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ENTIRE SOLUTIONS FOR NON-LINEAR DIFFERENTIAL-DIFFERENCE EQUATIONS

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Abstract. In this article, we investigate the entire solutions of the non-linear differential-difference equation

$$f^n(z) + \omega f^{n-1}(z)f'(z) + q(z)e^{Q(z)}\mathcal{D}(z,f) = p_1(z)e^{\lambda z} + p_2(z)e^{-\lambda z},$$

where $\mathcal{D}(z,f) = \sum_{i=0}^{k} b_i f^{(t_i)}(z+c_i) \not\equiv 0$, with $b_i, c_i \in \mathbb{C}$, t_i being non-negative integers, $c_0 = 0$, $t_0 = 0$. Here, n is an integer, λ, p_1, p_2 are non-zero constants, ω is a constant, and $q \not\equiv 0$, $Q(z)$ are polynomials such that $Q(z)$ is non-constant. Our results improve upon and generalize some previously established findings in this area.

1. Introduction

Assuming the reader's familiarity with conventional notation and core outcomes of Nevanlinna's theory on meromorphic functions [9], in this article, we consistently refer to meromorphic functions as those meromorphic in the entire complex plane $\mathbb{C}$. For a meromorphic function f and $a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, any z such that $f(z) = a$ is termed an a-point of f. In 1926, the Finnish mathematician Rolf Nevanlinna made a noteworthy breakthrough in complex analysis by investigating meromorphic functions over the complex plane. He demonstrated that a nonconstant function can be uniquely determined by five distinct pre-images, including infinity, without considering multiplicities. This finding is particularly interesting because it has no counterpart in the real function theory. Later, Nevanlinna went on to prove that when multiplicities are taken into account, four points are adequate for determining the uniqueness of a pair of meromorphic functions. In such cases, either the functions coincide, or one is a bilinear transformation of the other. These seminal discoveries marked the beginning of research into the uniqueness of pairs of meromorphic functions, especially when one function is related to the other. Two meromorphic functions $f(z)$ and $g(z)$ share a CM (Counting multiplicity) or IM (Ignoring multiplicity) if $f-a$ and $g-a$ have the same set of zeros counting multiplicities or ignoring multiplicities, respectively. Further recall that the order of f is defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

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The exponential of convergence of zeros of f is defined by

$$\lambda(f) = \limsup_{r\to\infty} \frac{\log N\big(r, \frac{1}{f}\big)}{\log r} = \limsup_{r\to\infty} \frac{\log n\big(r, \frac{1}{f}\big)}{\log r}.$$

Establishing the existence of solutions for complex differential equations poses a significant and challenging problem. The Nevanlinna theory has been widely used to analyze the characteristics of complex differential equations. In recent times, an increasing number of researchers have employed Nevanlinna theory to investigate the solutions of complex differential equations [2, 4, 12]. Additionally, the difference analogs of Nevanlinna theory have been applied to explore topics related to complex difference equations or complex nonlinear differential-difference equations [3, 22, 26]. In 1964, Hayman [9] examined the behavior of nonlinear differential equations of the form

$$f^n + P_d(z, f) = g(z), \tag{1.1}$$

where $P_d(z, f)$ is a differential polynomials in f of degree d with meromorphic coefficients of growth $S(r, f)$ and $n \geq 2$ is an integer.

Theorem 1.1 ([9]). *If non-constant meromorphic functions $f(z)$ and $g(z)$ satisify $N(r, f) + N\left(r, \frac{1}{g}\right) = S(r, f)$ and $d \leq n - 1$ in* (1.1)*, then $g(z) = (f(z) + \gamma(z))^n$, where $\gamma(z)$ is a meromorphic function and a small function of $f(z)$.*

Theorem 1.1 represents an expanded form of the Tumura-Clunie theory, which finds its foundation in a theorem initially proposed by Tumura [18]. However, the complete proof was later provided by Clunie [7]. Following its introduction, the nonlinear differential equation (1.1) has undergone extensive study over the years, as evidenced by the works [16, 17, 20, 24] and the references contained therein.

Li and Yang [11], in their 2006 study, explored the outline where the function $g(z)$ in equation (1.1) takes the specific form $p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}$. Their investigation led to the following results.

