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MULTIPLE SOLUTIONS FOR $p(x)$-KIRCHHOFF TYPE PROBLEMS WITH EXTENDED ROBIN BOUNDARY CONDITIONS

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Abstract. This article considers $p(x)$-Kirchhoff type problems with extended Robin boundary conditions. Using the mountain pass theorem, Ekeland's variational principle, and Krasnoselskii's genus theory, we prove the existence at least two, and infinitely many non-trivial weak solutions under some suitable conditions on the non-linearities. The main results improve and generalize the results introduced in [1].

1. Introduction

In this article, we study the existence of weak solutions for $p(x)$-Kirchhoff type problems with extended Robin boundary conditions

$$\begin{aligned} &- M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx \\ &+ \int_{\partial\Omega}\Big(\frac{\beta(x)}{p(x)}|u|^{p(x)} + G(x,u)\Big)\,d\sigma_x\Big)\operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) \\ &= f(x,u) + \lambda h(x), \quad x\in\Omega, \\ &|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} + \beta(x)|u|^{p(x)-2}u + g(x,u) = 0, \quad x\in\partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\frac{\partial u}{\partial\nu}$ is the outer normal derivative, $d\sigma_x$ is the measure on the boundary $\partial\Omega$, $\beta\in L^1(\partial\Omega)$, $\beta^- := \inf_{x\in\partial\Omega}\beta(x) > 0$, $g:\partial\Omega\times\mathbb{R}\to\mathbb{R}$ is a measurable function, with $G(x,t) := \int_0^t g(x,s)\,ds$, $p\in C_+(\bar\Omega)$,

$$1 < p^- := \inf_{x\in\bar\Omega} p(x) \le p^+ := \max_{x\in\bar\Omega} p(x) < N,$$

λ is a non-negative parameter, $f:\Omega\times\mathbb{R}\to\mathbb{R}$ and $M:\mathbb{R}^+ := [0,+\infty)\to\mathbb{R}^+$ are two continuous functions, $h:\Omega\to\mathbb{R}$ is a measurable function.

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We can prove the same results if $\beta(x) \equiv 0$. Then the problem (1.1) becomes

$$\begin{gathered} -M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega} G(x,u)\,d\sigma_x\Big)\operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) \\ = f(x,u) + \lambda h(x), \quad x \in \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} + g(x,u) = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.2}$$

Problem (1.1) is related to the stationary version of the Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2\,dx\Big)\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.3}$$

presented by Kirchhoff in 1883 as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings, see [24]. The parameters in (1.3) have the following meaning: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension. Problem (1.3) is often called a non-local problem because it contains an integral over Ω. This causes some mathematical difficulties which make the study of such a problem particularly interesting. The non-local problem models several physical and biological systems, where u describes a process which depends on the average of itself, such as the population density, see [9]. Kirchhoff type problems have been studied in many papers in the previous decades. In [8, 14, 23, 26, 29, 30], using various methods the authors study the existence and multiplicity of solutions for Kirchhoff type problems involving the p-Laplacian operator $-\Delta_p(\cdot) = -\operatorname{div}\big(|\nabla\cdot|^{p-2}\nabla\cdot\big)$. The $p(x)$-Laplacian operator where $p(\cdot)$ is a continuous function possesses more complicated properties than the p-Laplacian operator, mainly due to the fact that it is not homogeneous. The study of various mathematical problems with variable exponent are interesting in applications and raise many difficult mathematical problems, see [25, 27]. For this reason, ordinary differential and partial differential equations with non-standard growth conditions have received specific attention in recent years, we refer to some results on $p(x)$-Kirchhoff type problems with Dirichlet or Neumann boundary conditions [6, 10, 13, 15, 16, 22]. Relatively speaking, Kirchhoff type problems with Robin boundary conditions have rarely been considered. Robin boundary conditions are a weighted combination of Dirichlet and Neumann boundary conditions and it is also called impedance boundary conditions, from their application in electromagnetic problems or convective boundary conditions from their application in heat transfer problems. Moreover, Robin conditions are commonly used in solving Sturm-Liouville problems which appear in many contexts in sciences and engineering, see [17]. To the best of our knowledge, Allaoui [3] first introduced the $p(x)$-Kirchhoff type problems involving Robin boundary conditions and studied problem (1.1) in the case $\lambda = 0$ by using the mountain pass theorem, the fountain theorem and some properties of $(S)_+$type operator. Regarding the $p(x)$-Laplacian problems with the Robin boundary conditions in the local case when $M(t) \equiv 1$, we refer to [12, 4, 17, 21, 28], in which some existence and multiplicity results were obtained by using variational methods. Motivated by above mentioned papers and the results on the Kirchhoff type problem involving Laplace operator $-\Delta(\cdot)$ in [8], the purpose of this article is to consider Robin problem (1.1) with perturbation h and parameter λ. More precisely, under some suitable conditions on the nonlinear term f and the Kirchhoff function M, we prove that problem (1.1) has at least two weak solutions if $\lambda > 0$ small enough, see Theorem 3.1. In the case when $\lambda = 0$, we prove problem (1.1) with subcritical growth condition has infinitely many solutions, see Theorem 3.8. Our methodology relies fundamentally on the utilization of the mountain pass theorem [5], the Ekeland variational principle [20], and Krasnoselskii's genus theory [12], complemented by the approach outlined in the article by Afrouzi [1]. We highlight that the findings presented in this work are novel, extending to cases where $p(\cdot)$ is a constant. Furthermore, we do not require the non-degenerate condition on the Kirchhoff function M, as stipulated in [3, 8]; refer to the assumptions (A1) and (A2).

This article si structured as follows: In the preliminaries we define the functional space and give results that we need for the proofs. In the main result section, we prove the existence and the multiplicity of the solution. In the application section, we apply the results of section 2.

2. Preliminaries

We recall some definitions and basic properties of the generalized Lebesgue Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ where Ω is an open subset of $\mathbb{R}^N$. For this context, we refer to the books [18, 27] and the papers [2, 17, 21, 25]. We define the set

$$C_+(\bar{\Omega}) := \{h; h \in C(\bar{\Omega}), h(x) > 1 \text{ for all } x \in \bar{\Omega}\}$$

For each $h \in C_+(\bar{\Omega})$ we define

$$h^+ = \sup_{x\in\bar{\Omega}} h(x), \quad h^- = \inf_{x\in\bar{\Omega}} h(x).$$

For each $p(x) \in C_+(\bar{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \big\{u \text{ measurable real-valued functions such that } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\big\}$$

We recall the so-called Luxemburg norm on this space defined by the formula

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} := \inf\big\{\lambda > 0 : \int_\Omega \big|\frac{u(x)}{\lambda}\big|^{p(x)}\,dx \le 1\big\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces, the Hölder inequality holds, they are reflexive if and only if $1 < p^- \le p^+ < +\infty$ and continuous functions are dense if $p^+ < +\infty$. The inclusion between Lebesgue spaces also generalizes naturally: if $0 < |\Omega| < +\infty$ and p_1, p_2 are variable exponents so that $p_1(x) \le p_2(x)$ a.e. $x \in \Omega$ then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder inequalities

$$\big|\int_\Omega uv\,dx\big| \le \Big(\frac{1}{p^-} + \frac{1}{(p')^-}\Big)|u|_{p(x)}|v|_{p'(x)} \le 2|u|_{p(x)}|v|_{p'(x)}$$

hold. An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\,dx$$

If $u \in L^{p(x)}(\Omega)$ and $p^+ < +\infty$ then the following relations hold

$$\|u\|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le \|u\|_{p(x)}^{p^+},$$

provided $|u|_{p(x)} > 1$, while

$$\|u\|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le \|u\|_{p(x)}^{p^-},$$

provided $\|u\|_{p(x)} < 1$, and

$$\|u_n - u\|_{p(x)} \to 0 \Leftrightarrow \rho_{p(x)}(u_n - u) \to 0\,.$$

If $p \in C_+(\bar{\Omega})$ the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$, consisting of functions $u \in L^{p(x)}(\Omega)$ whose distributional gradient ∇u exists almost everywhere and belongs to $\big[L^{p(x)}(\Omega)\big]^N$, endowed with the norm

$$\|u\| := \inf\big\{\lambda > 0 : \int_\Omega \Big[\big|\frac{\nabla u(x)}{\lambda}\big|^{p(x)} + \big|\frac{u(x)}{\lambda}\big|^{p(x)}\Big]\,dx \le 1\big\}$$

or

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)},$$

is a separable and reflexive Banach space.

Proposition 2.1 ([31]). *Let*

$$\rho(u) = \int_\Omega |\nabla u|^{p(x)}\,dx\,.$$

Then

$$\|u\|^{p^-} \le \rho(u) \le \|u\|^{p^+}$$

provided $\|u\| > 1$*, while*

$$\|u\|^{p^+} \le \rho(u) \le \|u\|^{p^-}$$

provided $\|u\| < 1$*, and*

$$\|u_n - u\| \to 0 \Leftrightarrow \rho(u_n - u) \to 0.$$

The space of smooth functions are in general not dense in $W^{1,p(x)}(\Omega)$, but if the exponent $p \in C_+(\bar{\Omega})$ is logarithmic Hölder continuous, that is,

$$|p(x) - p(y)| \le -\frac{M}{\log(|x-y|)}, \quad \forall x, y \in \Omega,\ |x-y| \le \frac{1}{2},$$

then the smooth functions are dense in $W^{1,p(x)}(\Omega)$. The space $(W_0^{1,p(x)}(\Omega), \|\cdot\|)$ is a separable and Banach space. We note that if $s \in C_+(\bar{\Omega})$ and $s(x) < p^*(x)$ for all $x \in \bar{\Omega}$ then the embedding

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$$

is compact and continuous, where $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ or $p^*(x) = +\infty$ if $p(x) > N$.

Proposition 2.2 ([31]). *Let us define the functional* $L : W^{1,p(x)}(\Omega) \to \mathbb{R}$ *by*

$$L(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\,dx \tag{2.1}$$

for all $u \in W^{1,p(x)}(\Omega)$*. Then* $L \in C^1\big(W^{1,p(x)}(\Omega), \mathbb{R}\big)$ *and its derivative is*

$$L'(u)(v) = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v\,dx \tag{2.2}$$

Moreover, we have the following assertions

(i) $L' : W^{1,p(x)}(\Omega) \to W^{-1,p(x)}(\Omega)$ *is a continuous, bounded and strictly monotone operator;*

(ii) $L' : W^{1,p(x)}(\Omega) \to W^{-1,p(x)}(\Omega)$ *is a mapping of type* $(S)_+$, *i.e. if* $\{u_n\}$ *converges weakly to* u *in* $W^{1,p(x)}(\Omega)$ *and* $\limsup_{n\to\infty} L'(u_n)(u_n - u) \le 0$, *then* $\{u_n\}$ *converges strongly to* u *in* $W^{1,p(x)}(\Omega)$.

If $s \in C_+(\partial\Omega)$ and $s(x) < p_*(x)$ for all $x \in \partial\Omega$ then the trace embedding

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\partial\Omega)$$

is compact and continuous, where $p_*(x) = \frac{(N-1)p(x)}{N-p(x)}$ if $p(x) < N$ or $p_*(x) = +\infty$ if $p(x) > N$. Moreover, for any $u \in W^{1,p(x)}(\Omega)$, we define

$$\|u\|_\partial := |\nabla u|_{L^{p(x)}(\Omega)} + |u|_{L^{p(x)}(\partial\Omega)},$$

then $\|u\|_\partial$ is a norm on $W^{1,p(x)}(\Omega)$ which is equivalent to the norm $\|u\|$, see [17, Theorem 2.1].

