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EXISTENCE AND UNIQUENESS OF GENERALIZED NORMAL SOLUTIONS TO FIRST ORDER FRACTIONAL DIFFERENTIAL EQUATIONS AND APPLICATIONS

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Abstract. We study the existence and uniqueness of continuous generalized normal solutions to initial value problems of first order fractional differential equations. We use the Banach contraction principle and the Weissinger fixed point theorem to obtain our results. We assume that the absolute values of the nonlinearities have upper bound functions in a subspace of continuous functions. As an example, the results are applied to equations with nonlinearities arising in logistic type population models with heterogeneous environments, and to population models of Ricker type.

1. Introduction

We study the existence and uniqueness of generalized normal solutions, that is, solutions u satisfying $\overline{I}^{\alpha}_{a^+}u \in AC[a,b]$, in $C([a,b];J)$ of the initial value problems (IVPs) of the first order fractional differential equation (FDE)

$$\big(D^{1-\alpha}_{p,a^+}u\big)(x) := \big(I^{\alpha}_{a^+}(u-w_0)\big)'(x) = f(x,u(x)) \quad \text{for each } x \in (a,b] \tag{1.1}$$

subject to the initial condition (IC)

$$u(a) = w_0, \tag{1.2}$$

where $\alpha \in (0,1)$, $w_0 \in J$, $a, b \in \mathbb{R}$ with $a < b$, I^{α}_{a+} is the standard Riemann-Liouville (R-L) fractional integral operator, $f : (a,b] \times J \to \mathbb{R}$ is a continuous function satisfying suitable conditions and J is one of intervals: $[c,d]$ where $c, d \in \mathbb{R}$ with $c < d$, $\mathbb{R}_+$ or $\mathbb{R}$. We use u' to denote the first order derivative of a function u. We emphasize that the function f is not assumed to be defined at $\{a\} \times J$.

The symbols used in the introduction will be given later. The first order fractional derivatives $D^{1-\alpha}_{p,a^+}u$ in (1.1) and higher order fractional derivatives have been studied in [17, 19, 22, 23] and [16, 20], respectively.

The FDE (1.1) holds for all $x \in (a,b]$ and is different from the following FDE which holds for almost every (a.e.) $x \in [a,b]$:

$$\big(D^{1-\alpha}_{p,a^+}u\big)(x) := \big(I^{\alpha}_{a^+}(u-w_0)\big)'(x) = f(x,u(x)) \quad \text{for a.e. } x \in [a,b]. \tag{1.3}$$

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When the value $u(a)$ exists, one can consider the following fractional derivative

$$\big(D_{*a}^{1-\alpha}u\big)(x) := \big(I_{a^+}^{\alpha}(u-u(a))\big)'(x) \quad \text{for a.e. } x \in (a,b].$$

The operator $D_{*a}^{1-\alpha}$ is called the Caputo fractional operator in [7, Chapter 6] and the derivative $D_{*a}^{1-\alpha}u$ is called the modified Caputo derivative in [30]. It is easy to see that (1.3) with (1.2) is equivalent to the IVP of the FDE

$$\big(D_{*a}^{1-\alpha}u\big)(x) := \big(I_{a^+}^{\alpha}(u-u(a))\big)'(x) = f(x,u(x)) \quad \text{for a.e. } x \in [a,b] \tag{1.4}$$

subject to the IC (1.2). When other ICs such as (1.7) below are considered, the value $u(a)$ is not required to exist, so it is necessary to consider (1.3) instead of (1.4). We refer to [19, 22, 23] for the study on the existence of generalized normal solutions or upper or lower generalized normal solutions of (1.3) where the value $u(a)$ is not required to exist.

When $f:[a,b]\times J\to\mathbb{R}$ is an L^p-Carathéodory function for $p\in(\frac{1}{1-\alpha},\infty)$, that is, f satisfies the Carathéodory conditions and the absolute value $|f|$ has an upper bound function in $L_+^p(a,b)$ on any bounded subset of $[a,b]\times J$, the existence of generalized normal solutions in $C([a,b];J)$ of (1.3) with (1.2) or (1.4) with (1.2) were studied before via the corresponding integral equation

$$u(x) = w_0 + (\overline{I}_{a^+}^{1-\alpha}Fu)(x) := Au(x) \quad \text{for each } x \in [a,b]. \tag{1.5}$$

For example, the minimal and maximal generalized normal solutions in $C([a,b];J)$ of (1.3) with (1.2) were obtained in [23] by using the monotone iterative techniques for compact maps. When $a=0$, the existence of generalized normal solutions of (1.4) with (1.2) were studied in [24, 29] by using the Leray-Schauder theorem for compact maps. Some results on the existence and uniqueness of (1.4) with (1.2) and other ICs can be found in [9, 13, 14, 15, 28]. We refer to [32] for the study on the existence of solutions in $C[0,b]$ of first and higher order FDEs of Bagley-Torvik and Langevin type and to [9, 13, 14, 34, 35] for the study on the existence of global solutions of (1.4) with (1.7).

When $J=\mathbb{R}_+$ or $\mathbb{R}$ and $f:[a,b]\times J\to J$ satisfies the Carathéodory conditions and a suitable growth condition, the existence and uniqueness of generalized normal solutions in $L^p([a,b];J)$ of (1.3) subject to the initial condition initial condition

$$\lim_{x\to a^+}(I_{a^+}^{\alpha}(u-w_0))(x) = c_0 \tag{1.6}$$

were studied in [19] by using the Banach contraction principle. The conditions on f imply that f is an L^p-Carathéodory function for $p=\infty$ or $p\in[1,1/\alpha)$. When $f:(a,b]\times J\to J$ satisfies suitable continuity conditions, the existence and uniqueness of generalized normal solutions in $C_{-\alpha}((a,b]:J)$ of (1.3) with the initital condition

$$(\overline{I}_{a^+}^{\alpha}u)(a) = c_0 \tag{1.7}$$

were studied in [22, Section 7]. When $w_0=0$, the existence and uniqueness of generalized normal solutions in $C[a,b]$ or $C(a,b]$ of (1.4) subject to the initial condition

$$\lim_{x\to a^+}(x-a)^{\alpha}u(x) = \frac{c_0}{\Gamma(1-\alpha)} \tag{1.8}$$

were studied, for example, in [3, Theorem 6.2], [6, Theorem 4.3], [11, Lemma 9], [12], [29, Theorem 4.11], [30, Proposition 6.4], [31, section 7], and [37, 38, 39].

In this article, we study the existence and uniqueness of generalized normal solutions in $C([a,b];J)$ of (1.1)-(1.2) via the integral equation (1.5). The nonlinearity f

is required to be defined on $(a,b]\times J$. Since f is only defined on $(a,b]\times J$ instead of $[a,b]\times J$, it is unknown under what conditions on f, the operator F in (1.5) is continuous and bounded on $C([a,b];J)$ and the map A in (1.5) is compact from $C([a,b];J)$ into $C[a,b]$. Hence, the known fixed point existence theorems for compact maps in Banach spaces such as Schauder fixed point theorem and Leray-Schauder fixed theorem can not be applied, so we shall employ the Banach contraction principle and Weissinger fixed point theorem to obtain the existence and uniqueness of the generalized normal solutions of (1.1)-(1.2) in $C([a,b];J)$. We shall provide conditions on f such that the map A maps $C([a,b];J)$ to $C([a,b];J)$. These new conditions on f show that f is an L^p-Carathéodory function for $p\in[1,\frac{1}{1-\alpha})$, so the new results on the existence and uniqueness of (1.1)-(1.2) are different from those considered in [23, 24, 29, 32], where f is defined on $[a,b]\times J$ and is an L^p-Carathéodory function for $p\in(\frac{1}{1-\alpha},\infty)$.

In section 2, we recall basic properties of the generalized R-L fractional integral introduced in [22] and the Banach space $C_{-\alpha}(a,b]$. In section 3, we study the existence and uniqueness of generalized normal solutions in $C([a,b];J)$ of (1.1)-(1.2). In section 4, we apply the results obtained in section 3 to study the existence and uniqueness of nonnegative generalized normal solutions of the first order FDEs with nonlinearities arising in logistic type population models with heterogeneous environments, and the population models of Ricker type.

