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NORMALIZED SOLUTIONS FOR BIHARMONIC SCHRÖDINGER EQUATIONS WITH POTENTIAL AND GENERAL NONLINEARITY

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Abstract. We study the existence and non-existence of normalized solutions to the biharmonic equation

$$\Delta^2 u - \Delta u + V(x)u + \lambda u = f(u) \quad \text{in } \mathbb{R}^N.$$

where $0 \not\equiv V(x) \leq V_\infty := \lim_{|x|\to\infty} V(x) \in (-\infty, +\infty]$ and $f \in C(\mathbb{R}, \mathbb{R})$ is a nonlinearity. For the trapping case of $V_\infty = +\infty$, under some suitable assumptions on f, we prove that there exists a ground state as a global minimizer of the corresponding energy functional. For the case of $V_\infty < +\infty$, under some other assumptions on f, we prove that there exists $\bar{\alpha} \geq 0$ such that a global minimizer exists if $\alpha > \bar{\alpha}$ while no global minimizer exists if $\alpha < \bar{\alpha}$. Moreover, the size of $\bar{\alpha}$ is also explored, depending on the potential V.

1. Introduction

Our starting point is the biharmonic nonlinear Schrödinger (NLS) equations

$$\begin{gathered} i\psi_t - \gamma\Delta^2\psi + \beta\Delta\psi + |\psi|^{p-2}\psi = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^N, \\ \psi(x,t) = \psi_0(x), \end{gathered} \tag{1.1}$$

where $\psi(x,t) : \mathbb{R}^N \times [0,T) \to \mathbb{C}$ is a wave function, $\gamma, p > 0$ and $\beta \in \mathbb{R}$. This equation has been introduced by Karpman and Shagalov in [11, 12] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity, see also [6]. It has also been used to describe the motion of a vortex filament in an incompressible fluid [7]. Equation (1.1) is Hamiltonian, and the mass and energy are conserved by the flow.

Equation (1.1) has an important class of special solutions, i.e. the standing waves. A standing wave is a solution of the form $\psi(t,x) = e^{i\lambda t}u(x)$, where $\lambda \in \mathbb{R}$ is a frequency. Then the real valued function u satisfies the elliptic equation

$$\gamma\Delta^2 u - \beta\Delta u + \lambda u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

To study the solutions of (1.2), one can consider λ to be an unknown of the problem. Then λ appears as a Lagrange multiplier and L^2-norms of solutions are

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prescribed, i.e.

$$\int_{\mathbb{R}^N} |u|^2 dx = \alpha > 0,$$

which are usually called normalized solutions. This study seems to be particularly meaningful from the physical point of view, since standing waves of (1.1) conserve their mass along time. Normalized solutions to Eq. (1.2) can be identified with critical points of the energy functional $E_{\gamma,\beta} : H^2(\mathbb{R}^N) \to \mathbb{R}$ given by

$$E_{\gamma,\beta}(u) = \frac{1}{2}\int_{\mathbb{R}^N} (\gamma|\Delta u|^2 + \beta|\nabla u|^2)dx - \int_{\mathbb{R}^N} F(u)dx,$$

on the set

$$S_{\gamma,\beta}(\alpha) := \big\{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} u^2 dx = \alpha\big\},$$

The study of normalized solutions to biharmonic NLS equations (1.2) has attracted much attention in recent years. We refer the readers to [1, 2, 3, 5, 14, 15]. More precisely, when $\gamma > 0, \beta \le 0$ and $2 < p < p^* := 2 + \frac{8}{N}$, Bonheure et al. [2] proved the existence of a global minimizer by using the minimization method. Subsequently, when $\gamma > 0, \beta \le 0$ and $p^* < p < 4^* := \frac{2N}{N-4}$, the existence of a ground state and the multiplicity of radial solutions were obtained by the Pohozaev constraint method in [1]. Luo et al. [15] proved the existence of a global minimizer when $\gamma = 1, \beta \in \mathbb{R}$ and $2 < p \le p^*$ by using the profile decomposition of bounded sequences in $H^2(\mathbb{R}^N)$ established in [21]. In [3], Boussaid et al. studied (1.2) with $\gamma > 0$, $\beta > 0$ and $2 < p \le p^*$, which improved the results in [15] by relaxing the extra restriction on α and β. Very recently, Luo and Yang [14] obtained the existence of two normalized solutions for (1.2) with $\gamma > 0, \beta > 0$ and $p^* < p \le 4^*$.

To the best of our knowledge, there seems to be no any results on normalized solutions to biharmonic NLS equations with a potential and a general nonlinearity in existing literature so far. Inspired by this, in this paper we focus on this case and explore the effect of the potential on the number of normalized solutions. Specifically, for $\alpha > 0$, the problem considered in this study is as follows:

$$\begin{gathered} \Delta^2 u - \Delta u + V(x)u + \lambda u = f(u) \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 dx = \alpha > 0, \end{gathered} \tag{1.3}$$

where $N \ge 5$, V and f satisfy the following assumptions:

- (A1) $f \in C(\mathbb{R}, \mathbb{R})$, $f(0) = 0$ and there exists $\zeta > 0$ such that $F(\zeta) > 0$, where $F(s) = \int_0^s f(t)dt$ for $s \in \mathbb{R}$;
- (A2) $\lim_{s\to 0} f(s)/s = 0$ and $\limsup_{|s|\to\infty} |f(s)|/|s|^{4^*-1} < \infty$;
- (A3) $\limsup_{|s|\to\infty} f(s)s/|s|^{p^*} \le 0$;
- (A4) $f(s)s - 2F(s) > 0$ for $s \ne 0$;
- (A5) $F(\theta s) \ge \theta^{\frac{2(N+5)}{N+1}} F(s)$ for $\theta > 1$ and $s > 0$.
- (A6) $0 \ne V(x) \le V_\infty := \lim_{|x|\to\infty} V(x) \in (-\infty, +\infty]$;
- (A7) $\varepsilon^4 V(\varepsilon x) \le V(x)$ for $\varepsilon \in (0, 1)$ and $x \in \mathbb{R}^N$.

