Electronic Journal of Differential Equations, Vol. 2025 (2025), No. 07, pp. 1–19.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2025.07

INFINITELY MANY SIGN-CHANGING SOLUTIONS FOR AN ASYMPTOTICALLY LINEAR AND NONLOCAL SCHRÖDINGER EQUATION

RUOWEN QIU, RENQING YOU, FUKUN ZHAO

Abstract. In this article, we consider the nonlocal schrödinger equation

$$-\mathcal{L}_K u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N,$$

where $-\mathcal{L}_K$ is an integro-differential operator and V is coercive at infinity, and $f(x,u)$ is asymptotically linear for u at infinity. Combining minimax method and invariant set of descending flow, we prove that the problem possesses infinitely many sign-changing solutions.

1. Introduction

The existence and multiplicity of sign-changing solutions are interesting topics in the studies of nonlinear elliptic equations. Recently, much more attention has been paid to such topics of the following classical elliptic equations

$$-\Delta u = f(x,u), \; x \in \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. In fact, there are different ways to obtain sign-changing solutions of the (1.1). Via a variational argument and a version of deformation lemma, Castro, Cossio and Neuberger [10] proved that (1.1), on a bounded domain Ω, possesses a sign-changing solution which changes sign only once. Dancer and Du [12] considered the equation

$$\begin{gathered} -\Delta u = u|u|^{p-1} + g(u) \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered}$$

where $1 < p < 2^* - 1$, $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous and

$$\limsup_{u\to 0} \frac{g(u)}{u} < \lambda_1\,, \quad \limsup_{|u|\to\infty} \frac{g(u)}{|u|^p} = 0.$$

The authors showed that the above problem has at least one sign-changing solution, besides a positive solution and a negative solution, and their method based on a topological degree argument combined with an priori bound of solutions. Suppose $f'(0) < \lambda_2$ and f is superlinear but subcritical at infinity, Bartsch and Wang [3] showed that there exists a solution u_1 of (1.1) which changes sign, and if u_2 is a

2020 *Mathematics Subject Classification.* 35R11, 35A15, 35B28.
Key words and phrases. Sign-changing solution; integro-differential operator; invariant set; variational method.

Submitted September 9, 2024. Published January 15, 2025.

second nontrivial solution then $u_2 > u_1$ (respectively, $u_2 < u_1$) implies that u_2 is positive (respectively, negative). They found that there is a critical point u_1 whose critical group $C_k(J, u_1) := H_k(J^c, J^c - \{u_1\})$ is not trivial for some $k \geq 2$, then u_1 can be never positive nor negative. In [6], by constructing invariant sets of descent flow, Bartsch, Liu and Weth obtained a sign-changing solution with precisely two nodal domains and infinitely many nodal solutions for the Schrödinger equation

$$-\Delta u + V(x)u = f(x,u), \ \ x \in \mathbb{R}^N. \tag{1.2}$$

In addition, there are many useful results about this equation, such as [4, 8, 38]. In almost the above-mentioned papers, the (AR) condition which introduced by Ambrosetti and Rabinowitz [1] was imposed, i.e. for some $\mu > 2$,

$$0 \leq \mu F(x,u) \equiv \mu \int_0^u f(x,s)ds \leq f(x,u)u, \quad \forall (x,u) \in \mathbb{R}^N \times \{\mathbb{R} \setminus \{0\}\}. \tag{1.3}$$

We remark that inequality (1.3) is one of the main tools to prove the boundedness of the PS sequence. By a simple calculation, (1.3) shows that $f(x,u)$ must be superlinear with respect to u at infinity, that is

$$\lim_{u \to \infty} \frac{f(x,u)}{u} = +\infty.$$

However, the study of many practical problems such as the self-trapping of an electromagnetic wave, under some suitable assumptions, leads to some problems related to (1.2), in which $f(x,u)$ is asymptotically linear with respect to u at infinity, the background and the results for some typical models can be found in [31, 32].