2. Preliminaries

Throughout this articler, we assume $\alpha\in(0,1)$ and $a,b\in\mathbb{R}$ with $a<b$. For $u\in L^1[a,b]$, the Riemann-Liouville (R-L) fractional integral $I^\alpha_{a^+}u$ is defined by

$$(I^\alpha_{a^+}u)(x) := \frac{1}{\Gamma(\alpha)}\int_a^x \frac{u(y)}{(x-y)^{1-\alpha}}\,dy \tag{2.1}$$

for $x\in(a,b]$ such that the integral in (2.1) exists, where $\Gamma:(0,\infty)\to\mathbb{R}_+$ is the Gamma function defined by $\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}\,dx$ for $\alpha\in(0,\infty)$, see [7, p.13] and [27, p.33]. It is well known that $I^\alpha_{a^+}$ maps $L^1[a,b]$ to $L^1[a,b]$. Since $I^\alpha_{a^+}u\in L^1[a,b]$, there exists $E^*_u\subset[a,b]$ with $\operatorname{meas}(E^*_u)=0$ such that

$$(a,b]\setminus E^*_u=\{x\in(a,b]:(I^\alpha_{a^+}u)(x)\in\mathbb{R}\}. \tag{2.2}$$

The set $(a,b]\setminus E^*_u$ is treated as the natural domain of $I^\alpha_{a^+}u$ for each $u\in L^1[a,b]$.

Definition 2.1. We denote by $L^1_\alpha(a,b)$ the set of functions u in $L^1[a,b]$ where u satisfies that there exists $\delta_u\in(0,b-a]$ such that $I^\alpha_{a^+}u$ is well defined on $(a,a+\delta_u]$ and the limit $\lim_{x\to a^+}(I^\alpha_{a^+}u)(x)$ exists.

The space $L^1_\alpha(a,b)$ was introduced in [22, Definition 1] and is a linear space [22, Theorem 3].

Definition 2.2. The operator $\overline{I}^\alpha_{a^+}:L^1_\alpha(a,b)\to L^1[a,b]$ defined by

$$(\overline{I}^\alpha_{a^+}u)(x)=\begin{cases}(I^\alpha_{a^+}u)(x) & \text{for each } x\in(a,b]\setminus E^*_u,\\ \lim_{x\to a^+}(I^\alpha_{a^+}u)(x) & \text{if } x=a\end{cases}. \tag{2.3}$$

is called the generalized R-L fractional integral of order α.

The integral operator $\overline{I}^\alpha_{a^+}$ was introduced in [22, Definition 2]. In many cases, we need $(I^\alpha_{a^+}u)(x)$ to be well defined for each $x\in(a,b]$. A proper linear subspace of $L^1_\alpha(a,b)$ was introduced in [22, Definition 3], which is stated as follows.

Definition 2.3. We denote by $S_\alpha^1(a,b)$ the set of functions u in $L^1[a,b]$ where u satisfies that $I_{a^+}^\alpha u$ is well defined on $(a,b]$ and the limit $\lim_{x\to a^+}(I_{a^+}^\alpha u)(x)$ exists.

It is known that $S_\alpha^1(a,b)$ is a linear space [22, Theorem 4] and it is shown in [22, Propositions 2 and 3] and [22, Corollary 6] that the following inclusions hold.

$$\mathrm{L^p[a,b]} \subsetneqq \mathrm{S}_\alpha^1(\mathrm{a,b}) \subsetneqq \mathrm{L}_\alpha^1(\mathrm{a,b}) \subsetneqq \mathrm{L}^1[\mathrm{a,b}] \quad \text{for each } \mathrm{p} \in (\frac{1}{\alpha},\infty]. \tag{2.4}$$

For $u \in L^1[a,b]$ and $\beta \in [1,\infty)$, the R-L fractional integral $I_{a^+}^\beta u$ is defined by

$$(I_{a^+}^\beta u)(x) = \frac{1}{\Gamma(\beta)}\int_a^x (x-y)^{\beta-1}u(y)\,dy \quad \text{for each } x \in [a,b]. \tag{2.5}$$

It is known (see [22, Lemma 1 (2)]) that when $\beta \in [1,\infty)$, $I_{a^+}^\beta u \in C[a,b]$ for each $u \in L^1[a,b]$. Hence, for consistency with Definition 2.2, we can rewrite $I_{a^+}^\beta$ as $\overline{I}_{a^+}^\beta$, that is,

$$(I_{a^+}^\beta u) = (\overline{I}_{a^+}^\beta u)(x) := \begin{cases} (I_{a^+}^\beta u)(x) & \text{for each } x \in (a,b], \\ \lim_{x\to a^+}(I_{a^+}^\beta u)(x) & \text{if } x = a. \end{cases} \tag{2.6}$$

Moreover, if $u, v \in L^1[a,b]$ satisfy $u(x) = v(x)$ for a.e. $x \in [a,b]$, then by (2.5) and (2.6), we have for each $\beta \in [1,\infty)$,

$$(\overline{I}_{a^+}^\beta u)(x) = (I_{a^+}^\beta u)(x) = (I_{a^+}^\beta v)(x) = (\overline{I}_{a^+}^\beta v)(x) \quad \text{for each } x \in [a,b].$$

Lemma 2.4. (1) *If $\gamma \in [-\alpha,\infty)$, then*

$$\overline{I}_{a^+}^\alpha (x-a)^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\alpha+\gamma)}(x-a)^{\alpha+\gamma} \quad \text{for each } x \in [a,b].$$

(2) *Let $\gamma \in (0,\infty)$ and $u \in C[a,b]$. Then*

$$\overline{I}_{a^+}^\alpha(\overline{I}_{a^+}^\gamma u)(x) = (\overline{I}_{a^+}^{\alpha+\gamma} u)(x) \quad \text{for each } x \in [a,b].$$

(3) *Assume that $u \in S_\alpha^1(a,b)$ satisfies $\overline{I}_{a^+}^{1-\alpha} u \in C[a,b]$. Then*

$$(I_{a^+}^{2-\alpha} u)'(x) = (\overline{I}_{a^+}^{1-\alpha} u)(x) \quad \text{for each } x \in [a,b].$$

(4) *Let $w_0 \in \mathbb{R}$ and $u \in S_\alpha^1(a,b)$. Then $\overline{I}_{a^+}^\alpha(u - w_0) \in AC[a,b]$ if and only if $\overline{I}_{a^+}^\alpha u \in AC[a,b]$.*

(5) *For $p \in \left(\frac{1}{\alpha},\infty\right]$, $\overline{I}_{a^+}^\alpha$ maps $L^p[a,b]$ to $C[a,b]$ and $(\overline{I}_{a^+}^\alpha u)(a) = 0$ for each $u \in L^p[a,b]$.*

The results (1)–(5) can be found in Lemma 12, Lemma 14 and Proposition 5, Corollary 5 (2), Theorem 5 (1) in [22], respectively.

We define the linear space

$$C_{-\alpha}(a,b] = \{u \in C(a,b] : \lim_{x\to a^+}(x-a)^\alpha u(x) \in \mathbb{R}\}, \tag{2.7}$$

see [22, (3.21)]. Note that each function in $C_{-\alpha}(a,b]$ is not necessarily defined at a. When $a = 0$, the set on the right side of (2.7) is denoted by other notations, for example, $C_{-\alpha}[a,b]$ is used in [30], also see [2, Theorem 2], [6, p.612] and [11, p.370].

We introduce the following subset in $C_{-\alpha}(a,b]$.

$$C_{-\alpha}^0(a,b] = \{u \in C(a,b] : \lim_{x\to a^+}(x-a)^\alpha u(x) = 0\}. \tag{2.8}$$

It is easy to verify that

$$C_{-\alpha}(a,b) \subset C(a,b) \cap L^p[a,b] \quad \text{for } p \in [1, \frac{1}{\alpha}). \tag{2.9}$$

Lemma 2.5. (i) *$\overline{I}^{\alpha}_{a+}$ maps $C_{-\alpha}(a,b]$ into $C[a,b]$.*
(ii) *$\overline{I}^{\alpha}_{a+}$ maps $C^0_{-\alpha}(a,b]$ into $C[a,b]$ and $(\overline{I}^{\alpha}_{a+}u)(a) = 0$ for each $u \in C^0_{-\alpha}(a,b]$.*

Proof. The result (i) was proved in [22, Lemma 11 (i)]. By [22, Lemma 11 (ii)] and (2.8), we see that the result (ii) holds. □

3. Initial value problems of first order FDEs

We study the existence and uniqueness of generalized normal solutions or nonnegative generalized normal solutions in $C([a,b];J)$ of the IVPs for the nonlinear first order FDE

$$\big(D^{1-\alpha}_{p,a^+}u\big)(x) := \big(I^{\alpha}_{a^+}(u - w_0)\big)'(x) = f(x,u(x)) \quad \text{for each } x \in (a,b] \tag{3.1}$$

subject to the initial condition

$$u(a) = w_0, \tag{3.2}$$

where $\alpha \in (0,1)$, $w_0 \in J$ and $f : (a,b] \times J \to \mathbb{R}$ is a continuous function, J is one of the intervals $J = [c,d]$, $\mathbb{R}_+ := [0,\infty)$ or $\mathbb{R} := (-\infty,\infty)$ and $c,d \in \mathbb{R}$ with $c < d$.