It is clear that normalized solutions to (1.3) correspond to critical points of the energy functional $I : \mathcal{H} \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + V(x)u^2)dx - \int_{\mathbb{R}^N} F(u)dx,$$

on the constraint

$$S(\alpha) := \Big\{ u \in \mathcal{H} : \int_{\mathbb{R}^N} u^2 dx = \alpha \Big\},$$

where

$$\mathcal{H} := \big\{ u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x) u^2 dx < \infty \big\}.$$

Note that $\mathcal{H}$ is a Hilbert space, endowed with the norm

$$\|u\|_{\mathcal{H}} := \Big[\int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + V(x) u^2) dx \Big]^{1/2}.$$

To find solutions of (1.3), we consider the minimization problem

$$m_\alpha := \inf_{u \in S(\alpha)} I(u).$$

First of all, we study the case of $V_\infty = +\infty$ and obtain the following result.

Theorem 1.1. *Assume that conditions* (A1)–(A3) *hold and* $V \in C^1(\mathbb{R}^N)$ *satisfies condition* (A6) *with* $V_\infty = +\infty$. *Then* m_α *is attained by* $u \in S(\alpha)$ *for any* $\alpha > 0$, *which is a ground state of problem* (1.3).

Next, we turn to study the case of $V_\infty < +\infty$. Following from [17], we define

$$\bar{\alpha} = \inf\{\alpha > 0 : m_\alpha < 0\}, \tag{1.4}$$

and need the limit

$$\lim_{s \to 0} \frac{F(s)}{|s|^{2+\frac{8}{N}}} < +\infty. \tag{1.5}$$

Then we have the following results.

Theorem 1.2. *Assume that conditions* (A1)–(A4) *hold and* $V \in C^1(\mathbb{R}^N)$ *satisfies condition* (A6) *with* $V_\infty < +\infty$. *Then there exists a constant* $\bar{\alpha} \geq 0$ *such that* m_α *is attained by* $\overline{u} \in S(\alpha)$ *for* $\alpha > \bar{\alpha}$, *which is a ground state of problem* (1.3).

Theorem 1.3. *Assume that conditions* (A1)–(A3), (A5) *hold and* V *satisfies conditions* (A6) *with* $V_\infty < +\infty$ *and* (A7). *If in addition condition* (1.5) *holds and* $\bar{\alpha} \geq 0$ *is uniquely determined, then the following conclusions are true.*

(i) *If* $\alpha > \bar{\alpha}$, *there exists a global minimizer with respect to* m_α;
(ii) *If* $0 < \alpha < \bar{\alpha}$, *there is no global minimizer with respect to* m_α.

Remark 1.4. It is meaningful to point out that the condition (A4) is weaker than the well known Ambrosetti-Rabinowitz type condition:

(AR) There exists $\alpha > 2$ such that $f(s)s \geq \alpha F(s) > 0$, for all $s \neq 0$.

To obtain more information, we need a stronger condition (A5). In deed, by (A5),

$$\frac{F(\theta s)}{\theta^2 s^2} > \frac{F(s)}{s^2}, \quad \forall \theta > 1, \ s \neq 0.$$

This implies that $s \mapsto \frac{F(s)}{s^2}$ is strictly increasing for $s > 0$ and strictly decreasing for $s < 0$. Then we have

$$\frac{d}{ds}\Big(\frac{F(s)}{s^2}\Big) = \frac{f(s)s - 2F(s)}{s^3} > 0, \quad \text{for } s > 0,$$
$$\frac{d}{ds}\Big(\frac{F(s)}{s^2}\Big) = \frac{f(s)s - 2F(s)}{s^3} > 0, \quad \text{for } s < 0,$$

which means that condition (A4) holds.

Moreover, we have the following result.

Theorem 1.5. *Assume that conditions* (A1)–(A3), (A5) *hold and* V *satisfies conditions* (A6) *with* $V_\infty < +\infty$ *and* (A7). *If in addition condition* (1.5) *holds, then the following conclusions hold.*

(i) *Assume that there exists an* $s_0 > 0$ *such that*$f(s) \geq 0$ *in* $[0, s_0]$ *and*

$$\inf_{\|u\|_2=1} \int_{\mathbb{R}^N} \left(|\Delta u|^2 + |\nabla u|^2 + V(x)u^2\right) dx < 0, \tag{1.6}$$

Then $\bar{\alpha} = 0$.

(ii) *If* $V \in L^{N/4}(\mathbb{R}^N)$ *satisfies* $\|V\|_{N/4} < \mathcal{S}$, *then we have* $\bar{\alpha} > 0$. *Here,* $\mathcal{S}$ *is defined as a Sobolev constant, i.e.,*

$$\mathcal{S} := \inf_{u \in D^{2,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\|\Delta u\|_2^2}{\|u\|_{4^*}^2}.$$

We wish to point out that normalized solutions of NLS equations with potential and various types of nonlinearities has been studied by Ikoma and Miyamoto [8] and Yang et al. [20] recently. In this article our results can been viewed as an extension to the case of biharmonic NLS equations.

This article is structured as follows. We introduce some preliminary results in Section 2. We give the proofs of Theorems 1.1-1.5 in Section 3.

