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WEIGHTED (p,q)-EQUATIONS WITH GRADIENT DEPENDENT REACTION

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Abstract. We consider a weighted (p,q)-equation with a parametric reaction depending on the gradient. Using truncation and comparison techniques and the theory of nonlinear operators of monotone type, we show that for all small values of the parameter, the problem has a positive smooth solution.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this article we study the parametric Dirichlet problem

$$\begin{gathered} -\Delta_p^{a_1} u(z) - \Delta_q^{a_2} u(z) = f(z,u(z)) + \lambda |Du(z)|^{p-1} \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad 1 < q < p < N,\ \lambda > 0,\ u > 0. \end{gathered} \tag{1.1}$$

If $a \in C^{0,1}(\overline{\Omega})$ with $0 < \widehat{c} \le a(z)$ for all $z \in \overline{\Omega}$ and $s \in (1,\infty)$, then by Δ_s^a we denote the weighted s-Laplace differential operator defined by

$$\Delta_s^a u = \operatorname{div}(a(z)|Du|^{s-2}Du), \quad \text{for all } u \in W_0^{1,s}(\Omega).$$

In problem (1.1) the equation is driven by the sum of two such operators with different exponents $q < p$ and in general different weights (a weighted (p,q)-differential operator). So, the differential operator in (1.1) is not homogeneous. The reaction (right hand side) of (1.1) is gradient dependent. Combining varational tools from critical point theory, with the theory of nonlinear operators of monotone type, we show that for all small values of parameter $\lambda > 0$, problem (1.1) has a positive small solution.

Nonlinear elliptic equations with gradient dependence, such as (1.1), were examined by Candito-Gasiński-Papageorgiou [1], Faria-Miyagaki-Motreanu [4], Tanaka [16], Zeng-Papageorgiou [17]. All the aforementioned works deal with equations driven by autonomous differential operators and their method of proof is based on the fixed point theory. Our approach here is different and it is partly motivated by the work of Deuel-Hess [2] (see also Liu-Papageorgiou [10] on double phase equations).

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2. Mathematical background, hypotheses

In the analysis of (1.1) the main spaces are the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$. On account of the Poincaré inequality, on $W_0^{1,p}(\Omega)$ we can use the following equivalent norm

$$\|u\| = \|Du\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The space $C_0^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone $C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}$. This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \big\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial \mathbf{n}}|_{\partial\Omega} < 0\big\}$$

where $\frac{\partial u}{\partial \mathbf{n}} = (Du, \mathbf{n})_{\mathbb{R}^N}$ with $\mathbf{n}(\cdot)$ being the outward unit normal on $\partial\Omega$.

Let $a \in C^{0,1}(\overline{\Omega})$ with $a(z) \geq \widehat{c} > 0$ for all $z \in \overline{\Omega}$ and $1 < s < \infty$. We consider the following nonlinear eigenvalue problem

$$-\Delta_s^a u(z) = \widehat{\lambda}|u(z)|^{s-2}u(z) \quad \text{in } \Omega,\ u|_{\partial\Omega} = 0.$$

This problem was examined by Liu-Papageorgiou [11] (see the Appendix in [11]), who established the existence of a smallest eigenvalue $\widehat{\lambda}_1^a(s) > 0$ which has the following variational characterization

$$\widehat{\lambda}_1^a(s) = \inf\big\{\frac{\varrho_{\alpha,s}(Du)}{\|u\|_s^s} : u \in W_0^{1,s}(\Omega),\ u \neq 0\big\}, \tag{2.1}$$

where $\varrho_{\alpha,s}(Du) = \int_\Omega a(z)|Du|^s\,dz$ for all $u \in W_0^{1,s}(\Omega)$. This eigenvalue is isolated in the spectrum and simple (that is, if $\widehat{u}, \widehat{v} \in W_0^{1,s}(\Omega)$ are two eigenfunctions corresponding to $\widehat{\lambda}_1^a(s) > 0$, then $\widehat{u} = \vartheta\widehat{v}$ for some $\vartheta \in \mathbb{R}\backslash\{0\}$). So, the eigenspace corresponding to $\widehat{\lambda}_1^a(s)$ is one-dimensional and its elements have fixed sign. In fact $\widehat{\lambda}_1^a(s)$ is the only eigenvalue with eigenfunctions of constant sign. All the other eigenvalues have nodal (sign changing) eigenfunctions. The infimum in (2.1) is realized on the one dimensional eigenspace of $\widehat{\lambda}_1^a(s)$. By $\widehat{u}_1 = \widehat{u}_1(s) \in W_0^{1,s}(\Omega)$, we denote the positive, L^s-normalized (that is, $\|\widehat{u}_1\|_s = 1$) eigenfunction corresponding to $\widehat{\lambda}_1^a(s)$. The nonlinear regularity theory and the nonlinear maximum principle (see Gasiński-Papageorgiou ([5], p.738), imply that $\widehat{u}_1 \in \operatorname{int} C_+$.

These properties of the principal eigenvalue $\widehat{\lambda}_1^a(s) > 0$ and of its eigenfunctions, lead to the following result (see Liu-Papageorgiou [11, Proposition 4.2])).

Proposition 2.1. *If $\vartheta \in L^\infty(\Omega), \vartheta(z) \leq \widehat{\lambda}_1^a(s)$ for a.a. $z \in \Omega$ and $\vartheta \not\equiv \hat{\lambda}_1^\alpha(s)$, then there exists, $c_0 > 0$ such that*

$$c_0\|u\|^s \leq \varrho_{\alpha,s}(Du) - \int_\Omega \vartheta(z)|u|\,dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Suppose $u : \Omega \to \mathbb{R}$ is a measurable function. We define

$$u^+(z) = \max\{u(z), 0\}, \quad u^-(z) = \max\{-u(z), 0\} \quad \text{for all } z \in \Omega.$$

We have $u = u^+ - u^-$, $|u| = u^+ + u^-$ and if $u \in W_0^{1,s}(\Omega)$, then $u^\pm \in W_0^{1,s}(\Omega)$. Also by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$.

If $u, v : \Omega \to \mathbb{R}$ are measurable functions and $u(z) \leq v(z)$ for a.a. $z \in \Omega$, then we define

$$[u, v] = \{h \in W_0^{1,p}(\Omega) : u(z) \leq h(z) \leq v(z) \text{ a.a. } z \in \Omega\}.$$

If X is a Banach space and $\varphi \in C^1(X)$, then $K_\varphi = \{u \in X : \varphi'(u) = 0\}$ (the critical set of φ). Suppose X is a reflexive Banach space and $E : X \to X^*$ a bounded nonlinear map. We say that $E(\cdot)$ is "pseudomonotone", if it has the following property:

If $u_n \xrightarrow{w} u$ in X, $E(u_n) \xrightarrow{w} u^*$ in X^* and $\limsup_{n\to\infty}\langle E(u_n), u_n - u\rangle \le 0$, then $u^* = E(u)$ and $\langle E(u_n), u_n\rangle \to \langle E(u_n), u\rangle$.