Now, let us introduce a norm which will be used later. Let $\beta \in L^1(\partial\Omega)$ with $\beta^- = \inf_{x\in\partial\Omega} \beta(x) > 0$, and for any $u \in W^{1,p(x)}(\Omega)$, define

$$\|u\|_{\beta(x)} := \inf\{\lambda > 0 : \int_\Omega \big|\frac{\nabla u(x)}{\lambda}\big|^{p(x)}\,dx + \int_{\partial\Omega} \beta(x)\big|\frac{u(x)}{\lambda}\big|^{p(x)}\,d\sigma_x\,dx \le 1\}$$

where $d\sigma_x$ is the measure on the boundary $\partial\Omega$. Then $\|u\|_{\beta(x)}$ is also a norm on $W^{1,p(x)}(\Omega)$ which is equivalent to $\|\cdot\|$ and $\|\cdot\|_\partial$.

Proposition 2.3 ([1]). *Let*

$$\rho_{\beta(x)}(u) = \int_\Omega |\nabla u|^{p(x)}\,dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)}\,d\sigma_x,$$

we have

$$\|u\|^{p^-}_{\beta(x)} \le \rho_{\beta(x)}(u) \le \|u\|^{p^+}_{\beta(x)} \tag{2.3}$$

provided $\|u\|_{\beta(x)} > 1$, *while*

$$\|u\|^{p^+}_{\beta(x)} \le \rho_{\beta(x)}(u) \le \|u\|^{p^-}_{\beta(x)} \tag{2.4}$$

provided $\|u\|_{\beta(x)} < 1$, *and*

$$\|u_n - u\|_{\beta(x)} \to 0 \Leftrightarrow \rho_{\beta(x)}(u_n - u) \to 0. \tag{2.5}$$

Proposition 2.4 ([21]). *For* $\beta \in L^1(\partial\Omega)$ *with* $\beta^- := \inf_{x\in\partial\Omega} \beta(x) > 0$, *let us define the functional* $L_{\beta(x)} : W^{1,p(x)}(\Omega) \to \mathbb{R}$ *by*

$$L_{\beta(x)}(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u|^{p(x)}\,d\sigma_x \tag{2.6}$$

for all $u \in W^{1,p(x)}(\Omega)$. *Then* $L_{\beta(x)} \in C^1\left(W^{1,p(x)}(\Omega), \mathbb{R}\right)$ *and its derivative is*

$$L'_{\beta(x)}(u)(v) = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\,dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)-2}uv\,d\sigma_x \tag{2.7}$$

Moreover, we have the following assertions

(i) $L'_{\beta(x)} : W^{1,p(x)}(\Omega) \to W^{-1,p(x)}(\Omega)$ *is a continuous, bounded and strictly monotone operator;*

(ii) $L'_{\beta(x)} : W^{1,p(x)}(\Omega) \to W^{-1,p(x)}(\Omega)$ *is a mapping of type* $(S)_+$*, i.e. if* $\{u_n\}$ *converges weakly to* u *in* $W^{1,p(x)}(\Omega)$ *and* $\limsup_{n\to\infty} L'_{\beta(x)}(u_n)(u_n-u) \le 0$*, then* $\{u_n\}$ *converges strongly to* u *in* $W^{1,p(x)}(\Omega)$.

In the rest of this section, we introduce some notion and results on Krasnoselskii's genus theory, the readers can consult [7, 12]. Let Y be a real Banach space. Let us denote by $\mathcal{R}$ the class of all closed subsets $A \subset X\backslash\{0\}$ that are symmetric with respect to the origin, that is, $u \in A$ implies $-u \in A$, i.e.

$$\mathcal{R} = \{A \subset Y\backslash\{0\} : A \text{ is compact and } A = -A\}$$

Definition 2.5 ([1]). Let $A \in \mathcal{R}$ and $Y = \mathbb{R}^N$. The genus $\gamma(A)$ of A is defined by

$$\gamma(A) = \min\big\{k \ge 1 : \text{ there exists an odd continuous mapping } \phi : A \to \mathbb{R}^k\backslash\{0\}\big\}$$

If such a mapping ϕ does not exist for any $k > 0$, we set $\gamma(A) = +\infty$.

Note that if A is a subset that consists of finitely many pairs of points, then $\gamma(A) = 1$. Moreover, from the above definition, $\gamma(\emptyset) = 0$. A typical example of a set of genus k is a set, which is homeomorphic to a $(k-1)$ dimensional sphere via an odd map.

Proposition 2.6. [1] *Let* $Y = \mathbb{R}^N$ *and* $\partial\Omega$ *be the boundary of an open, symmetric and bounded subset* $\Omega \subset \mathbb{R}^N$ *with* $0 \in \Omega$*. Then we have* $\gamma(\partial\Omega) = N$.

Let us denote by S the unit sphere in Y. It follows from Proposition 2.6 that $\gamma\left(S^{N-1}\right) = N$. If Y is of infinite dimension and separable then $\gamma(S) = +\infty$. We now recall an application of Palais-Smale "compactness" criterion, which was introduced by Clark [12].

Proposition 2.7 ([1]). *Let* $J \in C^1(Y, \mathbb{R})$ *be a functional satisfying the Palais-Smale condition. Furthermore, let us suppose that*

(i) J *is bounded from below and even;*
(ii) *There is a compact set* $K \in \mathcal{R}$ *such that* $\gamma(K) = k$ *and* $\sup_{x\in K} J(x) < J(0)$.

Then J *possesses at least* k *pairs of distinct critical points, and their corresponding critical values are less than* $J(0)$.

3. Main results

3.1. **Existence of at least two solutions.** In this part, we consider problem (1.1) in the case when $\lambda > 0$. Under suitable conditions on the nonlinear term f and the Kirchhoff function M, we prove that (1.1) has at least two nontrivial weak solutions in the space $X = W^{1,p(x)}(\Omega)$. Our idea is to apply the mountain pass theorem in [5] combined with Ekeland's variational principle in [20] to the energy functional J_λ associated qith problem (1.1) when $\lambda > 0$ small enough. For this purpose, let us assume that $M : \mathbb{R}^+ \to \mathbb{R}^+$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ are continuous functions, and introduce the following asumptions:

(A1) There exists $\alpha \in (1, q^-/p+)$ such that

$$tM(t) \le \alpha\widehat{M}(t) = \alpha \int_0^t M(\tau)\, d\tau$$

for all $t \in \mathbb{R}_0^+$, where $q^- = \inf_{x\in\bar\Omega} q(x)$, $q \in C_+(\bar\Omega)$ is given by (A3);

(A2) For each $\tau > 0$ there exists $\kappa = \kappa(\tau) > 0$ such that $M(t) \ge \kappa$ for all $t \ge \tau$;

(A3) There exists a positive constant C_1 such that
$$|f(x,t)| \le C_1(1+|t|^{q(x)-1}) \quad \forall (x,t)\in \Omega\times\mathbb{R},$$
where $q\in C_+(\bar\Omega), p(x)<q(x)<p^*(x)=\frac{Np(x)}{N-p(x)}$ for all $x\in\bar\Omega$;

(A4) $f(x,t)=o\left(|t|^{\alpha p^+-1}\right), t\to 0$, uniformly a.e. $x\in\Omega$;

(A5) There exists a constant $\mu>\alpha p^+$ such that
$$\mu F(x,t):=\mu\int_0^t f(x,s)\,ds \le f(x,t)t, \quad \forall (x,t)\in\Omega\times\mathbb{R};$$

(A6) $\inf_{\{x\in\Omega;|t|=1\}} F(x,t)>0$.

(A7) $g(x,t)=o(|t|^{r_1(x)-1})$ uniformly a.e. $x\in\partial\Omega$, as $t\to 0$, where $r_1\in C_+(\partial\Omega)$, $\sup_{x\in\partial\Omega} r_1(x)=r_1^+<p^-\le p(x)$ for all $x\in\partial\Omega$;

(A8) $g(x,t)=o\big(|t|^{r_2(x)-1}\big)$, $t\to+\infty$, uniformly a.e. $x\in\partial\Omega$, where $r_2\in C_+(\partial\Omega)$, $\sup_{x\in\partial\Omega} r_2(x)=r_2^+<p^-\le p(x)$ for all $x\in\partial\Omega$;

(A9)
$$G(x,t):=\int_0^t g(x,s)\,ds\ge 0, \quad \forall (x,t)\in\partial\Omega\times\mathbb{R},$$
where α and μ are given in (A1) and (A5).

We say that $u\in W^{1,p(x)}(\Omega)$ is a weak solution of problem (1.1) if
$$\begin{aligned}
&M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx+\int_{\partial\Omega}(\frac{\beta(x)}{p(x)}|u|^{p(x)}+G(x,u))\,d\sigma_x\Big)\\
&\times\Big(\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v\,dx+\int_{\partial\Omega}\beta(x)|u|^{p(x)-2}uv\,d\sigma_x+\int_{\partial\Omega} g(x,u)v\,d\sigma_x\Big)\\
&-\int_\Omega f(x,u)v\,dx-\lambda\int_\Omega h(x)v\,dx=0
\end{aligned}$$
for all $v\in W^{1,p(x)}(\Omega)$.

The first results of this article reads as follows.

Theorem 3.1. *Suppose that $\beta\in L^1(\partial\Omega)$, $\beta^-:=\inf_{x\in\partial\Omega}\beta(x)>0$, $h\in L^{\frac{\alpha p^+}{\alpha p^+-1}}(\Omega)$ and $g\not\equiv 0$. Let $M(0)=0$ and the conditions* (A1)–(A9) *hold. Then there exists $\lambda^*>0$ such that* (1.1) *has at least two non-trivial weak solutions when $\lambda\in(0,\lambda^*)$.*

Theorem 3.2. *Suppose that $\beta(x)\equiv 0$, $h\in L^{\frac{\alpha p^+}{\alpha p^+-1}}(\Omega)$ and $g\not\equiv 0$. Let $M(0)=0$ and the conditions* (A1)–(A9) *hold. Then there exists $\lambda^*>0$ such that the problem* (1.2) *has at least two non-trivial weak solutions when $\lambda\in(0,\lambda^*)$.*

Let us denote by X the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ and consider the energy functional $I_\lambda: X\to\mathbb{R}$ given by
$$\begin{aligned}
I_\lambda(u)=\widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx+\int_{\partial\Omega}\frac{\beta(x)}{p(x)}|u|^{p(x)}\,d\sigma_x+\int_{\partial\Omega}G(x,u)\,d\sigma_x\Big)\\
-\int_\Omega F(x,u)\,dx-\lambda\int_\Omega h(x)u\,dx.
\end{aligned}$$
Then by (A3) and the continuous embeddings, we can show that the functional I_λ is well-defined on X and $I_\lambda\in C^1(X,\mathbb{R})$ with the derivative
$$I_\lambda'(u)(v)=M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx+\int_{\partial\Omega}(\frac{\beta(x)}{p(x)}|u|^{p(x)}+G(x,u))\,d\sigma_x\Big)$$

$$\times \Big(\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\,dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)-2}uv\,d\sigma_x + \int_{\partial\Omega} g(x,u)v\,d\sigma_x\Big)$$
$$- \int_\Omega f(x,u)v\,dx - \lambda \int_\Omega h(x)v\,dx$$

for all $u, v \in X$. Hence, we can find weak solutions of (1.1) as the critical points of the functional I_λ in the space X.