In the previus decades, some results about existence of positive solutions for elliptic problems that are asymptotically linear at infinity have been obtained. In [33], Stuart and Zhou obtained a positive radial solution by applying mountain pass theorem, where the equation is radially symmetric and V is a constant. Liu, Su and Weth [21] established the compactness of PS sequences for the associated energy functional under general spectral-theoretic assumptions, and obtained existence of three nontrivial solutions if the energy functional has a mountain pass geometry. Asymptotically linear problems with steep potential well have been studied in [35, 36], in which multiple solutions were constructed without giving nodal information about the solutions. Unlike with the superliear case, less was known about the sign-changing solutions to the asymptotically linear case. Maia, Miyagaki and Soares [22] investigated the problem

$$-\Delta u + \lambda u = f(u), \quad x \in \mathbb{R}^N, \tag{1.4}$$

where the nonlinearity f is asymptotically linear at infinity. The authors showed the existence of a sign-changing solution of (1.4), which changes sign exactly once.

In this article, we are concerned with the existence and multiplicity of sign-changing solutions of the nonlocal Schrödinger equation

$$-\mathcal{L}_K u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.5}$$

where $V(x)$ is a nonnegative potential function and $f(x,u)$ is asymptotically linear for u at infinity, and $\mathcal{L}_K$ is an integro-differential operator defined as follow

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^N} (u(x+y) + u(x-y) - 2u(x))K(y)dy, \quad x \in \mathbb{R}^N,$$

and the kernel K which is a measurable function and satisfies the following assumptions:

- (A1) there is $\theta > 0$ and $s \in (0,1)$ such that $K(x) \geq \theta |x|^{-(N+2s)}$ for any $x \in \mathbb{R}^N \backslash \{0\}$;
- (A2) $mK \in L^1(\mathbb{R}^N)$, where $m(x) := \min\{|x|^2, 1\}$.

If $K(x) = |x|^{-N-2s}$, then $-\mathcal{L}_K$ change into the fractional Laplacian operator $(-\Delta)^s$, and when $s \to 1^-$, $(-\Delta)^s \to -\Delta$, for more details we refer to the readers to [13, 24] and the references therein. From a physical point of view, the nonlocal operators play a crucial role in describing several different physical phenomena, such as in the anomalous diffusion [27, 34], in the fractional quantum mechanics [25] and so on.

Different from the operator $-\Delta$, the integro-differential operator $\mathcal{L}_K$ is nonlocal, which brings us some difficulties in applying variational methods. We refer the reader to [28], and [29] for the variational setting and the existence of nontrivial solution of such problem settled on a bounded domain of $\mathbb{R}^N$. It is worth mentioning that there also are many interesting results related to nonlocal elliptic equations with integro-differential operators in books [9, 24]. These result most focus on the existence and multiplicity of nontrivial solutions or positive solutions.

However, to the best of our knowledge, there are few results concerning the existence of sign-changing solution for the nonlocal schrödinger equation (1.5). When $K(x) = |x|^{-N-2s}$, Wang and Zhou [37] obtained a radial sign-changing solution of a fractional Schrödinger equation. Moreover, Chang and Wang [11] considered a fractional Laplacian equation and obtained the existence and multiplicity of sign-changing solutions via applying the Caffarelli-Silvestre extension method and invariant sets of descending flow. In [14, 15], by combining constraint variational method and quantitative deformation, the authors prove the the equation

$$
\begin{gathered}
-\mathcal{L}_K u = f(x,u), \quad \text{in } \Omega, \\
u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega
\end{gathered}
$$

possesses one least energy sign-changing solution and infinitely many sign-changing solutions. The case with potential function can be seen [16]. We mention that the above results are heavily based on the nonlinearity term f is superlinear and subcritical. Simultaneously, the problem settled on a bounded domain.

Recall that a solution of (1.5) is called sign-changing if $u^{\pm} \neq 0$, where

$$
u^+(x) = \max\big\{u(x), 0\big\} \quad \text{and} \quad u^-(x) = \min\big\{u(x), 0\big\}.
$$

To state our main results, we need the following assumptions on $V(x)$ and f:

- (A3) $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $V_0 := \inf_{x \in \mathbb{R}^N} V(x) > 0$;
- (A4) For each $M > 0$, there exists $r > 0$ such that

  $$
  \text{meas}(\{x \in B_r(y) : V(x) \leq M\}) \to 0 \quad \text{as } |y| \to \infty,
  $$

  where meas denotes for the Lebesgue measure, and $B_R(x)$ denotes an open ball of $\mathbb{R}^N$ centered at x and of radius $R > 0$, while we simply write B_R when $x = 0$.