(see Gasiński-Papageorgiou [5, p.330]). Here by $\langle \cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*, X). A maximal monotone operator is pseudomonotone. We say that $E(\cdot)$ is "strongly coercive", if

$$\frac{\langle E(u), u\rangle}{\|u\|_X} \to +\infty \quad \text{as } \|u\|_X \to +\infty.$$

The next theorem reveals the importance of pseudomonotone maps.

Theorem 2.2. *If X is a reflexive Banach space and $E : X \to X^*$ is a strongly coercive pseudomonotone map, then $E(\cdot)$ is surjective.*

Let $A_s^a : W_0^{1,s}(\Omega) \to W^{-1,s'}(\Omega) = W_0^{1,s}(\Omega)^* (\frac{1}{s} + \frac{1}{s'} = 1)$ be the nonlinear map defined by

$$\langle A_s^a(u), h\rangle = \int_\Omega a(z)|Du|^{s-2}(Du, Dh)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W_0^{1,s}(\Omega).$$

From Gasiński-Papageorgiou [6] (p.279), we have the following properties for this map.

Proposition 2.3. *$A_s^a : W_0^{1,s}(\Omega) \to W^{-1,s'}(\Omega)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (thus maximal monotone too) and of type $(S)_+$, that is: if $u_n \xrightarrow{w} u$ in $W_0^{1,s}(\Omega)$ and $\limsup_{n\to\infty}\langle A_s^a(u_n), u_n - u\rangle \le 0$, then $u_n \to u$ in $W_0^{1,s}(\Omega)$.*

Our hypotheses on the data of problem (1.1), are the following:

(H0) $a_1, a_2 \in C^{0,1}(\overline{\Omega})$ and $0 < \widehat{c} \le a_1(z), a_2(z)$ for all $z \in \overline{\Omega}$.

(H1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and
- (i) for every $\varrho > 0$, there exists $a_\varrho \in L^\infty(\Omega)$ such that
$$0 \le f(z,x) \le a_\varrho(z) \quad \text{for a. a. } z \in \Omega \text{ and all } 0 \le x \le \varrho;$$
- (ii) there exists a function $\widehat{\vartheta} \in L^\infty(\Omega)$ such that
$$\widehat{\vartheta}(z) \le \widehat{\lambda}_1^{a_1}(p) \text{ for a.a. } z \in \Omega,\ \widehat{\vartheta} \not\equiv \widehat{\lambda}_1^{a_1}(p)$$
$$\limsup_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} \le \widehat{\vartheta}(z) \quad \text{uniformly for a.a. } z \in \Omega;$$
- (iii) there exist $\tau \in (1,q)$ and $\delta > 0$ such that
$$cx^{\tau-1} \le f(z,x) \text{ for a.a. } z \in \Omega,\ \text{all } 0 \le x \le \delta,\ \text{some } c > 0.$$

Remark 2.4. Since we are looking for positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0,\infty)$ without any loss of generality we may assume that $f(z,x) = 0$ for a.a. $z \in \Omega$, all $x \le 0$.

3. Auxiliary problems

In this section we examine the following auxiliary parametric Dirichlet problem, with parameter $\mu > 0$

$$\begin{gathered} -\Delta_p^{a_1} u(z) - \Delta_q^{a_2} u(z) = \mu c u(z)^{\tau-1} \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad u > 0, \mu > 0, 1 < \tau < q < p < N. \end{gathered} \tag{3.1}$$

For this problem we have the following result.

Proposition 3.1. *If* (H0) *holds, then for every* $\mu > 0$ *problem* (3.1) *has a unique positive solution such that:* $\underline{u}_\mu \in \operatorname{int} C_+$, $\{\underline{u}_\mu\}_{\mu>0}$ *is nondecreasing, and*

$$\underline{u}_\mu \to 0 \quad \text{in } C^1(\overline{\Omega}) \quad \text{as } \mu \to 0^+.$$

Proof. Consider the C^1-functional $\psi_\mu : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\mu(u) = \frac{1}{p}\varrho_{a_1,p}(Du) + \frac{1}{q}\varrho_{a_2,q}(Du) - \frac{\mu c}{\tau}\|u^+\|_\tau^\tau \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Since $\tau < q < p$, we see that $\psi_\mu(\cdot)$ is coercive. Also, using the Sobolev embedding theorem, we see that $\psi_\mu(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $\underline{u}_\mu \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} &\psi_\mu'(\underline{u}_\mu) = \min\{\psi_\mu(u) : u \in W_0^{1,p}(\Omega)\} < 0 = \psi_\mu(0) \text{ (since } \tau < q < p) \\ &\Rightarrow \underline{u}_\mu \neq 0. \end{aligned} \tag{3.2}$$

Then we have

$$\begin{aligned} &\langle \psi_\mu(\underline{u}_\mu), h\rangle = 0 \quad \text{for all } h \in W_0^{1,p}(\Omega), \\ &\Rightarrow \langle V(\underline{u}_\mu), h\rangle = \int_\Omega \mu c (\underline{u}_\mu^+)^{\tau-1} h \, dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \end{aligned} \tag{3.3}$$

Here $V = A_p^{a_1} + A_q^{a_2} : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ and on account of Proposition 2.3 this map is bounded, continuous, strictly monotone (thus maximal monotone too) and of type $(S)_+$. In (3.3) we use the test function $h = -\underline{u}_\mu^- \in W_0^{1,p}(\Omega)$ and obtain

$$\varrho_{a_1,p}(D\underline{u}_\mu^-) \le 0 \implies \hat{c}\|D\underline{u}_\mu^-\|_p^p \le 0 \implies \underline{u}_\mu \ge 0, \ \underline{u}_\mu \neq 0.$$

From Ladyzhenskaya-Uraltseva [8, Theorem 7.1, p.286], we have $\underline{u}_\mu \in L^\infty(\Omega)$. Then the nonlinear regularity theory of Lieberman [9], implies that $\underline{u}_\mu \in C_+\backslash\{0\}$. Since $\Delta_p^{a_1}\underline{u}_\mu + \Delta_q^{a_2}\underline{u}_\mu \le 0$ in Ω, from the nonlinear maximum principle of Pucci-Serrin [15, pp.111, 120], we have that $\underline{u}_\mu \in$ int C_+.