2. Preliminary results

For sake of convenience, we set

$$A(u) := \int_{\mathbb{R}^N} |\Delta u|^2 dx \quad \text{and} \quad B(u) := \int_{\mathbb{R}^N} |\nabla u|^2 dx.$$

Then the energy functional I is rewritten as

$$I(u) = \frac{1}{2}A(u) + \frac{1}{2}B(u) + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 dx - \int_{\mathbb{R}^N} F(u) dx.$$

In view of the Gagliardo-Nirenberg inequality [14], for $2 < p < 4^*$, there exists a constant $\mathcal{C}_{N,p} > 0$ such that

$$\|u\|_p^p \leq \mathcal{C}_{N,p}^p \|\Delta u\|_2^{p\gamma_p} \|u\|_2^{p(1-\gamma_p)} \quad \text{for } u \in H^2(\mathbb{R}^N),$$

where $\gamma_p := \frac{N(p-2)}{4p}$.

Lemma 2.1. *Assume that conditions* (A1)–(A3) *hold. Then the following statements hold:*

(i) *For any bounded sequence* $\{u_n\} \subset \mathcal{H}$, *if* $\lim_{n\to\infty} \|u_n\|_\infty = 0$, *then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(u_n) dx = 0,$$

and if $\lim_{n\to\infty} \|u_n\|_{2+\frac{8}{N}} = 0$, *then*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} F(u_n) dx \leq 0.$$

(ii) *For any* $\alpha > 0$, *the energy functional* I *is bounded from below and coercive on* $S(\alpha)$.

Proof. (i) It can be found in [9, Lemma 2.1 (i)].

(ii) According to [9, Lemma 2.1 (ii)], since $V_0 := \inf_{x\in\mathbb{R}^N} V(x) > -\infty$, by using the Poincaré inequality, there exists a constant $C = C(f,\alpha,V_0) > 0$ such that

$$\begin{aligned}I(u) &= \frac{1}{2}A(u) + \frac{1}{2}B(u) + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2dx - \int_{\mathbb{R}^N} F(u)dx \\ &\geq \frac{1}{2}\|u\|_{\mathcal{H}}^2 - C(f,\alpha,V_0) \text{ for any } u \in S(\alpha),\end{aligned}$$

which implies that the functional I is bounded from below and coercive on $S(\alpha)$ for any $\alpha > 0$. □

Lemma 2.2. *Assume that conditions* (A1)–(A3) *hold. Let $\{u_n\}$ be a bounded sequence in $\mathcal{H}$ such that $u_n \rightharpoonup u$ in $\mathcal{H}$. Then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |F(u_n-u)+F(u)-F(u_n)|dx = 0.$$

The proof of the above lemma is similar to [10, Lemma 3.2], so we omit it here.

Lemma 2.3 ([13, Lemma I.1])**.** *Let $\{u_n\}$ be a bounded sequence in $\mathcal{H}$ satisfying*

$$\sup_{z\in\mathbb{R}^N}\int_{B(z,1)} |u_n|^2dx \to 0 \quad \text{as } n\to\infty.$$

Then for $q > 2$, we have $\|u_n\|_q \to 0$ as $n \to \infty$.

Lemma 2.4 ([16])**.** *Assume that $V_\infty = +\infty$. Then the embedding $\mathcal{H} \hookrightarrow L^q(\mathbb{R}^N)$ is compact for all $2 \leq q < 4^*$.*

3. Proof of the main theorems

3.1. Case $V_\infty = +\infty$.

Proof of Theorem 1.1. According to Lemma 2.1 (ii), there exists a minimizing sequence $\{u_n\} \subset S(\alpha)$ of I with respect to m_α such that $m_\alpha = \lim_{n\to\infty} I(u_n)$. Clearly, $\{u_n\}$ is bounded in $\mathcal{H}$. By Lemma 2.4, there exists $\bar{u} \in \mathcal{H}$ such that

$$\begin{aligned}u_n &\rightharpoonup \bar{u} \quad \text{in } \mathcal{H},\\ u_n &\to \bar{u} \quad \text{in } L^2(\mathbb{R}^N),\\ u_n &\to \bar{u} \quad \text{a.e. in } \mathbb{R}^N.\end{aligned}$$

This shows that $\bar{u} \in S(\alpha)$. Using Lemma 2.2 and the weak lower semi-continuity of the $\mathcal{H}$-norm, it follows that

$$I(\bar{u}) \leq \lim_{n\to\infty} I(u_n) = m_\alpha \leq I(\bar{u}),$$

which implies that $m_\alpha = I(\bar{u})$ and $u_n \to \bar{u}$ in $\mathcal{H}$ as $n \to \infty$. Therefore, $\bar{u} \in S(\alpha)$ is a ground state solution of problem (1.3). The proof is complete. □

3.2. **Case of $V_\infty < +\infty$.** Without loss of generality, in this subsection, we may assume that $V_\infty = 0$ in condition $(V1)$. If not, we may replace $(V(x), \lambda)$ by

$$(\widetilde{V}(x), \widetilde{\lambda}) := (V(x) - V_\infty, \lambda + V_\infty),$$

and problem (1.3) becomes the equivalent problem

$$\begin{gathered} \Delta^2 u - \Delta u + \widetilde{V}(x)u + \widetilde{\lambda}u = f(u), \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 dx = \alpha. \end{gathered}$$

Lemma 3.1. *Assume that conditions (A1)–(A3), (A6) with $V_\infty = 0$ hold. Then we have*

- (i) *$m_\alpha \le m_\alpha^\infty \le 0$ for any $\alpha \ge 0$,*
- (ii) *$m_\alpha \le m_\beta + m_{\alpha-\beta}$ for $\alpha > \beta > 0$,*
- (iii) *m_α^∞ and m_α are non-increasing on $\alpha \ge 0$,*
- (iv) *$\alpha \mapsto m_\alpha$ is continuous for $\alpha > 0$.*

The proof of the above lemma is almost the same as [8, Lemma 2.5], se we omit it here.

