Electronic Journal of Differential Equations, Vol. 2025 (2025), No. 02, pp. 1–19.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2025.02

TRAVELING WAVEFRONTS FOR A DISCRETE DIFFUSIVE LOTKA-VOLTERRA COMPETITION SYSTEM WITH NONLOCAL NONLINEARITIES

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Abstract. This article concerns the traveling wavefronts of a discrete diffusive Lotka-Volterra competition system with nonlocal nonlinearities. We first prove that there exists a $c_* > 0$ such that when the wave speed is large than or equals to c_*, the system admits an increasing traveling wavefront connecting two boundary equilibria by the upper-lower solutions method. Furthermore, we prove that (i) all traveling wavefronts with speed $c > c^*(> c_*)$ are globally stable with exponential convergence rate $t^{-1/2}e^{-\varepsilon_\tau \sigma t}$, where $\sigma > 0$ and $\varepsilon_\tau = \varepsilon(\tau) \in (0,1)$ is a decreasing function for the time delay $\tau > 0$; (ii) the traveling wavefronts with speed $c = c^*$ are globally algebraically stable in the algebraic form $t^{-1/2}$. The approaches are the weighted energy method, the comparison principle and Fourier transform.

1. Introduction

Consider the discrete diffusive Lotka-Volterra competition system with nonlocal nonlinearities

$$
\begin{aligned}
u_t(x,t) &= d_1\mathcal{D}[u](x,t) + r_1 u(x,t)\Big[1 - u(x,t) - b_1 \sum_{i\in\mathbb{Z}} g_1(i) v(x-i, t-\tau)\Big],\\
v_t(x,t) &= d_2\mathcal{D}[v](x,t) + r_2 v(x,t)\Big[1 - b_2 \sum_{i\in\mathbb{Z}} g_2(i) u(x-i, t-\tau) - v(x,t)\Big],
\end{aligned}
\tag{1.1}
$$

where $x \in \mathbb{R}$, $t > 0$, d_j, r_j, b_j $(j = 1,2)$, τ are positive constants, and

$$
\mathcal{D}[w](x,t) = w(x+1,t) - 2w(x,t) + w(x-1,t)
$$

with $w = u, v$. This model is often used to describe the competing interaction of two species. The unknown functions $u(x,t)$ and $v(x,t)$ stand for the population densities of two competitive species at location x and time t, respectively. The parameter d_j is the diffusion coefficient of species j, b_j is the inter-specific competition coefficient, r_j is the growth rate, and τ is time delay, $j = 1,2$. The kernel functions g_1 and g_2 are weight functions describing the distribution at past times of the individuals of the species u or v who are at position x at time t (see Guo and Lin [9]), and satisfy

(H1) $g_j(i) = g_j(-i) \geq 0$, $i \in \mathbb{Z}$, $\sum_{i\in\mathbb{Z}} g_j(i) = 1$, $j = 1,2$.

2020 *Mathematics Subject Classification.* 35K55, 35C07, 92D25.

Key words and phrases. Epidemic system; nonlocal dispersal; bistable traveling waves; stability; time delay.

Submitted July 23, 2024. Published January 4, 2025.

(H2) For every $\lambda > 0$, $\sum_{i\in\mathbb{Z}} g_j(i)e^{-\lambda i} < \infty$, $j = 1, 2$.

It is clear that the spatially homogeneous system of (1.1) is

$$\begin{aligned} u'(t) &= r_1 u(t)[1 - u(t) - b_1 v(t-\tau)], \\ v'(t) &= r_2 v(t)[1 - b_2 u(t-\tau) - v(t)]. \end{aligned} \tag{1.2}$$

System (1.2) always has three nonnegative equilibria

$$E^0 := (0,0), \quad E^1 := (0,1), \quad E^2 := (1,0).$$

When $b_1, b_2 < 1$ or $b_1, b_2 > 1$, there exists a unique coexistence equilibrium $E^* := (u^*, v^*) = (\frac{1-b_1}{1-b_1b_2}, \frac{1-b_2}{1-b_1b_2})$. In this article, we make the following assumption on the coefficients b_1 and b_2 of (1.1):

(H3) $0 < b_1 < 1 < b_2$.

By (H3), we obtain that E^1 is unstable and E^2 is stable (see [7, 19] for more details).

When $g_j(0) = 1$ and $g_j(i) = 0$ for all $i \neq 0$ and $\tau = 0$, system (1.1) is reduced to the classical Lotka-Volterra competitive system with discrete diffusion

$$\begin{aligned} u_t(x,t) &= d_1\mathcal{D}[u](x,t) + r_1 u(x,t)[1 - u(x,t) - b_1 v(x,t)], \\ v_t(x,t) &= d_2\mathcal{D}[v](x,t) + r_2 v(x,t)[1 - b_2 u(x,t) - v(x,t)], \end{aligned} \tag{1.3}$$

which has been studied in [7, 32]. More precisely, under assumption (H3), Guo and Wu [7] obtained the existence of monostable traveling wave solutions of (1.3) connecting E^1 and E^2. Later on, Tian and Zhang [32] further studied the stability of traveling wave solutions of system (1.3) with relatively large speed by the weighted energy method combining with the comparison principle.

In (1.1), we assume that the migration only happens to the nearest neighbors and the interaction happens with infinite range. As such, system (1.1) can help us understand the intricate cumulative effect due to an interaction between time delay and diffusion through the whole spatial location and the previous time over $[-\tau, 0]$. It is well known that traveling wave solutions can describe the transitions between different states of a physical system, propagation of patterns, and domain invasion of species in population biology (see, e.g., [6]). In mathematics, traveling wave solutions play an important role in the description of the long-term behaviour of solutions to initial value problems in reaction-diffusion equations, both in the spatially continuous cases and in spatially discrete situations[11, 12, 16, 17, 28, 33, 36]. In recent years, many efforts have been made to study the traveling wave solutions of discrete diffusion equations or systems, see [1, 2, 3, 4, 7, 8, 9, 10, 13, 15, 21] and references therein. Traveling wave solution of (1.1) is a special solution of the form $(u(x,t), v(x,t)) = (\phi(\xi), \varphi(\xi))$, $\xi := x + ct$, where $c > 0$ is wave speed, (ϕ, φ) is called wave profile. If ϕ and φ are monotone, then (ϕ, φ) is called a traveling wavefront. Substituting $(\phi(x+ct), \varphi(x+ct))$ into (1.1), we obtain the wave profile system

$$\begin{aligned} c\phi'(\xi) &= d_1\mathcal{D}[\phi](\xi) + r_1\phi(\xi)\Big[1 - \phi(\xi) - b_1\sum_{i\in\mathbb{Z}} g_1(i)\phi(\xi - i - c\tau)\Big], \\ c\varphi'(\xi) &= d_2\mathcal{D}[\varphi](\xi) + r_2\varphi(\xi)\Big[1 - b_2\sum_{i\in\mathbb{Z}} g_2(i)\varphi(\xi - i - c\tau) - \varphi(\xi)\Big], \end{aligned} \tag{1.4}$$

where $' = \frac{d}{d\xi}$ and $\mathcal{D}[\psi](\xi) = \psi(\xi+1) - 2\psi(\xi) + \psi(\xi-1)$ with $\psi = \phi$ or φ. With (1.4) we associate the boundary conditions

$$(\phi, \varphi)(-\infty) = E^1 \quad \text{and} \quad (\phi, \varphi)(+\infty) = E^2. \tag{1.5}$$

In this article, we shall study the existence and stability of traveling wavefronts of (1.1) connecting E^1 to E^2, i.e., solutions of (1.4) and(1.5). We first transform the competition system (1.1) into a cooperation system. Then, by using upper and lower solutions, monotone iteration method and a limiting argument, we can prove the existence of monostable traveling wavefronts for $c \geq c_*$, where c_* is some positive constant (see Lemma 2.1). The stability of traveling wave solutions for reaction-diffusion equations with and without time delays has been extensively investigated, see e.g., [20, 23, 26, 29, 30, 31, 34, 35]. Compared to the rich results for the classical reaction-diffusion equations, limited results exist for the spatial discrete diffusion equations, especially for discrete diffusion systems. Chen and Guo [4] employed the squeezing technique to prove the asymptotic stability of traveling waves for discrete quasilinear monostable equations without time delay. Guo and Zimmer [10] proved the global stability of traveling wavefronts for spatially discrete equations with nonlocal delay effects by using a combination of the weighted energy method and the Green function technique. Tian and Zhang [32] investigated the global stability of traveling wavefronts for a discrete diffusive Lotka-Volterra competition system with two species by the weighted energy method together with the comparison principle. Later on, Chen, Wu and Hsu [3] employed the similar method to show the global stability of traveling wavefronts for a discrete diffusive Lotka-Volterra competition system with three species. For other results on the stability of traveling wavefronts for a discrete diffusive equations, we refer the reader to [13, 14]. Note that the comparison principle also works for the transformed system of (1.1). Thus, the weighted energy method together with the comparison principle can still be used to prove the stability of traveling wavefronts of (1.1). However, since (1.1) is a system of two equations and contains nonlocal interaction terms, we can only obtain the stability of traveling wavefronts with large speed. Hence, in order to establish the stability of traveling wavefronts with relatively lower speed, in this paper, we shall employ the method of weighted energy combining with the comparison principle and Fourier's transform, which is different from that in [32], to study the stability of the traveling wavefronts connecting E^1 and E^2 for (1.1). Our result shows that all traveling wavefronts with speed $c > c^*$ are exponentially stable, where c^* is a positive constant larger than c_*, defined in Section 2, while the traveling wavefront with speed $c = c^*$ is algebraically stable with decay rate $t^{-1/2}$. In addition, the time delay τ will slow down the convergence of the solution $(u(x,t), v(x,t))$ to the traveling wavefronts $(\phi(x+ct), \varphi(x+ct))$ with speed $c > c^*$.