Next we show the uniqueness of this positive solution. So, let $\underline{v} \in W_0^{1,p}(\Omega)$ be another positive solution of (Q_μ). Again we have $\underline{v}_\mu \in$ int C_+ We introduce the integral functional $j : L^1(\Omega) \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \frac{1}{p}\varrho_{a_1,p}(Du^{1/q}) + \frac{1}{q}\varrho_{a_2,q}(Du^{1/q}) & \text{if } u \ge 0, \ u^{1/q} \in W_0^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

From Diaz-Saa [3], see also Papageorgiou-Rădulescu [12] (proof of Proposition 3.5), we know that $j(\cdot)$ is convex. Let $dom j = \{u \in L^1(\Omega) : j(u) < +\infty\}$ (the effective domain of $j(\cdot)$). Let $h = \underline{u}_\mu^q - \underline{v}_\mu^q \in C_0^1(\bar{\Omega})$. Since $\underline{u}_\mu, \underline{v}_\mu \in \operatorname{int} C_+$, using Proposition 4.1.22, p.274, of Papageorgiou-Rădulescu-Repovš [14], we have that

$$\frac{\underline{u}_\mu}{\underline{v}_\mu} \in L^\infty(\Omega) \quad \text{and} \quad \frac{\underline{v}_\mu}{\underline{u}_\mu} \in L^\infty(\Omega).$$

Then if $t \in (0,1)$ is small, we have

$$\underline{u}_\mu^q + th \in \operatorname{dom} j \quad \text{and} \quad \underline{v}_\mu^q + th \in \operatorname{dom} j.$$

Using the convexity of $j(\cdot)$, we can compute the directional derivatives of $j(\cdot)$ at $\underline{u}_\mu^q$ and at $\underline{v}_\mu^q$ in the direction h. A direct computation involving the nonlinear Green's identity (see [14, pp.34-35]) gives

$$\begin{aligned} j'(\underline{u}_\mu^q)(h) &= \frac{1}{q}\int_\Omega \frac{-\Delta_p^{a_1}\underline{u}_\mu - \Delta_q^{a_2}\underline{u}_\mu}{\underline{u}_\mu^{q-1}} h\,dz = \frac{1}{q}\int_\Omega \frac{\mu c}{\underline{u}_\mu^{q-\tau}} h\,dz, \\ j'(\underline{v}_\mu^q)(h) &= \frac{1}{q}\int_\Omega \frac{-\Delta_p^{a_1}\underline{v}_\mu - \Delta_q^{a_2}\underline{v}_\mu}{\underline{v}_\mu^{q-1}} h\,dz = \frac{1}{q}\int_\Omega \frac{\mu c}{\underline{v}_\mu^{q-\tau}} h\,dz. \end{aligned}$$

The convexity of $j(\cdot)$ implies the monotonicity of the directional derivative $j'(\cdot)$. So, we have

$$0 \le \int_\Omega \Big(\frac{1}{\underline{u}_\mu^{q-\tau}} - \frac{1}{\underline{v}_\mu^{q-\tau}}\Big)(\underline{u}_\mu^q - \underline{v}_\mu^q)\,dz \le 0, \implies \underline{u}_\mu = \underline{v}_\mu.$$

This proves the uniqueness of the positive solution $\underline{u}_\mu \in \operatorname{int} C_+$ of (Q_μ) for all $\mu > 0$.

Next we show the monotonicity of the family $\{\underline{u}_\mu\}_{\mu>0}$. So suppose that $0 < \mu < \eta$. We have

$$-\Delta_p^{a_1}\underline{u}_\eta - \Delta_q^{a_2}\underline{u}_\eta \ge \eta c \underline{u}_\eta^{\tau-1} \ge \mu c \underline{u}_\eta^{\tau-1} \quad \text{in } \Omega. \tag{3.4}$$

We introduce the Carathéodory function $g_\mu(z,x)$ defined by

$$g_\mu(z,x) = \begin{cases} \mu c (x^+)^{z-1} & \text{if } x \le \underline{u}_\eta(z) \\ \mu c \underline{u}_\eta(z)^{\tau-1} & \text{if } \underline{u}_\eta(z) < x. \end{cases} \tag{3.5}$$

We set $G_\mu(z,x) = \int_0^x g_\mu(z,x)ds$ and consider the C^1-functional $\sigma_\mu : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\sigma_\mu(u) = \frac{1}{p}\varrho_{a_1,p}(Du) + \frac{1}{q}\varrho_{a_2,q}(Du) - \int_\Omega G_\mu(z,u)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From (3.5) it is clear that $\sigma_\mu(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $u_\mu^* \in W_0^{1,p}(\Omega)$ such that

$$\sigma_\mu(u_\mu^*) = \inf\{\sigma_\mu(u) : u \in W_0^{1,p}(\Omega)\}. \tag{3.6}$$

Let $u \in C_+\backslash\{0\}$. Since $\underline{u}_\mu \in \operatorname{int} C_+$, we can find $t \in (0,1)$ small such that $0 \le tu \le \underline{u}_\mu$ (see [14, p.274]). Then

$$\sigma_\mu(tu) = \frac{t^p}{p}\varrho_{a_1,p}(Du) + \frac{t^q}{q}\varrho_{a_2,q}(Du) - \frac{t^\tau}{\tau}\mu c\|u\|_\tau^\tau.$$

Since $\tau < q < p$, choosing $t \in (0,1)$ even smaller if necessary, we have that

$$\begin{aligned} \sigma_\mu(tu) < 0 &\implies \sigma_\mu(u_\mu^*) < 0 = \sigma_\mu(0) \quad \text{(see (3.6))} \\ &\implies u_\mu^* \neq 0. \end{aligned}$$

From (3.6), we see that $u_\mu^* \in K_{\sigma_\mu}$ and so

$$\langle V(u_\mu^*), h\rangle = \int_\Omega g_\mu(z,u_\mu^*)h\,dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{3.7}$$

In (3.7). We choose the test function $h = -(u_\mu^*)^- \in W_0^{1,p}(\Omega)$ and obtain

$$\widehat{c}\|D(u_\mu^*)^-\|_p^p \le 0 \quad \text{(see hypotheses (H0))},$$

$$\implies u_\mu^* \geq 0,\ u_\mu^* \neq 0.$$

Also, we test (3.7) with $h = (u_\mu^* - \underline{u}_\eta)^+ \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned} \langle V(u_\mu^*), (u_\mu^* - \underline{u}_\eta)^+\rangle &= \int_\Omega \mu c \underline{u}_\eta^{\tau-1}(u_\mu^* - \underline{u}_\eta)^+\,dz \quad \text{(see (3.5))} \\ &\leq \langle V(\underline{u}_\eta), (u_\mu^* - \underline{u}_\eta)^+\rangle \quad \text{(see (3.4))}, \end{aligned}$$

which implies $u_\mu^* \leq \underline{u}_\eta$ (since $V(\cdot)$ is strictly monotone).