In what follows, we study the degenerate problem (1.1), and in passing we recall that (A1) is assumed throughout the paper.

Remark 3.3. Obviously, (A2) implies that $M(t) > 0$ as $t > 0$. Hence, if $M(t_0) > 0$ for some $t_0 > 0$, then (A1) yields that

$$t_0^\alpha \widehat{M}(t) \geq \widehat{M}(t_0)t^\alpha \quad \forall t \in [0, t_0], \tag{3.1}$$
$$t_0^\alpha \widehat{M}(t) \leq \widehat{M}(t_0)t^\alpha \quad \forall t \geq t_0. \tag{3.2}$$

assuming that $\{u_n\}_n \subset X$. If

$$\inf_{n\in\mathbb{N}} \Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u_n|^{p(x)}\,d\sigma_x + \int_{\partial\Omega} G(x,u_n)\,d\sigma_x\Big) > 0$$

Then from condition (A2) we can choose

$$\tau = \inf_{n\in\mathbb{N}} \Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u_n|^{p(x)}\,d\sigma_x + \int_{\partial\Omega} G(x,u_n)\,d\sigma_x\Big) > 0.$$

Then there exists $\kappa = \kappa(\tau) > 0$ such that

$$M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u_n|^{p(x)}\,d\sigma_x + \int_{\partial\Omega} G(x,u_n)\,d\sigma_x\Big) \geqslant \kappa \quad \forall n \in \mathbb{N}.$$

We will discuss the other case later.

Lemma 3.4. *Assume that* (A1)–(A5), (A9) *hold and that* $h \in L^{\frac{\alpha p^+}{\alpha p^+ - 1}}(\Omega)$. *Then there exist constants* $\rho, r, \lambda^* > 0$ *such that* $I_\lambda(u) \geq r$ *for all* $u \in X$ *with* $\|u\|_{\beta(x)} = \rho$, *when* $\lambda \in (0, \lambda^*)$.

Proof. Since $1 < r_1^- < p^- \leq p(x) \leq p^+ < \alpha p^+ < q^- \leq q(x) < p^*(x)$ for all $x \in \bar{\Omega}$, $1 < r_1^- < p^-$ and $1 < r_2^- < p^-$, the embeddings

$$X \hookrightarrow L^{p(x)}(\Omega), \quad X \hookrightarrow L^{\alpha p^+}(\Omega), \quad X \hookrightarrow L^{q(x)}(\Omega), \quad X \hookrightarrow L^{p(x)}(\partial\Omega)$$

are continuous, and there exists positive constants C_2, C_3, C_4, C_5 such that

$$\begin{gathered} \|u\|_{p(x)} \leq C_2\|u\|_{\beta(x)}, \quad \|u\|_{\alpha p^+} \leq C_3\|u\|_{\beta(x)}, \\ \|u\|_{q(x)} \leq C_4\|u\|_{\beta(x)}, \quad \|u\|_{\partial\Omega, p(x)} \leq C_5\|u\|_{\beta(x)}. \end{gathered} \tag{3.3}$$

Let $0 < \epsilon_1$. From the assumptions (A1) and (A2), there exists a constant $C(\epsilon_1)$ depending on ϵ_1 such that

$$|F(x,t)| \leq \epsilon_1|t|^{\alpha p^+} + C(\epsilon_1)|t|^{q(x)}, \quad \forall (x,t) \in \Omega \times \mathbb{R} \tag{3.4}$$

Let $u \in X$ with $\|u\|_{\beta(x)} < 1$ sufficiently small. From (A1), (A2), (2.4), (3.3), (3.4), and (A9) applying the Hölder inequality we have

$$I_\lambda(u)$$
$$= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega} (\frac{\beta(x)}{p(x)}|u|^{p(x)} + G(x,u))\,d\sigma_x\Big) - \int_\Omega F(x,u)\,dx$$

$$
\begin{aligned}
&- \lambda \int_\Omega h(x)u\,dx \\
&\geq \frac{1}{\alpha} M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega}(\frac{\beta(x)}{p(x)}|u|^{p(x)} + G(x,u))\,d\sigma_x\Big) \\
&\quad \times \Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega}(\frac{\beta(x)}{p(x)}|u|^{p(x)} + G(x,u))\,d\sigma_x\Big) - \int_\Omega F(x,u)\,dx \\
&\quad - \lambda \int_\Omega h(x)u\,dx \\
&\geq \frac{\kappa}{\alpha p^+}\Big(\rho_{\beta(x)}(u) + p^+\int_{\partial\Omega} G(x,u)d\sigma_x\Big) - \int_\Omega F(x,u)\,dx - \lambda\int_\Omega h(x)u\,dx \\
&\geq \frac{\kappa}{\alpha p^+}\|u\|_{\beta(x)}^{p^+} - \epsilon_1\int_\Omega |u|^{\alpha p^+}\,dx - C(\epsilon_1)\int_\Omega |u|^{q(x)}dx - \lambda\int_\Omega h(x)u\,dx \\
&\geq \frac{\kappa}{\alpha p^+}\|u\|_{\beta(x)}^{p^+} - \epsilon_1 C_3^{\alpha p^+}\|u\|_{\beta(x)}^{\alpha p^+} - C(\epsilon_1)C_4^{q^-}\|u\|_{\beta(x)}^{q^-} - \lambda C_3\|h\|_{\frac{\alpha p^+}{\alpha p^+-1}}\|u\|_{\beta(x)} \\
&\geq \Big(\frac{\kappa}{\alpha p^+}\|u\|_{\beta(x)}^{p^+-1} - \epsilon_1 C_3^{\alpha p^+}\|u\|_{\beta(x)}^{\alpha p^+-1} - C(\epsilon_1)C_4^{q^-}\|u\|_{\beta(x)}^{q^--1} - \lambda C_3\|h\|_{\frac{\alpha p^+}{\alpha p^+-1}}\Big)\|u\|_{\beta(x)},
\end{aligned}
$$

for ϵ_1 sufficiently small. Then

$$
I_\lambda(u) \geq \Big(\frac{\kappa}{2\alpha p^+}\|u\|_{\beta(x)}^{p^+-1} - C(\epsilon_1)C_4^{q^-}\|u\|_{\beta(x)}^{q^--1} - \lambda C_3\|h\|_{\frac{\alpha p^+}{\alpha p^+-1}}\Big)\|u\|_{\beta(x)}
$$

where $C_3, C_4 > 0$ are given by (3.3). Consider the functions $\gamma_1 : [0,+\infty) \to \mathbb{R}$ is given by

$$
\gamma_1(\tau) = \frac{\kappa}{2\alpha p^+}\tau^{p^+-1} - C(\epsilon_1)C_4^{q^-}\tau^{q^--1}
$$

Since $q^- > p^+$, there exists a constant $\tau = \rho > 0$ obeying the relationship $\gamma_1(\rho) = \max_{\tau\in[0,+\infty)}\gamma_1(\tau) > 0$. Taking $\lambda^* = \frac{\gamma_1(\rho)}{2C_3\|h\|_{\frac{\alpha p^+}{\alpha p^+-1}}} > 0$, it then follows that, if $\lambda \in (0,\lambda^*)$, we can choose r and $\rho > 0$ such that $I_\lambda(u) \geq r > 0$ for all $u \in X$ with $\|u\|_{\beta(x)} = \rho$. □

Lemma 3.5. *Assume that* (A1), (A2), (A5), (A6), (A8) *hold. Then there exists a function $e \in X$ with $\|e\|_{\beta(x)} > \rho$ such that $J_\lambda(e) < 0$, where ρ is given by Lemma 3.4.*

Proof. By (A8) for $\epsilon > 0$ there is a constant $M_\epsilon > 0$ such that

$$
|G(x,t)| \leq \epsilon|t|^{r_2(x)} + M_\epsilon \quad \forall (x,t) \in \partial\Omega\times\mathbb{R}.
$$

For each $x \in \Omega$ and $t \in \mathbb{R}$, we define the function $\gamma_2(\tau) = \tau^{-\mu}F(x,\tau t) - F(x,t)$ for all $\tau \geq 1$. Then we deduce from (A5) that

$$
\gamma_2'(\tau) = \tau^{-\mu-1}(f(x,\tau t)\tau t - \mu F(x,\tau t)) \geq 0, \quad \forall \tau \geq 1
$$

thus the function γ_2 is increasing on $[1,+\infty)$ and $\gamma_2(\tau) \geq \gamma_2(1) = 0$ for all $\tau \in [1,+\infty)$. Hence,

$$
F(x,\tau t) \geq \tau^\mu F(x,t), \quad \forall x \in \Omega, \quad t \in \mathbb{R},\ \tau \geq 1. \tag{3.5}
$$

Let $\varphi \in C_0^\infty(\Omega)$ and $\varphi \not\equiv 0$ such that $\int_\Omega F(x,\varphi)\,dx > 0$, by (3.2), (3.7) and (3.5) we have

$$
I_\lambda(\tau\varphi) = \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla\tau\varphi|^{p(x)}\,dx + \int_{\partial\Omega}(\frac{\beta(x)}{p(x)}|\tau\varphi|^{p(x)} + G(x,\tau\varphi))\,d\sigma_x\Big)
$$

$$
\begin{aligned}
&- \int_\Omega F(x,\tau\varphi)\,dx - \lambda \int_\Omega h(x)\tau\varphi\,dx \\
&\le \frac{m\tau^{\alpha p^+}}{(p^-)^\alpha}\Big(\int_\Omega |\nabla\varphi|^{p(x)}\,dx + \int_{\partial\Omega} (\beta(x)|\varphi|^{p(x)} \\
&+ p^-\tau^{-p^+} G(x,\tau\varphi))\,d\sigma_x\Big)^\alpha - \tau^\mu \int_\Omega F(x,\varphi)\,dx - \lambda\tau \int_\Omega h(x)\varphi\,dx \\
&\le \frac{m\tau^{\alpha p^+}}{(p^-)^\alpha}\Big(\int_\Omega |\nabla\varphi|^{p(x)}\,dx + \int_{\partial\Omega} \beta(x)|\varphi|^{p(x)}\,d\sigma_x \\
&+ p^-\epsilon \int_{\partial\Omega} (|u|^{r_2(x)}\,d\sigma_x + p^-\tau^{-p^+} M_\epsilon|\partial\Omega|)\Big)^\alpha \\
&- \tau^\mu \int_\Omega F(x,\varphi)\,dx - \lambda\tau \int_\Omega h(x)\varphi\,dx \to -\infty,
\end{aligned}
$$

as $\tau \to +\infty$, where $m = \widehat{M}(t_0)t_0^{-\alpha}$ since $\mu > \alpha p^+$. Therefore, there exists a constant $\tau_0 > 0$ such that $\|\tau_0\varphi\|_{\beta(x)} > \rho$ and $I_\lambda(\tau_0\varphi) < 0$. Letting $e = \tau_0\varphi$ we complete the proof. $\square$

Lemma 3.6. *Assume that* (A1)–(A5), (A7), (A8) *hold. Then the functional* I_λ *satisfies the Palais-Smale condition.*

Proof. Since $1 < r_2^+ < r_1^+ < p^- \le p(x)$ for all $x \in \partial\Omega$, the embeddings

$$
X \hookrightarrow L^{r_1(x)}(\Omega), \quad X \hookrightarrow L^{r_2(x)}(\Omega),
$$

are continuous, and there existspositive constants C_{r_1}, C_{r_2} such that

$$
\|u\|_{\partial\Omega, r_1(x)} \le C_{r_1}\|u\|_{\beta(x)}, \quad \|u\|_{\partial\Omega, r_2(x)} \le C_{r_2}\|u\|_{\beta(x)}
$$

Let $\{u_n\} \subset X$ be such that

$$
I_\lambda(u_n) \to c \in \mathbb{R}, \quad I'_\lambda(u_n) \to 0 \quad \text{in } X^* \tag{3.6}
$$

where X^* is the dual space of X.