The rest of this article is organized as follows. In Section 2, we first give some preliminaries and then present the main results. In Section 3, we show the existence of traveling wavefronts of (1.1). Section 4 is devoted to proving the stability of the traveling wavefronts of (1.1).

2. Preliminaries and main results

In this section, we first show the existence of traveling wavefronts of (1.1), then give the comparison principle of the corresponding initial value problem of (1.1), and finally state the main result on the global stability of traveling wavefronts of (1.1).

Letting $u = \overline{u}$ and $v = 1 - \overline{v}$, and dropping the bar for the sake of convenience, then the competitive system (1.1) becomes the following cooperative system

$$
\begin{aligned}
u_t(x,t) &= d_1\mathcal{D}[u](x,t) + r_1u(x,t)\Big[1 - u(x,t) - b_1 + b_1\sum_{i\in\mathbb{Z}} g_1(i)v(x-i,t-\tau)\Big],\\
v_t(x,t) &= d_2\mathcal{D}[u](x,t) + r_2(v(x,t)-1)\Big[v(x,t) - b_2\sum_{i\in\mathbb{Z}} g_2(i)u(x-i,t-\tau)\Big].
\end{aligned}
\tag{2.1}
$$

The equilibria E^1 and E^2 of system (1.1) are corresponding to the equilibria $E_- = (u_-, v_-) := (0,0)$ and $E_+ = (u_+, v_+) := (1,1)$ of system (2.1).

Substituting $(u(x,t), v(x,t)) = (\phi(\xi), \varphi(\xi))$, $\xi := x + ct$ into (2.1) leads to the following wave profile system with the asymptotic boundary conditions

$$
\begin{gathered}
c\phi'(\xi) = d_1\mathcal{D}[\phi](\xi) + r_1\phi(\xi)\Big[1 - \phi(\xi) - b_1 + b_1\sum_{i\in\mathbb{Z}} g_1(i)\varphi(\xi - i - c\tau)\Big],\\
c\varphi'(\xi) = d_2\mathcal{D}[\varphi](\xi) + r_2(\varphi(\xi)-1)\Big[\varphi(\xi) - b_2\sum_{i\in\mathbb{Z}} g_2(i)\phi(\xi - i - c\tau)\Big],\\
(\phi,\varphi)(-\infty) = E_-, \quad (\phi,\varphi)(+\infty) = E_+.
\end{gathered}
\tag{2.2}
$$

We define the function

$$
\Delta(\lambda, c) = d_1(e^{\lambda} + e^{-\lambda} - 2) - c\lambda + r_1(1 - b_1).
$$

One can easily show that the following result holds.

Lemma 2.1. *Assume that* (H1)–(H3) *hold. Then there exist* $\lambda_* > 0$ *and* $c_* > 0$ *such that*

$$
\Delta(\lambda_*, c_*) = 0 \quad \text{and} \quad \frac{\partial}{\partial\lambda}\Delta(\lambda, c_*)\Big|_{\lambda=\lambda_*} = 0.
$$

Furthermore,

- (i) *If* $c > c_*$*, then* $\Delta(\lambda, c) = 0$ *has two distinct positive real roots* $\lambda_1(c)$ *and* $\lambda_2(c)$ *with* $\lambda_1(c) < \lambda_* < \lambda_2(c)$*, and* $\Delta(\lambda, c) < 0$ *for* $\lambda \in (\lambda_1(c), \lambda_2(c))$*, and* $\Delta(\lambda, c) > 0$ *for* $\lambda \in (0, \lambda_1(c)) \cup (\lambda_2(c), +\infty)$*.*
- (ii) *If* $0 < c < c_*$*, then* $\Delta(\lambda, c) > 0$ *for all* $\lambda > 0$*.*

By the method of upper-lower solutions, we can obtain the existence of traveling wavefronts of system (2.1) under additional assumption

- (H4) $\sum_{i\in\mathbb{Z}} g_j(i)e^{-\lambda_1 i} \le 1$, $j = 1, 2$, and $b_1b_2 \le 1$.

Theorem 2.2 (Existence)**.** *Assume that* (H1)–(H4) *hold and* $d_1 \ge d_2$*. Then for any* $c \ge c_*$*, system* (2.1) *admits an increasing traveling wavefront* $(\phi(\xi), \varphi(\xi))$ *connecting* E_- *and* E_+*.*

Remark 2.3. Condition (H4) is a technical assumption, which is only used for verification of the upper solution constructed in Section 3. We should point out that (H4) can be replaced with $\sum_{i\in\mathbb{Z}} g_1(i)e^{-\lambda_1 i} \le 1$ and $b_1b_2\sum_{i\in\mathbb{Z}} g_2(i)e^{-\lambda_1 i} \le 1$, or $\tau > 0$ is suitable large.

Throughout this paper, we assume that (2.1) has the following initial data

$$
u(x,s) = u_0(x,s), \quad v(x,s) = v_0(x,s), \quad (x,s) \in \mathbb{R} \times [-\tau, 0]. \tag{2.3}
$$

Then following comparison principle to the initial value problem (2.1)-(2.3) can be proved by an argument similar to [19, Lemma 3.2]. Thus, we omit the proof here.

Lemma 2.4 (Comparison principle). *Assume* (H1)–(H3)*, and let* $(u^{\pm}, v^{\pm})(x,t)$ *be the solution of system* (2.1) *with the initial data* $(u_0^{\pm}, v_0^{\pm})(x,s)$, $(x,s) \in \mathbb{R} \times [-\tau, 0]$, *respectively. If*

$$E_- \le (u_0^-, v_0^-)(x,s) \le (u_0^+, v_0^+)(x,s) \le E_+$$

for $(x,s) \in \mathbb{R} \times [-\tau, 0]$*, then*

$$E_- \le (u^-, v^-)(x,t) \le (u^+, v^+)(x,t) \le E_+$$

for $(x,t) \in \mathbb{R} \times \mathbb{R}^+$.

Throughout this article, we introduce some necessary notions at first. $C > 0$ denotes a generic constant, while $C_i (i = 1, 2, \dots)$ represents a specific constant. Let $\|\cdot\|$ and $\|\cdot\|_\infty$ denote 1-norm and ∞-norm of the matrix (or vector), respectively. Let I be an interval, typically $I = \mathbb{R}$. Denote by $L^1(I)$ the space of integrable functions defined on I, and $W^{k,1}(I)(k \ge 0)$ the Sobolev space of the L^1-functions $f(x)$ defined on the interval I whose derivatives $\frac{d^n}{dx^n} f(n = 1, \dots, k)$ also belong to $L^1(I)$. Let $L^1_w(I)$ be the weighted L^1-space with a weight function $w(x) > 0$ and its norm is defined by

$$\|f\|_{L^1_w(I)} = \int_I w(x)|f(x)|dx,$$

$W^{k,1}_w(I)$ be the weighted Sobolev space with the norm

$$\|f\|_{W^{k,1}_w(I)} = \sum_{i=0}^{k} \int_I w(x) \left| \frac{d^i f(x)}{dx^i} \right| dx.$$

Let $T > 0$ be a number and $\mathcal{B}$ be a Banach space. We denote by $C([0,T]; \mathcal{B})$ the space of the $\mathcal{B}$-valued continuous functions on $[0,T]$, and by $L^1([0,T]; \mathcal{B})$ the space of the $\mathcal{B}$-valued L^1-functions on $[0,T]$. The corresponding spaces of the $\mathcal{B}$-valued functions on $[0, \infty)$ are defined similarly. For any function $f(x)$, its Fourier transform is

$$\mathcal{F}[f](\eta) = \hat{f}(\eta) = \int_{\mathbb{R}} e^{-\mathbf{i}x\eta} f(x) dx$$

and the inverse Fourier transform is

$$\mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{\mathbf{i}x\eta} \hat{f}(\eta) d\eta,$$

where $\mathbf{i}$ is the imaginary unit, $\mathbf{i}^2 = -1$.

To obtain the stability of traveling wavefronts, we need the following technical assumption.

(H5) $d_1 \ge d_2$, $r_1 > r_2$, $d_1 - d_2 \le \frac{r_1 - r_2}{2}$.