So, we have proved that

$$u_\mu^* \in [0, \underline{u}_\eta], u_\mu^* \neq 0, \implies u_\mu^* = \underline{u}_\mu \leq \underline{u}_\eta \quad \text{(see (3.6) and (3.7))}.$$

This proves the monotonicity of $\{\underline{u}_\mu\}_{\mu>0}$. Standard Moser iteration, gives

$$\|\underline{u}_\mu\|_\infty \leq c_1 \|\underline{u}_1\|_\infty^{\frac{1}{p-1}} \quad \text{for some } c_1 > 0,\ \text{all } \mu \in (0,1].$$

Then the nonlinear regularity theory of Lieberman [9], implies that there exist $\alpha \in (0,1)$ and $c_2 > 0$ such that

$$\underline{u}_\mu \in C_0^{1,\alpha}(\overline{\Omega}) = C^{1,\alpha}(\overline{\Omega}) \cap C_0^1(\overline{\Omega}), \quad \|\underline{u}_\mu\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq c_2 \quad \text{for all } \mu \in (0,1].$$

Recall that $C_0^{1,\alpha}(\overline{\Omega}) \hookrightarrow C_0^1(\overline{\Omega})$ compactly (Arzela-Ascoli theorem). Therefore,

$$\underline{u}_\mu \to 0 \text{ in } C_0^1(\overline{\Omega}) \quad \text{as } \mu \to 0^+.$$

□

Next we consider another auxiliary Dirichlet probem,

$$\begin{gathered} -\Delta_p^{a_1} u(z) - \Delta_q^{a_2} u(z) = f(z, u(z)) + 1 \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad u > 0. \end{gathered} \tag{3.8}$$

Proposition 3.2. *Under Assumptions* (H0) *and* (H1), *problem* (3.6) *has a smallest positive solution* $\overline{u} \in \operatorname{int} C_+$.

Proof. We consider the nonlinear map $E : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$ defined by

$$E(u) = V(u) - N_f(u^+) \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

with $N_f(v)(\cdot) = f(\cdot, v(\cdot))$ for all $v \in W_0^{1,p}(\Omega)$ (the Nemytski map corresponding to $f(z,x)$). Note that on account of hypotheses $H_1(i), (ii)$ $N_f(v) \in L^{p'}(\Omega)$ for all $v \in W_0^{1,p}(\Omega)$ and $L^{p'}(\Omega) \hookrightarrow W^{-1,p'}(\Omega)$ continuously, see [5, p. 141].

Evidently $E(\cdot)$ is bounded and continuous.

Claim 1: $E(\cdot)$ is pseudomonotone. We consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ which satisfies

$$\begin{gathered} u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega), \quad E(u_n) \xrightarrow{w} u^* \text{ in } W_0^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*, \\ \limsup_{n\to\infty} \langle E(u_n), u_n - u\rangle \leq 0. \end{gathered} \tag{3.9}$$

From (3.9) and since $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ compactly, we have that

$$\begin{aligned} &u_n \to u \text{ in } L^p(\Omega), \quad \text{as } n \to \infty, \\ &\implies \langle N_f(u_n^+), u_n - u\rangle = \int_\Omega f(z, u_n^+)(u_n - u)\,dz \to 0 \text{ as } n \to \infty \\ &\quad \text{(see (H1)(i), (H1)(ii))}, \end{aligned}$$

$$\begin{aligned}
&\implies \limsup_{n\to\infty}\langle V(u_n), u_n - u\rangle \le 0 (\text{see } (3.9)),\\
&\implies u_n \to u \text{ in } W_0^{1,p}(\Omega) \quad (\text{since } V(\cdot) \text{ is of type } (S)_+).
\end{aligned}$$

Therefore $u^* = E(u)$ (since $E(\cdot)$ is continuous) and $\langle E(u_n), u_n\rangle \to \langle E(u), u\rangle$. So, $E(\cdot)$ is pseudomonotone and this proves Claim 1.

Claim 2: $E(\cdot)$ is strongly coercive. Hypotheses (H1) (i), (ii) imply that given $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

$$0 \le f(z,x) \le (\hat\vartheta(z)+\varepsilon)x^{p-1} + c_\varepsilon \quad \text{for a.a. } z\in\Omega \text{ and all } x \ge 0. \tag{3.10}$$

We have that

$$\begin{aligned}
\langle E(u), u\rangle &= \langle V(u), u\rangle - \int_\Omega f(z,u^+)u\,dz\\
&\ge \varrho_{a_1,p}(Du) - \int_\Omega \hat\vartheta(z)|u|^p\,dz - \varepsilon\|u\|_p^p - c_\varepsilon|\Omega|_N \quad (\text{see } (3.10))\\
&\ge [c_0 - \frac{\varepsilon}{\hat\lambda_1^{a_1}(p)}]\|u\|^p - c_\varepsilon|\Omega|_N \quad (\text{see Proposition 2.1 and } (2.1)).
\end{aligned}$$

Choosing $\varepsilon \in (0, \widehat\lambda_1^{a_1}(p)c_0)$, we infer that

$$\langle E(u), u\rangle \ge c_3\|u\|^p - c_4 \quad \text{for some } c_3, c_4 > 0 \implies E(\cdot) \text{ is strongly coercive.}$$

This proves Claim 2.

Claims 1 and 2, permit the use of Theorem 2.2. So, we have that the map $E(\cdot)$ is surjective. Hence we can find $\overline{u} \in W_0^{1,p}(\Omega)\backslash\{0\}$ such that

$$E(\overline{u}) = 1 \text{ in } W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*.$$

With $-\overline{u}^- \in W_0^{1,p}(\Omega)$, we obtain

$$\widehat{c}\|D\overline{u}^-\|_p^p \le 0 \implies \overline{u} \ge 0, u \ne 0.$$

So $\overline{u} \in W_0^{1,p}(\Omega)$ is a solution of (3.8) and as before the nonlinear regularity theory and the nonlinear maximum principle (see [9], [15]), imply that $\overline{u} \in \operatorname{int} C_+$.