Theorem 1.2. *Let $n \geq 4$ be an integer and $P_d(f)$ denotes an algebraic differential polynomial in f of degree $d \leq n-3$. Let p_1, p_2 be two non zero polynomials, α_1 and α_2 be two non zero constants with $\frac{\alpha_1}{\alpha_2}$ not rational. Then, the differential equation*

$$f^n(z) + P_d(z, f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)} \tag{1.2}$$

has no transcendental entire solutions.

When $n = 3$, Yang and Li [21] determined the precise forms of solutions for (1.2) under specific conditions. Furthermore, Li [10] and Liao, Yang, and Zhang [12] obtained entire or meromorphic solutions for cases where $n \geq 2$ (resp. $n \geq 3$) and $d \leq n - 2$ in equation (1.2). Other research papers, such as [1] and [27], have also explored the structure of solutions for various differential equations.

In 2014, Liao and Ye [13] investigated the differential equation

$$f^n f' + P_d(z, f) = u(z)e^{v(z)}, \tag{1.3}$$

with non-zero rational function u and nonconstant polynomial v and obtained the following result.

Theorem 1.3 ([13]). *Suppose that f is a meromorphic solution of* (1.3)*, which has finitely many poles. Then*

$$P_d(f) \equiv 0, \quad f(z) = s(z)e^{v(z)/(n+1)}$$

for $n \geq d+1$ *and* s *is a rational function satisfying* $s^n[(n+1)s' + v's] = (n+1)u$.

Now, we define two classes of transcendental entire functions:

$$\Gamma_0(z) = \{e^{\alpha(z)} : \alpha(z) \text{ is a non constant polynomial}\}.$$

$$\Gamma_0(z) = \{e^{\alpha(z)} + d : \alpha(z) \text{ is a non constant polynomial and } d \in \mathbb{C}\}.$$

Exponential polynomials are crucial in exploring nonlinear complex differential equations, highlighting numerous intriguing properties. As an illustration, in 2012, Wen, Heittokangas, and Laine [19] conducted an investigation and classification of finite-order entire solutions $f(z)$ for the equation

$$f^n + q(z)e^{Q(z)}f(z+c) = P(z) \tag{1.4}$$

in terms of growth and zero distribution, where $n \geq 2$ is an integer, $q(z),\ P(z),\ Q(z)$ are polynomials and $c \in \mathbb{C} \setminus \{0\}$.

Following the aforementioned study, Liu [14] investigated the cases in which $f(z+c)$ in equation (1.4) was substituted by $f^{(k)}(z+c)$. Additionally, Liu, Mao, and Zheng [15] examined cases involving the replacement of $f(z+c)$ with $\Delta_c f(z)$. Their investigations led to specialized forms of solutions for the corresponding equations.

Upon examining the results above, it becomes evident that the left-hand side of all the above equations contains only one dominant term, f^n. Consequently, an exciting area of inquiry arises when studying equations that may have two dominant terms. Inspired by equation (1.4) and some of the previously discussed equations, the objective of this paper is to explore the finite-order entire solutions of the differential-difference equation

$$f^n(z) + \omega f^{n-1}(z)f'(z) + q(z)e^{Q(z)}\mathcal{D}(z,f) = p_1(z)e^{\lambda z} + p_2(z)e^{-\lambda z} \tag{1.5}$$

where $\mathcal{D}(z,f) = \sum_{i=0}^{k} b_i f^{(t_i)}(z+c_i)(\not\equiv 0)$, such that $b_i,\ c_i \in \mathbb{C},\ t_i$ are non negative integers, $c_0 = 0,\ t_0 = 0$, n is an integer, $\lambda,\ p_1,\ p_2$ are non zero constants ω is a constant, and $q \not\equiv 0$, $Q(z)$ are polynomials such that $Q(z)$ is not a constant.