We assume f satisfies the following assumptions:

- (A5) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and $f(x,u) = o(|u|)$ as $u \to 0$ uniformly in x;
- (A6) There is a constant $a \in (0, +\infty)$ such that $f(x,u)u^{-1} \to a$ as $u \to \infty$ uniformly in x and

  $$
  a > \inf \sigma(-\mathcal{L}_K + V(x)),
  $$

where $\sigma(-\mathcal{L}_K+V(x))$ denotes the spectrum of the operator $-\mathcal{L}_K+V(x)$;
- (A7) $f(x,u)/|u|$ is a increasing function of $u\neq 0$;
- (A8) $\inf_{x\in\mathbb{R}^N,u\in\mathbb{R}\setminus\{0\}}F(x,u)>0$;
- (A9) $\lim_{u\to\infty}(f(x,u)u-2F(x,u))=+\infty$.

Theorem 1.1. *If* (A3)–(A9) *hold, then* (1.5) *has infinitely many sign-changing solutions.*

Remark 1.2. Condition (A4), which is weaker than the coercive assumption: $V(x)\to\infty$ as $|x|\to\infty$, was firstly introduced by Bartsch and Wang in [2] to overcome the lack of compactness.

Remark 1.3. There are functions satisfying (A5)–(A9). For example, $f(x,u)=\frac{au^3}{1+u^2}$, where a is the constant can be found in (A6). By direct calculations, we know $F(x,u)=a[\frac{1}{2}u^2-\frac{1}{2}\ln(1+u^2)]$, it is easy to prove that function $f(x,u)$ satisfies the assumptions (A5)–(A9).

Notation. Throughout this paper, we denote by $|\cdot|_p$ the usual norm of the space $L^p(\mathbb{R}^N)$, $1\le p<\infty$. $u_n\rightharpoonup u$ and $u_n\to u$ mean the weak and strong convergence, respectively, as $n\to\infty$. $B_\rho=\{u\in E,||u||<\rho\}$.

2. Preliminary lemmas

First, we shall introduce some notation. For any $s\in(0,1)$, we define

$$X^s(\mathbb{R}^N)=\big\{u:\mathbb{R}^N\to\mathbb{R}: u \text{ is Lebesgue measurable } u\in L^2(\mathbb{R}^N) \text{ and the mapping } (x,y)\mapsto\big(u(x)-u(y)\big)\sqrt{K(x-y)}\in L^2(\mathbb{R}^N\times\mathbb{R}^N)\big\},$$

where the kernel K satisfies (A1) and (A2). The norm in $X^s(\mathbb{R}^N)$ is defined as

$$\|u\|_{X^s}:=\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u(x)-u(y)|^2K(x-y)\,dx\,dy+\int_{\mathbb{R}^N}|u|^2dx\Big)^{1/2}$$

and $\big(X^s(\mathbb{R}^N),\|\cdot\|_{X^s}\big)$ is a Hilbert space, we refer to [13, 29] for more properties of $X^s(\mathbb{R}^N)$. Since there is a potential functional $V(x)$ is involved in (1.5), we introduce the following subspace E of $X^s(\mathbb{R}^N)$

$$E:=\big\{u\in X^s(\mathbb{R}^N):\int_{\mathbb{R}^N}V(x)u^2dx<+\infty\big\},$$

which is a Hilbert space equipped with the inner product

$$(u,v)=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))(v(x)-v(y))K(x-y)\,dx\,dy+\int_{\mathbb{R}^N}V(x)uv\,dx.$$

The norm on E induced by the above inner productis denoted by $\|u\|$. We will look for solutions of (1.5) in the space E. We say that $u\in E$ is a weak solution of (1.5) if

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))(v(x)-v(y))K(x-y)\,dx\,dy+\int_{\mathbb{R}^N}V(x)uv\,dx\\&=\int_{\mathbb{R}^N}f(x,u)v\,dx\end{aligned}$$

for all $v\in E$. So the energy functional associated with (1.5) is

$$\Psi(u)=\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))^2K(x-y)\,dx\,dy+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2dx$$

$$- \int_{\mathbb{R}^N} F(x,u)\,dx, \quad u \in E.$$

Under our assumptions, it is standard to check that $\Psi \in C^1(E, \mathbb{R})$ and for $v \in E$, it holds

$$\begin{aligned}\langle \Psi'(u), v\rangle &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (u(x)-u(y))(v(x)-v(y))K(x-y)\,dx\,dy \\ &\quad + \int_{\mathbb{R}^N} V(x)uv\,dx - \int_{\mathbb{R}^N} f(x,u)v\,dx.\end{aligned}$$

Next we prove some preliminary lemmas, which are crucial for proving our main results. Firstly, to overcome the difficulties brought by the nonlocal feature of operator $\mathcal{L}_K$, we need the following embedding result.