We show that there is a smallest positive solution of (3.8). Let S_+ be the set of positive solutions of (3.8). We have just seen that

$$\emptyset \ne S_+ \subseteq \operatorname{int} C_+.$$

The set S_+ is downward directed (that is, if $u_1, u_2 \in S_+$, then we can find $u \in S_+$ such that $u \le u_1, u \le u_2$; see [13, Proposition 7]). So, by Theorem 5.109, p.308, of Hu-Papageorgiou [7], we can find a decreasing sequence $\{\overline{u}_n\}_{n\in\mathbb{N}} \subseteq S_+$ such that

$$\inf S_+ = \inf_{n\in\mathbb{N}} \overline{u}_n.$$

We have

$$\langle V(\overline{u}_n), h\rangle = \int_\Omega [f(z,\overline{u}_n)+1]h\,dz \quad \text{for all } h \in W_0^{1,p}(\Omega) \text{ and all } n\in\mathbb{N}, \tag{3.11}$$

$$0 \le \overline{u}_n \le \overline{u}_1. \tag{3.12}$$

If in (3.11), we choose $h = \overline{u}_n \in W_0^{1,p}(\Omega)$, then using (H0), (H1)(i) and (3.12), we obtain

$$\begin{gathered}
\widehat{c}\|\overline{u}_n\|^p \le c_5 \text{ for some } c_5 > 0, \text{ all } n\in\mathbb{N},\\
\implies \overline{u}_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}
\end{gathered}$$

As before, the nonlinear regularity theory of Lieberman [9] implies that we can assume that

$$\overline{u}_n \to \overline{u} \quad \text{in } C_0^1(\overline{\Omega}) \text{ as } n \to \infty.$$

If $\overline{u} = 0$, then we can find $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} &0 \le \overline{u}_n(z) \le \delta \quad \text{for all } z \in \overline{\Omega} \text{ and all } n \ge n_0, \\ &\implies c\overline{u}_n(z)^{\tau-1} \le f(z, \bar{u}_n(z)) \quad \text{for a.a. } x \in \Omega \text{ and all } n \ge n_0. \end{aligned} \tag{3.13}$$

We introduce the Carathéodory function $k_n(z,x)$ defined by

$$k_n(z,x) = \begin{cases} c(x^+)^{\tau-1} & \text{if } x \le \overline{u}_n(z),\ n \ge n_0. \\ c\overline{u}_n(z)^{\tau-1} & \text{if } \overline{u}_n(z) < x,\ n \ge n_0. \end{cases} \tag{3.14}$$

Now we consider the Dirichlet problem

$$\begin{gathered} -\Delta_{\alpha_1}^p u(z) - \Delta_{\alpha_2}^q u(z) = k_n(z, u(z)) \quad \text{in } \Omega \\ u|_{\partial\Omega} = 0,\ u > 0. \end{gathered} \tag{3.15}$$

As before, using the Weierstrass-Tonelli theorem and since $\tau < q < p$, we can find $\tilde{u}_1 \in \operatorname{int} C_+$ solution of (3.15) and $0 \le \tilde{u}_1 \le \overline{u}_n$ (see (3.14), (3.13)). Hence

$$\tilde{u}_1 = \underline{u}_1 \quad \text{(see Proposition 3.1)} \implies \underline{u}_1 \le u_n \text{ for all } n \ge n_0,$$

which contradicts our hypothesis that $\overline{u} = 0$. So, $\overline{u} \ne 0$ and then

$$\overline{u} \in S_+ \subset \operatorname{int} C_+, \overline{u} = \inf S_+. \qquad \square$$

4. Positive solutions

Let $\overline{u} \in \operatorname{int} C_+$ be the minimal positive solution of (3.8) produced in Proposition 3.2. We can find $\lambda_0 > 0$ such that $\lambda|D\overline{u}(z)| \le 1$ for all $z \in \overline{\Omega}$, all $0 < \lambda \le \lambda_0$. We have

$$-\Delta_p^{\alpha_1}\overline{u} - \Delta_q^{\alpha_2}\overline{u} = f(z,\overline{u}) + 1 \ge f(z,\overline{u}) + \lambda|D\overline{u}|^{p-1} \tag{4.1}$$

in Ω for all $0 < \lambda \le \lambda_0$.

Also, using Proposition 3.1 and the fact that $\overline{u} \in \operatorname{int} C_+$, we can find $\mu \in (0,1]$ small such that

$$0 \le \underline{u}_\mu \le \min\{\delta, \overline{u}(z)\} \quad \text{for all } z \in \overline{\Omega}. \tag{4.2}$$

So, we have

$$\begin{aligned} -\Delta_p^{\alpha_1}\underline{u}_\mu - \Delta_1^{\alpha_2}\underline{u}_\mu = \mu c\underline{u}_\mu^{\tau-1} &\le c\underline{u}_\mu^{\tau-1} \quad (\text{since } 0 < \mu \le 1) \\ &\le f(z,\underline{u}_\mu) + \lambda|D\underline{u}_\mu| \quad \text{in } \Omega \text{ for all } \lambda > 0 \end{aligned} \tag{4.3}$$

(see (4.2)).

Using that $\underline{u}_\mu \le \overline{u}$ (see (4.2)), we introduce the truncation map $\tau_0 : L^p(\Omega) \to L^p(\Omega)$ defined by

$$\tau_0(u)(z) = \begin{cases} \underline{u}_\mu(z) & \text{if } u(z) < \underline{u}_\mu(z), \\ u(z), & \text{if } \underline{u}_\mu(z) \le u(z) \le \overline{u}(z), \\ \overline{u}(z) & \text{if } \overline{u}(z) < u(z). \end{cases} \tag{4.4}$$

Clearly $\tau_0(\cdot)$ is bounded and continuous. In fact we can say more. Note that if $u \in W_0^{1,p}(\Omega)$, then $\tau_0(u) \in W_0^{1,p}(\Omega)$ (see Papageorgiou- Rădulescu-Repovš [14, Proposition 1.4.5, p 23]). So, we can consider the map $\tau_0 : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$.