Theorem 1.4. *If* $f(z)$ *is a transcendental entire solution with finite order of* (1.5) *then the following conclusions hold:*

1. *If* $n \geq 4$ *for* $\omega \neq 0$ *and* $n \geq 3$ *for* $\omega = 0$, *then every solution* f *satisfies* $\rho(f) = \deg(Q(z)) = 1$.
2. *If* $n \geq 1$ *and* f *is a solution of* (1.5) *which belongs to* Γ_0, *then*

$$f(z) = e^{\frac{-\lambda}{n}z+\mathcal{B}}, \quad Q(z) = \frac{(n+1)}{n}\lambda z + b$$

or

$$f(z) = e^{\frac{\lambda}{n}z+\mathcal{B}}, \quad Q(z) = \frac{-(n+1)}{n}\lambda z + b,$$

where $b,\ \mathcal{B} \in \mathbb{C}$

Theorem 1.4 represents a generalized and enhanced form of Theorem 1.3, originally established by Chen et al. [5]. To demonstrate the precision of our findings, we now present an illustrative example.

Example 1.5. Let $\mathcal{D}(z,f) = f^{(2)}(z+c)$. Then the function $f(z) = e^{2z}$ satisfies the equation

$$f^3 + f^2(z)f' + \frac{1}{4}e^{-8z}f^{(2)}(z+c) = 3e^{6z} + 4e^{-6z}.$$

Obviously, the conclusion $\rho(f) = \deg(Q(z)) = 1$ holds.

2. Preliminaries

To prove our results, we first give some Lemmas as follows: The first Lemma presents the difference analogs of the Logarithmic Derivative Lemma, a crucial tool in investigating complex difference equations. The following version represents a special case.

Lemma 2.1 ([6]). *Let $f(z)$ be a non constant meromorphic function with finite orderσ and $c_1,\ c_2$ be two complex numbers such that $c_1 \neq c_2$ then for each $\epsilon > 0$*

$$m\Big(r, \frac{f(z+c_1)}{f(z+c_2)}\Big) = O\left(r^{\sigma-1+\epsilon}\right).$$

Lemma 2.2 ([9]). *Let $f(z)$ be a nonconstant meromorphic function and let $k \geq 1$. Then, if the growth order of $f(z)$ is finite, we have*

$$m\Big(r, \frac{f^{(k)}}{f}\Big) = O(\log(r)).$$

nd if the growth order of $f(z)$ is infinite, we have

$$m\Big(r, \frac{f^{(k)}}{f}\Big) = O(\log(T(r,f)) + \log(r)), \quad \text{as } r \to \infty$$

possibly outside a set of finite linear measure.

Lemma 2.3 ([24]). *If $f_k(z)$, $1 \leq k \leq m$, and $g_k(z)$, $1 \leq k \leq m$, $m \geq 2$ are entire functions that meet conditions listed below:*

1. $\sum_{i=0}^{m} f_k(z)e^{g_k(z)} \equiv 0$,
2. *The orders of $f_k(z)$ are less than that of $e^{g_1(z)-g_n(z)}$ for $1 \leq k \leq m,\ 1 \leq k \leq l < n \leq m$, then $f_k \equiv 0$ for $1 \leq k \leq m$*

Lemma 2.4 ([8]). *Let f be a non constant meromorphic solution of $f^n(z)P(z,f) = Q(z,f)$, where P, Q are difference polynomials in f with small meromorphic coefficients and let $c \in \mathbb{C}$, $\delta < 1$. If the total degree of $Q(z,f)$ as a polynomial in f and its shifts are at most n, then*

$$m(r, P(z,f)) = o\Big(\frac{T(r+\mid c\mid, f)}{r^\delta}\Big) + o(T(r,f))$$

for all r outside a possible exceptional set with finite logarithmic measure.

Lemma 2.5 ([10]). *Suppose that $f(z)$ is a transcendental meromorphic function, $p,\ a,\ r$ and s are small functions of f with $prs \not\equiv 0$. If $pf^2 + aff' + r(f')^2 = s$, then*

$$r(a^2-4pr)\frac{s'}{s} + q(q^2-4rp) - r(a^2-4rp)' + (a^2-4rp)r' \equiv 0.$$

Lemma 2.6 ([23]). *Let $f_j(z),\ j = 1,2,3$ be meromorphic functions and $f_1(z)$ is not a constant. If $\sum_{j=1}^{3} f_j(z) \equiv 1$ and*

$$\sum_{j=1}^{3} N\Big(r, \frac{1}{f_j}\Big) + 2\sum_{j=1}^{3} \overline{N}(r, f_j) < (\lambda + o(1))T(r),\ r \in I,$$

where $\lambda < 1$, $T(r) = \max_{1\leq j\leq 3}\{T(r,f_j)\}$ and I represents a set of $r \in (o,\ \infty)$ with infinite linear measure. Then $f_2 \equiv 1$ or $f_3 \equiv 1$.

