chi(t),\\ y_{tt}(x,t) = (a(x)y_x(x,t))_x - \Sigma(x)\Upsilon\hat{K}\chi(t) - \Sigma(x)\hat{K}\dot{\chi}(t),\\ y_x(0,t) = my_t(0,t) - k_1\chi(t) - k_2(\hat{F}P^{-1}\hat{K}\chi(t)+\dot{\chi}(t)) + m\dot{\chi}(t),\\ y_x(1,t) = -ky_t(1,t) - y(1,t),\end{gathered} \tag{4.9}$$

where $\Sigma(x)$ is given by (3.3). The well-posedness and exponential stability of (y,y_t)-part has been shown in Lemma 3.2. Therefore, the state of the system (4.1) is uniformly bounded. According to (2.29) and transformation (3.2), we see that $e(t) = y(0,t) + \chi(t)$. This implies that $e(t)$ exponentially converges to zero as time goes to infinity.

When $d(t) \equiv 0$ and $r(t) \equiv 0$. $\hat{\Phi}(t)$ is given by

$$\dot{\hat{\Phi}}(t) = \Xi\hat{\Phi}(t) + \hat{K}\chi_1(1). \tag{4.10}$$

Since Ξ is Hurwitz and $\chi_1(t)$ is exponentially stable, we obtain $\hat{\Phi}(t)$ is exponentially stable. Similarly to Lemma 3.2, the $(\hat{\Gamma},\hat{\Gamma}_t)$-part of system admits a unique solution $(\hat{\Gamma},\hat{\Gamma}_t) \in C([0,\infty);\mathcal{H})$, and it follows from transformation (3.2) that $(\hat{\Gamma},\hat{\Gamma}_t)$ is exponentially stable. By (2.16), we conclude that $e(t)$ decays exponentially to zero. Therefore, the estimate (4.4) holds. The proof is complete. □

5. Numerical simulations

In this section, we present some numerical simulations to validate the theoretical results. The numerical results are programmed in MATLAB. The space step and time step are taken as $dx = 0.001$ and $dt = 0.999dx$, respectively. Let $a(x) = 1$ and the initial states are chosen as

$$\begin{gathered}\Gamma(x,0) = \cos(2\pi x) - 1,\\ \Gamma_t(x,0) = \hat{\Gamma}(x,0) = \hat{\Gamma}_t(x,0) = 0,\\ \hat{\Phi}(0) = 0,\\ v_0 = (1,1,1,1)^\top.\end{gathered} \tag{5.1}$$

The parameters are chosen as

$$m = 0.5, \quad k = 1, \quad k_1 = 1, \quad k_2 = 0.9, \quad \eta = (4,6,4,1), \tag{5.2}$$

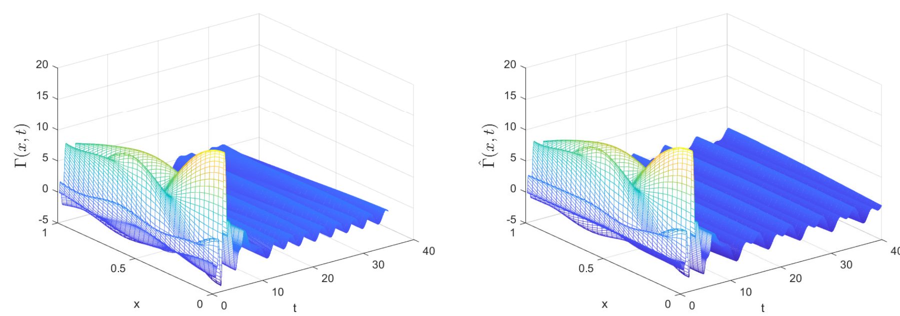
and the matrix Q is

$$Q = \operatorname{bdiag}\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix} \right\}. \tag{5.3}$$

In this case, the solution of equation (1.4) is $v(t) = (\sin t, \cos t, \sin 2t, \cos 2t)^\top$. The corresponding parameters are chosen as

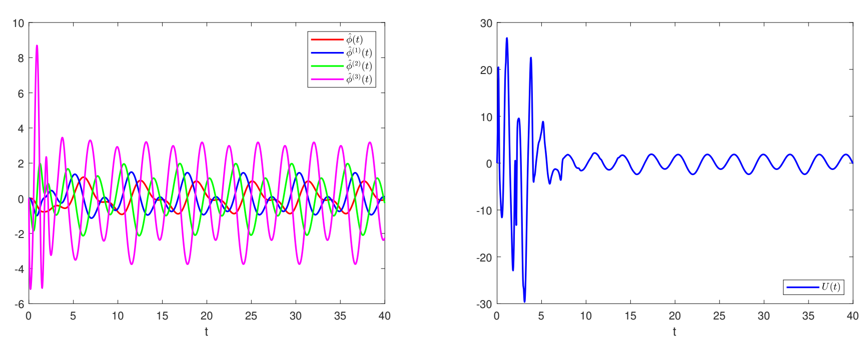
$$F_1 = (1, 1, 2, 1), \quad F_2 = (1, 2, 1, 2). \tag{5.4}$$

The solution of PDE-part of the closed-loop system (4.1) is depicted in Figure 1. The control law $\Phi(t)$ and the state $U(t)$ are plotted in Figure 2. It is seen that all states of the closed-loop system are uniformly bounded. Figure 3 shows that tracking error $e(t)$ converges to zero as time goes to infinity, and $\hat{F}P^{-1}\hat{\Phi}(t)$ can gradually synchronize with $p(t)$, respectively. Both of them converge effectively.



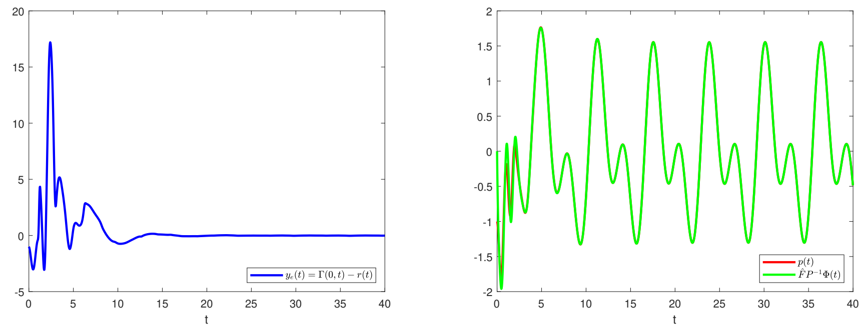
(a) state of $\Gamma(x,t)$ (b) state of $\hat{\Gamma}(x,t)$

Figure 1. Solution of closed-loop system (4.1)



(a) state of $\Phi(t)$ (b) state of $U(t)$

Figure 2. state of $\Phi(t)$ and $U(t)$



(a) tracking error $e(t)$ (b) state estimation $\hat{F}P^{-1}\hat{\Phi}(t) \to p(t)$

FIGURE 3. Tracking performance

6. Conclusions

In this article, we address the output tracking problem of a wave equation with variable coefficients subjected to boundary control matched with harmonic disturbances. The performance output is non-collocated with the control input. First, we establish an auxiliary system using measurable tracking error and its derivative. Subsequently, we construct an internal model dynamics to estimate the unknown disturbances. As a result, we reformulate the tracking error dynamics by integrating the internal model and the auxiliary system. Following this, we design an error-based feedback control mechanism for the auxiliary system, employing an invertible transformation for this purpose. Finally, we prove the boundedness of the closed-loop system and tracking error converges to zero exponentially, and demonstrate the effectiveness of our proposed controller by some numerical simulations.

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