Proposition 4.1. $\tau_0 : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ *is bounded and continuous.*

Proof. We know that

$$D\tau_0(u)(z) = \begin{cases} D\underline{u}_\mu(z) & \text{if } u(z) < \underline{u}_\mu(z), \\ Du(z), & \text{if } \underline{u}_\mu(z) \le u(z) \le \overline{u}(z) \text{ for all } u \in W_0^{1,p}(\Omega) \\ D\overline{u}(z) & \text{if } \overline{u}(z) < u(z). \end{cases} \tag{4.5}$$

(see [14, p.23]). From (4.5) it is clear that $\tau_0 : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ is bounded. We show that $\tau_0(\cdot)$ is also continuous. To this end let $\{u_n\}_{n\in\mathbb{N}}$ such taht $u_n \to u$ in $W_0^{1,p}(\Omega)$ as $n \to \infty$. So, we can assume that

$$\begin{gathered} u_n(z) \to u(z) \text{ in } \mathbb{R}, \quad Du_n(z) \to Du(z) \text{ in } \mathbb{R}^N \quad \text{for a.a. } z \in \Omega, \\ |u_n(z)|, |Du_n(z)| \le \widehat{h}(z) \quad \text{for a.a. } z \in \Omega, \text{ all } n \in \mathbb{N}, \text{ with } \widehat{h} \in L^p(\Omega). \end{gathered} \tag{4.6}$$

Then from (4.6) and (3.13), (3.14) it follows that

$$\tau_0(u_n)(z) \to \tau_0(u)(z) \text{ in } \mathbb{R}, \quad D\tau_0(u_n)(z) \to D\tau_0(u)(z) \text{ in } \mathbb{R}^N \tag{4.7}$$

for a.a. $z \in \Omega$ as $n \to \infty$.

Moreover, from Gasinski-Papageorgiou[6, Problem 1.4, p.35]) we have that

$$\{|u_n - u|^p\}_{n\in\mathbb{N}}, \quad \{|D(u_n - u)|^p\}_{n\in\mathbb{N}} \subseteq L^1(\Omega), \tag{4.8}$$

are both unifirmly integrable. Then (4.7), (4.8) and Vitali's theorem (see [7, Theorem 2.147, p.91]), imply that

$$\begin{aligned} &\tau_0(u_n) \to \tau_0(u) \text{ in } L^p(\Omega), D\tau_0(u_n) \to D\tau_0(u) \text{ in } L^p(\Omega, \mathbb{R}^N) \\ &\implies \tau_0(u_n) \to \tau_0(u) \text{ in } W_0^{1,p}(\Omega). \end{aligned}$$

So, we have proved the continuity of $\tau_0 : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$. □

The above result is due to Deuel-Hess[2, Lemma on p.53]. We have reproduced the proof, since in the proof of [2] there is a small gap, since they state that $|\tau_0(u_n)(z)| \le |u_n(z)|$ for a.a. $z \in \Omega$ (see [2, p.54]), which is not true in general.

For $\lambda > 0$, we introduce the Caratheodory function $\widehat{f}_\lambda(z,x,y)$ defined by

$$\widehat{f}_\lambda(z,x,y) = f(z,x) + \lambda|y|^{p-1} \quad \text{for all } x \in \Omega, \text{ all } x \in \mathbb{R}, \text{ and all } y \in \mathbb{R}^N.$$

Also let $\widehat{\tau}_0 : W_0^{1,p}(\Omega) \to L^p(\Omega) \times L^p(\Omega, \mathbb{R}^N)$ be defined by

$$\widehat{\tau}_0(u) = (\tau_0(u), D\tau_0(u)) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

By Proposition 3.2, $\widehat{\tau}_0(\cdot)$ is bounded and continuous. Then we consider $K_\lambda : W_0^{1,p}(\Omega) \to L^{p'}(\Omega) \hookrightarrow W^{-1,p'}(\Omega)^*$ defined by

$$K_\lambda(u) = (N_{\widehat{f}_\lambda} \circ \widehat{\tau}_0)(u) \text{ for all } u \in W_0^{1,p}(\Omega).$$

Then $K_\lambda(\cdot)$ is bounded and continuous.

Following Deuel-Hess [2], we introduce also the Caratheodory function

$$b(z,x) = \begin{cases} -(\underline{u}_\mu(z) - x)^{p-1} & \text{if } x < \underline{u}_\mu(z) \\ 0, & \text{if } \underline{u}_\mu(z) \le x \le \overline{u}(z) \\ (x - \overline{u}(z))^{p-1} & \text{if } \overline{u}(z) < x. \end{cases} \tag{4.9}$$

Let $N_b : L^p(\Omega) \to L^{p'}(\Omega)$ be the Nemytski map corresponding to $b(z,x)$, that is,

$$N_b(u)(\cdot) = b(\cdot, u(\cdot)) \text{ for all } u \in L^p(\Omega).$$

Evidently $N_b(\cdot)$ is bounded and continuous. In what follows by $(\cdot,\cdot)_{pp'}$, we denote the duality brackets for the dual pair $(L^{p'}(\Omega), L^p(\Omega))$. We state the next proposition in a more general setting than the one we have here and so the estimate can be used in other more general situations. So, for the purposes of the next proposition, $\underline{u}_\mu$ and $\overline{u}$ are general functions in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ not necessarily positive (as the case here), which satisfy $\underline{u}_\mu \le \overline{u}$.

Proposition 4.2. $(N_b(u),u)_{pp'} = \int_\Omega b(z,u)u\,dz \ge c_6\|u\|_p^p - c_7$ *for some* $c_6, c_7 > 0$ *all* $u \in W_0^{1,p}(\Omega)$.

Proof. We have

$$(N_b(u),u)_{pp'} = \int_{\{u>\overline{u}\}} (u-\overline{u})^{p-1}u\,dz - \int_{\{u<\underline{u}_\mu\}} (\underline{u}_\mu - u)^{p-1}udz.$$

On $\{u > \overline{u}\}$ we have

$$|u|^{p-1} = |(u-\overline{u}) + \overline{u}|^{p-1} \le [(u-\overline{u}) + |\overline{u}|]^{p-1} \le \widehat{c}_1[(u-\overline{u})^{p-1} + |\overline{u}|^{p-1}]$$

for some $\widehat{c}_1 > 0$. This implies

$$\frac{1}{\widehat{c}_1}|u|^{p-1} - |\overline{u}|^{p-1} \le (u-\overline{u})^{p-1}.$$

If $u(z) > 0$, then on $\{u > \overline{u}\}$ we have

$$(u-\overline{u})^{p-1}u \ge \frac{1}{\widehat{c}_1}|u|^p - |\overline{u}|^{p-1}|u| \quad (\text{since } |u(z)| = u(z) > 0). \tag{4.10}$$

If $u(z) < 0$, then on $\{u > \overline{u}\}$ we have $\overline{u} < u < 0 \Rightarrow |u| < |\overline{u}|$.