3. Proof of main results

Proof of Theorem refTh1. Let us consider a case where f is a transcendental entire solution of finite order for equation (1.5). The following discussion will establish our conclusion (1) stated in Theorem 1.4. To begin, we shall examine the case where $\omega \neq 0$.

Case 1. If $\rho(f) < 1$. Using Lemma 2.1 and Lemma 2.2 and from (1.5), we obtain

$$\begin{aligned}
T(r, e^{Q(z)}) &= m(r, e^{Q(z)}) \\
&= m\Big(r, \frac{p_1(z)e^{\lambda z} + p_2(z)e^{-\lambda z} - f^n(z) - \omega f^{n-1}(z)f'(z)}{q(z)\mathcal{D}(z,f)}\Big) \\
&\leq m\Big(r, \frac{1}{q(z)e^{Q(z)}\mathcal{D}(z,f))}\Big) + m(r, p_1(z)e^{\lambda z} + p_2(z)e^{-\lambda z}) \\
&\quad + m\left(r, f^n(z) + \omega f^{n-1}(z)f'(z)\right) + O(1) \\
&\leq 2T(r, e^{\lambda z}) + (n+1)T(r,f) + S(r, e^{\lambda z}),
\end{aligned}$$

then $\deg(Q(z)) \leq 1$, and Observing that $\deg(Q(z)) \geq 1$, we can deduce that $\deg(Q(z)) = 1$. Let us represent $Q(z)$ as $Q(z) = az + b$, where $a \in \mathbb{C} \setminus 0$ and $b \in \mathbb{C}$. With this representation, we can rewrite equation (1.5) in the form

$$f^n(z) + \omega f^{n-1}(z)f'(z) + q(z)e^{az+b)}\mathcal{D}(z,f) = p_1(z)e^{\lambda z} + p_2(z)e^{-\lambda z}. \tag{3.1}$$

Differentiating (3.1), we obtain

$$\begin{aligned}
&nf^{n-1}f' + (n-1)\omega f^{n-2}(f'(z))^2 + \omega f^{n-1}f^{(2)} + \alpha(z)e^{az+b} \\
&= \lambda(p_1(z)e^{\lambda z} - p_2(z)e^{-\lambda z}).
\end{aligned} \tag{3.2}$$

Eliminating $e^{\lambda z}$ and $e^{-\lambda z}$ from (3.1) and (3.2), yields

$$\begin{aligned}
&(n-\lambda\omega)f^{n-1}f' + (n-1)\omega f^{n-2}(f')^2 + \omega f^{n-1}f^{(2)} \\
&- \lambda f^n + [\alpha(z) - \lambda q(z)\mathcal{D}(z,f)]e^{az+b} \\
&= 2\lambda p_1 e^{\lambda z}
\end{aligned} \tag{3.3}$$

Subcase 1.1 If $a \neq \lambda$, by (3.3) and Lemma 2.3, we have $2\lambda \equiv 0$, which is a contradiction.

Subcase 1.2 If $a = \lambda$, by (3.3) we have

$$\begin{aligned}
&(n-\lambda\omega)f^{n-1}f' + (n-1)\omega f^{n-2}(f')^2 + \omega f^{n-1}f^{(2)} \\
&- \lambda f^n + [(\alpha(z) - \lambda q(z)\mathcal{D}(z,f))e^b + 2\lambda p_1]e^{\lambda z} = 0.
\end{aligned} \tag{3.4}$$

From (3.4) and Lemma 2.3, we have

$$(n-\lambda\omega)f^{n-1}f' + (n-1)\omega f^{n-2}(f')^2 + \omega f^{n-1}f^{(2)} - \lambda f^n = 0. \tag{3.5}$$