Theorem 2.1. *If* (A3), (A4) *hold, then the embeddings $E \hookrightarrow L^p(\mathbb{R}^N)$ are continuous for $p \in [2, 2_s^*]$ and compact for $p \in [2, 2_s^*)$, where $2_s^* = \frac{2N}{N-2s}$ is the fractional Sobolev critical exponent.*

Proof. First, we show that $E \hookrightarrow L^p(\mathbb{R}^N)$ for $p \in [2, 2_s^*]$. In fact, by (A3) we know that $E \hookrightarrow X^s(\mathbb{R}^N)$ is continuous, from the [13, Theorem 6.5] and (A1) we obtain $X^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for $p \in [2, 2_s^*]$. Then the conclusion follows.

Next we show that $E \hookrightarrow L^p(\mathbb{R}^N)$ is compact for $p \in [2, 2_s^*)$. In fact, let $\{u_n\} \subset E$ be a bounded sequence of E, going if necessary to a subsequence we have $u_n \rightharpoonup u$ in E and $u_n \to u$ in $L^p_{\rm loc}(\mathbb{R}^N)$, $p \in [2, 2_s^*)$ and $\|u_n(x)\| + \|u(x)\| \le \bar{C}$, where $\bar{C}$ is a positive constant. We first prove that $u_n \to u$ in $L^2(\mathbb{R}^N)$, it suffices to prove that $|u_n|_2 \to |u|_2$.

Fix $M > 0$ and set $A_M(y) := \{x \in \mathbb{R}^N : V(x) \le M\} \cap B_r(y)$, where $r > 0$ is given by (A4). When $p = 2$, $u_n \to u$ in $L^2_{\rm loc}(\mathbb{R}^N)$ implies that $u_n \to u$ in $L^2(B_R)$ for any $R > 0$. Now, we choose $\{y_n\} \subset \mathbb{R}^N$ such that $\mathbb{R}^N \subset \bigcup_{i=1}^\infty B_r(y_i)$ and each $x \in \mathbb{R}^N$ is covered by at most 2^N balls. Denote the set $C_M(y_i) := \{x \in \mathbb{R}^N : V(x) > M\} \cap B_r(y_i)$, we have

$$\begin{aligned}&\int_{\mathbb{R}^N\setminus B_R} |u_n(x)-u(x)|^2\,dx \\ &\le \sum_{|y_i|\ge R-r}^{\infty} \int_{B_r(y_i)} |u_n(x)-u(x)|^2\,dx \\ &= \sum_{|y_i|\ge R-r}^{\infty} \Big(\int_{C_M(y_i)} |u_n(x)-u(x)|^2\,dx + \int_{A_M(y_i)} |u_n(x)-u(x)|^2\,dx\Big).\end{aligned}$$

Using the definition of $C_M(y_i)$ and Hölder inequality we obtain

$$\int_{C_M(y_i)} |u_n(x)-u(x)|^2\,dx \le \frac{1}{M}\int_{B_r(y_i)} V(x)|u_n(x)-u(x)|^2\,dx,$$

$$\begin{aligned}\int_{A_M(y_i)} |u_n(x)-u(x)|^2dx &\le \Big(\int_{A_M(y_i)} |u_n(x)-u(x)|^{2t}dx\Big)^{1/t}\Big(\int_{A_M(y_i)} 1^{t'}\Big)^{1/t'} \\ &= |u_n(x)-u(x)|^2_{L^{2t}(A_M(y_i))}\big(\operatorname{meas} A_M(y_i)\big)^{1/t'},\end{aligned}$$