It follows that

$$\begin{aligned}
&\Big(\frac{|\overline{u}|}{|u|}\Big)^{p-1} - \Big(\frac{|\overline{u}|}{|u|} - 1\Big)^{p-1} \ge \widehat{c}_2 > 0, \\
&\implies \frac{|\overline{u}|^{p-1}}{|u|^{p-1}} - \frac{(u-\overline{u})^{p-1}}{|u|^{p-1}} \ge \widehat{c}_2 > 0 \\
&\qquad (\text{since } (|\overline{u}| - |u|)^{p-1} = (u-\overline{u})^{p-1} \text{ on } \{\overline{u} < u < 0\}) \\
&\implies |\overline{u}|^{p-1} - (u-\overline{u})^{p-1} \ge \widehat{c}_2|u|^{p-1} \\
&\implies \widehat{c}_2|u|^p - |\overline{u}|^{p-1}|u| \le (u-\overline{u})^{p-1}u \quad \text{on } \{\overline{u} < u < 0\}.
\end{aligned} \tag{4.11}$$

From (4.10) and (4.11), we see that

$$\int_{\{u>\overline{u}\}} (u-\overline{u})^{p-1}u\,dz \ge \int_{\{u>\overline{u}\}} [\widehat{c}_3|u|^p - |\overline{u}|^{p-1}|u|]dz \quad \text{for some } \widehat{c}_3 > 0. \tag{4.12}$$

Next we estimate the set $\{u < \underline{u}_\mu\}$. If $u(z) < 0$, then on $\{u < \underline{u}_\mu\}$ we have

$$\begin{aligned}
-(\underline{u}_\mu - u)^{p-1}u &= (\underline{u}_\mu - u)^{p-1}|u| \\
&= |\underline{u}_\mu - u|^{p-1}|u| \\
&\ge \widehat{c}_4|u|^p - \widehat{c}_5|\underline{u}_\mu|^{p-1}|u| \quad \text{for some } \widehat{c}_4, \widehat{c}_5 > 0.
\end{aligned} \tag{4.13}$$

If $u(z) > 0$, then $\underline{u}_\mu > u > 0$ on $\{u < \underline{u}_\mu\}$ and so

$$-(\underline{u}_\mu - u)^{p-1}u \ge u^p - \widehat{c}_7\underline{u}_\mu^{p-1}u \text{ with } \widehat{c}_7 > 0. \tag{4.14}$$

From (4.13) and (4.14), it follows that

$$\int_{\{u<\underline{u}_\mu\}} b(z,u)udz \geq \int_{\{u<\underline{u}_\mu\}} [\widehat{c}_8|u|^p - \widehat{c}_9|\underline{u}_\mu|^{p-1}|u|]dz \tag{4.15}$$

for some $\widehat{c}_8, \widehat{c}_9 > 0$. Using (4.12) and (4.15), we see that

$$\begin{aligned}
&\int_\Omega b(z,u)\,dz \\
&\geq \int_{\{u>\overline{u}\}} [\widehat{c}_3|u|^p - |\overline{u}|^{p-1}|u|]dz + \int_{\{u<\underline{u}_\mu\}} [\widehat{c}_8|u|^p - \widehat{c}_9|\underline{u}_\mu|^{p-1}|u|]dz \\
&\geq \widehat{c}_{10}\|u\|_p^p - \widehat{c}_{10}\int_{\{\underline{u}_\mu\leq u\leq\overline{u}\}} |u|^p\,dz - \widehat{c}_{11}\int_\Omega [|\underline{u}_\mu|^{p-1} + |\overline{u}|^{p-1}]|u|dz \\
&\quad \text{for some } \widehat{c}_{10}, \widehat{c}_{11} > 0 \\
&\geq \widehat{c}_{10}\|u\|_p^p - \widehat{c}_{12}\|u\| - \widehat{c}_{13} \quad \text{for some } \widehat{c}_{12}, \widehat{c}_{13} > 0.
\end{aligned} \tag{4.16}$$

Using Young's inequality with $\varepsilon \in (0, \rho\widehat{c}_{10})$, from (4.16) we obtain

$$(N_b(u), u)_{pp'} \geq c_6\|u\|_\rho^p - c_7 \quad \text{for some } c_6, c_7 > 0. \qquad \square$$

We introduce the map $G_\lambda : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)^*$ defined by $G_\lambda(u) = V(u) + \theta N_b(u) - K_\lambda(u)$ for all $u \in W_0^{1,p}(\Omega)$, all $0 < \lambda \leq \lambda_0, \theta > 0$. Evidently $G_\lambda(\cdot)$ is bounded and continuous.

Proposition 4.3. *If* (H0), (H1), $0 < \lambda \leq \lambda_0$ *and* $\theta > 0$ *hold, then* $G_\lambda(\cdot)$ *is pseudomonotone.*

Proof. We consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ such that

$$\begin{gathered}
u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega), \quad G_\lambda(u_n) \xrightarrow{w} u^* \text{ in } W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^* \\
\limsup_{n\to\infty} \langle G_\lambda(u_n), u_n - u\rangle \leq 0.
\end{gathered} \tag{4.17}$$

From this and the fact that $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ compactly, we have $u_n \to u$ in $L^p(\Omega)$ as $n \to \infty$.

Then using Hölder's inequality, we obtain

$$\begin{gathered}
\int_\Omega b(z,u_n)(u_n - u)dz \to 0 \quad \text{as } n \to \infty, \\
\int_\Omega \widehat{f}_\lambda(z, \tau_0(u_n), D\tau_0(u_n))(u_n - u)dz \to 0 \quad \text{as } n \to \infty.
\end{gathered}$$

From (4.17) it follows that

$$\begin{aligned}
&\limsup_{n\to\infty} \langle V(u_n), u_n - u\rangle \leq 0 \\
&\implies u_n \to u \quad \text{in } W_0^{1,p}(\Omega) \quad (\text{since } V(\cdot) \text{ is an } (S)_+\text{-map}).
\end{aligned}$$

Therefore, $u^* = G_\lambda(u)$ and $\langle G_\lambda(u_n), u_n\rangle \to \langle G_\lambda(u), u\rangle$. This proves that $G_\lambda(\cdot)$ is a pseudomonotone map. $\square$

Proposition 4.4. *If* (H0), (H1) *hold and* $0 < \lambda \leq \lambda_0$, *then for* $\theta > 0$ *large the map* $G_\lambda(\cdot)$ *is strongly coercive.*