We note that the function f is required to be defined only on $(a,b] \times J$ and the FDE (3.1) holds for all $x \in (a,b]$ instead of for almost every (a.e.) $x \in [a,b]$. Let

$$C([a,b];J) = \{u \in C[a,b] : u(x) \in J \text{ for each } x \in [a,b]\}. \tag{3.3}$$

By (2.4), $C([a,b];J) \subset S^1_{\alpha}(a,b)$. It is known that $C([a,b];[c,d])$ is a bounded closed convex subset of $C[a,b]$, $C([a,b];\mathbb{R}_+) = C_+[a,b]$, the cone of nonnegative continuous functions defined on $[a,b]$ and $C([a,b];\mathbb{R}) = C[a,b]$.

The existence of nonnegative generalized normal solutions in $C([a,b];J)$ of the IVPs for the nonlinear first order FDE

$$\big(D^{1-\alpha}_{*a}u\big)(x) := \big(I^{\alpha}_{a^+}(u - u(a))\big)'(x) = f(x,u(x)) \quad \text{for a.e. } x \in [a,b] \tag{3.4}$$

subject to (3.2) was studied in [24], where $a = 0$, $J = \mathbb{R}_+$ or $J = \mathbb{R}$ and the function $f : [a,b] \times J \to J$ is an L^p-Carathéodory function for $p \in (\frac{1}{1-\alpha}, \infty)$.

When $f(x,u) = x^{\gamma}h(x,u)$ for $x \in [0,b] \times \mathbb{R}_+$, h is continuous and $\gamma \in [0, 1-\alpha)$, the existence and uniqueness of generalized normal solutions of (3.4) with (3.2) was obtained in [29, Theorem 4.8]. As mentioned in the Introduction, when f is defined on $(a,b] \times J$, (3.1) with other ICs such as (1.7) and (1.8) were studied in [3, 6, 11, 22, 29, 30, 31, 37, 38, 39]. The following definition can be found in [22, Definition 5].

Definition 3.1. A function $u \in S^1_{\alpha}(a,b)$ is said to be a generalized normal solution of (3.1)-(3.2) if $\overline{I}^{\alpha}_{a+}u \in AC[a,b]$ and u satisfies (3.1) and (3.2).

Definition 3.2. A function $f : (a,b] \times [c,d] \to \mathbb{R}$ is continuous at $x_0 \in (a,b]$ uniformly with respect to $[c,d]$ if for $\varepsilon > 0$ there exists $\delta > 0$ such that when $|x - x_0| < \delta$,

$$|f(x,u) - f(x_0,u)| < \varepsilon \quad \text{for all } u \in [c,d].$$

Definition 3.2 was introduced in [22, Definition 12]. The following result gives its relation with continuity of f.

Lemma 3.3. *The following assertions are equivalent.*

(1) *$f:(a,b]\times J\to\mathbb{R}$ is continuous*
(2) *$f:(a,b]\times J\to\mathbb{R}$ is continuous at each $x\in(a,b]$ uniformly with respect to any interval $[c,d]\subset J$ and $f(x,\cdot):J\to\mathbb{R}$ is continuous for each $x\in(a,b]$.*

Proof. It is sufficient to prove the result when $J=[c,d]$.

(1) implies (2): Since f is continuous, it follows from [22, Lemma 19] that f is continuous at each $x\in(a,b]$ uniformly with respect to $u\in[c,d]$. It is obvious that $f(x,\cdot):[c,d]\to\mathbb{R}$ is continuous for each $x\in(a,b]$.

(2) impliesg (1): Let $(x_0,u_0)\in(a,b]\times[c,d]$ and $\varepsilon>0$. Since $f(x_0,\cdot):[c,d]\to\mathbb{R}$ is continuous, there exists $\delta_0>0$ such that

$$|f(x_0,u)-f(x_0,u_0)|\le\varepsilon/2 \quad \text{for all } u\in[c,d] \text{ with } |u-u_0|<\delta_0. \tag{3.5}$$

Since f is continuous at x_0 uniformly with respect to $u\in[c,d]$, by Definition 3.2 there exists $\delta\in(0,\delta_0)$ such that when $|x-x_0|<\delta$,

$$|f(x,u)-f(x_0,u)|<\varepsilon/2 \quad \text{for all } u\in[c,d].$$

This with (3.5) implies that when $|x-x_0|<\delta$ and $|u-u_0|<\delta$,

$$|f(x,u)-f(x_0,u_0)|\le|f(x,u)-f(x_0,u)|+|f(x_0,u)-f(x_0,u_0)|\le\varepsilon/2+\varepsilon/2=\varepsilon$$

and f is continuous. □

When $J=\mathbb{R}_+$ or $\mathbb{R}$, the condition (2) of Lemma 3.3 first appeared in the condition $(H_1)_J$ in [22] and was used to prove [22, Lemma 20].

We write by $C^0_{-(1-\alpha),+}(a,b]$ for the set of nonnegative functions in $C^0_{-(1-\alpha)}(a,b]$ defined in (2.8), that is,

$$C^0_{-(1-\alpha),+}(a,b]=\{u\in C^0_{-(1-\alpha)}(a,b]:u(x)\ge 0 \text{ for all } x\in(a,b]\}. \tag{3.6}$$

Definition 3.4. A function $f:(a,b]\times J\to\mathbb{R}$ is said to be a $C^0_{-(1-\alpha)}$-function if for any bounded closed interval $[c_1,d_1]$ in J, there exists $g\in C^0_{-(1-\alpha),+}(a,b]$ such that

$$|f(x,u)|\le g(x) \quad \text{for all } x\in(a,b] \text{ and } u\in[c_1,d_1]. \tag{3.7}$$

A function $f:(a,b]\times J\to\mathbb{R}$ is said to be a $C^0_{-(1-\alpha)}$-continuous function if f is a $C^0_{-(1-\alpha)}$-function and is continuous on $(a,b]\times J$.

When $J=[c,d]$, then (3.7) holds on $[c,d]$. In this case, if we want to emphasize the function g in (3.7), then we say that f is a $C^0_{-(1-\alpha),+}$-function with function g. By (2.9), $C_{-(1-\alpha)}(a,b]\subset L^p[a,b]$ for each $p\in[1,\frac{1}{1-\alpha})$, Hence, a $C^0_{-(1-\alpha)}$-functionf on $(a,b]\times J$ is an L^p-Carathéodory function for each $p\in[1,\frac{1}{1-\alpha})$, that is, f satisfies the Carathéodory conditions and (3.7) holds with $g\in L^p_+[a,b]$. See [21, Definition 4.1], [23, Definition 3.3] and [24, Definition 2] for the definition of an L^p-Carathéodory function.

We define an operator F on $C((a,b];J)$, called the Nemytskii operator, by

$$(Fu)(x)=f(x,u(x)) \quad \text{for each } x\in(a,b]. \tag{3.8}$$

We study generalized normal solutions of (3.1)-(3.2) in $C([a,b];J)$ via the fixed points of the map A defined by

$$Au(x)=w_0+(\overline{I}^{1-\alpha}_{a^+}Fu)(x) \quad \text{for each } x\in[a,b], \tag{3.9}$$

that is, solutions in $C([a,b];J)$ of the integral equation

$$u(x)=w_0+(\overline{I}^{1-\alpha}_{a^+}Fu)(x)=Au(x) \quad \text{for each } x\in[a,b]. \tag{3.10}$$

Definition 3.5. A function $u : [a, b] \to J$ is said to be a solution of (3.10) if $u \in C([a, b]; J)$ and u satisfies (3.10).