Lemma 3.2. *Assume that conditions* (A1)–(A3), (A6) *hold. Then there exists $\alpha^* > 0$ such that $m_\alpha < 0$ for all $\alpha > \alpha^*$. Assume that in addition condition* (1.5) *holds, for $\alpha > 0$ small enough, we have $m_\alpha = 0$.*

Proof. From condition (A1), there exists $u \in H^2(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} F(u)dx > 0$. For any $\alpha > 0$, set $u_\alpha := u(\alpha^{-1/N} \cdot \|u\|_2^{2/N} \cdot x) \in S_\alpha$. Since

$$\begin{aligned} &I^\infty(u_\alpha) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\Delta u_\alpha|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_\alpha|^2 dx - \int_{\mathbb{R}^N} F(u_\alpha)dx \\ &= \frac{\alpha^{\frac{N-4}{N}}}{2\|u\|_2^{2(N-4)/N}} \int_{\mathbb{R}^N} |\Delta u|^2 dx + \frac{\alpha^{\frac{N-2}{N}}}{2\|u\|_2^{2(N-2)/N}} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\alpha}{\|u\|_2^2}\int_{\mathbb{R}^N} F(u)dx \\ &=: C_1\alpha^{\frac{N-4}{N}} + C_2\alpha^{\frac{N-2}{N}} - C_3\alpha =: g(\alpha), \end{aligned}$$

it follows that $m_\alpha^\infty \le I^\infty(u_\alpha) = g(\alpha) < 0$ for sufficiently large $\alpha > 0$.

By condition (1.5), there exists $C_f > 0$ such that $F(s) \le C_f|s|^{2+8/N}$ for any $s \in \mathbb{R}$. By the Gagliardo-Nirenberg inequality,

$$\int_{\mathbb{R}^N} F(u)dx \le C_f \mathcal{C}_{N,p^*}^{p^*} \alpha^{8/N} \|\Delta u\|_2^2 \quad \text{for all } u \in S_\alpha.$$

For any $\alpha > 0$ small enough such that $C_f \mathcal{C}_{N,p^*}^{p^*} \alpha^{8/N} \le 1/4$, we have

$$I(u) \ge \frac{1}{4}\|\Delta u\|_2^2 > 0,$$

and thus $m_\alpha \ge 0$. It follows that $m_\alpha = 0$ for $\alpha > 0$ small enough. The proof is complete. □

For $u \in S(1)$, we set

$$u^t(x) := t^{N/2} u(tx).$$

It is clear that $u^t \in S(1)$ and $\sqrt{\alpha}u^t \in S(\alpha)$. Define the fibering map $g_{\alpha,u}(t) : (0,+\infty) \to \mathbb{R}$ by

$$\begin{aligned} g_{\alpha,u}(t) &:= I(\sqrt{\alpha}u^t) \\ &= \frac{\alpha t^2}{2}A(u) + \frac{\alpha^2 t}{2}B(u) + \frac{\alpha}{2}\int_{\mathbb{R}^N} V(x/t)|u|^2dx - \frac{1}{t^N}\int_{\mathbb{R}^N} F\left(\sqrt{\alpha t^N}u\right)dx. \end{aligned}$$

By calculating the first derivative of $g_{\alpha,u}$, we have

$$\begin{aligned} g'_{\alpha,u}(t) &= t\alpha A(u) + \alpha B(u) - \frac{\alpha}{2t^2}\int_{\mathbb{R}^N} \langle \nabla V(x/t), x\rangle |u|^2dx \\ &\quad - \frac{N}{2t^{N+1}}\int_{\mathbb{R}^N}\left[f\left(\sqrt{\alpha t^N}u\right)\sqrt{\alpha t^N}u - 2F\left(\sqrt{\alpha t^N}u\right)\right]dx, \end{aligned}$$

and

$$\begin{aligned} g'_{\alpha,u}(1) &= \alpha A(u) + \alpha B(u) - \frac{\alpha}{2}\int_{\mathbb{R}^N} \langle \nabla V(x), x\rangle u^2dx \\ &\quad - \frac{N}{2}\int_{\mathbb{R}^N}[f(\sqrt{\alpha}u)\sqrt{\alpha}u - 2F(\sqrt{\alpha}u)]dx \\ &=: Q_\alpha(u). \end{aligned}$$

Then we have the Pohozaev identity.

Lemma 3.3. *Let $u \in S(1)$ be a critical point of the functional I restricted to $S(1)$. Then*

$$2A(u) + B(u) - \frac{1}{2}\int_{\mathbb{R}^N}\langle \nabla V(x), x\rangle u^2dx - \frac{N}{2}\int_{\mathbb{R}^N}[f(u)u - 2F(u)]dx = 0. \tag{3.1}$$

Proof. Since $u \in S(1)$ is a critical point of I restricted to $S(1)$, there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$\Delta^2 u - \Delta u + V(x)u + \lambda u = f(u). \tag{3.2}$$

Multiplying (3.2) by u and integrating, we obtain

$$\int_{\mathbb{R}^N}(|\Delta u|^2 + |\nabla u|^2 + (V(x)+\lambda)u^2)dx = \int_{\mathbb{R}^N} f(u)udx.$$

From [19, Lemma 2.2], we have

$$\begin{aligned} &\frac{N-4}{2}\int_{\mathbb{R}^N}|\Delta u|^2dx + \frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx + \frac{N}{2}\int_{\mathbb{R}^N}(V(x) + N\langle \nabla V(x), x\rangle + \lambda)u^2dx \\ &= N\int_{\mathbb{R}^N}F(u)dx. \end{aligned}$$

Therefore, combining these two equalities above, we obtain (3.1). The proof is complete. □

Now, we define the set

$$\mathcal{M}_\alpha := \{u \in S(1) : Q_\alpha(u) = 0\},$$

and the functional $h(\alpha) : [0,+\infty) \to \mathbb{R}$ by

$$h(\alpha) := \frac{\alpha}{2}A(u) + \frac{\alpha}{2}B(u) + \frac{\alpha}{2}\int_{\mathbb{R}^N} V(x)u^2dx - \int_{\mathbb{R}^N} F(\sqrt{\alpha}u)dx$$

for $u \in \mathcal{M}_\alpha$. Then we have the following lemmas.