We define the function

$$\tilde{\Delta}(\lambda, c) = d_1(e^{\lambda} + e^{-\lambda} - 2) - c\lambda + r_1 + r_1 b_1 q G_1(\lambda),$$

where

$$G_1(\lambda) = \sum_{i \in \mathbb{Z}} g_1(i) e^{-\lambda(i + c\tau)} < \infty, \quad q = \max\{1, \frac{r_2 b_2}{r_1 b_1}\}.$$

By computations, we have

$$\begin{gathered}
\tilde{\Delta}(0,c) = r_1 + r_1 b_1 q \sum_{i\in\mathbb{Z}} g_1(i) = r_1 + r_1 b_1 q > 0 \text{ for all } c,\\
\frac{\partial \tilde{\Delta}(\lambda,c)}{\partial c} = -\lambda - \lambda\tau r_1 b_1 q \sum_{i\in\mathbb{Z}} g_1(i) e^{-\lambda(i+c\tau)} < 0 \text{ for all } \lambda > 0,\\
\frac{\partial \tilde{\Delta}(\lambda,c)}{\partial \lambda}\Big|_{\lambda=0} = -c - c\tau r_1 b_1 q < 0,\ c > 0,\\
\frac{\partial^2 \tilde{\Delta}(\lambda,c)}{\partial \lambda^2} = d_1(e^{\lambda} + e^{-\lambda}) + c^2\tau^2 r_1 b_1 q \sum_{i\in\mathbb{Z}} g_1(i) e^{-\lambda(i+c\tau)} > 0.
\end{gathered}$$

Similar to Lemma 2.1, there exist $\lambda^* > 0$ and $c^* > 0$ such that $\tilde{\Delta}(\lambda^*, c^*) = 0$ and $\frac{\partial}{\partial\lambda}\tilde{\Delta}(\lambda, c^*)\big|_{\lambda=\lambda^*} = 0$. Furthermore, when $c > c^*$, $\tilde{\Delta}(\lambda, c) = 0$ has two distinct positive real roots $\lambda_1^\natural(c)$ and $\lambda_2^\natural(c)$ with $\lambda_1^\natural(c) < \lambda^* < \lambda_2^\natural(c)$, and $\tilde{\Delta}(\lambda,c) < 0$ for $\lambda \in (\lambda_1^\natural(c), \lambda_2^\natural(c))$, and $\tilde{\Delta}(\lambda,c) > 0$ for $\lambda \in (0, \lambda_1^\natural(c)) \cup (\lambda_2^\natural(c), +\infty)$. When $0 < c < c^*$, $\tilde{\Delta}(\lambda, c) > 0$ for all $\lambda > 0$. By computation, we see that

$$\begin{aligned}
\tilde{\Delta}(\lambda, c_*) &= d_1(e^{\lambda} + e^{-\lambda} - 2) - c_*\lambda + r_1 + r_1 b_1 q G_1(\lambda)\\
&\geq -r_1(1-b_1) + r_1 + r_1 b_1 q G_1(\lambda) > 0 \quad \text{for } \lambda > 0.
\end{aligned}$$

Thus, $c_* < c^*$.

We define the weight function

$$\omega(\xi) := \begin{cases} e^{-\lambda^*(\xi-\xi_0)}, & \xi \leq \xi_0,\\ 1, & \xi > \xi_0, \end{cases}$$

where $\xi_0 > 0$ is a large enough constant defined in the proof of Lemma 4.7 and λ^* is defined above.

Theorem 2.5 (Stability)**.** *Assume that* (H1)–(H3), (H5) *hold and* $g_1(\cdot) = g_2(\cdot)$*. Let* $(\phi(x+ct), \varphi(x+ct))$ *be the traveling wavefront connecting* E_- *and* E_+ *with* $c \geq c^*$*. If the initial data satisfy*

$$(0,0) \leq (u_0(x,s), v_0(x,s)) \leq (1,1), \quad (x,s) \in \mathbb{R}\times[-\tau,0]$$

and the initial perturbation

$$\begin{gathered}
u_0(x,s) - \phi(x+cs) \in C([-\tau,0]; W^{1,1}_\omega(\mathbb{R})),\\
v_0(x,s) - \varphi(x+cs) \in C([-\tau,0]; W^{1,1}_\omega(\mathbb{R})),\\
\partial_s(u_0 - \phi) \in L^1([-\tau,0]; L^1_\omega(\mathbb{R})), \quad \partial_s(v_0 - \varphi) \in L^1([-\tau,0]; L^1_\omega(\mathbb{R})),
\end{gathered}$$

then the solution $(u(x,t), v(x,t))$ *of the Cauchy problem* (2.1) *and* (2.3) *uniquely exists and satisfies*

$$(0,0) \leq (u(x,t), v(x,t)) \leq (1,1), \quad (x,t) \in \mathbb{R}\times[0,+\infty)$$

and

$$\begin{gathered}
u(x,t) - \phi(x+ct) \in C([0,+\infty); W^{1,1}_\omega(\mathbb{R})),\\
v(x,t) - \varphi(x+ct) \in C([0,+\infty); W^{1,1}_\omega(\mathbb{R})).
\end{gathered}$$

Furthermore, when $c > c^*$, the solution $(u(x,t), v(x,t))$ converges to the traveling wavefront $(\phi(x+ct), \varphi(x+ct))$ as follows:

$$\sup_{x\in\mathbb{R}} |u(x,t) - \phi(x+ct)| \le Ct^{-1/2} e^{-\varepsilon_\tau \sigma t}, \quad \forall t > 0,$$
$$\sup_{x\in\mathbb{R}} |v(x,t) - \varphi(x+ct)| \le Ct^{-1/2} e^{-\varepsilon_\tau \sigma t}, \quad \forall t > 0,$$

where σ, C are positive numbers and $\varepsilon_\tau \in (0,1)$. When $c = c^*$, the solution $(u(x,t), v(x,t))$ converges to the traveling wavefront $(\phi(x+ct), \varphi(x+ct))$ as follows:

$$\sup_{x\in\mathbb{R}} |u(x,t) - \phi(x+ct)| \le Ct^{-1/2}, \quad \forall t > 0,$$
$$\sup_{x\in\mathbb{R}} |v(x,t) - \varphi(x+ct)| \le Ct^{-1/2}, \quad \forall t > 0.$$

3. Existence

In this section, we shall prove the existence of traveling wavefronts of (1.1) by upper-lower solutions method. We first define the notion of upper-lower solutions.

Definition 3.1. A continuous function (ϕ, φ) from $\mathbb{R}$ to $[0,1]$ is called an upper solution (or a lower solution) of (2.2), if each ϕ and φ are continuously differentiable in $\mathbb{R}$ except at finite points and satisfy

$$c\phi'(\xi) \ge (\le) d_1 \mathcal{D}[\phi](\xi) + r_1 \phi(\xi) \Big[1 - \phi(\xi) - b_1 + b_1 \sum_{i\in\mathbb{Z}} g_1(i) \varphi(\xi - i - c\tau)\Big],$$
$$c\varphi'(\xi) \ge (\le) d_2 \mathcal{D}[\varphi](\xi) + r_2 (\varphi(\xi) - 1) \Big[\varphi(\xi) - b_2 \sum_{i\in\mathbb{Z}} g_2(i) \phi(\xi - i - c\tau)\Big],$$

a.e. $\xi \in \mathbb{R}$.

We define two continuous functions as follows:

$$\bar\phi(\xi) = \begin{cases} e^{\lambda_1 \xi}, & \xi \le 0, \\ 1, & \xi > 0, \end{cases} \quad \text{and} \quad \bar\varphi(\xi) = \begin{cases} e^{\lambda_1 \xi}/b_1, & \xi \le \xi_1, \\ 1, & \xi > \xi_1. \end{cases}$$

Lemma 3.2. *Assume that $c > c_*$ and $d_1 \ge d_2$. Then $(\bar\phi(\xi), \bar\varphi(\xi))$ is an upper solution of (2.2).*

Proof. When $\xi > 0$, $\bar\phi(\xi) = 1$. Note that $\bar\varphi(\xi) \le 1$ for all $\xi \in \mathbb{R}$. Then we obtain

$$\begin{aligned} &c\bar\phi'(\xi) - d_1 \mathcal{D}[\bar\phi](\xi) - r_1 \bar\phi(\xi) \Big[1 - \bar\phi(\xi) - b_1 + b_1 \sum_{i\in\mathbb{Z}} g_1(i) \bar\varphi(\xi - i - c\tau)\Big] \\ &\ge r_1 \Big[b_1 - b_1 \sum_{i\in\mathbb{Z}} g_1(i) \bar\varphi(\xi - i - c\tau)\Big] \ge 0, \end{aligned}$$

because of (H1). When $\xi < 0$, $\bar\phi(\xi) = e^{\lambda_1 \xi}$. In view of $\bar\varphi(\xi) \le e^{\lambda_1 \xi}/b_1$ for all $\xi \in \mathbb{R}$, we have

$$\begin{aligned} &c\bar\phi'(\xi) - d_1 \mathcal{D}[\bar\phi](\xi) - r_1 \bar\phi(\xi) \Big[1 - \bar\phi(\xi) - b_1 + b_1 \sum_{i\in\mathbb{Z}} g_1(i) \bar\varphi(\xi - i - c\tau)\Big] \\ &\ge c\lambda_1 e^{\lambda_1 \xi} - d_1 e^{\lambda_1 \xi} (e^{\lambda_1} + e^{-\lambda_1} - 2) - r_1 e^{\lambda_1 \xi} \Big[1 - e^{\lambda_1 \xi} - b_1 \\ &\quad + b_1 \sum_{i\in\mathbb{Z}} g_1(i) \bar\varphi(\xi - i - c\tau)\Big] \end{aligned}$$

$$
\begin{aligned}
&= e^{\lambda_1\xi}\Big[c\lambda_1 - d_1(e^{\lambda_1}+e^{-\lambda_1}-2) - r_1(1-b_1)\Big]\\
&\quad - r_1 e^{\lambda_1\xi}\Big[-e^{\lambda_1\xi} + b_1\sum_{i\in\mathbb{Z}} g_1(i)\bar{\varphi}(\xi-i-c\tau)\Big]\\
&= e^{\lambda_1\xi}\Delta(\lambda_1,c) + r_1 e^{\lambda_1\xi}\Big[e^{\lambda_1\xi} - b_1\sum_{i\in\mathbb{Z}} g_1(i)\bar{\varphi}(\xi-i-c\tau)\Big]\\
&\ge r_1 e^{2\lambda_1\xi}\Big[1 - e^{-\lambda_1 c\tau}\sum_{i\in\mathbb{Z}} g_1(i)e^{-\lambda_1 i}\Big] \ge 0,
\end{aligned}
$$