Lemma 3.6. (1) *If $f : (a, b] \times J \to \mathbb{R}$ is continuous, then F maps $C((a, b]; J)$ to $C(a, b]$.*
(2) *If $f : (a, b] \times J \to \mathbb{R}$ is a $C^0_{-(1-\alpha)}$-continuous function, then F maps $C([a, b]; J)$ to $C^0_{-(1-\alpha)}(a, b]$.*
(3) *If $f : (a, b] \times J \to \mathbb{R}$ is a $C^0_{-(1-\alpha)}$-continuous function, then the map A maps $C([a, b]; J)$ to $C[a, b]$ and $(Au)(a) = w_0$.*

Proof. (1) When $J = \mathbb{R}_+$ or $J = \mathbb{R}$, Lemma 3.6 (1) was proved in [22, Lemma 20]. We only prove the case $J = [c, d]$. Let $u \in C((a, b]; [c, d])$ and $x_0 \in (a, b]$. Then $u(x_0) \in [c, d]$. We consider the following two cases.

(i) When $u(x_0) \in (c, d]$, since $f(x_0, \cdot) : [c, d] \to \mathbb{R}$ is continuous at $u(x_0)$, for $\varepsilon > 0$, there exists $\omega_0 \in (0, u(x_0) - c]$ such that when $v \in [c, d]$ with $|v - u(x_0)| < \omega_0$,

$$|f(x_0, v) - f(x_0, u(x_0))| < \varepsilon/2. \tag{3.11}$$

Since u is continuous at x_0, there exists $\delta_1 \in (0, \min\{x_0 - a, b - x_0\})$ such that when $x \in (a, b]$ with $|x - x_0| < \delta_1$,

$$|u(x) - u(x_0)| < \omega_0.$$

This with (3.11) implies that when $x \in (a, b]$ with $|x - x_0| < \delta_1$,

$$|f(x_0, u(x)) - f(x_0, u(x_0))| < \varepsilon/2. \tag{3.12}$$

Since $f : (a, b] \times J \to \mathbb{R}$ is continuous, by (1) implying (2) of Lemma 3.3, $f : (a, b] \times [c, d] \to \mathbb{R}$ is continuous at $x_0 \in (a, b]$ uniformly with respect to $u \in [c, d]$. Hence, by Definition 3.2 there exists $\delta_0 \in (0, \delta_1)$ such that when $|x - x_0| < \delta_0$,

$$|f(x, u(x)) - f(x_0, u(x))| < \varepsilon/2.$$

This with (3.12) implies that when $|x - x_0| < \delta_0$,

$$\begin{aligned}
&|(Fu)(x) - (Fu)(x_0)| \\
&= |f(x, u(x)) - f(x_0, u(x_0))| \\
&\le |f(x, u(x)) - f(x_0, u(x)) + |f(x_0, u(x)) - f(x_0, u(x_0))| < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

It follows that Fu is continuous at x_0.

(ii) When $u(x_0) = c$, since $f(x_0, \cdot) : [c, d] \to \mathbb{R}$ is continuous at c, for $\varepsilon > 0$, there exists $\omega_0 \in (0, d - c]$ such that when $v \in [c, w_0 + c]$,

$$|f(x_0, v) - f(x_0, u(x_0))| = |f(x_0, v) - f(x_0, c)| < \varepsilon/2.$$

The rest of the proof is the same as that after (3.11).

(2) Let $u \in C([a, b]; J)$. Since f is continuous, by the result (1) we have $Fu \in C(a, b]$. If $J = [c, d]$, then $u(x) \in J$ for each $x \in [a, b]$. If $J = \mathbb{R}_+$ or $J = \mathbb{R}$, then $u(x) \in [0, \rho]$ for each $x \in [a, b]$, where $\rho = \|u\|$. Since f is a $C^0_{-(1-\alpha)}$-function, there exists $g \in C^0_{-(1-\alpha),+}(a, b]$ such that

$$|(Fu)(x)| = |f(x, u(x))| \le g(x) \quad \text{for each } x \in (a, b].$$

Multiplying both sides of the above inequality by $(x - a)^{(1-\alpha)}$ implies

$$(x - a)^{1-\alpha}|(Fu)(x)| \le (x - a)^{1-\alpha} g(x) \quad \text{for each } x \in (a, b].$$

Taking limit superiors on both sides of the above inequality implies

$$\limsup_{x\to a^+}(x-a)^{1-\alpha}|(Fu)(x)| \le \limsup_{x\to a^+}(x-a)^{1-\alpha}g(x), \tag{3.13}$$

see [18] for the theory of limit inferior and superior. Since $g \in C^0_{-(1-\alpha),+}(a,b]$, by (2.8) and (3.6) we have

$$\limsup_{x\to a^+}(x-a)^{1-\alpha}g(x) = \lim_{x\to a^+}(x-a)^{1-\alpha}g(x) = 0.$$

This with (3.13) implies

$$\lim_{x\to a^+}(x-a)^{1-\alpha}|(Fu)(x)| = 0$$

and $Fu \in C^0_{-(1-\alpha)}(a,b]$.

(3) By result (2), F maps $C([a,b];J)$ to $C^0_{-(1-\alpha)}(a,b]$. By Lemma 2.5 (i), $\overline{I}^{1-\alpha}_{a^+}$ maps $C_{-(1-\alpha)}(a,b]$ into $C[a,b]$. Since $C^0_{-\alpha}(a,b] \subset C_{-\alpha}(a,b]$, we have for $u \in C([a,b];J)$, $\overline{I}^{1-\alpha}_{a^+}Fu \in C[a,b]$ and $Au = w_0 + (\overline{I}^{1-\alpha}_{a^+}Fu) \in C[a,b]$. Hence, A maps $C([a,b];J)$ to $C[a,b]$. For $u \in C([a,b];J)$, since $Fu \in C^0_{-(1-\alpha)}(a,b]$, by Lemma 2.5 (ii), we have $(\overline{I}^{1-\alpha}_{a^+}Fu)(a) = 0$ and $(Au)(a) = w_0$. □

Theorem 3.7. *Assume that $f : (a,b] \times J \to \mathbb{R}$ is a $C^0_{-(1-\alpha)}$-continuous function. Then the following assertions are equivalent.*

1. *$u \in C([a,b];J)$ is a generalized normal solution of* (3.1)-(3.2)*.*
2. *$u \in C([a,b];J)$ is a solution of* (3.10)*.*

Proof. (1) implies (2): Assume that (1) holds. Since $u \in C([a,b];J)$, by Lemma 2.4 (5), we have $\overline{I}^{\alpha}_{a^+}(u - w_0)(a) = 0$. Since $\overline{I}^{\alpha}_{a^+}u \in AC[a,b]$, it follows from Lemma 2.4 (4) that $\overline{I}^{\alpha}_{a^+}(u - w_0) \in AC[a,b]$. Since

$$\overline{I}^{\alpha}_{a^+}(u-w_0)(x) = I^{\alpha}_{a^+}(u-w_0)(x) \quad \text{for each } x \in (a,b],$$

by (3.1), we obtain

$$\big(\overline{I}^{\alpha}_{a^+}(u-w_0)\big)'(x) = f(x,u(x)) \quad \text{for each } x \in (a,b]. \tag{3.14}$$

Since $\overline{I}^{\alpha}_{a^+}(u-w_0) \in AC[a,b]$, integrating (3.14) from a to x and applying $\overline{I}^{\alpha}_{a^+}(u - w_0)(a) = 0$, we have

$$\overline{I}^{\alpha}_{a^+}(u-w_0)(x) = (\overline{I}^{1}_{a^+}Fu)(x) \quad \text{for each } x \in [a,b].$$

Applying $\overline{I}^{1-\alpha}_{a^+}$ to both sides of the above equation and using Lemma 2.4 (2) imply that for each $x \in [a,b]$,

$$\overline{I}^{1}_{a^+}(u-w_0)(x) = \overline{I}^{1-\alpha}_{a^+}\overline{I}^{\alpha}_{a^+}(u-w_0)(x) = \overline{I}^{1-\alpha}_{a^+}(\overline{I}^{1}_{a^+}Fu)(x) = (\overline{I}^{2-\alpha}_{a^+}Fu)(x).$$

Since $\overline{I}^{1-\alpha}_{a^+}Fu \in C[a,b]$, by Lemma 2.4 (3) and differentiating both sides of the above equation, we have, for each $x \in [a,b]$,

$$u(x) - w_0 = \big(\overline{I}^{1}_{a^+}(u-w_0)\big)'(x) = \big(\overline{I}^{2-\alpha}_{a^+}Fu\big)'(x) = (\overline{I}^{1-\alpha}_{a^+}Fu)(x).$$

and the result (2) holds.