Proof. For every $u \in W_0^{1,p}(\Omega)$, we have

$$
\begin{aligned}
&|\langle K_\lambda(u), u\rangle| \\
&= |(K_\lambda(u), u)_{pp'}| \\
&= \Big|\int_\Omega [f(z,\tau_0(u)) + \lambda|D\tau_0(u)|^{p-1}]u\,dz\Big| \\
&\le \int_\Omega [f(z,\tau_0(u)) + \lambda_0|D\tau_0(u)|^{p-1}]|u|\,dz \\
&\le c_8[\|u\|_p + \lambda_0\|D\tau_0(u)\|_p^{p-1}\|u\|_p] \quad \text{for some } c_8>0 \quad \text{(use hölder's inequality)} \\
&\le c_9[\|u\| + \lambda_0(\|u\|^{p-1}+1)\|u\|_p] \quad \text{for some } c_9>0 \\
&\quad \text{(recall that } W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \text{ continuously and see (4.5))} \\
&\le c_{10}[\|u\| + \lambda_0\|u\|^{p-1}\|u\|_p] \quad \text{for some } c_{10}>0 \\
&\le c_{11}[\|u\| + \lambda_0(\varepsilon\|u\|^p + \frac{1}{\varepsilon}\|u\|_p^p)] \quad \text{for some } c_{11}>0 \\
&\quad \text{(use Young's inequality with } \varepsilon>0).
\end{aligned}
\tag{4.18}
$$

From Proposition 4.2 we have

$$
\theta\langle N_b(u), u\rangle = \theta(N_b(u), u)_{pp'} \ge \theta c_6\|u\|_p^p - \theta c_7 \quad \text{for all } u \in W_0^{1,p}(\Omega). \tag{4.19}
$$

Using (4.18), (4.19), and hypotheses(H0), we have

$$
\langle G_\lambda(u), u\rangle \ge [\widehat{c} - \lambda_0 c_{11}\varepsilon]\|u\|^\rho + [\theta c_6 - \frac{\lambda_0 c_{11}}{\varepsilon}]\|u\|_p^p - \theta c_7. \tag{4.20}
$$

First we choose $\varepsilon \in (0, \frac{\widehat{c}}{\lambda_0 c_{11}})$. So, $\widehat{c} - \lambda c_{11}\varepsilon > 0$. Then using this choice of $\varepsilon > 0$, we choose $\theta > \frac{\lambda_0 c_{11}}{\varepsilon c_6}$. From (4.20), we see that

$$
\langle G_\lambda(u), u\rangle \ge c_{12}\|u\|^\rho - \theta c_7 \quad \text{for some } c_{12} > 0 \text{ and all } u \in W_0^{1,p}(\Omega)
$$

implies that $G_\lambda(\cdot)$ is strongly coercive. ∎

We can now state and prove the existence theorem for problem (1.1)

Theorem 4.5. *If* (H0), (H1) *hold and* $0 < \lambda \le \lambda_0$, *then problem* (1.1) *has a positive solution* $\widehat{u}_\lambda \in \operatorname{int} C_+$.

Proof. Proposition 4.3 and 4.4 permit the use of Theorem 2.2. So $G_\lambda(\cdot)$ is surjective and we can find $\widehat{u}_\lambda \in W_0^{1,p}(\Omega)$ such that

$$
\begin{aligned}
&G_\lambda(\widehat{u}_\lambda) = 0 \quad \text{in } W^{-1,p'}(\Omega)^* = W_0^{1,p}(\Omega)^* \\
&\implies \langle V(\widehat{u}_\lambda), h\rangle + \theta\int_\Omega b(z,\widehat{u}_\lambda)h\,dz = \int_\Omega [f(z,\tau_0(u)) + \lambda|D\tau_0(u)|^{p-1}]h\,dz \\
&\quad \text{for all } h \in W_0^{1,p}(\Omega).
\end{aligned}
\tag{4.21}
$$

In (4.21) first we used the test function $h = (\widehat{u}_\lambda - \overline{u})^+ \in W_0^{1,p}(\Omega)$. Then

$$
\begin{aligned}
&\langle V(\widehat{u}_\lambda), (\widehat{u}_\lambda - \overline{u})^+\rangle + \theta\int_\Omega (\widehat{u}_\lambda - \overline{u})^{p-1}(\widehat{u}_\lambda - \overline{u})^+\,dz \\
&= \int_\Omega [f(z,\overline{u}) + \lambda|D\overline{u}|^{p-1}](\widehat{u}_\lambda - \overline{u})^+ dz \text{ (see (4.9), (4.4), (4.5))} \\
&\le \langle V(\overline{u}), (\widehat{u}_\lambda - \overline{u})^+\rangle \quad \text{(see (4.1))}
\end{aligned}
$$

which implies

$$\langle V(\widehat{u}_\lambda) - V(\overline{u}), (\widehat{u}_\lambda - \overline{u})^+\rangle \le -\theta \int_\Omega (\widehat{u}_\lambda - \overline{u})^{p-1}(\widehat{u}_\lambda - \overline{u})^+ \, dz \le 0,$$

which in turn implies $\widehat{u}_\lambda \le \overline{u}$.

Next, we test (4.21) with $h = (\underline{u}_\mu - \widehat{u}_\lambda)^+ \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned}
&\langle V(\widehat{u}_\lambda), (\underline{u}_\mu - \overline{u})^+\rangle - \theta \int_\Omega (\underline{u}_\mu - \overline{u})^{p-1}(\underline{u}_\mu - \widehat{u}_\lambda)^+ \, dz \\
&= \int_\Omega [f(z, \underline{u}_\mu) + \lambda |D\underline{u}_\mu|^{p-1}](\underline{u}_\mu - \widehat{u}_\lambda)^+ dz \quad \text{(see (4.9), (4.4), (4.5))} \\
&\ge \langle V(\underline{u}_\mu), (\underline{u}_\mu - \widehat{u}_\lambda)^+\rangle \text{ (see (4.3))}
\end{aligned}$$

which implies

$$\langle V(V(\underline{u}_\mu) - \widehat{u}_\lambda), (\underline{u}_\mu - \widehat{u}_\lambda)^+\rangle \le -\theta \int_\Omega (\underline{u}_\mu) - \widehat{u}_\lambda)^{p-1}(\underline{u}_\mu) - \widehat{u}_\lambda)^+ \, dz \le 0,$$

which in turn implies $\underline{u}_\mu \le \widehat{u}_\lambda$. So, we have proved that

$$\widehat{u}_\lambda \in [\underline{u}_\mu, \overline{u}]. \tag{4.22}$$

From (4.22), (4.9), (4.4), (4.5), and (4.21), it follows that

$$-\Delta^p_{\alpha_1} \widehat{u}_\lambda - \Delta^1_{\alpha_2} \widehat{u}_\lambda = f(z, \widehat{u}_\lambda) + \lambda |D\widehat{u}_\lambda|^{p-1} \quad \text{in } \Omega$$

implies $\widehat{u}_\lambda \in \inf C_+$ (see [9], (4.22) and recall that $\underline{u}_\mu \in \operatorname{int} C_+$). □

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