Dividing by f^n on both sides, we obtain

$$(n-\lambda\omega)\frac{f'}{f} + (n-1)\omega\Big(\frac{f'}{f}\Big)^2 + \omega\Big(\frac{f''}{f}\Big) - \lambda = 0. \tag{3.6}$$

Since $\frac{f''}{f} = \Big(\frac{f'}{f}\Big)' + \Big(\frac{f'}{f}\Big)^2$, we obtain a Riccati differential equation

$$(n-\lambda\omega)t + \omega t' + n\omega t^2 - \lambda = 0, \tag{3.7}$$

where $t = f'/f$. Through simple computations, we derive

$$t = (\frac{1}{n})\frac{[\frac{-n}{\lambda+n/\omega}e^{-(\lambda+n/\omega)z}+C_1]'}{\frac{-n}{\lambda+n/\omega}e^{-(\lambda+n/\omega)z}+C_1}+\frac{\lambda}{n},$$

which consequently leads to

$$f^n = C_2\Big[\frac{-n}{\lambda+n/\omega}e^{-(\lambda+n/\omega)z}+C_1\Big]e^{\lambda z},$$

where $C_1,\ C_2$ are constants.

Considering the case where $C_2 = 0$, we arrive at $f^n = 0$, which presents a contradiction. On the other hand, if $C_2 \neq 0$, then we obtain $\rho(f) = 1$, contradicting the given condition that $\rho(f) < 1$.

Case 2. Let us consider the case where $\rho(f) > 1$. We denote $\mathcal{P}(z) = p_1(z)e^{\lambda z} + p_2(z)e^{-\lambda z}$ and $\mathcal{H}(z) = q(z)\mathcal{D}(z, f)$. It is evident that $\rho(f) = 1$, implying $T(r, \mathcal{P}) = S(r, f)$. Consequently, equation (1.5) can be rewritten as

$$f^n(z) + \omega f^{n-1}(z)f'(z) + \mathcal{H}(z)e^{Q(z)} = \mathcal{P}(z). \tag{3.8}$$

Differentiating (3.8), yields

$$nf^{n-1}f' + \omega(n-1)f^{n-2}(f')^2 + \omega f^{n-1}f^{(2)} + \mathcal{G}(z)e^{Q(z)} = \mathcal{P}'(z), \tag{3.9}$$

where $\mathcal{G}(z) = \mathcal{H}' + Q'\mathcal{H}$. Eliminating $e^{Q(z)}$ from (3.8) and (3.9), we obtain

$$f^{n-2}\Big[\mathcal{G}f^2 + (\omega\mathcal{G} - n\mathcal{H})ff' - \omega\mathcal{H}ff^{(2)} - (n-1)\omega\mathcal{H}(f')^2\Big] = \mathcal{P}\mathcal{G} - \mathcal{P}'\mathcal{H}. \tag{3.10}$$

It is important to note that $n - 2 \geq 2$, and $\mathcal{P}\mathcal{G} - \mathcal{P}'\mathcal{H}$ represents a differential-difference polynomial in f with a total degree of at most 1. By Lemma 2.4, we obtain

$$m(r, \mathcal{G}f^2 + (\omega\mathcal{G} - n\mathcal{H})ff' - \omega\mathcal{H}ff^{(2)} - (n-1)\omega\mathcal{H}(f')^2) = S(r, f)$$

and

$$m(r, f[\mathcal{G}f^2 + (\omega\mathcal{G} - n\mathcal{H})ff' - \omega\mathcal{H}ff^{(2)} - (n-1)\omega\mathcal{H}(f')^2]) + S(r, f).$$

If $\mathcal{G}f^2 + (\omega\mathcal{G} - n\mathcal{H})ff' - \omega\mathcal{H}ff^{(2)} - (n-1)\omega\mathcal{H}(f')^2 \not\equiv 0$, since f is transcendental entire function, then

$$\begin{aligned}
T(r, f) &= m(r, f)\\
&\leq m\Big(f[\mathcal{G}f^2 + (\omega\mathcal{G} - n\mathcal{H})ff' - \omega\mathcal{H}ff^{(2)} - (n-1)\omega\mathcal{H}(f')^2]\Big)\\
&\quad + m\Big(r, \frac{1}{\mathcal{G}f^2 + (\omega\mathcal{G} - n\mathcal{H})ff' - \omega\mathcal{H}ff^{(2)} - (n-1)\omega\mathcal{H}(f')^2}\Big)\\
&\leq T(r, \mathcal{G}f^2 + (\omega\mathcal{G} - n\mathcal{H})ff' - \omega\mathcal{H}ff^{(2)} - (n-1)\omega\mathcal{H}(f')^2) + S(r, f)\\
&= S(r, f),
\end{aligned}$$

which yields a contradiction.