(2) implies (1): Assume that $u \in C([a,b];J)$ is a solution of (3.10). By Lemma 3.6 (3), we have $\overline{I}_{a^+}^{1-\alpha} Fu \in C[a,b]$. By Lemma 2.4 (5), $\overline{I}_{a^+}^{\alpha}(u-w_0) \in C[a,b]$. Applying $\overline{I}_{a^+}^{\alpha}$ to both sides of (3.10) and using Lemma 2.4 (2), we have

$$\overline{I}_{a^+}^{\alpha}(u-w_0)(x) = \overline{I}_{a^+}^{\alpha}(\overline{I}_{a^+}^{1-\alpha} Fu)(x) = (\overline{I}_{a^+}^{1} Fu)(x) \quad \text{for each } x \in [a,b]. \tag{3.15}$$

Since $\overline{I}_{a^+}^{1} Fu \in AC[a,b]$, it follows from (3.15) that $\overline{I}_{a^+}^{\alpha}(u-w_0) \in AC[a,b]$. By Lemma 2.4 (4), we have $\overline{I}_{a^+}^{\alpha} u \in AC[a,b]$. Differentiating both sides of (3.15) implies that

$$\big(\overline{I}_{a^+}^{\alpha}(u-w_0)\big)'(x) = \big(\overline{I}_{a^+}^{1} Fu\big)'(x) = (Fu)(x) \quad \text{for each } x \in [a,b]. \tag{3.16}$$

Since

$$\big(I_{a^+}^{\alpha}(u-w_0)\big)'(x) = (\overline{I}_{a^+}^{\alpha}(u-w_0))'(x) \quad \text{for } x \in (a,b],$$

it follows from (3.16) that

$$\big(D_{p,a^+}^{1-\alpha} u\big)(x) = \big(I_{a^+}^{\alpha}(u-w_0)\big)'(x) = f(x,u(x)) \quad \text{for each } x \in (a,b]$$

and u is a generalized normal solution of (3.1). By Lemma 3.6 (3),

$$u(a) = (Au)(a) = w_0$$

and (3.2) holds. □

Let X be a Banach space. Recall that a map $A : D \to X$ is said to be a Lipschitz map with Lipschitz constant L, if

$$\|Ax - Ay\| \le L\|x-y\| \quad \text{for } x, y \in D.$$

A Lipschitz map with Lipschitz constant $L < 1$ is said to be a contractive map (with contraction constant L).

The following Banach contraction principle can be found for example in [10].

Lemma 3.8. *Let K be a nonempty closed convex set in X. Assume that $A : K \to K$ is a contractive map with contraction constant L. Then the following assertions hold.*

- (a) *A has a unique fixed point u^* in K.*
- (b) *For each u_1 in K, the sequence $\{u_{n+1}\}$ defined by $u_{n+1} = Au_n$ for each $n \in \mathbb{N}$ converges to u^*.*
- (c) $\|u_n - u^*\| \le \frac{L^n}{1-L}\|u_1 - u_2\|$.

Another useful fixed point theorem on uniqueness is the Weissinger fixed point theorem. The following definition was introduced in [21, Definition 2.1].

Definition 3.9. Let K be a nonempty closed subset in X. A map $A : K \to K$ is said to be a Weissinger map (with a sequence of Lipschitz constants $\{k_n\}$) if for each $n \in \mathbb{N}$, $A^n : K \to K$ is a Lipschitz map with Lipschitz constant $k_n \in (0,\infty)$ and $\sum_{n=1}^{\infty} k_n < \infty$.

It is easy to see that if $A : K \to K$ is a contractive map with contractive constant $L < 1$, then A is a Weissinger map with $k_n = L^n$.

Lemma 3.10. *Let K be a nonempty closed subset in X. Assume that $A : K \to K$ is a Weissinger map with Lipschitz constants $\{k_n\}$. Then A has a unique fixed point u_* in K and for each $n \in \mathbb{N}$,*

$$\|u_{n+1} - u_*\| \le \sum_{i=n}^{\infty} k_i \|u_2 - u_1\| \quad \textit{for each } u_1 \in K, \tag{3.17}$$

where $u_{n+1} = Au_n$ *for each* $n \in \mathbb{N}$.

Lemma 3.10 was given in [33] whose English version can be found in [7, Theorem D.7] or [8, Theorem 2.3]. A proof in English is given in [21, Lemma 2.4].

By [7, Definition 4.1] and [7, Theorem 4.1], the Mittag-Leffler function $E_{(1-\alpha)}$ satisfies

$$E_{(1-\alpha)}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(1+(1-\alpha)n)} \quad \text{for each } x \in \mathbb{R}. \tag{3.18}$$

It is known that $E_{(1-\alpha)}(x) \in \mathbb{R}$ for each $x \in \mathbb{R}$.

Recall that a function $f : (a,b] \times J \to \mathbb{R}$ is said to satisfy a Lipschitz condition in the second variable with function $\phi : (a,b] \to \mathbb{R}_+$ if

$$|f(x,u) - f(x,v)| \le \phi(x)|u-v| \quad \text{for each } x \in (a,b] \text{ and all } u,v \in J.$$

If $\phi(x) \equiv L > 0$ for each $x \in [a,b]$, then L is said to be a Lipschitz constant.

In the next theorem we use the following conditions:

- (H1) If $J = \mathbb{R}_+$ or $\mathbb{R}$, assume that $f : (a,b] \times J \to J$ is a $C^0_{-(1-\alpha)}$-function and if $J = [c,d]$, assume that $f : (a,b] \times [c,d] \to \mathbb{R}$ is a $C^0_{-(1-\alpha)}$-function with function g in $C^0_{-(1-\alpha),+}(a,b]$ satisfying
$$\|\overline{I}^{1-\alpha}_{a^+} g\| \le \min\{w_0 - c, d - w_0\}. \tag{3.19}$$
- (H2) f satisfies a Lipschitz condition in the second variable with function $\phi \in C_{-(1-\alpha),+}(a,b]$ and $\|\overline{I}^{1-\alpha}_{a^+} \phi\| < 1$.

Theorem 3.11. *Under assumptions* (H1) *and* (H2)*, then the following assertions hold.*

1. (3.10) *has a unique solution* u^* *in* $C([a,b];J)$.
2. *For each* $u_1 \in C([a,b];J)$*, the sequence* $\{u_n\}$ *defined by* $u_{n+1} = Au_n$ *satisfies*
$$\lim_{n\to\infty} \|u_n - u^*\| = 0 \quad \textit{and} \quad \|u_n - u^*\| \le \frac{\|\overline{I}^{1-\alpha}_{a^+} \phi\|^n}{1 - \|\overline{I}^{1-\alpha}_{a^+} \phi\|} \|u_1 - u_2\|$$
for each $n \in \mathbb{N}$.
3. u^* *is a unique generalized normal solution of* (3.1)-(3.2) *in* $C([a,b];J)$.

Proof. Since $f : (a,b] \times J \to \mathbb{R}$ is a $C^0_{-(1-\alpha)}$-function with function g, by Lemma 3.6 (3), the map A defined in (3.9) maps $C([a,b];J)$ to $C[a,b]$. We prove that the map A maps $C([a,b];J)$ to $C([a,b];J)$. Indeed, if $J = \mathbb{R}_+$ or $\mathbb{R}$, then since f maps $(a,b] \times J$ to J, it is easy to see that the map A maps $C([a,b];J)$ to $C([a,b];J)$. If $J = [c,d]$, then by Lemma 2.5 (i), $\overline{I}^{1-\alpha}_{a^+} g \in C[a,b]$. By (3.19), we have for each $x \in [a,b]$,

$$-\min\{w_0 - c, d - w_0\} \le (\overline{I}^{1-\alpha}_{a^+} g)(x) \le \min\{w_0 - c, d - w_0\}.$$

This implies that for each $x \in [a,b]$,

$$-(w_0 - c) \le (\overline{I}^{1-\alpha}_{a^+} g)(x) \quad \text{and} \quad I^{1-\alpha}_{a^+} g(x) \le d - w_0. \tag{3.20}$$

Let $u \in C([a,b];[c,d])$. By (3.20), we have that for each $x \in [a,b]$,

$$c - w_0 \le (\overline{I}^{1-\alpha}_{a^+}(-g))(x) \le (\overline{I}^{1-\alpha}_{a^+} Fu)(x) \le (\overline{I}^{1-\alpha}_{a^+} g)(x) \le d - w_0.$$

This with (3.9) implies

$$c \le Au(x) \le d \quad \text{for each } x \in [a,b]$$

and $Au \in C([a,b];[c,d])$. Hence, the map A maps $C([a,b];[c,d])$ to $C([a,b];[c,d])$.