where $t \in (1, \frac{N}{N-2s})$ and $\frac{1}{t} + \frac{1}{t'} = 1$. Hence,

$$\int_{\mathbb{R}^N\setminus B_R} |u_n(x)-u(x)|^2dx$$

$$
\begin{aligned}
&\le \sum_{|y_i|>R-r} \Big(\frac{1}{M} \int_{B_r(y_i)} V(x)|u_n(x)-u(x)|^2 dx \\
&\quad + \sup_{|y_i|>R-r} \Big(\operatorname{meas} A_M(y_i) \Big)^{1/t'} |u_n(x)-u(x)|^2_{L^{2t}(A_M(y_i))} \Big) \\
&\le \frac{2^N}{M} \int_{\mathbb{R}^N \setminus B_{R-2r}} V(x)|u_n(x)-u(x)|^2 dx \\
&\quad + 2^N C_2^2 \sup_{|y_i|>R-r} \Big(\operatorname{meas} A_M(y_i) \Big)^{1/t'} \|u_n(x)-u(x)\|^2 \\
&\le \frac{2^N \bar{C}^2}{M} + 2^N (C_2 \bar{C})^2 \sup_{|y_i|>R-r} \Big(\operatorname{meas} A_M(y_i) \Big)^{1/t'},
\end{aligned}
$$

where $C_2 > 0$ is the embedding constant. Now, for any $\varepsilon > 0$ we choose $M > 0$ so large that

$$
\frac{2^{N+1}\bar{C}^2}{M} < \varepsilon. \tag{2.1}
$$

For fixed $M > 0$, there exists $R_M > 0$ such that

$$
2^{N+1}(C_2\bar{C})^2 \sup_{|y_i|>R_M-r} \Big(\operatorname{meas} A_M(y_i) \Big)^{1/t'} < \varepsilon, \tag{2.2}
$$

since

$$
\sup_{|y_i|\ge R-r} \big(\operatorname{meas} A_M(y_i) \big)^{1/t'} \to 0 \quad \text{as } R \to \infty.
$$

For such R_M, by (2.1) and (2.2) we have

$$
\int_{\mathbb{R}^N \setminus B_{R_M}} |u_n(x)-u(x)|^2 dx \le \varepsilon.
$$

Hence,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |u_n(x)-u(x)|^2 dx \\
&= \int_{B(0,R_M)} |u_n(x)-u(x)|^2 dx + \int_{\mathbb{R}^N \setminus B_{R_M}} |u_n(x)-u(x)|^2 dx \le 2\varepsilon,
\end{aligned} \tag{2.3}
$$

this proves that $|u_n|_2 \to |u|_2$ in $L^2(\mathbb{R}^N)$. Finally, by the Interpolation inequality we have (up to renaming C)

$$
\begin{aligned}
|u_n - u|_p &\le C|u_n - u|_2^{\theta} \, |u_n - u|_{2_s^*}^{1-\theta} \\
&\le C|u_n - u|_2^{\theta} ||u_n - u||^{1-\theta} \\
&\le C|u_n - u|_2^{\theta} (||u_n|| + ||u||)^{1-\theta},
\end{aligned} \tag{2.4}
$$

where $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{2_s^*}$ and $\theta \in (0,1)$. Hence the right hand of (2.4) is small enough, therefore, $u_n \to u$ in $L^p(\mathbb{R}^N)$ for $p \in (2, 2_s^*)$. □

Next, we consider the eigenvalues problem. As in [5, 29], we have the following results.

$$
-\mathcal{L}_K u + Vu = \lambda u, \quad x \in \mathbb{R}^N. \tag{2.5}
$$

Proposition 2.2. *Let $s \in (0,1)$, $N > 2s$, and $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ be a function satisfying assumptions (A1) and (A2). Then*

(1) (2.5) *admits an eigenvalue* λ_1 *that is positive, simple and that can be characterized as follows:*

$$\begin{aligned}\lambda_1 = \inf_{u\in E,\, |u|_2=1} \Big\{ &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^2 K(x-y)\,dx\,dy \\ &+ \int_{\mathbb{R}^N} V(x)u^2(x)\,dx\Big\}.\end{aligned} \tag{2.6}$$

(2) *The set of the eigenvalues of* (2.5) *consists of a sequence* $\{\lambda_k\}_{k\in\mathbb{N}}$ *with*

$$0<\lambda_1<\lambda_2\le\cdots\le\lambda_k\le\lambda_{k+1}\le\cdots$$

and $\lambda_k\to+\infty$ *as* $k\to+\infty$. *Moreover, for each* $k\in\mathbb{N}$, *the eigenvalues can be characterized as*

$$\begin{aligned}\lambda_{k+1} = \inf_{u\in X_k^{\perp},\, |u|_2=1} \Big\{ &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^2 K(x-y)\,dx\,dy \\ &+ \int_{\mathbb{R}^N} V(x)u^2(x)\,dx\Big\},\end{aligned} \tag{2.7}$$

where $X_k := \operatorname{span}\{e_1, e_2, \cdots, e_k\}$.