If $\mathcal{G}f^2 + (\omega\mathcal{G} - n\mathcal{H})ff' - \omega\mathcal{H}ff^{(2)} - (n-1)\omega\mathcal{H}(f')^2 \equiv 0$, from (3.10), we have $\mathcal{P}\mathcal{G} - \mathcal{P}'\mathcal{H} \equiv 0$. Then

$$\frac{\mathcal{P}'}{\mathcal{P}} = \frac{q'}{q} + \frac{\mathcal{D}'(z, f)}{\mathcal{D}(z, f)} + Q'.$$

Through the process of integration, it becomes evident that there exists a non-zero constant $C_3 \in \mathbb{C} \setminus 0$ such that

$$\mathcal{P} = C_3 q \mathcal{D}(z, f) e^{Q}. \tag{3.11}$$

Substituting (3.11) into (1.5), yields

$$f^n + \omega f^{n-1} f' = \Big(1 - \frac{1}{C_3}\Big) \left[p_1(z)e^{\lambda z} + p_2(z)e^{-\lambda z}\right]. \tag{3.12}$$

Given that f is a transcendental entire function with $\rho_1(f) < \rho(f)$, the Hadamard Decomposition Theorem allows us to express f in the form

$$f(z) = \Pi(z) e^{h(z)}. \tag{3.13}$$

In this representation, $\Pi(z)$ denotes the canonical product constructed from the zeros of $f(z)$, while $h(z)$ is a non-constant polynomial satisfying the condition

$$\deg(h) = \rho(f) > 1. \tag{3.14}$$

Substituting (3.13) into (3.12), we have

$$\begin{aligned} &e^{nh}\Pi^{n-1}(z)\left[\Pi(z) + \omega\Pi'(z) + \omega h'\Pi(z)\right] \\ &= \Big(1 - \frac{1}{C_3}\Big)\left(p_1(z)e^{\lambda z} + p_2(z)e^{-\lambda z}\right). \end{aligned} \tag{3.15}$$

By combining (3.14) and (3.15), we observe that the left-hand order of (3.15) exceeds 1, whereas the right-hand order is 1, which is a contradiction. Therefore $\rho(f) = 1$. From (1.5) and Lemma 2.4, we obtain

$$\begin{aligned} T(r, e^{Q(z)}) &= m(r, e^{Q(z)}) \\ &= m\Big(r, \frac{p_1(z)e^{\lambda z} + p_2(z)e^{-\lambda z} - f^n(z) - \omega f^{n-1}(z) f'(z)}{q(z)\mathcal{D}(z, f)}\Big) \\ &\le m\left(r, p_1(z)e^{\lambda z} + p_2(z)e^{-\lambda z}\right) + m\left(r, f^n(z) + \omega f^{n-1}(z) f'(z)\right) \\ &\quad + m\Big(r, \frac{1}{q(z)\mathcal{D}(z, f)}\Big) \\ &\le (n + k + 1)T(r, f) + 2T\left(r, e^{\lambda z}\right) + S\left(r, e^{\lambda z}\right). \end{aligned}$$

Note that $\deg(Q(z)) > 1$. Then

$$1 \le \deg(q) = \rho(e^{Q}) \le \max\{\rho(e^{\lambda z}),\ \rho(f)\} = 1,$$

i.e., $\rho(f) = \deg(Q) = 1$. The conclusion (1) is proved.