Now we prove that A is a contractive map with Lipschitz constant $\|\overline{I}_{a^+}^{1-\alpha}\phi\| < 1$. Indeed, since $\phi \in C_{-(1-\alpha)}(a,b]$, it follows from Lemma 2.5 (i) that $\overline{I}_{a^+}^{1-\alpha}\phi \in C[a,b]$. By (3.10), we have that for $u_1, u_2 \in C([a,b];J)$,

$$\begin{aligned} |(Au_1)(x) - (Au_2)(x)| &= |(\overline{I}_{a^+}^{1-\alpha}(Fu_1 - Fu_2)(x)| \le \overline{I}_{a^+}^{1-\alpha}[\phi|u_1 - u_2|](x) \\ &\le \|\overline{I}_{a^+}^{1-\alpha}\phi\|\|u_1 - u_2\| \quad \text{for each } x \in [a,b]. \end{aligned}$$

Taking maximum on the above inequality yields

$$\|Au_1 - Au_2\| \le \|\overline{I}_{a^+}^{1-\alpha}\phi\|\|u_1 - u_2\|.$$

Since $\|\overline{I}_{a^+}^{1-\alpha}\phi\| < 1$, it follows from Lemma 3.8 that (3.10) has a unique solution u^* in $C([a,b];J)$ and the results (1) and (2) hold. By (2) implying (1) of Theorem 3.7, u^* is a unique generalized normal solution of (3.1)-(3.2) in $C([a,b];[c,d])$ and the result (3) holds. □

Let $L \in (0,\infty)$ and $\phi(x) = L$ for each $x \in [a,b]$, then by Lemma 2.4 (1),

$$(\overline{I}_{a^+}^{1-\alpha}\phi)(x) = \frac{L}{\Gamma(2-\alpha)}(x-a)^{1-\alpha} \quad \text{for each } x \in [a,b]$$

and

$$\|\overline{I}_{a^+}^{1-\alpha}\phi\| = \frac{L(b-a)^{1-\alpha}}{\Gamma(2-\alpha)}. \tag{3.21}$$

If $L \ge \frac{\Gamma(2-\alpha)}{(b-a)^{1-\alpha}}$, then by (3.21) we have $\|\overline{I}_{a^+}^{1-\alpha}\phi\| \ge 1$, so Theorem 3.11 can not be applied. The following result can be used to deal with the case $L \ge \frac{\Gamma(2-\alpha)}{(b-a)^{1-\alpha}}$.

Theorem 3.12. *Assume that condition* (H1) *holds and f satisfies a Lipschitz condition in the second variable with a Lipschitz constant $L > 0$. Then the following assertions hold.*

(1) (3.10) *has a unique solution u^* in $C([a,b];J)$.*
(2) *For each $u_1 \in C([a,b];J)$, the sequence $\{u_n\}$ defined by $u_{n+1} = Au_n$ for each $n \in \mathbb{N}$ satisfies*

$$\|u_{n+1} - u^*\| \le \sum_{i=n}^{\infty} k_i\|u_2 - u_1\| \quad \textit{for each } n \in \mathbb{N},$$

where

$$k_n = \frac{L^n(b-a)^{n(1-\alpha)}}{\Gamma(1+n(1-\alpha))} \quad \textit{for each } n \in \mathbb{N}. \tag{3.22}$$

(3) *u^* is a unique generalized normal solution of* (3.1)-(3.2) *in $C([a,b];J)$.*

Proof. By the proof of Theorem 3.11, we see that the map A defined in (3.9) maps $C([a,b];J)$ to $C([a,b];J)$. Since f satisfies a Lipschitz condition with Lipschitz constant L, by (3.10), we have for $u_1, u_2 \in C([a,b];J)$ and $x \in [a,b]$,

$$|Au_1(x) - Au_2(x)| \le (\overline{I}_{a^+}^{1-\alpha}(Fu_1 - Fu_2))(x) \le (\overline{I}_{a^+}^{1-\alpha}L|u_1 - u_2|)(x).$$

This with Lemma 2.4 (2) implies that for each $x \in [a,b]$,

$$\begin{aligned}|A^2u_1(x)-A^2u_2(x)| &\le |A(Au_1)(x)-A(Au_2)(x)|\\ &\le \big(\overline{I}_{a^+}^{1-\alpha}L|Au_1-Au_2|\big)(x)\\ &\le \big(\overline{I}_{a^+}^{1-\alpha}L\big[(\overline{I}_{a^+}^{1-\alpha}L|u_1-u_2|)\big]\big)(x)\\ &= L^2\big(\overline{I}_{a^+}^{1-\alpha}\big(\overline{I}_{a^+}^{1-\alpha}|u_1-u_2|\big)\big)(x)\\ &= L^2\big(\overline{I}_{a^+}^{2(1-\alpha)}|u_1-u_2|\big)(x).\end{aligned}$$

Repeating the process and using Lemmas 2.4 (1) and (2) imply for $n \in \mathbb{N}$,

$$\begin{aligned}|A^nu_1(x)-A^nu_2(x)| &\le L^n\big(\overline{I}_{a^+}^{n(1-\alpha)}|u_1-u_2|\big)(x)\\ &\le L^n\|u_1-u_2\|(\overline{I}_{a^+}^{n(1-\alpha)}1)(x)\\ &= L^n\|u_1-u_2\|\big(\overline{I}_{a^+}^{n(1-\alpha)}(x-a)^0\big)\\ &= L^n\|u_1-u_2\|\frac{\Gamma(1)(x-a)^{n(1-\alpha)}}{\Gamma(1+n(1-\alpha))}\\ &= \frac{L^n(x-a)^{n(1-\alpha)}}{\Gamma(1+n(1-\alpha))}\|u_1-u_2\|\\ &\le \frac{L^n(b-a)^{n(1-\alpha)}}{\Gamma(1+n(1-\alpha))}\|u_1-u_2\|\\ &= k_n\|u_1-u_2\| \quad \text{for each } x\in[a,b].\end{aligned} \tag{3.23}$$

By (3.22) and (3.18), we have

$$\sum_{n=0}^{\infty}k_n=\sum_{n=0}^{\infty}\frac{[L(b-a)^{1-\alpha}]^n}{\Gamma(1+n(1-\alpha))}=E_{1-\alpha}(L(b-a)^{1-\alpha})<\infty.$$

This with (3.23) implies that A is a Weissinger map with Lipschitz constants $\{k_n\}$. Results (1) and (2) follow from Lemma 3.10. By (2) implying (1) of Theorem 3.7, u^* is a unique generalized normal solution of (3.1)-(3.2) in $C([a,b];J)$ and the result (3) holds. □

4. Examples

Motivated by some population models with heterogeneous environments, we give two examples of the IVPs of the first order FDEs with nonlinearities arising from population models to exhibit the applications of Theorem 3.11.

The first example is to study the existence and uniqueness of nonnegative generalized normal solutions of first order FDEs of the form

$$\big(D_{p,a^+}^{1-\alpha}u\big)(x):=\big(I_{a^+}^{\alpha}(u-w_0)\big)'(x)=\lambda u(x)\big[\psi_1(x)-\psi_2(x)u^r(x)\big] \tag{4.1}$$

for each $x \in (a,b]$ subject to the IC $u(a)=w_0$, where $w_0 \in (0,\infty)$, $\lambda > 0$, $r > 0$ and $\psi_1,\psi_2 : (a,b] \to \mathbb{R}$ are continuous functions. The nonlinearities arise from logistic type population models with heterogeneous environments and the population models of Ricker type of a single species. We remark that the derivative $D_{p,a^+}^{1-\alpha}u$ in (4.1) possibly has biological interpretations but not as growth rates in population models. We refer to [4, 5, 21] for the interpretations of the various terms on the right-side of (4.1).