(3) *The sequence* $\{e_k\}_{k\in\mathbb{N}}$ *of eigenfunctions corresponding to* λ_k *is an orthonormal basis of* $L^2(\mathbb{R}^N)$ *and an orthogonal basis of* E.

We give some properties of $f(x,u)$ and $F(x,u)$ in the following lemma.

Lemma 2.3. (1) *If (A5) and (A6) hold, then for any* $p\in(2,2_s^*)$ *and* $\varepsilon>0$, *there exists* $C_\varepsilon>0$ *such that for all* $u\in\mathbb{R}$,

$$|f(x,u)|\le\varepsilon|u|+C_\varepsilon|u|^{p-1}\quad |F(x,u)|\le\varepsilon u^2+C_\varepsilon|u|^p.$$

(2) *If (A7) hold, then for any* $(x,u)\in\mathbb{R}^N\times(\mathbb{R}\setminus\{0\})$, *we have*

$$\frac{1}{2}f(x,u)u-F(x,u)\ge 0,$$

where $F(x,u)=\int_0^u f(x,s)ds$.

Proof. Conclusion (1) is easy, so we omit it here. It follows from (A7) that, for any $t\ge 0$, $u\in\mathbb{R}\setminus\{0\}$, one has

$$\frac{1-t^2}{2}uf(x,u)+F(x,tu)-F(x,u)=\int_t^1\Big[\frac{f(x,u)}{u}-\frac{f(x,su)}{su}\Big]su^2ds\ge 0. \tag{2.8}$$

Taking $t=0$ in (2.8), we obtain for any $(x,u)\in\mathbb{R}^N\times(\mathbb{R}\setminus\{0\})$,

$$\frac{1}{2}f(x,u)u\ge F(x,u). \tag{2.9}$$

This completes the proof of conclusion (2). □

Under conditions (A5)–(A7), we will show that $\Psi(u)$ has a mountain pass geometry, see Lemma 2.4 and Lemma 2.5, where Lemma 2.4 can be directly obtained from the embedding result Theorem of 2.1 and Lemma 2.3.

Lemma 2.4. *Suppose A3)–(A6) hold. Then* $\Psi(u)=\frac{1}{2}\|u\|^2+o(\|u\|^2)$, $\langle\Psi'(u),u\rangle=\|u\|^2+o(\|u\|^2)$ *as* $u\to 0$ *in* E.

Lemma 2.5 ([17]). *Suppose* (A3)–(A8) *hold. Then there is a* $v\in E$ *with* $v\ne 0$ *such that* $\Psi(v)<0$.

Proof. By Proposition 2.2 we know that

$$\begin{aligned}&\inf \sigma(-\mathcal{L}_K + V(x))\\&= \inf_{u\in E,\,|u|_2=1} \Big\{ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^2 K(x-y)\,dx\,dy + \int_{\mathbb{R}^N} V(x)u^2(x)\,dx \Big\}.\end{aligned}$$

By (A6), we have an $\widetilde{u} \in E$ such that $|\widetilde{u}|_2 = 1$ and $\|\tilde{u}\|^2 < a$. Replacing $\tilde{u}$ by $|\tilde{u}|$ (still renaming $\tilde{u}$), we can suppose that $\tilde{u} \geq 0$ a.e. on $\mathbb{R}^N$. To prove the Lemma, it suffices to show that

$$\lim_{t\to+\infty} \frac{\Psi(t\tilde{u})}{t^2} < 0. \tag{2.10}$$

First, we claim that

$$\lim_{t\to+\infty} \int_{\mathbb{R}^N} \frac{F(x,t\tilde{u})}{t^2}\ \mathrm{d}x = \frac{1}{2}a. \tag{2.11}$$

To prove (2.11), without loss of generality we can assume that $\tilde{u}$ is defined everywhere on $\mathbb{R}^N$ and divide the argument into two situations: $\tilde{u}(x) > 0$ and $\tilde{u}(x) = 0$. When $\tilde{u}(x) > 0$. By $(f2)$, we obtain

$$\lim_{t\to+\infty} \frac{F(x,t\tilde{u})}{t^2} = \lim_{t\to+\infty} \frac{F(x,t\tilde{u})}{(t\tilde{u})^2}(\tilde{u})^2 = \frac{1}{2}a(\tilde{u})^2. \tag{2.12}$$