Next, we establish the conclusion (2). Suppose $f \in \Gamma_0$, and observe that $\rho(f) = \deg(Q) = 1$. In this case, we define $f(z) = e^{\gamma(z)}$, where $\gamma(z)$ is a non-constant polynomial. Substituting this representation of $f(z)$ into equation (1.5), we obtain

$$e^{n\gamma}\left[1 + \omega\gamma'\right] + q e^{Q(z) + k\gamma(z)} \Big(\sum_{i=0}^{k} b_i e^{\Delta_{c_i}\gamma(z)}\Big) = p_1(z)e^{\lambda z} + p_2(z)e^{-\lambda z} \tag{3.16}$$

Dividing both sides by $p_2 e^{-\lambda z}$, we obtain

$$\Big(\frac{1 + \omega\gamma'}{p_2}\Big) e^{n\gamma + \lambda z} + \big(\frac{q}{p_2}\big) e^{Q(k+1)\gamma(z) + \lambda z} \Big(\sum_{i=0}^{k} b_i e^{\Delta_{c_i}\gamma(z)}\Big) - \frac{p_1}{p_2} e^{2\lambda z} = 1. \tag{3.17}$$

It is evident that the expression $-\frac{p_1}{p_2}e^{2\lambda z}$ is not a constant. Consequently, by Lemma 2.6, we can deduce that

$$\frac{1+\omega\gamma'}{p_2} = 1 \quad \text{or} \quad \big(\frac{q}{p_2}\big)\sum_{i=0}^{k}\Big(b_i e^{Q(z)+\Delta_{c_i}\gamma(z)+(k+1)\gamma(z)+\lambda z}\Big) = 1.$$

We will now examine two separate cases.

If $\frac{1+\omega\gamma'}{p_2} = 1$, then it is straightforward to observe that $\gamma = \frac{-\lambda z}{n} + \mathcal{B}$, where $\mathcal{B}$ is a constant. Furthermore, we have

$$\big(\frac{q}{p_2}\big)\sum_{i=0}^{k}\Big(b_i e^{Q(z)+\Delta_{c_i}\gamma(z)+(k+1)\gamma(z)+\lambda z}\Big) = \frac{p_1}{p_2}e^{2\lambda z}, \tag{3.18}$$

which implies that

$$Q = \Big(\frac{(n+1)\lambda}{n}\Big)z + b,$$

where b is a constant.

If

$$\big(\frac{q}{p_2}\big)\sum_{i=0}^{k}\Big(b_i e^{Q(z)+\Delta_{c_i}\gamma(z)+(k+1)\gamma(z)+\lambda z}\Big) = 1$$

by (3.17), we have

$$\frac{1+\omega\gamma'}{p_2}e^{n\gamma+\lambda z} = \frac{p_1}{p_2}e^{2\lambda z}.$$

This case leads to the conclusion that $\gamma = \frac{\lambda z}{n} + \mathcal{B}$, where $\mathcal{B}$ is a constant.

Moreover, we can further deduce that

$$Q = \frac{-(n+1)\lambda}{n}z + b,$$

where b is a constant. The above analysis fully establishes the proof of conclusion (2). □

Limitations

First, our theoretical framework is predominantly applicable to meromorphic functions of finite order, which creates inherent restrictions when considering functions exhibiting infinite order growth behavior. Additionally, our analysis is confined to one-dimensional equations, leaving open questions about the behavior of multi-dimensional systems and coupled differential-difference equations that might demonstrate fundamentally different characteristics. Furthermore, while our research addresses particular non-linear forms, there exists a broader spectrum of non-linear equations that may require alternative analytical methods and approaches.

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References

[1] Alotaibi, A.; Langley, J.K.; The separation of zeros of solutions of higher order linear differential equations with entire coefficients, *Results Math.*, **63** (2013), 1365–1373.

[2] Chen, M. F.; Cui, N.; On zeros and growth of solutions of complex difference equations, *Adv. Difference Equ.*, **2021** (2021), 16.

[3] Chen, J.-F.; Ynag, Y.-Y.; On Meromorphic Solutions of Nonlinear Complex Differential Equations, *Analysis Mathematica*, **49** (2023), 699-719.

[4] Chen, J.-F.; Lian G.; Expressions of meromorphic solutions of a certain type of nonlinear complex differential equations, *Bull. Korean Math. Soc.*, **57** (2020), 1061–1073. DOI: 10.4134/BKMS.b190744.