Because $\Delta(\lambda_1,c)=0$ for $c>c_*$ and (H4). Analogously, when $\xi>\xi_1$, $\bar{\varphi}(\xi)=1$. It is clear that

$$
c\bar{\varphi}'(\xi) - d_2\mathcal{D}[\bar{\varphi}](\xi) - r_2(\bar{\varphi}(\xi)-1)\Big[\bar{\varphi}(\xi) - b_2\sum_{i\in\mathbb{Z}} g_2(i)\bar{\phi}(\xi-i-c\tau)\Big] \ge 0.
$$

When $\xi<\xi_1$, $\bar{\varphi}(\xi)=e^{\lambda_1\xi}/b_1$. Thus, we can derive that

$$
\begin{aligned}
&c\bar{\varphi}'(\xi) - d_2\mathcal{D}[\bar{\varphi}](\xi) - r_2(\bar{\varphi}(\xi)-1)\Big[\bar{\varphi}(\xi) - b_2\sum_{i\in\mathbb{Z}} g_2(i)\bar{\phi}(\xi-i-c\tau)\Big]\\
&\ge c\lambda_1\frac{1}{b_1}e^{\lambda_1\xi} - d_1\frac{1}{b_1}e^{\lambda_1\xi}(e^{\lambda_1}+e^{-\lambda_1}-2)\\
&\quad - r_2(\frac{1}{b_1}e^{\lambda_1\xi}-1)\Big[\frac{1}{b_1}e^{\lambda_1\xi} - b_2\sum_{i\in\mathbb{Z}} g_2(i)\bar{\phi}(\xi-i-c\tau)\Big]\\
&\ge \frac{1}{b_1}e^{\lambda_1\xi}[c\lambda_1 - d_1(e^{\lambda_1}+e^{-\lambda_1}-2)]\\
&\quad + r_2(1-\frac{1}{b_1}e^{\lambda_1\xi})\Big[\frac{1}{b_1}e^{\lambda_1\xi} - b_2 e^{\lambda_1\xi}\sum_{i\in\mathbb{Z}} g_2(i)e^{-\lambda_1(i+c\tau)}\Big]\\
&= \frac{1}{b_1}e^{\lambda_1\xi}\Big[r_1(1-b_1) + r_2(1-\frac{1}{b_1}e^{\lambda_1\xi})\Big(1 - b_1b_2e^{-\lambda_1 c\tau}\sum_{i\in\mathbb{Z}} g_2(i)e^{-\lambda_1 i}\Big)\Big] \ge 0,
\end{aligned}
$$

since $d_1\ge d_2$ and (H4) holds. Thus, we prove $(\bar{\phi}(\xi),\bar{\varphi}(\xi))$ is an upper solution of (2.2). The proof is complete. ☐

Let $\underline{\varphi}(\xi)=0$ for all $\xi\in\mathbb{R}$, and $\underline{\phi}(\xi)$ be positive and satisfy

$$
c\phi'(\xi) = d_1\mathcal{D}[\phi](\xi) + r_1\phi(\xi)[1-b_1-\phi(\xi)]
$$

and

$$
\lim_{\xi\to-\infty}\phi(\xi)e^{-\lambda_1\xi} = 1.
$$

For the existence of $\underline{\phi}(\xi)$, we refer readers to Chen and Guo [4, 5]. We can prove that $(\underline{\varphi}(\xi),\underline{\phi}(\xi))$ is a lower solution of (2.2). Since the proof is easy, we omit it here.

Lemma 3.3. $(\underline{\phi}(\xi),\underline{\varphi}(\xi))$ *is a lower solution of* (2.2).

Based on Lemmas 3.2 and 3.3, Theorem 2.2 can be easily obtained. More precisely, the existence of traveling wavefronts with speed $c>c_*$ can be proved by the monotone iteration method [5, 14], Schauder's fixed point theorem [22], or truncated method [7]. By applying the limiting arguments, we can obtain the existence of traveling wavefronts with speed $c=c_*$. We refer the reader to [14, 27], and here we skip the details.

4. Global stability

In this section, we are going to prove the stability of the traveling wavefronts by the weighted energy method together with the comparison principle and Fourier's transform. The global existence and uniqueness of the solution for Cauchy problem (2.1)-(2.3) can be proved by the standard energy method and continuity extension method [25] or the theory of abstract functional differential equations[24]. Therefore, we present the following proposition and omit the proof.

Proposition 4.1. *Assume that* (H1)–(H3), (H5) *hold. Let* $(\phi(x+ct), \varphi(x+ct)) = (\phi(\xi), \varphi(\xi))$ *be the traveling wavefront of* (2.1) *connecting* E_- *and* E_+ *with* $c \geq c^*$. *If the initial data satisfy*

$$(0,0) \leq (u_0(x,s), v_0(x,s)) \leq (1,1), \quad (x,s) \in \mathbb{R} \times [-\tau, 0].$$

then the Cauchy problem (2.1) *and* (2.3) *admits a unique solution* $(u(x,t), v(x,t))$ *satisfying*

$$(0,0) \leq (u(x,t), v(x,t)) \leq (1,1), \quad (x,t) \in \mathbb{R} \times [0, +\infty).$$

If, in addition, the initial perturbation satisfies

$$u_0(x,s) - \phi(x+cs) \in C([-\tau,0]; W^{1,1}_\omega(\mathbb{R})), v_0(x,s) - \varphi(x+cs) \in C([-\tau,0]; W^{1,1}_\omega(\mathbb{R})),$$

then

$$u(x,t) - \phi(x+ct) \in C([0,+\infty); W^{1,1}_\omega(\mathbb{R})), v(x,t) - \varphi(x+ct) \in C([0,+\infty); W^{1,1}_\omega(\mathbb{R})).$$

Before proving the stability of the traveling wavefronts, we recall some properties of the solutions to linear delayed differential system.

Lemma 4.2 ([18, Theorem 1])**.** *Let* $z(t)$ *be the solution to the linear delayed differential system*

$$\begin{gathered} \frac{d}{dt} z(t) = Az(t) + Bz(t-\tau), \quad t \geq 0, \ \tau > 0, \\ z(s) = z_0(s), \quad s \in [-\tau, 0]. \end{gathered} \tag{4.1}$$

where $A, B \in \mathbb{C}^{N \times N}$, $N \geq 2$, *and* $z_0(s) \in C^1([-\tau, 0], \mathbb{C}^N)$. *Then*

$$z(t) = e^{A(t+\tau)} e_\tau^{\bar{B}t} z_0(-\tau) + \int_{-\tau}^0 e^{A(t-s)} e_\tau^{\bar{B}(t-\tau-s)} [z_0'(s) + Az_0(s)] ds,$$

where $\bar{B} = Be^{A\tau}$ *and* $e_\tau^{\bar{B}t}$ *is the so-called delayed exponential function in the form*

$$e_\tau^{\bar{B}t} = \begin{cases} 0, & -\infty < t < -\tau, \\ I, & -\tau \leq t < 0, \\ I + \bar{B}\frac{t}{1!}, & 0 \leq t < \tau, \\ I + \bar{B}\frac{t}{1!} + \bar{B}^2 \frac{(t-\tau)^2}{2!}, & \tau \leq t < 2\tau, \\ \vdots & \vdots \\ I + \bar{B}\frac{t}{1!} + \bar{B}^2 \frac{(t-\tau)^2}{2!} + \dots + \bar{B}^m \frac{[t-(m-1)\tau]^m}{m!}, & (m-1)\tau \leq t < m\tau, \\ \vdots & \vdots \end{cases}$$

where $0, I \in \mathbb{C}^{N \times N}$, *and* 0 *is zero matrix and* I *is the unit matrix.*

Lemma 4.3 ([23, Theorem 3.1]). *Suppose* $\mu(A) := \frac{\mu_1(A)+\mu_\infty(A)}{2} < 0$, *where* $\mu_1(A)$ *and* $\mu_\infty(A)$ *denote the matrix measure of* A *induced by matrix* 1*-norm* $\|\cdot\|_1$ *and* ∞*-norm* $\|\cdot\|_\infty$*, respectively. If* $\nu(B) := \frac{\mu_1(B)+\mu_\infty(B)}{2} \le -\mu(A)$*, then there exists a decreasing function* $\varepsilon_\tau = \varepsilon(\tau) \in (0,1)$ *for* $\tau > 0$ *such that any solution of system* (4.1) *satisfies*

$$\|z(t)\| \le C_0 e^{-\varepsilon_\tau \sigma t}, \quad t > 0,$$

where C_0 *is a positive constant depending on initial data* $z_0(s)$*,* $s \in [-\tau, 0]$ *and* $\sigma = |\mu(A)| - \nu(B)$*. In particular,*

$$\|e^{At} e_\tau^{\bar{B}t}\| \le C_0 e^{-\varepsilon_\tau \sigma t}, \quad t > 0,$$

where $e_\tau^{\bar{B}t}$ *is defined in Lemma 4.2.*

Remark 4.4. It can be seen from the proof of [23, Theome 3.1] that

$$\mu_1(A) = \lim_{\theta\to 0^+} \frac{\|I+\theta A\| - 1}{\theta} = \max_{1\le j\le N} \Big[\operatorname{Re}(a_{jj}) + \sum_{j\ne i}^{N} |a_{ij}|\Big]$$

and

$$\mu_\infty(A) = \lim_{\theta\to 0^+} \frac{\|I+\theta A\|_\infty - 1}{\theta} = \max_{1\le i\le N} \Big[\operatorname{Re}(a_{ii}) + \sum_{i\ne j}^{N} |a_{ij}|\Big].$$

The following lemma can be found in [10, Lemma 3.2].