We will prove that $\{u_n\}$ is bounded in X. Indeed, assume by contradiction that $\|u_n\|_{\beta(x)} \to +\infty$ as $n \to \infty$. By (A7) and (A8) for $0 < \epsilon_2, \epsilon_3$ there exist a constant $C(\epsilon_2)$ depending on ϵ_2 and a constant $C(\epsilon_3)$ depending on ϵ_3 such that

$$
|G(x,t)| \le \epsilon_2|t|^{r_1(x)} + C(\epsilon_2)|t|^{r_2(x)}, \quad \forall (x,t) \in \partial\Omega \times \mathbb{R}, \tag{3.7}
$$

$$
|g(x,t)| \le \epsilon_3|t|^{r_1(x)-1} + C(\epsilon_3)|t|^{r_2(x)-1}, \quad \forall (x,t) \in \partial\Omega \times \mathbb{R} \tag{3.8}
$$

By assumptions (A1), (A2), (A5), (A7), (A8) and (2.5), (3.6), applying the Hölder inequality we deduce for n large enough that $\|u_n\|_{\beta(x)} > 1$, we have

$$
\begin{aligned}
&c + 1 + \|u_n\|_{\beta(x)} \\
&\ge I_\lambda(u_n) - \frac{1}{\mu}\langle I'_\lambda u_n, u_n\rangle \\
&= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega} (\frac{\beta(x)}{p(x)}|u_n|^{p(x)} + G(x,u_n))\,d\sigma_x\Big) \\
&\quad - \int_\Omega F(x,u_n)\,dx - \lambda\int_\Omega h(x)u_n\,dx - \frac{1}{\mu}M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx \\
&\quad + \int_{\partial\Omega} (\frac{\beta(x)}{p(x)}|u_n|^{p(x)} + G(x,u_n))\,d\sigma_x\Big)\Big(\int_\Omega |\nabla u_n|^{p(x)}\,dx
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_{\partial\Omega} (\beta(x)|u_n|^{p(x)} + g(x,u_n)u_n)\, d\sigma_x\Big) + \frac{1}{\mu}\int_\Omega f(x,u_n)u_n\, dx \\
&+ \frac{\lambda}{\mu}\int_\Omega h(x)u_n\, dx \\
&\geq \frac{1}{\alpha} M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\, dx + \int_{\partial\Omega}(\frac{\beta(x)}{p(x)}|u_n|^{p(x)} + G(x,u_n))\, d\sigma_x\Big) \\
&\times \Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\, dx + \int_{\partial\Omega}(\frac{\beta(x)}{p(x)}|u_n|^{p(x)} + G(x,u_n))\, d\sigma_x\Big) - \int_\Omega F(x,u_n)\, dx \\
&- \lambda\int_\Omega h(x)u_n\, dx - \frac{1}{\mu} M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\, d + \int_{\partial\Omega}(\frac{\beta(x)}{p(x)}|u_n|^{p(x)} \\
&+ G(x,u_n))\, d\sigma_x\Big)\Big(\int_\Omega |\nabla u_n|^{p(x)}\, dx + \int_{\partial\Omega}(\beta(x)|u_n|^{p(x)} + g(x,u_n)u_n)\, d\sigma_x\Big) \\
&+ \frac{1}{\mu}\int_\Omega f(x,u_n)u_n\, dx + \frac{\lambda}{\mu}\int_\Omega h(x)u_n\, dx \\
&\geq M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\, dx + \int_{\partial\Omega}(\frac{\beta(x)}{p(x)}|u_n|^{p(x)} + G(x,u_n))\, d\sigma_x\Big) \\
&\times \Big(\frac{1}{\alpha}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\, dx + \int_{\partial\Omega}(\frac{\beta(x)}{p(x)}|u_n|^{p(x)} + G(x,u_n))\, d\sigma_x\Big) \\
&- \frac{1}{\mu}\Big(\int_\Omega |\nabla u_n|^{p(x)}\, dx + \int_{\partial\Omega}(\beta(x)|u_n|^{p(x)} + g(x,u_n)u_n)\, d\sigma_x\Big)\Big) - \int_\Omega F(x,u_n)\, dx \\
&- \lambda\int_\Omega h(x)u_n\, dx + \frac{1}{\mu}\int_\Omega f(x,u_n)u_n\, dx + \frac{\lambda}{\mu}\int_\Omega h(x)u_n\, dx \\
&\geq M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\, dx + \int_{\partial\Omega}(\frac{\beta(x)}{p(x)}|u_n|^{p(x)} + G(x,u_n))\, d\sigma_x\Big) \\
&\times \Big(\frac{1}{\alpha}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\, dx + \int_{\partial\Omega}(\frac{\beta(x)}{p(x)}|u_n|^{p(x)} + G(x,u_n))\, d\sigma_x\Big) \\
&- \frac{1}{\mu}\Big(\int_\Omega |\nabla u_n|^{p(x)}\, dx + \int_{\partial\Omega}(\beta(x)|u_n|^{p(x)} + g(x,u_n)u_n)\, d\sigma_x\Big)\Big) \\
&+ \int_\Omega \Big(\frac{1}{\mu}f(x,u_n)u_n - F(x,u_n)\Big)\, dx - \lambda\big(1-\frac{1}{\mu}\big)\int_\Omega h(x)u_n\, dx \\
&\geq \kappa\Big(\frac{1}{\alpha p^+}\rho_{\beta(x)}(u_n) + \frac{1}{\alpha}\int_{\partial\Omega} G(x,u_n)\, d\sigma_x - \frac{1}{\mu}\rho_{\beta(x)}(u_n) - \frac{1}{\mu}\int_{\partial\Omega} g(x,u_n)u_n\, d\sigma_x\Big) \\
&+ \int_\Omega \big(\frac{1}{\mu}f(x,u_n)u_n - F(x,u_n)\big)\, dx - \lambda\big(1-\frac{1}{\mu}\big)\int_\Omega h(x)u_n\, dx \\
&\geq \kappa\Big[\big(\frac{1}{\alpha p^+} - \frac{1}{\mu}\big)\rho_{\beta(x)}(u_n) - \frac{1}{\alpha}\epsilon_2\int_{\partial\Omega}|u_n|^{r_1(x)}d\sigma_x - \frac{1}{\alpha}C(\epsilon_2)\int_{\partial\Omega}|u_n|^{r_2(x)}d\sigma_x \\
&- \frac{1}{\mu}\epsilon_3\int_{\partial\Omega}|u_n|^{r_2(x)-1}d\sigma_x - \frac{1}{\mu}C(\epsilon_3)\int_{\partial\Omega}|u_n|^{r_2(x)}d\sigma_x\Big] \\
&+ \int_\Omega \Big(\frac{1}{\mu}f(x,u_n)u_n - F(x,u_n)\Big)\, dx - \lambda\big(1-\frac{1}{\mu}\big)\int_\Omega h(x)u_n\, dx \\
&\geq \kappa\big(\frac{1}{\alpha p^+} - \frac{\kappa}{\mu}\big)\|u_n\|_{\beta(x)}^{p^-} - \frac{\kappa}{\alpha}\epsilon_2 C_{r_1}^{r_1^-}\|u_n\|_{\beta(x)}^{r_1^-} - \frac{\kappa}{\alpha}C(\epsilon_2)C_{r_2}^{r_2^-}\|u_n\|_{\beta(x)}^{r_1^-}
\end{aligned}
$$

$$-\frac{\kappa}{\mu}\epsilon_3 C_{r_1}^{r_1^-}\|u_n\|_{\beta(x)}^{r_1^-}-\frac{\kappa}{\mu}C_{r_2}^{r_2^-}(\epsilon_3)C\|u_n\|_{\beta(x)}^{r_2^-}-\lambda C_3\big(1-\frac{1}{\mu}\big)\|h\|_{\frac{\alpha p^+}{\alpha p^+-1}}\|u_n\|_{\beta(x)},$$

where $\mu > \alpha p^+$ and $\kappa > 0$. Dividing by $\|u\|_{\beta(x)}^{p^-}$in the above inequality and passing to the limit as $n \to \infty$, we obtain a contradiction. This follows that the sequence $\{u_n\}$ is bounded in X.

Now, since the Banach space X is reflexive, there exists $u \in X$ such that passing to a subsequence, still denoted by $\{u_n\}$, it converges weakly to u in X and converges strongly to u in the spaces $L^{q(x)}(\Omega)$. Using the condition (A3) and Hölder inequality, we have

$$\begin{aligned}
\big|\int_\Omega f(x,u_n)(u_n-u)\,dx\big| &\le \int_\Omega |f(x,u_n)|\,|u_n-u|\ dx\\
&\le C_1\int_\Omega \Big(1+|u_n|^{q(x)-1}\Big)\,|u_n-u|\ dx\\
&\le 2C_1\Big(1+\||u_n|^{q(x)-1}\|_{\frac{q(x)}{q(x)-1}}\Big)\|u_n-u\|_{q(x)}\\
&\to 0 \quad \text{as } n\to\infty
\end{aligned}$$

which yields

$$\lim_{m\to\infty}\int_\Omega f(x,u_n)(u_n-u)\,dx = 0\,. \tag{3.9}$$

Moreover,

$$\begin{aligned}
\big|\int_\Omega h(x)(u_n-u)\,dx\big| &\le \int_\Omega |h(x)|\,|u_n-u|\ dx\\
&\le 2\|h\|_{\frac{\alpha p^+}{\alpha p^+-1}}\|u_n-u\|_{\alpha p^+}\to 0 \quad \text{as } n\to\infty.
\end{aligned} \tag{3.10}$$

Since $\{u_n\}$ converges weakly to u in X, by (3.6) we have $I'_\lambda(u_n)(u_n-u)\to 0$ as $n\to\infty$ or

$$\begin{aligned}
&M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx+\int_{\partial\Omega}(\frac{\beta(x)}{p(x)}|u_n|^{p(x)}+G(x,u_n))\,d\sigma_x\Big)\\
&\times\Big(\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\,(\nabla u_n-\nabla u)\ dx+\int_{\partial\Omega}\beta(x)|u_n|^{p(x)-2}u_n(u_n-u)\,d\sigma_x\\
&\quad+\int_{\partial\Omega} g(x,u_n)(u_n-u)\,d\sigma_x\Big)-\int_\Omega f(x,u_n)(u_n-u)\,dx-\lambda\int_\Omega h(x)(u_n-u)\,dx\\
&\to 0,
\end{aligned}$$

which from (3.9) and (3.10) leads to

$$\begin{aligned}
&M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx+\int_{\partial\Omega}\frac{\beta(x)}{p(x)}|u_n|^{p(x)}\,d\sigma_x+\int_{\partial\Omega}G(x,u_n)\,d\sigma_x\Big)\\
&\times\Big(\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\,(\nabla u_n-\nabla u)\ dx+\int_{\partial\Omega}\beta(x)|u_n|^{p(x)-2}u_n(u_n-u)\,d\sigma_x\\
&+\int_{\partial\Omega} g(x,u_n)(u_n-u)\,d\sigma_x\Big)\to 0
\end{aligned}$$