Lemma 3.4. *For each $\alpha > 0$, it holds*

$$\widetilde{m}_\alpha := \inf_{u\in\mathcal{M}_\alpha} I(\sqrt{\alpha}u) = m_\alpha = \inf_{u\in S(\alpha)} I(u).$$

Proof. According to the definition of $\widetilde{m}_\alpha$, obviously $m_\alpha \le \widetilde{m}_\alpha$. In addition, for any $u \in S(\alpha)$ with $Q_\alpha(\frac{1}{\sqrt{\alpha}}u) = 0$, we have $\widetilde{m}_\alpha \le I(\sqrt{\alpha}\frac{1}{\sqrt{\alpha}}u) = I(u)$. Taking the infimum, we obtain $\widetilde{m}_\alpha \le m_\alpha$. Therefore, $\widetilde{m}_\alpha = m_\alpha$. The proof is complete. □

Lemma 3.5. *Assume that conditions* (A1)–(A4) *hold. In addition, let $V \in C^1(\mathbb{R}^N)$ satisfy condition (A6). Then the following conclusions are true:*

(i) *the function $\alpha \mapsto \frac{h(\alpha)}{\alpha}$ is decreasing for all $\alpha > 0$,*
(ii) *if m_α is achieved for $\alpha > 0$, then the function $\alpha \mapsto \frac{m_\alpha}{\alpha}$ is decreasing for all $\alpha > 0$.*

Proof. (i) For $u \in \mathcal{M}_\alpha$ fixed, define

$$J(\alpha) := \frac{h(\alpha)}{\alpha} \text{ for } \alpha > 0.$$

Then $J(\alpha) \in C^1(\mathbb{R})$ and $J'(\alpha) = \frac{h'(\alpha)\alpha - h(\alpha)}{\alpha^2}$. By calculating the first derivative of $h(\alpha)$ one has

$$h'(\alpha) = \frac{1}{2}A(u) + \frac{1}{2}B(u) + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u|^2 dx - \frac{1}{2\sqrt{\alpha}}\int_{\mathbb{R}^N} f(\sqrt{\alpha}u)u dx,$$

which implies that

$$h'(\alpha)\alpha - h(\alpha) = -\frac{1}{2}\int_{\mathbb{R}^N}\left[f(\sqrt{\alpha}u)\sqrt{\alpha}u - 2F(\sqrt{\alpha}u)\right] dx. \tag{3.3}$$

By condition (A4), we obtain

$$f(\sqrt{\alpha}u)\sqrt{\alpha}u - 2F(\sqrt{\alpha}u) > 0. \tag{3.4}$$

Thus, it follows from (3.3) and (3.4) that $h'(\alpha)\alpha - h(\alpha) < 0$, so (i) is valid.

(ii) Fix $0 < \alpha_1 < \alpha_2$, and let $u_i \in \mathcal{M}_{\alpha_i}$ satisfy $m_{\alpha_i} = I(\sqrt{\alpha_i}u_i)$ for $i = 1, 2$. Then from (i) and the definition of m_α, it follows that

$$\frac{m_{\alpha_2}}{\alpha_2} \le \frac{I(\sqrt{\alpha_2}u_2)}{\alpha_2} = \frac{h(\alpha_2)}{\alpha_2} < \frac{h(\alpha_1)}{\alpha_1} = \frac{I(\sqrt{\alpha_1}u_1)}{\alpha_1} = \frac{m_{\alpha_1}}{\alpha_1}.$$

This indicates that the function $\alpha \mapsto \frac{m_\alpha}{\alpha}$ is decreasing for all $\alpha > 0$. The proof is complete. □

As a direct consequence of Lemma 3.5, we have the following lemma.

Lemma 3.6. *Assume that conditions* (A1)–(A4) *hold. In addition, let $V \in C^1(\mathbb{R}^N)$ satisfy condition* (A6). *If m_α is attained for some $\alpha > 0$, then for any $\alpha_1, \alpha_2 \in (\bar{\alpha}, +\infty)$, we have*

$$m_{\alpha_2} < m_{\alpha_1} + m_{\alpha_2 - \alpha_1}.$$

Lemma 3.7. *Assume that* ((A1)–(A4) *hold. In addition, let $V \in C^1(\mathbb{R}^N)$ satisfy condition* (A6). *Let $\{u_n\} \subset S(\alpha)$ be a minimizing sequence of I with respect to $m(\alpha)$ for $\alpha > \bar{\alpha}$, then one of the following conclusions hold:*

(i)

$$\limsup_{n\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B(z,1)} |u_n|^2 dx = 0;$$

(ii) *Taking a sequence if necessary, there exist $u \in S(\alpha)$ and a family $\{y_n\} \subset \mathbb{R}^N$ such that*
$$u_n(\cdot - y_n) \to u \quad \text{in } \mathcal{H} \text{ as } n \to \infty.$$