Lemma 4.5. *For* $t > 0$*,*

$$\frac{1}{2\pi}\int_{\mathbb{R}} \exp\{-2td\epsilon_\tau \cosh(\lambda^*)(1-\cos\eta)\}d\eta \le \sqrt{\frac{\pi}{dt\epsilon_\tau}},$$

where $\cosh(\lambda^*) = (e^{\lambda^*} + e^{-\lambda^*})/2$.

Now we are ready to prove the stability of traveling wavefronts to (2.1) with a specific convergence rate. For any $c \ge c^*$, we define the functions

$$\tilde{U}_0^+(x,s) = \max\{u_0(x,s), \phi(x+cs)\}, \quad \tilde{V}_0^+(x,s) = \max\{v_0(x,s), \varphi(x+cs)\},$$
$$\tilde{U}_0^-(x,s) = \min\{u_0(x,s), \phi(x+cs)\}, \quad \tilde{V}_0^-(x,s) = \min\{v_0(x,s), \varphi(x+cs)\}$$

for $(x,s) \in \mathbb{R}\times[-\tau,0]$. It is easy to see that

$$0 = u^- \le \tilde{U}_0^-(x,s) \le u_0(x,s) \le \tilde{U}_0^+(x,s) \le u^+ = 1,$$
$$0 = u^- \le \tilde{U}_0^-(x,s) \le \phi(x+cs) \le \tilde{U}_0^+(x,s) \le u^+ = 1,$$
$$0 = v^- \le \tilde{V}_0^-(x,s) \le v_0(x,s) \le \tilde{V}_0^+(x,s) \le v^+ = 1,$$
$$0 = v^- \le \tilde{V}_0^-(x,s) \le \varphi(x+cs) \le \tilde{V}_0^+(x,s) \le v^+ = 1$$

for $(x,s) \in \mathbb{R}\times[-\tau,0]$.

It is clear that the initial data $(\tilde{U}_0^\pm(x,s), \tilde{V}_0^\pm(x,s))$ are piecewise continuous and have a poor regularity, which may also cause the absence of regularity for the corresponding solutions. To overcome such a shortcoming, we choose smooth functions $(U_0^\pm(x,s), V_0^\pm(x,s))$ instead of these initial data as the new initial data such that

$$\begin{aligned}(0,0) &\le (U_0^-(x,s), V_0^-(x,s)) \le (\tilde{U}_0^-(x,s), \tilde{V}_0^-(x,s)) \le (u_0(x,s), v_0(x,s))\\ &\le (\tilde{U}_0^+(x,s), \tilde{V}_0^+(x,s)) \le (U_0^+(x,s), V_0^+(x,s)) \le (1,1).\end{aligned}$$

We denote $(U^+(x,t),V^+(x,t))$ and $(U^-(x,t),V^-(x,t))$ as the corresponding solutions of (2.1) with respect to the above mentioned initial data

$$(U_0^+(x,s),V_0^+(x,s)),\quad (U_0^-(x,s),V_0^-(x,s)),$$

respectively. It then follows from Lemma 2.4 that

$$\begin{gathered}
0\le U^-(x,t)\le u(x,t)\le U^+(x,t)\le 1,\\
0\le U^-(x,t)\le \phi(x+ct)\le U^+(x,t)\le 1,\\
0\le V^-(x,t)\le v(x,t)\le V^+(x,t)\le 1,\\
0\le V^-(x,t)\le \varphi(x+ct)\le V^+(x,t)\le 1
\end{gathered}$$

for $(x,t)\in\mathbb{R}\times[0,+\infty)$.

We shall complete the proof of Theorem 2.5 in the following three steps:

(1) $(U^+(x,t),V^+(x,t))$ converges to $(\phi(x+ct),\varphi(x+ct))$.
(2) $(U^-(x,t),V^-(x,t))$ converges to $(\phi(x+ct),\varphi(x+ct))$.
(3) $(u(x,t),v(x,t))$ converges to $(\phi(x+ct),\varphi(x+ct))$.

We do only Step 1, since Step 2 can be done by a similar way. Using a squeezing technique, we can easily get the conclusion of Step 3, which implies Theorem 2.5.

Let $\xi=x+ct$ and

$$U_1(\xi,t):=U^+(x,t)-\phi(x+ct),\quad V_1(\xi,t):=V^+(x,t)-\varphi(x+ct)$$

with the initial data

$$U_{10}(\xi,s):=U^+(x,s)-\phi(x+cs),\quad V_{10}(\xi,s):=V^+(x,s)-\varphi(x+cs) \tag{4.2}$$

for $t>0$, $s\in[-\tau,0]$, $\xi\in\mathbb{R}$. It is easy to see that

$$U_1(\xi,t)\ge 0,\quad V_1(\xi,t)\ge 0,\quad \forall(\xi,t)\in\mathbb{R}\times[0,+\infty).$$

By (2.1) and (2.2), we can verify that $(U_1(\xi,t),V_1(\xi,t))$ satisfies

$$\begin{aligned}
U_{1t}+cU_{1\xi}&=d_1\mathcal{D}[U_1]+r_1M(\phi,\varphi)U_1-r_1U_1^2+r_1b_1U_1\sum_{i\in\mathbb{Z}}g_1(i)V_1^\tau\\
&\quad +r_1b_1\phi\sum_{i\in\mathbb{Z}}g_1(i)V_1^\tau,\\
V_{1t}+cV_{1\xi}&=d_2\mathcal{D}[V_1]-r_2N(\phi,\varphi)V_1+r_2V_1^2-r_2b_2V_1\sum_{i\in\mathbb{Z}}g_2(i)U_1^\tau\\
&\quad +r_2b_2(1-\varphi)\sum_{i\in\mathbb{Z}}g_2(i)U_1^\tau,
\end{aligned}\tag{4.3}$$

where

$$\begin{gathered}
M(\phi,\varphi)=1-2\phi-b_1+b_1\sum_{i\in\mathbb{Z}}g_1(i)\varphi^\tau,\quad N(\phi,\varphi)=1-2\varphi+b_2\sum_{i\in\mathbb{Z}}g_2(i)\phi^\tau,\\
\phi^\tau=\phi(\xi-i-c\tau),\quad \varphi^\tau=\varphi(\xi-i-c\tau),\\
U_1^\tau=U_1(\xi-i-c\tau,t-\tau),\quad V_1^\tau=V_1(\xi-i-c\tau,t-\tau).
\end{gathered}$$

Furthermore, system (4.3) can be rewritten as

$$
\begin{aligned}
U_{1t}+cU_{1\xi}&=d_1\mathcal{D}[U_1]+r_1U_1\Big[1-b_1-(U_1+\phi)+b_1\sum_{i\in\mathbb{Z}}g_1(i)(V_1^\tau+\varphi^\tau)\Big]\\
&\quad+r_1\phi\Big[b_1\sum_{i\in\mathbb{Z}}g_1(i)V_1^\tau-U_1\Big],\\
V_{1t}+cV_{1\xi}&=d_2\mathcal{D}[V_1]+r_2V_1\Big[(V_1+\varphi)-b_2\sum_{i\in\mathbb{Z}}g_2(i)(U_1^\tau+\phi^\tau)\Big]\\
&\quad+r_2(1-\varphi)\Big[b_2\sum_{i\in\mathbb{Z}}g_2(i)U_1^\tau-V_1\Big].
\end{aligned}\tag{4.4}
$$

Note that $U_1\ge 0$, $V_1\ge 0$, $\phi,\varphi\in(0,1)$, $U_1+\phi\in(0,1)$ and $V_1+\varphi\in(0,1)$. Then we obtain

$$
\begin{aligned}
&r_1U_1\Big[1-b_1-(U_1+\phi)+b_1\sum_{i\in\mathbb{Z}}g_1(i)(V_1^\tau+\varphi^\tau)\Big]+r_1\phi\Big[b_1\sum_{i\in\mathbb{Z}}g_1(i)V_1^\tau-U_1\Big]\\
&\le r_1U_1+r_1b_1\sum_{i\in\mathbb{Z}}g_1(i)V_1^\tau
\end{aligned}\tag{4.5}
$$

and

$$
\begin{aligned}
&r_2V_1\Big[(V_1+\varphi)-b_2\sum_{i\in\mathbb{Z}}g_2(i)(U_1^\tau+\phi^\tau)\Big]+r_2(1-\varphi)\Big[b_2\sum_{i\in\mathbb{Z}}g_2(i)U_1^\tau-V_1\Big]\\
&\le r_2V_1+r_2b_2\sum_{i\in\mathbb{Z}}g_2(i)U_1^\tau.
\end{aligned}\tag{4.6}
$$

Using (4.5) and (4.6) in (4.4), we have

$$
\begin{aligned}
U_{1t}+cU_{1\xi}&\le d_1\mathcal{D}[U_1]+r_1U_1+r_1b_1\sum_{i\in\mathbb{Z}}g_1(i)V_1^\tau,\\
V_{1t}+cV_{1\xi}&\le d_2\mathcal{D}[V_1]+r_2V_1+r_2b_2\sum_{i\in\mathbb{Z}}g_2(i)U_1^\tau
\end{aligned}
$$

for all $(\xi,t)\in\mathbb{R}\times\mathbb{R}^+$.