If

$$\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx+\int_{\partial\Omega}\frac{\beta(x)}{p(x)}|u_n|^{p(x)}\,d\sigma_x+\int_{\partial\Omega}G(x,u_n)\,d\sigma_x\Big)\to 0,$$

then by (A9),

$$\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u_n|^{p(x)}\sigma_x\Big) \to 0.$$

This implies that

$$\int_\Omega |\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega} \beta(x)|u_n|^{p(x)}\,d\sigma_x \to 0$$

as $n \to \infty$ and thus $u_n \to 0$ strongly in X as $n \to \infty$. If

$$\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u_n|^{p(x)}\,d\sigma_x + \int_{\partial\Omega} G(x,u_n)\,d\sigma_x\Big) \to t_1 > 0$$

as $n \to \infty$ then from the continuity of M it follows that

$$M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u_n|^{p(x)}\,d\sigma_x + \int_{\partial\Omega} G(x,u_n)\,d\sigma_x\Big) \to M(t_1) > 0,$$

so that

$$M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u_n|^{p(x)}\,d\sigma_x + \int_{\partial\Omega} G(x,u_n)\,d\sigma_x\Big) \ge \frac{1}{2}M(t_1) > 0$$

for all n large enough. Therefore,

$$\begin{aligned}&\Big(\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\left(\nabla u_n - \nabla u\right)\,dx + \int_{\partial\Omega} \beta(x)|u_n|^{p(x)-2}u_n(u_n-u)\,d\sigma_x \\ &+ \int_{\partial\Omega} g(x,u_n)(u_n-u)\,d\sigma_x\Big) \to 0\end{aligned}$$

by (3.8) and Hölder inequaly, we have

$$\begin{aligned}&|\int_{\partial\Omega} g(x,u_n)(u_n-u)\,dx| \\ &\le \int_{\partial\Omega} |g(x,u_n)|\,|u_n-u|\,dx \\ &\le \int_{\partial\Omega} \Big(\epsilon_3|t|^{r_1(x)-1} + C(\epsilon_3)|t|^{r_2(x)-1}\Big)\,|u_n-u|\,dx \\ &\le 2\epsilon_3\||u_n|^{r_1(x)-1}\|_{\frac{r_1(x)}{r_1(x)-1}}\|u_n-u\|_{r_1(x)} + 2C(\epsilon_3)\||u_n|^{r_2(x)-1}\|_{\frac{r_2(x)}{r_2(x)-1}}\|u_n-u\|_{r_2(x)} \\ &\to 0 \quad \text{as } n\to\infty\,.\end{aligned}$$

Then

$$\lim_{n\to\infty}\Big(\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\left(\nabla u_n - \nabla u\right)\,dx + \int_{\partial\Omega} \beta(x)|u_n|^{p(x)-2}u_n(u_n-u)\,d\sigma_x\Big) = 0$$

or

$$\lim_{n\to\infty} L'_{\beta(x)}(u_n)(u_n-u) = 0,$$

where $L_{\beta(x)}$ and $L'_{\beta(x)}$ are given by (2.6) and (2.7). From Proposition 2.4 the sequence $\{u_n\}$ converges strongly to u as $n \to \infty$. Thus, the functional I_λ satisfies the Palais-Smale condition. □

Lemma 3.7. *Assume that $h \in L^{\frac{\alpha p^+}{\alpha p^+-1}}(\Omega)$ with $h \not\equiv 0$, and that* (A1)–(A8) *hold. Then there exists a function $\psi \in X, \psi \not\equiv 0$ such that $I_\lambda(\tau\psi) < 0$ for all $\tau > 0$ small enough.*

Proof. For $(x,t)\in\Omega\times\mathbb{R}$, set $\gamma_3(\tau)=F\left(x,\tau^{-1}t\right)\tau^{\mu},\tau\geq 1$. By (A5), we have

$$\begin{aligned}\gamma_3'(\tau)&=f\left(x,\tau^{-1}t\right)\left(-\frac{t}{\tau^2}\right)\tau^{\mu}+F\left(x,\tau^{-1}t\right)\mu\tau^{\mu-1}\\&=\tau^{\mu-1}\left[\mu F\left(x,\tau^{-1}t\right)-\tau^{-1}tf\left(x,\tau^{-1}t\right)\right]\leq 0\end{aligned}$$

so, $\gamma_3(t)$ is non-increasing. Thus, for any $|t|\geq 1$, we have $\gamma_3(1)\geq\gamma_3(|t|)$, that is

$$F(x,t)\geq F\big(x,|t|^{-1}t\big)|t|^{\mu}\geq C_6|t|^{\mu} \tag{3.11}$$

where $C_6=\inf_{x\in\Omega,|t|=1}F(x,t)>0$ by (A6). From (A4), there exists a constant $\eta>0$ such that

$$\Big|\frac{f(x,t)t}{|t|^{\alpha p^+}}\Big|=\Big|\frac{f(x,t)}{|t|^{\alpha p^+-1}}\Big|\leq 1 \tag{3.12}$$

for all $x\in\Omega$ and all $0<|t|\leq\eta$. By (A3), for all $x\in\Omega$ and all $\eta\leq|t|\leq 1$, there exists $C_7>0$ such that

$$\Big|\frac{f(x,t)t}{|t|^{\alpha p^+}}\Big|\leq\frac{C\left(1+|t|^{q(x)-1}\right)|t|}{|t|^{\alpha p^+}}\leq C_7 \tag{3.13}$$

From (3.12) and (3.13), we deduce that

$$f(x,t)t\geq-\left(C_7+1\right)|t|^{\alpha p^+}$$

for all $x\in\Omega$ and all $|t|\in[0,1]$. Using the equality $F(x,t)=\int_0^1 f(x,\tau t)td\tau$, we obtain

$$F(x,t)\geq-\frac{1}{\alpha p^+}\left(C_7+1\right)|t|^{\alpha p^+} \tag{3.14}$$

for all $x\in\Omega$ and all $|t|\in[0,1]$. Taking $C_8=\frac{1}{\alpha p^+}\left(C_7+1\right)+C_6$, then from (3.11) and (3.14) we obtain that

$$F(x,t)\geq C_6|t|^{\mu}-C_8|t|^{\alpha p^+} \tag{3.15}$$

for all $x\in\Omega$ and all $t\in\mathbb{R}$.

We now prove that there exists a function $\psi\in X$ such that $J_\lambda(\tau\psi)<0$ for all $\tau>0$ small enough. Since $h\in L^{\frac{\alpha p^+}{\alpha p^+-1}}(\Omega)$ and $h\not\equiv 0$, we can choose a function $\psi\in X$ be such that

$$\int_\Omega h(x)\psi(x)\,dx>0\,.$$

If

$$\int_\Omega\frac{1}{p(x)}|\nabla\tau\psi|^{p(x)}\,dx+\int_{\partial\Omega}\big(\frac{\beta(x)}{p(x)}|\tau\psi|^{p(x)}+G(x,\tau\psi)\big)\,d\sigma_x\geq t_0$$

with t_0 defined in (3.1) and (3.2), then by (3.15) we have

$$\begin{aligned}I_\lambda(\tau\psi)&=\widehat{M}\Big(\int_\Omega\frac{1}{p(x)}|\nabla\tau\psi|^{p(x)}\,dx+\int_{\partial\Omega}\big(\frac{\beta(x)}{p(x)}|\tau\psi|^{p(x)}+G(x,\tau\psi)\big)\,d\sigma_x\Big)\\&\quad-\int_\Omega F(x,\tau\psi)\,dx-\lambda\int_\Omega h(x)\tau\psi\,dx\\&\leq m\Big(\int_\Omega\frac{1}{p(x)}|\nabla\tau\psi|^{p(x)}\,dx+\int_{\partial\Omega}\big(\frac{\beta(x)}{p(x)}|\tau\psi|^{p(x)}+G(x,\tau\psi)\big)\,d\sigma_x\Big)^{\alpha}\\&\quad-C_8\tau^{\mu}\int_\Omega|\psi|^{\mu}\,dx+C_6\tau^{\alpha p^+}\int_\Omega|\psi|^{\alpha p^+}\,dx-\lambda\tau\int_\Omega h(x)\psi\,dx\end{aligned}$$

$$
\begin{aligned}
&\le \frac{m\tau^{\alpha p^+}}{(p^-)^\alpha}\Big(\int_\Omega |\nabla\varphi|^{p(x)}\,dx + \int_{\partial\Omega}(\beta(x)|\varphi|^{p(x)} + p^-\tau^{-p^+}G(x,\tau\varphi))\,d\sigma_x\Big)^\alpha \\
&\quad - C_8\tau^\mu\int_\Omega |\psi|^\mu\,dx + C_6\tau^{\alpha p^+}\int_\Omega|\psi|^{\alpha p^+}\,dx - \lambda\tau\int_\Omega h(x)\psi\,dx \\
&\le \frac{m\tau^{\alpha p^+}}{(p^-)^\alpha}\Big[\int_\Omega |\nabla\varphi|^{p(x)}\,dx + \int_{\partial\Omega}\beta(x)|\varphi|^{p(x)}\,d\sigma_x + p^-\tau^{-p^+}\epsilon_2\int_{\partial\Omega}|u|^{r_1(x)}\,d\sigma_x \\
&\quad + p^-\tau^{-p^+}C(\epsilon_2)\int_{\partial\Omega}|u|^{r_2(x)}\,d\sigma_x\Big]^\alpha - C_8\tau^\mu\int_\Omega|\psi|^\mu\,dx \\
&\quad + C_6\tau^{\alpha p^+}\int_\Omega|\psi|^{\alpha p^+}\,dx - \lambda\tau\int_\Omega h(x)\psi\,dx < 0
\end{aligned}
$$

for all $\tau > 0$ and ϵ_2 small enough. If

$$
\int_\Omega \frac{1}{p(x)}|\nabla\tau\psi|^{p(x)}\,dx + \int_{\partial\Omega}(\frac{\beta(x)}{p(x)}|\tau\psi|^{p(x)} + G(x,\tau\psi))\,d\sigma_x \le t_0
$$

by $M(0) = 0$ and the continuity of M, there exists $m_0 > 0$ such that $M(t) \le m_0$ for all $t \in [0, t_0]$. Then