Proof. Assume that (i) does not hold. Then
$$0 < \limsup_{n\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B(z,1)} |u_n|^2 dx \le \alpha < \infty.$$
Taking a subsequence if necessary, there exists a family $\{y_n\} \subset \mathbb{R}^N$ such that
$$0 < \lim_{n\to\infty} \int_{B(0,1)} |u_n(x - y_n)|^2 dx < \infty.$$
By Lemma 2.1, $\{u_n\}$ is a bounded sequence in $\mathcal{H}$. Thus, up to a subsequence, there exists $u \in \mathcal{H}$ such that
$$\begin{gathered} u_n(\cdot - y_n) \rightharpoonup u \quad \text{in } \mathcal{H}, \\ u_n(\cdot - y_n) \to u \quad \text{in } L^2_{\rm loc}(\mathbb{R}^N), \\ u_n(\cdot - y_n) \to u \quad \text{a.e. in } \mathbb{R}^N, \end{gathered}$$
which implies that $0 < \|u\|_2^2 \le \alpha$. Set $\eta := \|u\|_2^2$ and $v_n := u_n(\cdot - y_n) - u$. It is clear that $v_n \rightharpoonup 0$ in $\mathcal{H}$ as $n \to \infty$. Moreover, it follows from Brezis-Lieb theorem [4] and Lemma 2.2 that
$$I(u_n) = I(u) + I(v_n) + o(1).$$
Next, we prove that $\eta = \alpha$. Otherwise, if $0 < \eta < \alpha$, using Brezis-Lieb theorem [4] again, we have
$$\|u_n\|_2^2 = \|u + v_n\|_2^2 = \|u\|_2^2 + \|v_n\|_2^2 + o(1), \tag{3.5}$$
which implies that $\|v_n\|_2^2 = \alpha - \eta + o(1) > 0$. To obtain a contradiction, we consider two separate cases. If m_η is not attained by u, then by Lemma 3.1, we have
$$\begin{aligned} m_\alpha &= I(u_n) + o(1) \\ &= I(u) + I(v_n) + o(1) \\ &> m_\eta + I(v_n) + o(1) \\ &\ge m_\eta + m_{\alpha-\eta} \ge m_\alpha, \end{aligned}$$
which is a contradiction. If m_η is attained by u, then from Lemma 3.6, it follows that
$$\begin{aligned} m_\alpha &= I(u) + I(v_n) + o(1) \\ &= m_\eta + I(v_n) + o(1) \\ &\ge m_\eta + m_{\alpha-\eta} > m_\alpha, \end{aligned}$$
which is also a contradiction. Hence, $\eta = \|u\|_2^2 = \alpha$ and $u \in S(\alpha)$.

Note that
$$I(u_n) = I(u) + I(v_n) + o(1) \ge m_\alpha + I(v_n) + o(1),$$
which means that
$$\lim_{n\to\infty} I(v_n) \le 0. \tag{3.6}$$

By (3.5), $\|v_n\|_2^2 \to 0$ as $n \to \infty$. Then for any $\varepsilon > 0$, there exists $R > 0$ such that

$$\begin{aligned}
|\int_{\mathbb{R}^N} V(x)v_n^2 dx| &\le \int_{B_R(0)} V(x)v_n^2 dx + \int_{\mathbb{R}^N\setminus B_R(0)} V(x)v_n^2 dx \\
&\le \sup_{B_R(0)} |V(x)| \int_{B_R} v_n^2 dx + \sup_{\mathbb{R}^N\setminus B_R(0)} |V(x)| \int_{\mathbb{R}^N} v_n^2 dx \\
&\le C\varepsilon,
\end{aligned}$$

where $C > 0$ is a constant. This indicates that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V(x)v_n^2 dx = 0. \tag{3.7}$$

Applying Lemmas 2.1 and 2.3 leads to

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(v_n) dx \le 0. \tag{3.8}$$

So, by (3.6)–(3.8), we have

$$\begin{aligned}
&\frac{1}{2} \lim_{n\to\infty} \int_{\mathbb{R}^N} (|\Delta v_n|^2 + |\nabla v_n|^2) dx \\
&\le \lim_{n\to\infty} I(v_n) + \lim_{n\to\infty} \Big(\int_{\mathbb{R}^N} F(v_n) dx - \int_{\mathbb{R}^N} V(x)v_n^2 dx \Big) \le 0,
\end{aligned}$$

which implies that $v_n \to 0$ in $\mathcal{H}$. Therefore, $\lim_{n\to\infty} u_n(\cdot - y_n) = u$ in $\mathcal{H}$. The proof is complete. □

Proof of Theorem 1.2. By Lemma 2.1, let $\{u_n\} \subset S(\alpha)$ be a bounded minimizing sequence of I with respect to m_α. It is sufficient to show that $\{u_n\}$ satisfies Lemma 3.7 (ii). Otherwise,

$$\limsup_{n\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B(z,1)} u_n^2 dx = 0. \tag{3.9}$$

By (3.9) and Lemma 2.3, we have

$$u_n \to 0 \text{ in } L^p(\mathbb{R}^N) \text{ for } 2 < p < 4^*. \tag{3.10}$$

Then it follows from Lemma 2.1 that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(u_n) dx \le 0. \tag{3.11}$$

In addition, since $\lim_{|x|\to\infty} V(x) = 0$, for each $\varepsilon > 0$, there exists a $M > 0$ such that

$$|V(x)| < \varepsilon \quad \text{for } |x| > M. \tag{3.12}$$

According to (3.10), we know that

$$u_n \to 0 \quad \text{in } L^2(B_M(0)). \tag{3.13}$$

Then combining (3.12) and (3.13), we deduce that

$$|\int_{\mathbb{R}^N} V(x)u_n^2 dx| \le \sup_{B_M(0)} |V(x)| \int_{B_M(0)} u_n^2 dx + \sup_{\mathbb{R}^N\setminus B_M(0)} |V(x)| \int_{\mathbb{R}^N} u_n^2 dx \le \varepsilon,$$

which means that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V(x)u_n^2 dx = 0. \tag{3.14}$$

Hence, by (3.11) and (3.14) one has

$$m_\alpha = \lim_{n\to\infty} I(u_n) \ge \lim_{n\to\infty} \Big(\frac{1}{2}\int_{\mathbb{R}^N} V(x)u_n^2 dx - \int_{\mathbb{R}^N} F(u_n)dx\Big) \ge 0,$$

contradicting to $m_\alpha < 0$. Therefore, there exists a global minimizer u such that $I(u) = m_\alpha$, that is, $u \in S(\alpha)$ is a ground state solution of problem (1.3). The proof is complete. □