Let $(U_1^+(\xi,t),V_1^+(\xi,t))$ be the solution of the initial value problem

$$
\begin{aligned}
U_{1t}^+(\xi,t)+cU_{1\xi}^+(\xi,t)&=d_1\mathcal{D}[U_1^+](\xi,t)+r_1U_1^+(\xi,t)\\
&\quad+r_1b_1\sum_{i\in\mathbb{Z}}g_1(i)V_1^+(\xi-i-c\tau,t-\tau),\\
V_{1t}^+(\xi,t)+cV_{1\xi}^+(\xi,t)&=d_2\mathcal{D}[V_1^+](\xi,t)+r_2V_1^+(\xi,t)\\
&\quad+r_2b_2\sum_{i\in\mathbb{Z}}g_2(i)U_1^+(\xi-i-c\tau,t-\tau),\\
(U_1^+(\xi,s),V_1^+(\xi,s))&=(U_{10}(\xi,s),V_{10}(\xi,s))
\end{aligned}\tag{4.7}
$$

for $(\xi,t)\in\mathbb{R}\times\mathbb{R}^+$, where $(U_{10}(\xi,s),V_{10}(\xi,s))$ is defined in (4.2). Then, by the comparison principle, we have

$$
(U_1(\xi,t),V_1(\xi,t))\le(U_1^+(\xi,t),V_1^+(\xi,t)),\quad\forall(\xi,t)\in\mathbb{R}\times\mathbb{R}^+.\tag{4.8}
$$

We introduce the transformation

$$
U(\xi,t)=e^{-\lambda^*(\xi-\xi_0)}U_1^+(\xi,t),\quad V(\xi,t)=e^{-\lambda^*(\xi-\xi_0)}V_1^+(\xi,t),
$$

where ξ_0 is a large enough positive constant. Then by (4.7), $(U(\xi,t),V(\xi,t))$ satisfies

$$\begin{aligned} U_t(\xi,t)+cU_\xi(\xi,t) &= d_1[e^{\lambda^*}U(\xi+1,t)+e^{-\lambda^*}U(\xi-1,t)]+k_1U(\xi,t)\\ &\quad + r_1b_1\sum_{i\in\mathbb{Z}} g_1(i)e^{-\lambda^*(i+c\tau)}V(\xi-i-c\tau,t-\tau),\\ V_t(\xi,t)+cV_\xi(\xi,t) &= d_2[e^{\lambda^*}V(\xi+1,t)+e^{-\lambda^*}V(\xi-1,t)]+k_2V(\xi,t)\\ &\quad + r_2b_2\sum_{i\in\mathbb{Z}} g_2(i)e^{-\lambda^*(i+c\tau)}U(\xi-i-c\tau,t-\tau), \end{aligned} \tag{4.9}$$

where

$$k_1 := r_1 - c\lambda^* - 2d_1, \quad k_2 := r_2 - c\lambda^* - 2d_2.$$

Taking the Fourier transform to system (4.9) and denoting the Fourier transform of $Z(\xi,t) := (U(\xi,t),V(\xi,t))^T$ by $\widehat{Z}(\eta,t) := (\widehat{U}(\eta,t),\widehat{V}(\eta,t))^T$, we obtain

$$\begin{aligned} \widehat{U}_t(\eta,t) &= [d_1(e^{\lambda^*+\mathbf{i}\eta}+e^{-(\lambda^*+\mathbf{i}\eta)})-\mathbf{i}c\eta+k_1]\widehat{U}(\eta,t)\\ &\quad + r_1b_1\sum_{i\in\mathbb{Z}} g_1(i)e^{-\lambda^*(i+c\tau)}e^{-(i+c\tau)\mathbf{i}\eta}\widehat{V}(\eta,t-\tau),\\ \widehat{V}_t(\eta,t) &= [d_2(e^{\lambda^*+\mathbf{i}\eta}+e^{-(\lambda^*+\mathbf{i}\eta)})-\mathbf{i}c\eta+k_2]\widehat{V}(\eta,t)\\ &\quad + r_2b_2\sum_{i\in\mathbb{Z}} g_2(i)e^{-\lambda^*(i+c\tau)}e^{-(i+c\tau)\mathbf{i}\eta}\widehat{U}(\eta,t-\tau). \end{aligned} \tag{4.10}$$

Let

$$A(\eta)=\begin{pmatrix} d_1(e^{\lambda^*+\mathbf{i}\eta}+e^{-(\lambda^*+\mathbf{i}\eta)})-\mathbf{i}c\eta+k_1 & 0\\ 0 & d_2(e^{\lambda^*+\mathbf{i}\eta}+e^{-(\lambda^*+\mathbf{i}\eta)})-\mathbf{i}c\eta+k_2 \end{pmatrix}$$

and

$$B(\eta)=\begin{pmatrix} 0 & r_1b_1\sum_{i\in\mathbb{Z}} e^{-\mathbf{i}(i+c\tau)\eta}g_1(i)e^{-\lambda^*(i+c\tau)}\\ r_2b_2\sum_{i\in\mathbb{Z}} e^{-\mathbf{i}(i+c\tau)\eta}g_2(i)e^{-\lambda^*(i+c\tau)} & 0 \end{pmatrix}.$$

Then system (4.10) can be rewritten as

$$\frac{d}{dt}\widehat{Z}(\eta,t)=A(\eta)\widehat{Z}(\eta,t)+B(\eta)\widehat{Z}(\eta,t-\tau). \tag{4.11}$$

By using the solution formula of (4.1) in Lemma 4.2, one can solve the linear time-delayed ordinary differential system (4.11) as follows

$$\begin{aligned} \widehat{Z}(\eta,t) &= e^{A(\eta)(t+\tau)}e_\tau^{\overline{B}(\eta)t}\widehat{Z}_0(\eta,-\tau)\\ &\quad + \int_{-\tau}^0 e^{A(\eta)(t-s)}e_\tau^{\overline{B}(\eta)(t-s-\tau)}\Big[\partial_s\widehat{Z}_0(\eta,s)-A(\eta)\widehat{Z}_0(\eta,s)\Big]ds\\ &=: J_1(\eta,t)+\int_{-\tau}^0 J_2(\eta,t-s)ds, \end{aligned} \tag{4.12}$$

where $\overline{B}(\eta) = B(\eta)e^{A(\eta)\tau}$. Taking the inverse Fourier transform to (4.12), one has

$$\begin{aligned}
&Z(\xi,t)\\
&= \mathcal{F}^{-1}[J_1](\xi,t) + \int_{-\tau}^{0} \mathcal{F}^{-1}[J_2](\xi,t-s)ds\\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\mathbf{i}\xi\eta} e^{A(\eta)(t+\tau)} e_{\tau}^{\overline{B}(\eta)t} \widehat{Z}_0(\eta,-\tau)d\eta\\
&\quad + \frac{1}{2\pi}\int_{-\tau}^{0}\int_{-\infty}^{+\infty} e^{\mathbf{i}\xi\eta} e^{A(\eta)(t-s)} e_{\tau}^{\overline{B}(\eta)(t-s-\tau)} \big[\partial_s \widehat{Z}_0(\eta,s) - A(\eta)\widehat{Z}_0(\eta,s)\big] d\eta ds.
\end{aligned} \tag{4.13}$$

Now we present the lemma about the decay rate of $U_1(\xi,t)$ and $V_1(\xi,t)$ for $\xi \in I := (-\infty,\xi_0]$.