$$
\widehat{M}(t) \le m_0 t \quad \forall t \le t_0. \tag{3.16}
$$

Then by (3.15) and (3.16), we have

$$
\begin{aligned}
I_\lambda(\tau\psi) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla\tau\psi|^{p(x)}\,dx + \int_{\partial\Omega}(\frac{\beta(x)}{p(x)}|\tau\psi|^{p(x)} + G(x,\tau\psi))\,d\sigma_x\Big) \\
&\quad - \int_\Omega F(x,\tau\psi)\,dx - \lambda\int_\Omega h(x)\tau\psi\,dx \\
&\le m_0\Big(\int_\Omega \frac{1}{p(x)}|\nabla\tau\psi|^{p(x)}\,dx + \int_{\partial\Omega}(\frac{\beta(x)}{p(x)}|\tau\psi|^{p(x)} + G(x,\tau\psi))\,d\sigma_x\Big) \\
&\quad - C_8\tau^\mu\int_\Omega|\psi|^\mu\,dx + C_6\tau^{\alpha p^+}\int_\Omega|\psi|^{\alpha p^+}\,dx - \lambda\tau\int_\Omega h(x)\psi\,dx \\
&\le \frac{m_0\tau^{p^+}}{p^-}\Big(\int_\Omega |\nabla\varphi|^{p(x)}\,dx + \int_{\partial\Omega}(\beta(x)|\varphi|^{p(x)} + p^-\tau^{-p^+}G(x,\tau\varphi))\,d\sigma_x\Big) \\
&\quad - C_8\tau^\mu\int_\Omega|\psi|^\mu\,dx + C_6\tau^{\alpha p^+}\int_\Omega|\psi|^{\alpha p^+}\,dx - \lambda\tau\int_\Omega h(x)\psi\,dx \\
&\le \frac{m_0\tau^{p^+}}{p^-}\Big(\int_\Omega |\nabla\varphi|^{p(x)}\,dx + \int_{\partial\Omega}\beta(x)|\varphi|^{p(x)}\,d\sigma_x + p^-\tau^{-p^+}\epsilon_2\int_{\partial\Omega}|u|^{r_1(x)}\,d\sigma_x \\
&\quad + p^-\tau^{-p^+}C(\epsilon_2)\int_{\partial\Omega}|u|^{r_2(x)}\,d\sigma_x\Big) - C_8\tau^\mu\int_\Omega|\psi|^\mu\,dx + C_6\tau^{\alpha p^+}\int_\Omega|\psi|^{\alpha p^+}\,dx \\
&\quad - \lambda\tau\int_\Omega h(x)\psi\,dx < 0
\end{aligned}
$$

for all $\tau > 0$ and ϵ_2 smalls enough. □

Proof of Theorem 3.1. By Lemmas 3.4–3.7, there exists $\lambda^* > 0$ such that for if $\lambda \in (0, \lambda^*)$, all assumptions of the mountain pass theorem by Ambrosetti-Rabinowitz [5] hold. Then, there exists a critical point $u_1 \in X$ of the functional I_λ, i.e. $I'_\lambda(u_1) = 0$ and thus, problem (1.1) has a nontrivial weak solution $u_1 \in X$ with positive energy

$$
I_\lambda(u_1) = \bar{c} := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda(\gamma(t)) > 0
$$

where $\Gamma := \{\gamma \in C([0,1],X) : \gamma(0)=0, \gamma(1)=e\}$ and the function e is given by Lemma 3.5. In the case

$$\inf_{n\in\mathbb{N}} \Big(\int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} |u_n|^{p(x)}\,d\sigma_x + \int_{\partial\Omega} G(x,u_n)\,d\sigma_x \Big) = 0,$$

by (A9),

$$\inf_{n\in\mathbb{N}} \Big(\int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} |u_n|^{p(x)}\,d\sigma_x \Big) = 0\,.$$

Here, either 0 is an accumulation point for the real sequence

$$\Big\{ \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} |u_n|^{p(x)}\,d\sigma_x \Big\}_n = \{v_n\}_n$$

and so there is a subsequence of $\{v_n\}_n$ strongly converging to $v=0$ and by (2.4) there exists a subcequence of $\{\|u_n\|_{\beta(x)}\}_n$ strongly converging to $u=0$, or 0 is an isolated point of

$$\Big\{ \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} |u_n|^{p(x)}\,d\sigma_x + \int_{\partial\Omega} G(x,u_{n_k})\,d\sigma_x \Big\}_n.$$

The first case can not occur since it implies that the trivial solution is a critical point at level $\bar{c}$. This is impossible, being $0 = I_\lambda(0) = \bar{c} > 0$. Hence only the latter case can occur, so that there is a subsequence, denoted by

$$\Big\{ \int_\Omega \frac{1}{p(x)} |\nabla u_{n_k}|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} |u_{n_k}|^{p(x)}\,d\sigma_x + \int_{\partial\Omega} G(x,u_{n_k})\,d\sigma_x \Big\}_k,$$

such that

$$\inf_{k\in\mathbb{N}} \Big\{ \int_\Omega \frac{1}{p(x)} |\nabla u_{n_k}|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} |u_{n_k}|^{p(x)}\,d\sigma_x + \int_{\partial\Omega} G(x,u_{n_k})\,d\sigma_x \Big\} > 0$$

and we can proceed as before. □

Proof of Theorem 3.2. If $\beta(x) \equiv 0$, then we can proceed as before using the usual norm on $W^{1,p(x)}(\Omega)$ instead of $\|\cdot\|_{\beta(x)}$, and the propositions 2.1 and 2.2 instead of the proposition 2.3 and 2.4. □

Next we show the existence of the second nontrivial weak solution $u_2 \in X$ and $u_2 \neq u_1$ by using the Ekeland variational principle. Indeed, by Lemma 3.4 it follows that on the boundary of the ball centered at the origin and of radius ρ in X, denoted by $B_\rho(0)$, we have

$$\inf_{u\in\partial B_\rho(0)} I_\lambda(u) > 0$$

On the other hand, by Lemma 3.4 again, the functional I_λ is bounded from below on $B_\rho(0)$. Moreover, by Lemma 3.7 there exists $\varphi \in X$ such that $J_\lambda(\tau\varphi) < 0$ for all τ small enough. It follows that

$$-\infty < \underline{c} = \inf_{u\in\bar{B}_\rho(0)} I_\lambda(u) < 0$$

Let us choose $\epsilon > 0$ such that

$$0 < \epsilon < \inf_{u\in\partial B_\rho(0)} I_\lambda(u) - \inf_{u\in\bar{B}_\rho(0)} I_\lambda(u)$$

Applying the Ekeland variational principle in [20] to the functional $I_\lambda : \bar{B}_\rho(0) \to \mathbb{R}$, it follows that there exists $u_\epsilon \in \bar{B}_\rho(0)$ such that

$$I_\lambda(u_\epsilon) < \inf_{u \in \bar{B}_\rho(0)} I_\lambda(u) + \epsilon$$

$$I_\lambda(u_\epsilon) < I_\lambda(u) + \epsilon \|u - u_\epsilon\|_{\beta(x)}, \quad u \neq u_\epsilon.$$

Then, we have $I_\lambda(u_\epsilon) < \inf_{u \in \partial B(0)} I_\lambda(u)$ and thus, $u_\epsilon \in B_\rho(0)$.

Now, we define the functional $J_\lambda : \bar{B}_\rho(0) \to \mathbb{R}$ by $J_\lambda(u) = I_\lambda(u) + \epsilon\|u - u_\epsilon\|_{\beta(x)}$. It is clear that u_ϵ is a minimum point of J_λ and thus

$$\frac{J_\lambda(u_\epsilon + \tau v) - J_\lambda(u_\epsilon)}{t} \geq 0$$

for all $\tau > 0$ small enough and all $v \in B_\rho(0)$. The above information shows that

$$\frac{I_\lambda(u_\epsilon + \tau v) - I_\lambda(u_\epsilon)}{\tau} + \epsilon\|v\|_{\beta(x)} \geq 0$$

Letting $\tau \to 0^+$, we deduce that

$$\langle I'_\lambda(u_\epsilon), v\rangle \geq -\epsilon\|v\|_{\beta(x)}$$

It should be noticed that $-v$ also belongs to $B_\rho(0)$, so replacing v by $-v$, we obtain

$$\langle I'_\lambda(u_\epsilon), -v\rangle \geq -\epsilon\|-v\|_{\beta(x)}$$

or

$$\langle I'_\lambda(u_\epsilon), v\rangle \leq \epsilon\|v\|_{\beta(x)}$$

which helps us to deduce that $\|I'_\lambda(u_\epsilon)\|_{X^*} \leq \epsilon$. Therefore, there exists a sequence $\{u_n\} \subset B_\rho(0)$ such that

$$I_\lambda(u_n) \to \underline{c} = \inf_{u \in \bar{B}_\rho(0)} I_\lambda(u) < 0 \quad \text{and} \quad I'_\lambda(u_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty. \tag{3.17}$$

Based on Lemma 3.7 the sequence $\{u_n\}$ converges strongly to some u_2 as $n \to \infty$. Moreover, since $I_\lambda \in C^1(X, \mathbb{R})$, by (3.17) it follows that $I'_\lambda(u_2) = 0$. Thus, u_2 is a non-trivial weak solution of problem (1.1) with negative energy $I_\lambda(u_2) = \underline{c} < 0$.

Finally, we point out that $u_1 \neq u_2$ since $I_\lambda(u_1) = \bar{c} > 0 > \underline{c} = I_\lambda(u_2)$. The proof is complete.

We can do the same for the problem (1.2).

3.2. **Existence of infinitely many solutions.** The purpose of this part is to consider problem (1.1) in the case $\lambda = 0$. Under some suitable conditions on M and f, we prove the existence of infinitely many solutions for (1.1) by using the Krasnoselskii's genus theory [12, Prop 1.4]. We introduce the following assumptions:

(A10) $f : \Omega \to \mathbb{R}$ is a continuous function such that

$$A_1 k(x)|t|^{s(x)-1} \leq f(x,t) \leq A_2 k(x)|t|^{s(x)-1}, \quad \forall (x,t) \in \Omega \times \mathbb{R}^+,$$

where $A_1, A_2 > 0$ are positive constants and $s \in C_+(\bar{\Omega})$ such that $1 < s(x) < p^*(x)$ for all $x \in \bar{\Omega}$, the function $k \equiv 1$ if $p(x) \leq s(x) < p^*(x)$ for all $x \in \bar{\Omega}$ while $k \in L_+^{s_0(x)}(\Omega)$ with $s_0(x) = \frac{p(x)}{p(x)-s(x)}$ if $1 < s(x) < p(x)$ for all $x \in \bar{\Omega}$

(A11) $f(x,-t) = -f(x,t)$ for all $(x,t) \in \Omega \times \mathbb{R}$;

(A12) $g(x,-t) = -g(x,t)$ for all $(x,t) \in \partial\Omega \times \mathbb{R}$;

Then we have the following result.

Theorem 3.8. *Let $M(0)=0$ and* (A1)–(A12) *hold. Then* (1.1) *with $\lambda=0$ has infinitely many weak solutions.*

With similar arguments as those used in the proof of Theorem 3.1, by assumption (A10), we can show that the functional $I_0 : X \to \mathbb{R}$ defined by

$$I_0(u)=\widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx+\int_{\partial\Omega}(\frac{\beta(x)}{p(x)}|u|^{p(x)}+G(x,u))\,d\sigma_x\Big)-\int_\Omega F(x,u)\,dx$$

is of class C^1 on X and its derivative is

$$\begin{aligned}I_0'(u)(v)&=M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx+\int_{\partial\Omega}(\frac{\beta(x)}{p(x)}|u|^{p(x)}+G(x,u))d\sigma_x\Big)\\&\quad\times\Big(\int_\Omega|\nabla u|^{p(x)-2}\nabla u\nabla v\,dx+\int_{\partial\Omega}\beta(x)|u|^{p(x)-2}uv\,d\sigma_x+\int_{\partial\Omega}g(x,u)u\,d\sigma_x\Big)\\&\quad-\int_\Omega f(x,u)v\,dx\end{aligned}$$

for all $u,v\in X$. Thus, weak solutions of (1.1) with $\lambda=0$ are exactly the critical points of I_0.