Example 4.1. Let $\rho > w_0 > 0$ and $r > 0$. If $\psi_1, \psi_2 \in C^0_{-(1-\alpha)}(a,b] \setminus \{0\}$. Then (4.1) has a unique nonnegative generalized normal solution from $[a,b]$ to $[0,\rho]$ for each $\lambda \in (0, \min\{\lambda_1(\rho), \lambda_2(\rho)\})$, where

$$\lambda_1(\rho) = \frac{1}{\|\overline{I}^{1-\alpha}_{a^+}|\psi_1|\| + (1+r)\rho^r \|\overline{I}^{1-\alpha}_{a^+}|\psi_2|\|}, \tag{4.2}$$

$$\lambda_2(\rho) = \frac{\min\{w_0, \rho - w_0\}}{\rho\|\overline{I}^{1-\alpha}_{a^+}|\psi_1|\| + \rho^r \|\overline{I}^{1-\alpha}_{a^+}|\psi_2|\|}. \tag{4.3}$$

Proof. Since $\psi_1, \psi_2 \in C^0_{-(1-\alpha)}(a,b] \setminus \{0\}$, $\|\overline{I}^{1-\alpha}_{a^+}|\psi_1|\| \neq 0$ and $\|\overline{I}^{1-\alpha}_{a^+}|\psi_2|\| \neq 0$. Hence, $\lambda_1(\rho)$ in (4.2) and $\lambda_2(\rho)$ in (4.3) are well defined. Let $\rho > w_0$ and $\lambda > 0$. We define a function $f : (a,b] \times [0,\rho] \to \mathbb{R}$ by

$$f(x,u) = \lambda u\big[\psi_1(x) - \psi_2(x)u^r\big]. \tag{4.4}$$

Since $\psi_1, \psi_2 \in C(a,b]$, $f : (a,b] \times [0,\rho] \to \mathbb{R}$ is continuous. We define a function $\phi : (a,b] \to \mathbb{R}$ by

$$\phi(x) = \lambda\big[|\psi_1(x)| + (1+r)|\psi_2(x)|\rho^r\big]. \tag{4.5}$$

Since $\psi_1, \psi_2 \in C^0_{-(1-\alpha)}(a,b]$, we have $\phi \in C^0_{-(1-\alpha)}(a,b]$ and for each $x \in (a,b]$,

$$\Big|\frac{\partial f(x,u)}{\partial u}\Big| = \Big|\lambda\big[\psi_1(x) - (1+r)\psi_2(x)u^r\big]\Big| \le \phi(x) \quad \text{for each } u \in [c,d].$$

For each $x \in (a,b]$ and $u, v \in [0,\rho]$ with $v < u$, there exists $\xi \in (u,v)$ such that

$$|f(x,u) - f(x,v)| = \big|\frac{\partial f(x,\xi)}{\partial u}(u-v)\big| \le \phi(x)|u-v|.$$

Hence, f satisfies a Lipschitz condition in the second variable with the function $\phi \in C^0_{-(1-\alpha)}(a,b]$. By (4.5), we have

$$(\overline{I}^{1-\alpha}_{a^+}\phi)(x)\| = \lambda\big[(\overline{I}^{1-\alpha}_{a^+}|\psi_1|)(x) + (1+r)\rho^r(\overline{I}^{1-\alpha}_{a^+}|\psi_2|)(x)\big]$$

and by (4.2) and $\lambda < \lambda_1(\rho)$,

$$\|\overline{I}^{1-\alpha}_{a^+}\phi\| \le \lambda\big[\|\overline{I}^{1-\alpha}_{a^+}|\psi_1|\| + (1+r)\rho^r\|\overline{I}^{1-\alpha}_{a^+}|\psi_2|\|\big] < 1.$$

By (4.4), we have for $(x,u) \in (a,b] \times [0,\rho]$,

$$|f(x,u)| \le \lambda u\big[|\psi_1(x)| + |\psi_2(x)|u^r\big] \le \lambda\rho\big[|\psi_1(x)| + |\psi_2(x)|\rho^r\big] = g(x).$$

Since $\psi_1, \psi_2 \in C^0_{-(1-\alpha)}(a,b]$, $g \in C^0_{-(1-\alpha)}(a,b]$. Since $\lambda \in (0, \lambda_2(\rho))$, we have

$$\begin{aligned}
\|\overline{I}^{1-\alpha}_{a^+}g\| &\le \lambda\rho\big[(\overline{I}^{1-\alpha}_{a^+}|\psi_1|)(x) + (\overline{I}^{1-\alpha}_{a^+}|\psi_2|)(x)\rho^r\big] \\
&\le \lambda\rho\big[\|\overline{I}^{1-\alpha}_{a^+}|\psi_1|\| + \|\overline{I}^{1-\alpha}_{a^+}|\psi_2|\|\rho^r\big] \\
&\le \lambda_2(\rho)\rho\big[\|\overline{I}^{1-\alpha}_{a^+}|\psi_1|\| + \|\overline{I}^{1-\alpha}_{a^+}|\psi_2|\|\rho^r\big] \\
&\le \min\{w_0, \rho - w_0\}.
\end{aligned}$$

The result follows from Theorem 3.11 with $J = [0,\rho]$. □

The second example is to study the existence and uniqueness of nonnegative generalized normal solutions in $C[a,b]$ of the IVP for the nonlinear first order FDE

$$\big(D^{1-\alpha}_{p,a^+}u\big)(x) = \big(I^{\alpha}_{a^+}(u - w_0)\big)'(x) = \xi(x) + \varphi(x)u(x)e^{r - \delta u(x) - \frac{m}{1+ku(x)}} \tag{4.6}$$

for each $x \in (a,b]$ subject to the IC $u(a) = w_0$, where $w_0, m \in \mathbb{R}_+$, $r, \delta, k > 0$ and $\xi, \varphi : (a,b] \to \mathbb{R}_+$ are continuous functions.

The FDE (4.6) is motivated by the population models with the growth rates of Ricker type governed by difference equations and first order ordinary differential equations in [1, 25, 26, 36]. The existence and uniqueness of generalized normal solutions in $L^p([a,b];J)$ of the IVP (4.6) were studied in [19, Theorem 12], where $\xi \in L^p_+[a,b]$ and $\phi \in L^\infty[a,b]$.

Example 4.2. Assume that $\xi, \varphi \in C^0_{-(1-\alpha),+}(a,b]$ and ϕ satisfies

$$\|\overline{I}^{1-\alpha}_{a^+}\varphi\| < e^{-r}\big(1+\frac{m}{4}\big)^{-1}. \tag{4.7}$$

Then (4.6) has a unique generalized normal solution u^* in $C_+[a,b]$.

Proof. We define a function $f:[a,b]\times\mathbb{R}_+\to\mathbb{R}_+$ by

$$f(x,u)=\xi(x)+\varphi(x)h(u), \tag{4.8}$$

where $h:\mathbb{R}_+\to\mathbb{R}_+$ is defined by

$$h(u)=ue^{r-\delta u-\frac{m}{1+ku}}. \tag{4.9}$$

Since $\xi,\varphi\in C(a,b]$, $f:(a,b]\times\mathbb{R}_+\to\mathbb{R}_+$ is continuous. It is easy to see that

$$f(x,u)=\xi(x)+\varphi(x)ue^{r-\delta u-\frac{m}{1+ku}}\le g_\rho(x)\quad\text{for each } x\in(a,b] \text{ and } u\in[0,\rho],$$

where $g_\rho(x)=\xi(x)+\varphi(x)ue^r$ for each $x\in(a,b]$. Since $\xi,\varphi\in C^0_{-(1-\alpha),+}(a,b]$, we obtain $g_\rho\in C^0_{-(1-\alpha),+}(a,b]$. Hence, $f:(a,b]\times\mathbb{R}_+\to\mathbb{R}_+$ is a $C^0_{-(1-\alpha)}$-continuous function. By (4.9), we have

$$h'(u)=e^{r-\delta u-\frac{m}{1+ku}}\Big[1-\delta u+\frac{mku}{(1+ku)^2}\Big]\quad\text{for each } u\in\mathbb{R}_+.$$

It follows that for each $u\in\mathbb{R}_+$,

$$h'(u)\le e^{r-\delta u}|1-\delta u|+e^r\frac{mku}{(1+ku)^2}\le e^r\Big[e^{-\delta u}|1-\delta u|+\frac{m}{4}\Big]. \tag{4.10}$$

We show that

$$e^{-\delta u}|1-\delta u|\le 1\quad\text{for each } u\in\mathbb{R}_+. \tag{4.11}$$

Indeed, (4.11) holds if and only if

$$-e^{\delta u}\le 1-\delta u\le e^{\delta u}\quad\text{for each } u\in\mathbb{R}_+. \tag{4.12}$$

Since $e^{\delta u}\ge 1+\delta u$ for each $u\in\mathbb{R}_+$, we have

$$e^{\delta u}+1-\delta u\ge 1+\delta u+1-\delta u=2\ge 0\quad\text{for each } u\in\mathbb{R}_+$$

and the first inequality of (4.12) holds. It is obvious that the second inequality of (4.12) holds. By (4.10) and (4.11), we have

$$h'(u)\le e^r\big[e^{-\delta u}|1-\delta u|+\frac{mk}{4k}\big]\le e^r\big[1+\frac{m}{4}\big] \tag{4.13}$$