Lemma 4.6. *Let assumptions in Theorem 2.5 hold. Then there exists a decreasing function $\varepsilon_\tau = \varepsilon(\tau) \in (0,1)$ such that*

$$\|U_1(\cdot,t)\|_{L^\infty(I)} + \|V_1(\cdot,t)\|_{L^\infty(I)} \le Ct^{-1/2}e^{-\varepsilon_\tau\sigma_1 t}, \quad \forall t>0, \tag{4.14}$$

where C is a positive constant and

$$\sigma_1 := -\tilde{\Delta}(\lambda^*,c)\begin{cases} >0 & \text{for } c>c^*,\\ =0 & \text{for } c=c^*. \end{cases} \tag{4.15}$$

Proof. By (4.13), it suffices to estimate $\mathcal{F}^{-1}[J_1](\xi,t)$ and $\int_{-\tau}^{0}\mathcal{F}^{-1}[J_2](\xi,t-s)ds$, respectively. By the definition of $\mu(A)$ and $\nu(B)$, we have

$$\begin{aligned}
\mu(A) &= \frac{\mu_1(A)+\mu_\infty(A)}{2}\\
&= \max\Big\{d_1(e^{\lambda^*}\cos\eta + e^{-\lambda^*}\cos\eta) + k_1, d_2(e^{\lambda^*}\cos\eta + e^{-\lambda^*}\cos\eta) + k_2\Big\}\\
&= d_1(e^{\lambda^*}+e^{-\lambda^*})\cos\eta + k_1\\
&= d_1(e^{\lambda^*}+e^{-\lambda^*}-2) - c\lambda^* + r_1 - m(\eta),
\end{aligned}$$

where

$$m(\eta) = d_1(e^{\lambda^*}+e^{-\lambda^*})(1-\cos\eta) \ge 0,$$

since $d_1 \ge d_2$, $r_1 > r_2$, $d_1 - d_2 \le \frac{r_1-r_2}{2}$ by (H5), and

$$\nu(B) \le \max\{r_1b_1G_1(\lambda^*), r_2b_2G_2(\lambda^*)\},$$

since

$$\begin{aligned}
\Big|r_jb_j\sum_{i\in\mathbb{Z}} e^{-\mathbf{i}(i+c\tau)\eta}g_j(i)e^{-\lambda^*(i+c\tau)}\Big| &\le r_jb_j\sum_{i\in\mathbb{Z}}|e^{-\mathbf{i}(i+c\tau)\eta}g_j(i)e^{-\lambda^*(i+c\tau)}|\\
&= r_jb_j\sum_{i\in\mathbb{Z}} g_j(i)e^{-\lambda^*(i+c\tau)}, \quad j=1,2.
\end{aligned}$$

Note that $\tilde{\Delta}(\lambda^*,c) = d_1(e^{\lambda^*}+e^{-\lambda^*}-2) - c\lambda^* + r_1 + r_1b_1qG_1(\lambda^*) \le 0$ for $c \ge c^*$. It then follows that $d_1(e^{\lambda^*}+e^{-\lambda^*}-2) - c\lambda^* + r_1 \le -r_1b_1qG_1(\lambda^*) < 0$. Thus, we can derive that $\mu(A) < 0$ and

$$\begin{aligned}
\mu(A)+\nu(B) &\le d_1(e^{\lambda^*}+e^{-\lambda^*}-2) - c\lambda^* + r_1 - m(\eta)\\
&\quad + \max\{r_1b_1G_1(\lambda^*), r_2b_2G_2(\lambda^*)\}\\
&= d_1(e^{\lambda^*}+e^{-\lambda^*}-2) - c\lambda^* + r_1 + r_1b_1qG_1(\lambda^*) - m(\eta)
\end{aligned}$$

$$= \tilde{\Delta}(\lambda^*, c) - m(\eta) \le 0,$$

because $g_1(\cdot) = g_2(\cdot)$, where $q = \max\{1, \frac{r_2 b_2}{r_1 b_1}\}$. Furthermore,

$$|\mu(A)| - \nu(B) = -\mu(A) - \nu(B) = -\tilde{\Delta}(\lambda^*, c) + m(\eta) \ge 0,$$

i.e. $|\mu(A)| \ge \nu(B)$. Then by Lemma 4.3, there exists a decreasing function $\varepsilon_\tau = \varepsilon(\tau) \in (0,1)$ such that

$$\left|e^{A(\eta)(t+\tau)} e_\tau^{\bar{B}(\eta)t}\right| \le Ce^{-\varepsilon_\tau(|\mu(A)|-\nu(B))t} \le Ce^{-\varepsilon_\tau \sigma_1 t} e^{-\varepsilon_\tau m(\eta)t}, \tag{4.16}$$

where C is a positive constant and σ_1 is defined in (4.15).

By the definition of Fourier's transform, we obtain

$$\sup_{\eta\in\mathbb{R}} \|\widehat{Z}_0(\eta, -\tau)\| \le \int_{-\infty}^{\infty} \|Z_0(\xi, -\tau)\| d\xi = \sum_{i=1}^{2} \|Z_{i0}(\cdot, -\tau)\|_{L^1(\mathbb{R})}.$$

Applying (4.16) and in view of Lemma 4.5, we have

$$\begin{aligned}
\sup_{\xi\in\mathbb{R}} \|\mathcal{F}^{-1}[J_1](\xi, t)\| &= \sup_{\xi\in\mathbb{R}} \Big\|\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\mathbf{i}\xi\eta} e^{A(\eta)(t+\tau)} e_\tau^{\overline{B}(\eta)t} \widehat{Z}_0(\eta, -\tau) d\eta\Big\| \\
&\le Ce^{-\varepsilon_\tau\sigma_1 t} \sup_{\eta\in\mathbb{R}} \|\widehat{Z}_0(\eta, -\tau)\| \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\varepsilon_\tau m(\eta)t} d\eta \\
&\le Ct^{-1/2} e^{-\varepsilon_\tau\sigma_1 t} \sum_{i=1}^{2} \|Z_{i0}(\cdot, -\tau)\|_{L^1(\mathbb{R})}.
\end{aligned} \tag{4.17}$$

Note that

$$\sup_{\eta\in\mathbb{R}} \|A(\eta)\widehat{Z}_0(\eta, s)\| \le C \sum_{i=1}^{2} \|Z_{i0}(\cdot, s)\|_{W^{1,1}(\mathbb{R})}.$$

Similarly, we can obtain

$$\begin{aligned}
&\sup_{\xi\in\mathbb{R}} \|\mathcal{F}^{-1}[J_2](\xi, t-s) d\eta\| \\
&= \sup_{\xi\in\mathbb{R}} \Big\|\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\mathbf{i}\xi\eta} e^{A(\eta)(t-s)} e_\tau^{\overline{B}(\eta)(t-s-\tau)} \big[\partial_s \widehat{Z}_0(\eta, s) - A(\eta)\widehat{Z}_0(\eta, s)\big] d\eta\Big\| \\
&\le C\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\varepsilon_\tau\sigma_1(t-s)} e^{-\varepsilon_\tau m(\eta)(t-s)} \|\partial_s \widehat{Z}_0(\eta, s) - A(\eta)\widehat{Z}_0(\eta, s)\| d\eta \\
&\le Ce^{-\varepsilon_\tau\sigma_1 t} e^{\varepsilon_\tau\sigma_1 s} \frac{1}{2\pi} \sup_{\eta\in\mathbb{R}} \|\partial_s \widehat{Z}_0(\eta, s) - A(\eta)\widehat{Z}_0(\eta, s)\| \int_{-\infty}^{+\infty} e^{-\varepsilon_\tau m(\eta)(t-s)} d\eta.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\int_{-\tau}^{0} \sup_{\xi\in\mathbb{R}} \|\mathcal{F}^{-1}[J_2](\xi, t-s) ds\| \\
&\le Ce^{-\varepsilon_\tau\sigma_1 t} \frac{1}{2\pi}\int_{-\tau}^{0} e^{\varepsilon_\tau\sigma_1 s} \sup_{\eta\in\mathbb{R}} \|\partial_s \widehat{Z}_0(\eta, s) - A(\eta)\widehat{Z}_0(\eta, s)\| \int_{-\infty}^{+\infty} e^{-\varepsilon_\tau m(\eta)(t-s)} d\eta ds \\
&\le Ct^{-1/2} e^{-\varepsilon_\tau\sigma_1 t} \int_{-\tau}^{0} \left(\|\partial_s Z_0(\eta, s)\|_{L^1(\mathbb{R})} + \|Z_0(\eta, s)\|_{W^{1,1}(\mathbb{R})}\right) ds \\
&\le Ct^{-1/2} e^{-\varepsilon_\tau\sigma_1 t} \left(\|\partial_s Z_0(\eta, s)\|_{L^1([-\tau,0];L^1(\mathbb{R}))} + \|Z_0(\eta, s)\|_{L^1([-\tau,0];W^{1,1}(\mathbb{R}))}\right).
\end{aligned} \tag{4.18}$$

Substituting (4.17) and (4.18) into (4.13), we obtain the decay rate

$$\|U(\cdot,t)\|_{L^\infty(\mathbb{R})}+\|V(\cdot,t)\|_{L^\infty(\mathbb{R})}\le Ct^{-1/2}e^{-\varepsilon_\tau\sigma_1 t}\quad\text{for } c\ge c^*. \tag{4.19}$$

When $\xi\le\xi_0$, one can see that $e^{\lambda^*(\xi-\xi_0)}\le 1$. This together with (4.8) yields

$$0\le U_1(\xi,t)\le U_1^+(\xi,t)=e^{\lambda^*(\xi-\xi_0)}U(\xi,t)\le U(\xi,t), \tag{4.20}$$

$$0\le V_1(\xi,t)\le V_1^+(\xi,t)=e^{\lambda^*(\xi-\xi_0)}V(\xi,t)\le V(\xi,t) \tag{4.21}$$

for all $\xi\in I$ and $t>0$. It then follows from (4.19), (4.20) and (4.21) that (4.14) holds. The proof is complete. □

Next, we present the decay rate for $U_1(\xi,t)$ and $V_1(\xi,t)$ in $\mathbb{R}\setminus I=(\xi_0,+\infty)$.