Lemma 3.9. *Assume that* (A1)–(A6), (A9), (A10) *hold. Then the functional I_0 is bounded from below on X and satisfies the Palais-Smale condition.*

Proof. Since $1<s(x)<p^+(x)$ for all $x\in\bar{\Omega}$, the embedding $X\hookrightarrow L^{s(x)}(\Omega)$ is continuous and compact, then there exists $C_9>0$ such that

$$\|u\|_{s(x)}\le C_9\|u\|_{\beta(x)},\quad \forall u\in X$$

By (A1), (A2), (A9), (A10) and the Hölder iniquality, it follows from the definition of the functional $I_0,(\lambda=0)$ that

$$\begin{aligned}I_0(u)&=\widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx+\int_{\partial\Omega}(\frac{\beta(x)}{p(x)}|u|^{p(x)}+G(x,u))\,d\sigma_x\Big)-\int_\Omega F(x,u)\,dx\\&\ge\frac{1}{\alpha}M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx+\int_{\partial\Omega}(\frac{\beta(x)}{p(x)}|u|^{p(x)}+G(x,u))\,d\sigma_x\Big)\\&\quad\times\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx+\int_{\partial\Omega}(\frac{\beta(x)}{p(x)}|u|^{p(x)}+G(x,u))\,d\sigma_x\Big)-\int_\Omega F(x,u)\,dx\\&\ge\frac{\kappa}{\alpha}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx+\int_{\partial\Omega}\frac{\beta(x)}{p(x)}|u|^{p(x)}\,d\sigma_x\Big)-\int_\Omega F(x,u)\,dx\\&\ge\frac{\kappa}{\alpha p^+}\|u\|_{\beta(x)}^{p^-}-\frac{A_2C_9^{s^+}}{s^-}\|k\|_{s_0}\|u\|_{\beta(x)}^{s^+},\end{aligned}$$

for all $u\in X$ with $\|u\|_{\beta(x)}>1$ large enough. Since we always have that $s^+<p^-$, I_0 is coercive, i.e. $I_0(u)\to+\infty$ as $\|u\|_{\beta(x)}\to+\infty$ and bounded from below on X.

From these statements, if $\{u_n\}$ is a Palais-Smale sequence for the functional I_0, i.e. $I(u_n)\to\bar{c}$, $I'(u_n)\to 0$ in X^*. Then $\{u_n\}$ is bounded in X. Since X is a reflexive Banach space, $\{u_n\}$ has a subsequence, still denoted by $\{u_n\}$, that converges weakly to some $u\in X$. Moreover, the embedding $X\hookrightarrow L^{s(x)}(\Omega)$ is continuous and compact, using (A10) and the Hölder inequality, we have

$$\Big|\int_\Omega f(x,u_n)(u_n-u)\,dx\Big|\le\int_\Omega|f(x,u_n)|\,|u_n-u|\,dx$$

$$
\begin{aligned}
&\le A_2 \int_\Omega h(x)|u_n|^{s(x)-1}\,|u_n-u|\,dx\\
&\le 2A_2\|h\|_{s_0(x)}\||u_n|^{s(x)-1}\|_{\frac{s(x)}{s(x)-1}}\|u_n-u\|_{s(x)}\\
&\to 0 \quad \text{as } n\to\infty.
\end{aligned}
$$

Then

$$
\lim_{m\to\infty}\int_\Omega f(x,u_n)(u_n-u)\,dx = 0\,. \tag{3.18}
$$

From (3.18), with similar arguments as those presented in the proof of Lemma 3.7, we can show that $\{u_n\}$ converges strongly to $u\in X$ and thus, the functional I_0 satisfies the Palais-Smale condition. □

Proof of Theorem 3.8. We known that for $p\in C_+(\bar\Omega)$ and $1<p^-\le p^+<N$, $X=W^{1,p(x)}(\Omega)$ is a separable and reflexive Banach space. Then there exist $\{e_n\}\subset X$ and $\{e_n^*\}\subset X^*$ such that

$$
\langle e_i^*, e_j\rangle = \begin{cases} 1, & i=j\\ 0, & i\ne j\end{cases},
$$

$$
X=\overline{\operatorname{span}\{e_n: n=1,2,\dots\}},\quad X^*=\overline{\operatorname{span}\{e_n^*: n=1,2,\dots\}}.
$$

For each $k\in\mathbb{N}$, consider $X_j=\operatorname{span}\{e_1,e_2,\dots,e_j\}$, the subspace if X spanned by the vectors $e_1,e_2,\dots,e_j$. Let $k\in L^{s_0(x)}(\Omega)$ for all $x\in\bar\Omega$, we define a norm $\|\cdot\|_{L^{s(x)}(\Omega,k(x))}$ on the space X_j sa follows

$$
\|u\|_{L^{s(x)}(\Omega,k(x))} := \inf\{\lambda>0; \int_\Omega k(x)\Big|\frac{u(x)}{\lambda}\Big|^{s(x)}\,dx\le 1\}.
$$

Note that the embedding $X_j\hookrightarrow L^{s(x)}(\Omega)$, $1<s(x)<p(x)$ is continuous. Since all norms on the finite dimensional space X_j are equivalent, so are the norms $\|\cdot\|_{\beta(x)}$ and $\|\cdot\|_{L^{s(x)}(\Omega,k(x))}$. Moreover, for any $u\in X_j$, it follows that if

$$
\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega}\big(\frac{\beta(x)}{p(x)}|u|^{p(x)} + G(x,u)\big)\,d\sigma_x \ge t_0
$$

with t_0 defined in (3.1) and (3.2), by (3.2) and (3.7) for ϵ_2 small enough we have

$$
\begin{aligned}
I_0(u) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega}\big(\frac{\beta(x)}{p(x)}|u|^{p(x)} + G(x,u)\big)\,d\sigma_x\Big) - \int_\Omega F(x,u)\,dx\\
&\le m\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega}\big(\frac{\beta(x)}{p(x)}|u|^{p(x)} + G(x,u)\big)\,d\sigma_x\Big)^\alpha\\
&\quad - \frac{A_1}{s^+}\int_\Omega k(x)|u|^{s(x)}\,dx\\
&\le \frac{m}{(p^-)^\alpha}\Big(\int_\Omega |\nabla u|^{p(x)}\,dx + \int_{\partial\Omega}\beta(x)|u|^{p(x)}\,d\sigma_x + p^-C(\epsilon_2)\int_{\partial\Omega}|u|^{r_2(x)}d\sigma_x\Big)^\alpha\\
&\quad \times \frac{A_1}{s^+}\int_\Omega k(x)|u|^{s(x)}\,dx\\
&\le \frac{m}{(p^-)^\alpha}(\|u\|_{\beta(x)}^{p^-} + C_{10}(\epsilon_2)\|u\|^{r_2^+})^\alpha - \frac{A_1}{s^+}C(j)\|u\|_{\beta(x)}^{s^+}\\
&= \frac{m}{(p^-)^\alpha}\|u\|_{\beta(x)}^{\alpha r_2^+}\Big(\|u\|_{\beta(x)}^{p^- - r_2^+} + C_{10}(\epsilon_2)\Big)^\alpha - \frac{A_1}{s^+}C(j)\|u\|_{\beta(x)}^{s^+}
\end{aligned}
$$

$$= \|u\|_{\beta(x)}^{s^+}\Big(\frac{m}{(p^-)^\alpha}\|u\|_{\beta(x)}^{\alpha r_2^+ - s^+}(\|u\|_{\beta(x)}^{p^- - r_2^+} + C_{10}(\epsilon_2))^\alpha - \frac{A_1}{s^+}C(j)\Big),$$

where $C(j)$ is a positive constant depending on j. For each $j \in \mathbb{N}$ as before, let us denote by R_j the positive constant such that

$$\frac{m}{(p^-)^\alpha}\rho_j^{\alpha r_2^+ - s^+}(\rho_j^{p^- - r_2^+} - C_{10}(\epsilon_2))^\alpha < \frac{A_1}{s^+}C(j)$$

then, for all $0 < \rho_j < R_j$, and $u \in S_{\rho_j} := \{u \in X_j : \|u\|_{\beta(x)} = \rho_j\}, S_{\rho_j}$ is a closed subset of $X\backslash\{0\}$ that is symmetric with respect to the origin. We obtain

$$\begin{aligned} I_0(u) &\le \rho_j^{s^+}\Big(\frac{m}{(p^-)^\alpha}\rho_j^{\alpha r_2^+ - s^+}(\rho_j^{p^- - r_2^+} - C_{10}(\epsilon_2))^\alpha - \frac{A_1}{s^+}C(j)\Big) \\ &\le R_j^{s^+}\Big(\frac{m}{(p^-)^\alpha}R_j^{\alpha r_2^+ - s^+}(R_j^{p^- - r_2^+} - C_{10}(\epsilon_2))^\alpha - \frac{A_1}{s^+}C(j)\Big) < 0 = I_0(0). \end{aligned}$$

If

$$\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega}(\frac{\beta(x)}{p(x)}|u|^{p(x)} + G(x,u))\,d\sigma_x \le t_0,$$

then by (3.16),

$$\begin{aligned} I_0(u) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega}(\frac{\beta(x)}{p(x)}|u|^{p(x)} + G(x,u))\,d\sigma_x\Big) - \int_\Omega F(x,u)\,dx \\ &\le m_0\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega}(\frac{\beta(x)}{p(x)}|u|^{p(x)} + G(x,u))\,d\sigma_x\Big) \\ &\quad - \frac{A_1}{s^+}\int_\Omega k(x)|u|^{s(x)}\,dx \\ &\le \frac{m}{(p^-)^\alpha}(\|u\|_{\beta(x)}^{p^-} + C_{10}(\epsilon_2)\|u\|^{r_2^+}) - \frac{A_1}{s^+}C(j)\|u\|_{\beta(x)}^{s^+} \\ &\le \|u\|_{\beta(x)}^{s^+}\Big(\frac{m_0}{(p^-)^\alpha}(\|u\|_{\beta(x)}^{p^- - s^+} + C_{10}(\epsilon_2)\|u\|^{r_2^+ - s^+}) - \frac{A_1}{s^+}C(j)\Big) \end{aligned}$$

where $C(j)$ is a positive constant depending on j. For each $j \in \mathbb{N}$ as before, let us denote by R'_j the positive constant such that

$$\frac{m_0}{(p^-)^\alpha}(\rho'_j)^{p^- - s^+} - C_{10}(\epsilon_2)(\rho'_j)^{r^+ - s^+}) < \frac{A_1}{s^+}C(j).$$

Then, for all $0 < \rho'_j < R'_j$, and $u \in S_{\rho'_j} := \{u \in X_j : \|u\|_{\beta(x)} = \rho'_j\}$, $S_{\rho'_j}$ is a closed subset of $X\backslash\{0\}$ that is symmetric with respect to the origin. We obtain

$$\begin{aligned} I_0(u) &\le (\rho'_j)^{s^+}\Big(\frac{m_0}{(p^-)^\alpha}(\rho'_j)^{p^- - s^+} - C_{10}(\epsilon_2)(\rho'_j)^{r_2^+ - s^+} - \frac{A_1}{s^+}C(j)\Big) \\ &\le (R'_j)^{s^+}\Big(\frac{m_0}{(p^-)^\alpha}(R'_j)^{p^- - s^+} - C_{10}(\epsilon_2)(R'_j)^{r_2^+ - s^+} - \frac{A_1}{s^+}C(j)\Big) < 0 = I_0(0). \end{aligned}$$

Then in both cases there exists a sphere S_j such that

$$\sup_{u \in S_j} I_0(u) < I_0(0).$$

Because X_j and $\mathbb{R}^j$ are isomorphic and S_j and S^{j-1} are homeomorphic, we conclude that $\gamma(S_j) = j$. Moreover, by (A11) and (A12), I_0 is even. By Proposition 2.7 the functional I_0 has at least j pair of different critical points. Since j is arbitrary, we obtain infinitely many critical points of I_0 and thus problem (1.1) with $\lambda = 0$ has infinitely many weak solutions.