[5] Chen, Min-Feng; Gao Zong-Sheng; Zhang, Ji-Long; Entire solutions of certain type of nonlinear difference equations, *Computational Methods and Function Theory*, **19** (2019), 17-36.

[6] Chiang, Yik-Man; Feng Shao-Ji; On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane, *The Ramanujan Journal*, **16** (2008), 105-129.

[7] Clunie, J.; On integral and meromorphic functions, *J. Lond. Math. Soc.*, **37** (1962), 17–27.

[8] Halburd, R. G.; Korhonen, R. J.; Difference analogue of the lemma on the logarithmic derivative with applications to difference equations, *J. Math. Anal. Appl.*, **314** (2006), 477–487.

[9] Hayman, W. K.; Meromorphic functions, *Oxford Mathematical Monographs, Clarendon Press, Oxford*, 1964.

[10] Li, P.; Entire solutions of certain type of differential equations II, *J. Math. Anal. Appl.*, **375** (2011), 310–319.

[11] Li, P.; Yang, C. C.; On the nonexistence of entire solutions of certain type of nonlinear differential equations, *J. Math. Anal. Appl.*, **320** (2006), 827–835.

[12] Liao, L. W.; Ynag, C. -C.; Zhang, J.-J.; On meromorphic solutions of certain type of nonlinear differential equations, *Ann. Acad. Sci. Fenn. Math.*, **38** (2013), 581–593.

[13] Liao, L. W.; Ye, Z.; On solutions to nonhomogeneous algebraic differential equations and their application, *J. Aust. Math. Soc.*, **97** (2014), 391–403.

[14] Liu, K.; Exponential polynomials as solutions of differential-difference equations of certain types, *Mediterr. J. Math.*, **13** (2016), 3015–3027.

[15] Liu, H. F.; Mao, Z. Q. Mao; Zheng D.; Meromorphic solutions of certain non-linear difference equations, *open Math.*, **18** (2020), 1292-1201.

[16] Mues, E.; Steinmetz, N.; The theorem of Tumura–Clunie for meromorphic functions, *J. Lond. Math. Soc.*, **23** (1981), 113–122.

[17] Reyzl, I.; On the theorem of Tumura–Clunie, *Complex Var. Theory Appl.*, **28** (1995), 175–188.

[18] Tumura, Y.; On the extensions of Borel's theorem and Saxer–Csillag's theorem, *Proc. Phys. Math. Soc. Jpn.*, **19** (1937), 29–35.

[19] Wen, Z. T.; Heittokangas, J.; Laine, I.; Exponential polynomials as solutions of certain nonlinear difference equations, *Acta Math. Sci. Ser. B*, **28** (2012), 1295–1306.

[20] Yang, C. C.; Applications of the Tumura–Clunie theorem, *Trans. Am. Math. Soc.*, **151** (1970), 659–662.

[21] Yang, C. C.; Li, P.; On the transcendental solutions of a certain type of nonlinear differential equations, *Arch. Math.*, **82** (2004), 442–448.

[22] Yang, Shuang-Shuang; Dong, Xian-Jing; Liao, Liang-Wen; Meromorphic Solutions of Nonlinear Differential-Difference Equations Involving Periodic Functions, *Bulletin of the Malaysian Mathematical Sciences Society*, **47** (2024), 1-18.

[23] Yang, C. C.; Yi, H. X.; Uniqueness Theory of Meromorphic Functions, *Springer Science & Business Media*, **557** (2004).

[24] Yi, Hong-Xun; Uniqueness theory of meromorphic functions, *Pure Appl. Math. Monographs*, (1995).

[25] Yi, H. X.; On a theorem of Tumura–Clunie for a differential polynomial, *Bull. Lond. Math. Soc.*, **20**, (1988) 593–596.

[26] Zhao, MingXin; Huang, Zhigang; On Meromorphic Solutions of Non-linear Differential-Difference Equations, *Journal of Nonlinear Mathematical Physics*, **30** (2023) 1-23.

[27] Zhang, J. J.; Xu, X. P.; Liao, L. W.; Meromorphic solutions of nonlinear complex differential equations (in Chinese), *Sci. Sin. Math.*, **47** (2017) 919–932.

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