By (4.8) and (4.13), for $u,v\in\mathbb{R}_+$, there exists $\mu\in[u,v]$ such that for each $x\in(a,b]$,

$$|f(x,u)-f(x,v)|=\varphi(x)|h(u)-h(v)|=\varphi(x)h'(\mu)|u-v|\le\phi(x)|u-v|,$$

where

$$\phi(x)=\varphi(x)e^r\big(1+\frac{m}{4}\big)\quad\text{for each } x\in(a,b]. \tag{4.14}$$

Since $\varphi \in C^0_{-(1-\alpha),+}(a,b]$, we obtain $\phi \in C^0_{-(1-\alpha),+}(a,b]$. Hence, $f:(a,b]\times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies a Lipschitz condition in the second variable with function $\phi \in C_{-(1-\alpha)}(a,b]$. By (4.7) and (4.14), we have

$$\|\overline{I}^{1-\alpha}_{a^+}\phi\| = e^r\big(1+\frac{m}{4}\big)\|\overline{I}^{1-\alpha}_{a^+}\varphi\| < 1.$$

The results follow from Theorem 3.11 with $J=\mathbb{R}_+$. □

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References

[1] A. S. Ackleh, P. L. Salceanu; Competitive exclusion and coexistence in n-species Ricker model, *J. Biol. Dyn.*, **9** (2015), Suppl. 1, 321-331.

[2] L. C. Becker, T. A. Burton, I. K. Purnaras; An inversion of a fractional differential equation and fixed points, *Nonlinear Dynam. and Syst. Theory*, **15** (2015) (3), 242-271.

[3] L. C. Becker, T. A. Burton, I. K. Purnaras; Complementary equations: A fractional differential equation and a Volterra integral equation, *Electron. J. Qual. Theory Differ. Equ.*, (2015), No 12, 24 pp.

[4] R. S. Cantrell, C. Cosner; Diffusive logistic equations with indefinite weights: population models in disrupted environments, *Proc. Roy. Soc. Edinburgh*, Sect. A, **112** (1989), 293-318.

[5] R. S. Cantrell, C. Cosner; The effects of spatial heterogeneity in population dynamics, *J. Math. Biol.*, **29** (1991), 315-338.

[6] D. Delbosco, L. Rodino; Existence and uniqueness for a nonlinear fractional differential equation, *J. Math. Anal. Appl.*, **204** (1996), 609-625.

[7] K. Diethelm; *The analysis of fractional differential equations. An application-oriented exposition using differential operators of Caputo type.*, Lecture Notes in Mathematics, No. 2004. Springer-Verlag, Berlin, 2010.

[8] K. Diethelm, N. J. Ford; Analysis of fractional differential equations, *J. Math. Anal. Appl.* **265** (2002), 229-248.

[9] P. W. Eloe, T. Masthay; Initial value problems for Caputo fractional differential equations, *J. Fract. Calc. Appl.* **9** (2018), 178-195.

[10] A. Granas, J. Dugundji; Fixed Point Theory, Springer, New York, 2010.

[11] A. A. Kilbas, M. Rivero, J. J. Trujillo; Existence and uniqueness theorems for differential equations of fractional order in weighted spaces of continuous functions, *Fract. Calc. Appl. Anal.*, **6** (2003) (4), 363-399.

[12] V. Lakshmikantham, S. Leela, J. Vasundhara Devi; *Theory of fractional dynamic system*, Cambridge Scientific Publishers, 2009.

[13] V. Lakshmikantham, A. S. Vatsala; Theory of fractional differential inequalities and applications, *Commun. Appl. Anal.*, **11** (2007), 395-402.

[14] V. Lakshmikantham, A. S. Vatsala, Basic theory of fractional differential equations, *Nonlinear Anal.*, **69** (2008), 2677-2682.

[15] V. Lakshmikantham, A. S. Vatsala; General uniqueness and monotone iterative technique for fractional differential equations, *Appl. Math. Lett.*, **21** (2008), 828-834.

[16] K. Q. Lan; Equivalence of higher order linear Riemann-Liouville fractional differential and integral equations, *Proc. Amer. Math. Soc.*, **148** (2020) (12), 5225-5234.

[17] K. Q. Lan; Linear first order Riemann-Liouville fractional differential and perturbed Abel's integral equations, *J. Differential Equations*, **306** (2022), 28-59.

[18] K. Q. Lan; Nonzero nonnegative solutions of population models of Ricker types and theory of limit inferiors and superiors, *Math. Meth. Appl. Sci.*, (2022), 1-43, DOI 10.1002/mma.8724

[19] K. Q. Lan; Existence and uniqueness of solutions of nonlinear Cauchy-type problems for first-order fractional differential equations, *Math. Meth. Appl. Sci.*, (2023), 1-21, DOI 10.1002/mma.9669.

[20] K. Q. Lan; Linear higher-order fractional differential and integral equations, *Electr. J. Differ. Equ.* (2023), No. 01, pp. 1-20.

[21] K. Q. Lan; A basic theory for initial value problems of first order ordinary differential equations with L^p-Carathéodory functions and applications, *J. Differential Equations*, **386** (2024), 368-403.

[22] K. Q. Lan; Generalizations of Riemann-Liouville fractional integrals and applications, *Math. Meth. Appl. Sci.* (2024), 1-38, DOI 10.1002/mma.10183.

[23] K. Q. Lan; Initial value problems of first order fractional differential equations via monotone iterative techniques, *Discrete Contin. Dyn. Syst.* Series S, (2024), DOI: 10.3934/dcdss.2024169.

[24] K. Q. Lan, J. R. L. Webb; A new Bihari inequality and initial value problems of first order fractional differential equations, *Fract. Calc. Appl. Anal.*, **26** (2023), 962-988. DOI: 10.1007/s13540-023-00152-5.

[25] F. Lutscher, T. Iljon; Competition, facilitation and the Allee effect, *Oikos* **122** (2013), 621-631.

[26] W. E. Ricker; Stock and recruitment, *Journal of the Fisheries Research Board of Canada*, **11** (1954), 559-623.

[27] S. G. Samko, A. A. Kilbas, O. I. Marichev; *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, Yverdon, 1993.

[28] C. C. Tisdell, Basic existence and a priori bound results for solutions to systems of boundary value problems for fractional differential equations, *Electron. J. Differential Equations*, 2016, Paper No. 84, 9 pp.

[29] J. R. L. Webb; Weakly singular Gronwall inequalities and applications to fractional differential equations, *J. Math. Anal. Appl.*, **471** (2019), 692-711.

[30] J. R. L. Webb; Initial value problems for Caputo fractional equations with singular nonlinearities, *Electr. J. Differ. Equ.* (2019), No. 117, 1-32.

[31] J. R. L. Webb; Compactness of nonlinear integral operators with discontinuous and with singular kernels, *J. Math. Anal. Appl.*, **509** (2022), 126000.

[32] J. R. L. Webb, K. Q. Lan; Fractional differential equations of Bagley-Torvik and Langevin type, *Fract. Calc. Appl. Anal.*, (2024), No. 4, 1639-1669. DOI: 10.1007/s13540-024-00292-2

[33] J. Weissinger; Zur theorie und anwendung des iterationsverfahrens. *Math. Nachr.* **8** (1952), 193-212.

[34] C. Wu; The continuous dependence of global solutions to Caputo fractional order systems, *J. Integral Equations Appl.*, **33** (2021) (3), 371-379.

[35] C. Wu, X. Z. Liu; The continuation of solutions to systems of Caputo fractional order differential equations, *Fract. Calc. Appl. Anal.* **23** (2020) (2), 591-599.

[36] C. W. Zhen, F. Q. Zhang, J. Q. Li; Stability analysis of a population model with maturation delay and Ricker birth function, *Abst. Appl. Anal.* 2014, Article ID 136707.

[37] T. Zhu; Weakly singular integral inequalities and global solutions for fractional differential equations of Riemann-Liouville type, *Mediterr. J. Math.* **18** (2021), no. 5, Paper No. 184, 17pp.

[38] T. Zhu; Attractivity of solutions of Riemann-Liouville fractional differential equations, *Electron. J. Qual. Theory Differ. Equ.*, (2022), Paper No. 52, 12pp.

[39] T. Zhu; Global attractivity for fractional differential equations of Riemann-Liouville type, *Fract. Calc. Appl. Anal.*, **26** (2023), 2264-2280.

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