We can prove the same result for $\beta(x) \equiv 0$. □

4. Applications

In this section we give two examples of problems that satisfy the hypotheses of Theorem 3.1 and Theorem 3.8. The first is an example of a non-linear problem with variable exponent and the second is an example of the linear case where $p(x)$ is a constant equal to 2.

4.1. **Non-linear case.** Let us consider the problem

$$\begin{gathered} - a\alpha\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega}\Big(\frac{\beta(x)}{p(x)}|u|^{p(x)} + \frac{\gamma(x)}{r(x)}|u|^{r(x)}\Big)\,d\sigma_x\Big)^{\alpha-1}\Delta_{p(x)}u \\ = k(x)|u|^{s(x)-2}u + \lambda h(x), \quad x \in \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} + \beta(x)|u|^{p(x)-2}u + \gamma(x)|u|^{r(x)-2}u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{4.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\frac{\partial u}{\partial\nu}$ is the outer normal derivative, $d\sigma_x$ is the measure on the boundary $\partial\Omega$, $\beta \in L^1(\partial\Omega)$, $\inf_{x\in\partial\Omega}\beta(x) > 0$, $\gamma \in L^1(\partial\Omega)$, $\inf_{x\in\partial\Omega}\gamma(x) > 0$, $p \in C_+(\bar\Omega)$, $1 < p^- := \inf_{x\in\bar\Omega} p(x) \le p^+ := \max_{x\in\bar\Omega} p(x) < s(x) < p^*(x)$, a and λ are non-negative parameters, $\alpha \ge 1$ and $h : \Omega \to \mathbb{R}$ is a measurable function. Let

$$M(t) = a\alpha t^{\alpha-1}, \quad f(x,t) = |t|^{s(x)-2}t, \quad g(x,t) = \gamma(x)|t|^{r(x)-2}t,$$

with $G(x,t) = \frac{\gamma(x)}{r(x)}|u|^{r(x)}$. It is clear that M satisfies (A1) and (A2); f satisfies (A3)–(A6), (A10), (A11); and g satisfies (A7) and (A8) for every $r(x)$ such that $r_1(x) < r(x) < r_2(x) < p(x)$, and g satisfies (A9) and (A12). Then we can deduce the next propositions

Proposition 4.1. *There exists $\lambda^* > 0$ such that problem* (4.1) *has at least two non-trivial weak solutions when $\lambda \in (0, \lambda^*)$.*

Proposition 4.2. *Problem* (4.1) *with $\lambda = 0$ has infinitely many weak solutions.*

4.2. **Linear case.** Let us consider the problem

$$\begin{gathered} -a\Big(\|u\|^2_{\beta(x)} + \int_{\partial\Omega} e^{-u^2}\,d\sigma_x\Big)\Delta u = |u|u + \lambda h(x), \quad x \in \Omega, \\ \frac{\partial u}{\partial\nu} + \beta(x)u - ue^{-u^2} = 0, \quad x \in \partial\Omega, \end{gathered} \tag{4.2}$$

where Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, $\frac{\partial u}{\partial\nu}$ is the outer normal derivative, $d\sigma_x$ is the measure on the boundary $\partial\Omega$, $\beta \in L^1(\partial\Omega)$, $\inf_{x\in\partial\Omega}\beta(x) > 0$, a and λ are non-negative parameters, $\alpha \ge 1$, and $h : \Omega \to \mathbb{R}$ is a measurable function. Let $M(t) = 2at$,

$$f(x,t) = |t|t \quad t(k \equiv 1 \text{ and } s(x) = 3)$$

and $g(x,t) = -te^{-t^2}$, with $G(x,t) = \frac{1}{2}e^{-t^2}$. Iti s clear that M satisfies (A1) and (A2); f satisfies (A3)–(A6), (A10), (A11); and g satisfies (A7) and (A8) for every $r_1(x) = r_2(x) = r = \frac{3}{2}$, and g satisfies (A9) and (A12).

Proposition 4.3. *There exists $\lambda^* > 0$ such that problem (4.2) has at least two non-trivial weak solutions when $\lambda \in (0, \lambda^*)$.*

Proposition 4.4. *Problem (4.2) with $\lambda = 0$ has infinitely many weak solutions.*

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References

[1] Afrouzi, G. A.; Chung, T. N.; Naghizadeh, Z.; *Muliple solutions for p(x)-Kirchhoff type problems with Robin Boundary conditions*, Electronic Journal of Differential Equations, Vol. 2022 (2022), No. 24, pp. 1-16.

[2] Allaoui, M.; *Existence of solutions for a robin problem involving the $p(x)$-Laplace operator*, Abstr. Apl. Anal., 2016 (2016), 1-8.

[3] Allaoui, M.; *Existence results for a class of $p(x)$-Kirchhoff problems*, Studia Sci. Math. Hungarica, 54 (3) (2017), 316-331.

[4] Allaoui, M.; *Robin problems involving the $p(x)$-Laplacian*, Appl. Math. Comput., 332(2018), 457-468.

[5] Ambrosetti, A.; Rabinowitz, P. H.; *Dual variational methods in critical points theory and applications*, J. Funct. Anal., 14 (1973), 349-381.

[6] Avci, M.; Cekic, B.; Mashiyev, R. A.; *Existence and multiplicity of the solutions of the $p(x)$-Kirchhoff type equation via genus theory*, Math. Methods Appl. Sci., 34 (14) (2011), 1751-1759.

[7] Chang, K. C.; *Critical point theory and applications*, Shanghai Scientific and Technology press: Shanghai, 1986

[8] Chen, S. J .; Li, L.; *Multiple solutions for the nonhomogeneous Kirchhoff equation on $\mathbb{R}^N$*, Nonlinear Anal. (RWA), 14 (2013), 1477-1486.

[9] Chipot, M.; Lovat, B.; *Some remarks on nonlocal elliptic and parabolic problems*, Nonlinear Anal. (TMA), 30 (7) (1997), 4619-4627.

[10] Chung, N. T.; *Multiple solutions for a $p(x)$-Kirchhoff-type equation with sign-changing nonlinearities*, Complex Var. Elliptic Equa., 58 (12) (2013), 1637-1646.

[11] Chung, N. T.; *Multiple solutions for a class of $p(x)$-Kirchhoff type problems with Neumann boundary conditions*, Adv. Pure Appl. Math., 4 (2) (2013), 165-177.

[12] Clarke, D. C.; *A variant of the Lusternik-Schnirelman theory*, Indiana Univ. Math. J., 22 (1972), 65-74.

[13] Colasuonno, F.; Pucci, P.; *Multiplicity of solutions for $p(x)$-polyharmonic Kirchhoff equations*, Nonlinear Anal. (TMA), 74 (2) (2011), 5962-5974.

[14] Correa, F. J. S. A; Figueiredo, G. M.; *On an elliptic equation of p-Kirchhoff type via variational methods*, Bull. Aust. Math. Soc., 74 (2006), 263-277.

[15] Dai, G.; *Three solutions for a nonlocal Dirichlet boundary value problem involving the $p(x)$ Laplacian*, Appl. Anal., 92 (1) (2013), 191-210.

[16] Dai, G.; Hao, R.; *Existence of solutions for a $p(x)$-Kirchhoff-type equation*, J. Math. Anal. Appl., 359 (2009), 275-284.

[17] Deng, S. G.; *Positive solutions for Robin problem involving the $p(x)$-Laplacian*, J. Math. Anal. Appl., 360 (2009), 548-560.

[18] Diening, L.; Harjulehto, P.; Hasto, P.; Ruzicka, M.; *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes, vol. 2017, Springer-Verlag, Berlin, 2011.

[19] J. Doumate, A. Marcos; *Weighted Steklov Problem Under Nonresonance Conditions*, Bol. Soc. Paran. Mat. v. 36 (4) (2018): 87–105.

[20] Ekeland, I.; *On the variational principle*, J. Math. Anal. Appl., 47 (1974), 324-353.

[21] Ge, B.; Zhou, Q. M.; *Multiple solutions for a Robin-type differential inclusion problem involving the $p(x)$-Laplacian*, Math. Meth. Appl. Sci., 40 (18) (2017), 6229-6238.

[22] Guo, E.; Zhao, P. H.; *Existence and multiplicity of solutions for nonlocal $p(x)$-Laplacian equations with nonlinear Neumann boundary conditions*, J. Math. Anal. Appl., 2012 (2012): 1.

[23] Lei, C. Y.; Liu, G. S.; Guo, L. T.; *Multiple positive solutions for a Kirchhoff type problem with a critical nonlinearity*, Nonlinear Anal. (RWA), 31 (2016), 343-355.

[24] Kirchhoff, G.; *Mechanik*,Teubner, Leipzig, Germany, 1883.
[25] Kovacik, O.; Rakosnik, J.; *On spaces $L^{p(x)}$ and $W^{1,p(x)}$*, Czechoslovak Math. J., 41 (1991), 592-618.
[26] Molica Bisci, G.; Radulescu, V. D.; *Applications of local linking to nonlocal Neumann problems*, Commun. Contemp. Math., 17 (1) (2015), 1450001.
[27] Ruzicka, M.; *Electrorheological Fluids: Modeling and Mathematical Theory*, Springer-Verlag, Berlin, 2002.
[28] Tsouli, N.; Darhouche, O.; *Existence and multiplicity results for nonlinear problems involving the $p(x)$-Laplace operator*, Opuscula Math., 34 (3) (2014), 621-638.
[29] Wang, W. B.; Tang, W.; *Infinitely many solutions for Kirchhoff type problems with nonlinear Neumann boundary conditions*, Electron. J. Diff. Equ., 2016 (188) (2016), 1-9.
[30] Wang, L.; Xie, K.; Zhang, B.; *Existence and multiplicity of solutions for critical Kirchhofftype p-Laplacian problems*, J. Math. Anal. Appl., 458 (1) (2018), 361-378.
[31] Z. Yucedag; *Infinitely many solutions for a p(x)-Kirchhoff type equation with Steklov boundary value*, J. Miskolc Mathematical Notes, 23 (2022), 987–999.

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