Lemma 3.8. *Assume that conditions* (A1)–(A3), (A5)–(A7) *and* (1.5) *hold. Then for any $\alpha > \bar{\alpha}$, we have*

(i) *$m_{l\alpha} \le l m_\alpha$ for any $l > 1$,*
(ii) *if m_α is attained, then $m_{l\alpha} < l m_\alpha$ for all $l > 1$.*

Proof. (i) For each $\varepsilon > 0$, there exists $u \in S(\alpha)$ such that $I(u) < m_\alpha + \varepsilon$. Set $u_l := l^{\frac{N+1}{2}} u(lx)$, we have

$$\int_{\mathbb{R}^N} u_l^2 dx = l \int_{\mathbb{R}^N} u^2 dx = l\alpha,$$

which implies that $u_l \in S(l\alpha)$. Then it follows from (A5) and (A7) that for all $l > 1$,

$$\begin{aligned}
m_{l\alpha} &\le I(u_l)\\
&= \frac{l^5}{2}\|\Delta u\|_2^2 + \frac{l^3}{2}\|\nabla u\|_2^2 + \frac{l}{2}\int_{\mathbb{R}^N} V(x/l)u^2 dx - \frac{1}{l^N}\int_{\mathbb{R}^N} F(l^{\frac{N+1}{2}}u)dx\\
&= l^5\Big(\frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{2l^2}\|\nabla u\|_2^2 + \frac{1}{2l^4}\int_{\mathbb{R}^N} V(x/l)u^2 dx - \frac{1}{l^{N+5}}\int_{\mathbb{R}^N} F(l^{\frac{N+1}{2}}u)dx\Big)\\
&= l^5\Big[I(u) + \frac{1}{2}(\frac{1}{l^2} - 1)\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} \Big(\frac{1}{l^4}V(x/l) - V(x)\Big)u^2 dx\\
&\quad + \int_{\mathbb{R}^N} \Big(F(u) - \frac{1}{l^{N+5}}F(l^{\frac{N+1}{2}}u)\Big)dx\Big]\\
&< l^5 I(u) < l^5(m_\alpha + \varepsilon).
\end{aligned}$$

This implies that $m_{l\alpha} \le l^5 m_\alpha \le l m_\alpha$ for all $l > 1$, since $\varepsilon > 0$ is arbitrary.

(ii) Let m_α be attained by some $u \in S(\alpha)$, i.e. $I(u) = m_\alpha$. According to (i), we have

$$m_{l\alpha} < l m_\alpha \text{ for any } l > 1.$$

The proof is complete. □

As an immediate consequence of Lemma 3.8, we have the following lemma.

Lemma 3.9. *Assume that conditions* (A1)–(A3), (A5)–(A7) *and* (1.5) *hold. If m_α is attained for $\alpha > \bar{\alpha}$, then for each $\alpha_1, \alpha_2 \in (\bar{\alpha}, \infty)$,*

$$m_{\alpha_2} < m_{\alpha_1} + m_{\alpha_2 - \alpha_1}.$$

Proof of Theorem 1.3. (i) Suppose by contradiction that there exists a global minimizer of the energy functional I with respect to m_α for $0 < \alpha < \bar{\alpha}$. According to (1.4), $m_\alpha = 0$ when $0 < \alpha < \bar{\alpha}$. Then we infer from Lemma 3.8 (ii) that

$$0 = m_\alpha > m_{\bar{\alpha}},$$

which contradicts with Lemma 3.1 (iii)-(iv). Hence, m_α is not attained for $0 < \alpha < \bar{\alpha}$.

(ii) By Lemma 3.2, we have $m_\alpha < 0$ for $\alpha > \bar{\alpha}$. It follows from Lemma 2.1 that there exists a minimizing sequence $\{u_n\} \subset S(\alpha)$ such that $\lim_{n\to\infty} I(u_n) = m_\alpha$. Next, applying the argument of Theorem 1.2, by Lemmas 3.1, 2.3 and 3.9, there exists a global minimizer u such that $I(u) = m_\alpha$. The proof is complete. □

Proof of Theorem 1.5. (i) By (1.6), there exists a $u \in C_0^\infty(\mathbb{R}^N)$ such that $\|u\|_2^2 = 1$ and

$$\int_{\mathbb{R}^N} \left(|\Delta u|^2 + |\nabla u|^2 + V(x)u^2\right) dx < 0.$$

Replacing with $|u|$ and thus we can assume that u is non-negative. Let $\alpha \in (0, s_0^2/\|u\|_\infty^2)$. Clearly, $\sqrt{\alpha}u \in S(\alpha)$ and $F(\sqrt{\alpha}u) \geq 0$. Then there exists $\alpha_0 \in (0, s_0^2/\|u\|_\infty^2)$ such that for $\alpha < \alpha_0$,

$$m_\alpha \leq I(\sqrt{\alpha}u) \leq \frac{\alpha}{2} \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + V(x)u^2) dx < 0.$$

By the monotonicity of m_α in Lemma 3.1, we obtain that $m_\alpha < 0$ for all $\alpha > 0$, so $\bar{\alpha} = 0$.