Lemma 4.7. *It holds that*

$$\|U_1(\cdot,t)\|_{L^\infty(\mathbb{R}\setminus I)}+\|V_1(\cdot,t)\|_{L^\infty(\mathbb{R}\setminus I)}\le Ct^{-1/2}e^{-\gamma t}\quad\textit{for } t>0,$$

where $\gamma>0$ is a small constant satisfying $0<\gamma<\min\{\varepsilon_\tau\sigma_1,\delta_1\}$ with

$$\delta_1=\min\big\{r_1(1-b_1),r_2(b_2-1),\frac{1}{\tau}\ln\frac{1}{b_1}\big\}, \tag{4.22}$$

when $c>c^*$, and $\gamma=0$, when $c=c^*$.

Proof. By (4.3), we can see that (U_1,V_1) for $c\ge c^*$ satisfies

$$\begin{gathered}
U_{1t}+cU_{1\xi}\le d_1\mathcal{D}[U_1]+r_1(1-\phi)U_1+r_1\phi(b_1\sum_{i\in\mathbb{Z}}g_1(i)V_1^\tau-U_1),\\
\xi\in\mathbb{R}\setminus I,\ t>0,\\
V_{1t}+cV_{1\xi}\le d_2\mathcal{D}[V_1]+r_2(1-b_2\sum_{i\in\mathbb{Z}}g_2(i)\phi^\tau)V_1+r_2(1-\varphi)(b_2\sum_{i\in\mathbb{Z}}g_1(i)U_1^\tau-V_1),\\
\xi\in\mathbb{R}\setminus I,\ t>0,\\
U_1|_{\xi=\xi_0}\le C_0(1+t)^{-1/2}e^{-\varepsilon_\tau\sigma_1 t},\quad t>0,\\
V_1|_{\xi=\xi_0}\le C_0(1+t)^{-1/2}e^{-\varepsilon_\tau\sigma_1 t},\quad t>0,\\
U_1|_{t=s}=U_{10}(\xi,s),\quad V_1|_{t=s}=V_{10}(\xi,s),\quad \xi\in\mathbb{R}\setminus I,\ s\in[-\tau,0].
\end{gathered} \tag{4.23}$$

where σ_1 is defined in (4.15), and $C_0=C_1$ when $c>c^*$, and $C_0=C_2$ when $c=c^*$. We choose ξ_0 and t_* large enough and γ small enough satisfying $0<\gamma<\min\{\varepsilon_\tau\sigma_1,\delta_1\}$ such that

$$\begin{gathered}
r_1\phi(\xi_1)\Big(1-b_1\big(1+\frac{\tau}{1+t}\big)^{1/2}e^{\gamma\tau}\Big)+r_1(\phi(\xi_1)-1)-\frac{1}{2}(1+t+\tau)^{-1}-\gamma\ge 0,\\
r_2(\varphi(\xi_1)-1)\Big(b_2\big(1+\frac{\tau}{1+t}\big)^{1/2}e^{\gamma\tau}-1\Big)\\
+r_2(b_2\phi^\tau(\xi_1)-1)-\frac{1}{2}(1+t+\tau)^{-1}-\gamma\ge 0
\end{gathered}$$

for $\xi_1\ge\xi_0$ and $t>t_*$. When $c>c^*$, we let

$$\overline{U}_1(\xi,t)=\overline{V}_1(\xi,t)=\widetilde{C}(1+t+\tau)^{-1/2}e^{-\gamma t}\quad\text{for } t>0,$$

where $\widetilde{C}$ is large enough such that $(\overline{U}_1(\xi,t),\overline{V}_1(\xi,t))\ge(U_1(\xi,t),V_1(\xi,t))$ for $(\xi,t)\in\mathbb{R}\times[0,t_*]$. By a direct computation, we can verify that $(\overline{U}_1,\overline{V}_1)$ is an upper solution to (4.23), i.e.,

$$\begin{aligned}
&\overline{U}_{1t} + c\overline{U}_{1\xi} \geq d_1\mathcal{D}[\overline{U}_1] + r_1(1-\phi)\overline{U}_1 + r_1\phi(b_1\sum_{i\in\mathbb{Z}} g_1(i)\overline{V}_1^{\tau} - \overline{U}_1),\\
&\qquad \xi\in\mathbb{R}\setminus I,\ t>0,\\
&\overline{V}_{1t} + c\overline{V}_{1\xi} \geq d_2\mathcal{D}[\overline{V}_1] + r_2(1-b_1\sum_{i\in\mathbb{Z}} g_2(i)\phi^{\tau})\overline{V}_1\\
&\qquad + r_2(1-\varphi)(b_2\sum_{i\in\mathbb{Z}} g_1(i)\overline{U}_1^{\tau} - \overline{V}_1), \quad \xi\in\mathbb{R}\setminus I,\ t>0,\\
&\overline{U}_1|_{\xi=\xi_0} \geq C_1(1+t)^{-1/2}e^{-\varepsilon_\tau\sigma_1 t}, \quad t>0,\\
&\overline{V}_1|_{\xi=\xi_0} \geq C_1(1+t)^{-1/2}e^{-\varepsilon_\tau\sigma_1 t}, \quad t>0,\\
&\overline{U}_1|_{t=s} \geq \overline{U}_{10}(\xi,s), \overline{V}_1|_{t=s} \geq \overline{V}_{10}(\xi,s), \quad \xi\in\mathbb{R}\setminus I,\ s\in[-\tau,0].
\end{aligned}$$

Hence, for $c>c^*$, we obtain

$$0\leq U_1(\xi,t)\leq \overline{U}_1(\xi,t) = \widetilde{C}(1+t+\tau)^{-1/2}e^{-\gamma t} \quad \text{for } t>0,\ \xi\in\mathbb{R}\setminus I, \tag{4.24}$$

$$0\leq V_1(\xi,t)\leq \overline{V}_1(\xi,t) = \widetilde{C}(1+t+\tau)^{-1/2}e^{-\gamma t} \quad \text{for } t>0,\ \xi\in\mathbb{R}\setminus I. \tag{4.25}$$

When $c=c^*$, we let

$$\overline{U}_1(\xi,t) = \overline{V}_1(\xi,t) = \widetilde{C}(1+t+\tau)^{-1/2} \quad \text{for } t>0.$$

Similarly, we can obtain that for $c=c^*$,

$$0\leq U_1(\xi,t)\leq \overline{U}_1(\xi,t) = \widetilde{C}(1+t+\tau)^{-1/2} \quad \text{for } t>0,\ \xi\in\mathbb{R}\setminus I, \tag{4.26}$$

$$0\leq V_1(\xi,t)\leq \overline{V}_1(\xi,t) = \widetilde{C}(1+t+\tau)^{-1/2} \quad \text{for } t>0,\ \xi\in\mathbb{R}\setminus I. \tag{4.27}$$

Thus, Lemma 4.7 can be immediately obtained by (4.24)-(4.27). The proof is complete. ∎

Lemma 4.8. *It holds that*

$$\|U_1(\cdot,t)\|_{L^\infty(\mathbb{R})} + \|V_1(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq Ct^{-1/2}e^{-\varepsilon_\tau\sigma t} \quad \textit{for } t>0,\ c>c^*,$$

$$\|U_1(\cdot,t)\|_{L^\infty(\mathbb{R})} + \|V_1(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq Ct^{-1/2} \quad \textit{for } t>0,\ c=c^*,$$

where $0<\sigma<\min\{\sigma_1,\delta_1/\varepsilon_\tau\}$, $\sigma_1 := -\Delta(\lambda^*,c)>0$ *with* $c>c^*$, *and* δ_1 *is given by* (4.22).

According to the definition of U_1 and V_1, we have the following convergence of the solution $(U^+(x,t),V^+(x,t))$ to $(\phi(x+ct),\varphi(x+ct))$.

Lemma 4.9. *It holds that for* $c>c^*$,

$$\sup_{x\in\mathbb{R}} |U^+(x,t)-\phi(x+ct)| \leq Ct^{-1/2}e^{-\varepsilon_\tau\sigma t}, \quad \forall t>0,$$

$$\sup_{x\in\mathbb{R}} |V^+(x,t)-\varphi(x+ct)| \leq Ct^{-1/2}e^{-\varepsilon_\tau\sigma t}, \quad \forall t>0,$$

and for $c=c^*$,

$$\sup_{x\in\mathbb{R}} |U^+(x,t)-\phi(x+ct)| \leq Ct^{-1/2}, \quad \forall t>0,$$

$$\sup_{x\in\mathbb{R}} |V^+(x,t)-\varphi(x+ct)| \leq Ct^{-1/2}, \quad \forall t>0,$$

where $0 < \sigma < \min\{\sigma_1, \delta_1/\varepsilon_\tau\}$, $\sigma_1 := -\Delta(\lambda^*, c) > 0$ *with* $c > c^*$, *and* δ_1 *is given by* (4.22).

Acknowledgments. B. G. Zhang was supported by the NSF of China [12261081], the NSF of Gansu Province [21JR7RA121], and the Outstanding Youth Science Fund of Gansu Province [20JR10RA111].

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