(ii) By condition (1.5), there exists $C_f > 0$ such that $F(t) \leq C_f |t|^{2+\frac{8}{N}}$ for any $t \in \mathbb{R}$. According to the Gagliardo-Nirenberg inequality, we have

$$\int_{\mathbb{R}^N} F(u) dx \leq C_f \mathcal{C}_{N,2+\frac{8}{N}}^{2+\frac{8}{N}} \alpha^{4/N} \|\Delta u\|_2^2 \quad \text{for all } u \in S(\alpha). \tag{3.15}$$

By [18, Lemma 2.2], we recall that

$$\Big| \int_{\mathbb{R}^N} V(x)u^2 dx \Big| \leq \|V\|_{N/4} \|u\|_{4^*}^2 \leq U^{-1} \|V\|_{N/4} \|\Delta u\|_2^2, \tag{3.16}$$

where $\mathcal{S}$ is defined as a Sobolev constant, namely,

$$\mathcal{S} := \inf_{u \in D^{2,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\|\Delta u\|_2^2}{\|u\|_{4^*}^2}. \tag{3.17}$$

Define

$$\alpha_1 := \Big(\frac{1 - \mathcal{S}^{-1} \|V\|_{N/4}}{2C_f \mathcal{C}_{N,2+\frac{8}{N}}^{2+\frac{8}{N}}} \Big)^{N/4}.$$

It is clear that $\alpha_1 > 0$ when $\|V\|_{N/4} < \mathcal{S}$. Then for $u \in S(\alpha)$ with $\alpha \in (0, \alpha_1)$, it follows from (3.15) and (3.16) that

$$\begin{aligned} I(u) &= \frac{1}{2}A(u) + \frac{1}{2}B(u) + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 dx - \int_{\mathbb{R}^N} F(u) dx \\ &\geq \frac{1}{2}B(u) + \frac{1}{2}\left(1 - \mathcal{S}^{-1}\|V\|_{\frac{N}{4}} - 2C_f \mathcal{C}_{N,2+\frac{8}{N}}^{2+\frac{8}{N}} \alpha^{\frac{4}{N}}\right) A(u) \geq 0, \end{aligned}$$

which indicates that $m_\alpha = 0$ and $\bar{\alpha} \geq \alpha > 0$ from the monotonicity of m_α. The proof is complete. □

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References

[1] D. Bonheure, J. Casteras, T. Gou, L. Jeanjean; *Normalized solutions to the mixed dispersion nonlinear Schrödinger equation in the mass critical and supercritical regime*, Trans. Amer. Math. Soc., 372 (2019), 2167–2212.

[2] D. Bonheure, J. Casteras, E. M. D. Santos, R. Nascimento; *Orbitally stable standing waves of a mixed dispersion nonlinear Schrö dinger equation*, SIAM J. Math. Anal., 50 (2018), 5027–5071.

[3] N. Boussaid, A. Fernandez, L. Jeanjean; *Some remarks on a minimization problem associated to a fourth order nonlinear Schrödinger equations*, ArXiv: 1910.13177, (2019).

[4] H. Brézis, E. Lieb; *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc., 88 (1983), 486–490.

[5] A. J. Fernández, L. Jeanjean, R. Mandel, M. Maris; *Non-homogeneous Gagliardo-Nirenberg inequalities in $\mathbb{R}^N$ and application to a biharmonic non-linear Schrödinger equation*, J. Differential Equations, 330 (2022) 1–65.

[6] G. Fibich, B. Ilan, G. Papanicolaou; *Self-focusing with fourth-order dispersion*, SIAM J. Appl. Math., 62 (2002), 1437–1462.

[7] Y. Fukumoto, H. K. Mofatt; *Motion and expansion of a viscous vortex ring: I. A higher-order asymptotic formula for the velocity*, J. Fluid Mech., 417 (2000), 1–45.

[8] N. Ikoma, Y. Miyamoto; *Stable standing waves of nonlinear Schrödinger equations with potentials and general nonlinearities*, Calc. Var., (2020), 59–48.

[9] L. Jeanjean, S. S. Lu; *On global minimizers for a mass constrained problem*, Calc. Var., (2021) 61–214.

[10] L. Jeanjean, S. S. Lu; *Nonradial normalized solutions for nonlinear scalar field equations*, Nonlinearity, 32 (2019), 4942–4966.

[11] V. I. Karpman; *Stabilization of soliton instabilities by higher-order dispersion: fourth-order nonlinear Schrödinger type equations*, Phys. Rev. E, 53 (1996) R1336–R1339, American Physical Society.

[12] V. I. Karpman, A. G. Shagalov; *Stability of solitons described by nonlinear Schrödinger-type equations with higher order dispersion*, Phys. D, 144 (2000) 194–210.

[13] P.-L. Lions; *The concentration-compactness principle in the Calculus of Variations. The locally compact case I, part I and II*, Ann. Inst. Henri Poincaré Anal. Non Linéaire, 1 (1984), 109–145 and 223–283.

[14] X. Luo, T. Yang; *Normalized solutions for a fourth-order Schrödinger equation with a positive second-order dispersion coefficient*, Sci. China Math., 66 (2023), 1237–1262.

[15] T. Luo, S. Zheng, S. Zhu; *The existence and stability of normalized solutions for a biharmonic nonlinear Schrödinger equation with mixed dispersion.* Acta. Math. Sci., 43 (2023), 539–563.

[16] P. H. Rabinowitz; *On a class of nonlinear Schrödinger equations.* Z. Angew. Math. Phys., 43 (1992), 270–291.

[17] M. Shibata; *Stable standing waves of nonlinear Schrödinger equations with a general nonlinear term*, Manuscr. Math., 143 (2014), 221–237.

[18] R. Molle, G. Riey, G. Verzini; *Existence of normalized solutions to mass supercritical Schrödinger equations with negative potential*, J. Differential Equations, 333 (2022), 302–331.

[19] L. Xu, H. Chen; *Existence of positive ground solutions for biharmonic equations via Pohozaev-Nehari manifold*, J. Differential Equations, (2018) 541–560.

[20] Z. Yang, S. Qi, W. Zou; *Normalized solutions of nonlinear Schrödinger equations with potentials and non-autonomous nonlinearities*, J. Geom. Anal. (2022) 32:159.

[21] S. Zhu, J. Zhang, H. Yang; *Limiting profile of the blow-up solutions for the fourth-order nonlinear Schrödinger equation*, Dyn. Partial Differential Equations, 7 (2010), 187–205.

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