When $\tilde{u}(x) = 0$, for all $t > 0$

$$\frac{F(x,t\tilde{u})}{t^2} = 0 = \frac{1}{2}a(\tilde{u})^2. \tag{2.13}$$

In view of (2.12) and (2.13) we know that

$$\lim_{t\to+\infty} \frac{F(x,t\tilde{u})}{t^2} = \frac{1}{2}a(\tilde{u})^2 \quad \text{a.e. on } \mathbb{R}^N. \tag{2.14}$$

On the other hand, by (A6)–(A8), there exists a $C > 0$ such that

$$0 \leq \frac{f(x,u)}{u} \leq C \quad \text{for all } u \in \mathbb{R}\setminus\{0\},$$

and thus

$$0 \leq \frac{F(x,u)}{u^2} \leq \frac{C}{2} \quad \text{for all } u \in \mathbb{R}\setminus\{0\}.$$

Therefore,

$$0 \leq \frac{F(x,t\tilde{u})}{t^2} \leq \frac{C}{2}(\tilde{u})^2 \quad \text{for all } u \in \mathbb{R}\setminus\{0\}. \tag{2.15}$$

Equations (2.14) and (2.15) allow us to apply Lebesgue dominated convergence theorem to obtain

$$\lim_{t\to+\infty} \int_{\mathbb{R}^N} \frac{F(x,t\tilde{u})}{t^2}dx = \frac{a}{2}\int_{\mathbb{R}^N} (\tilde{u})^2 dx = \frac{a}{2},$$

that is claim (2.10). According to (2.10), we easily obtain that

$$\lim_{t\to+\infty} \frac{\Psi(t\tilde{u})}{t^2} = \frac{1}{2}\|\tilde{u}\|^2 - \lim_{t\to+\infty}\int_{\mathbb{R}^N} \frac{F(x,t\tilde{u})}{t^2}\ \mathrm{d}x = \frac{1}{2}\big(\|\tilde{u}\|^2 - a\big) < 0,$$

so the Lemma is proved. □

Another difficulty that needs to be overcome is the lack of boundedness for Palais-Smale sequences when the (AR) condition (1.3) is does not satisfy. As in [33] our proof of the boundedness of $\{u_n\}$ relies on the work of Lions [19, 20] on the concentration compactness principle.

Lemma 2.6. *If* (A3)–(A9) *hold, and* $\{u_n\} \subset \mathcal{N}$ *is a sequence such that* $\{\Psi(u_n)\}$ *is bounded, then* $\{u_n\}$ *is bounded.*

Proof. Assuming that the statement does not hold, then we can suppose that there exists a subsequence again denoted by $\{u_n\}$ such that $\{\Psi(u_n)\}$ is bounded but $\|u_n\| \to \infty$ when $n \to \infty$. By the definition of $\mathcal{N}$, for all $u \in \mathcal{N}$ we obtain that

$$\begin{aligned} \Psi(u) &= \Psi(u) - \frac{1}{2}\langle \Psi'(u), u\rangle \\ &= \int_{\mathbb{R}^N} \Big(\frac{1}{2} f(x,u) - F(x,u)\Big)\, dx \geq 0. \end{aligned} \tag{2.16}$$

Then, up to a subsequence, $\Psi(u_n) \to l \geq 0$ by (2.16). If $l > 0$, we define $v_n := \frac{2\sqrt{l}u_n}{\|u_n\|}$, and $\|v_n\| = 2\sqrt{l}$. If $l = 0$, we define $v_n := \frac{u_n}{\|u_n\|}$, so that $\|v_n\| = 1$.

Now, we prove the following Claim.

Claim: There exist $r, d > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \int_{B_r(y_n)} v_n^2 dx \geq d > 0. \tag{2.17}$$

If the claim is not true, then by the nonlocal type Lions lemma, $v_n \to 0$ in $L^p\left(\mathbb{R}^N\right)$, where $2 < p < 2_s^*$. Using Lemma 2.3 we obtain

$$\Big| \int_{\mathbb{R}^N} F(x, v_n)\, dx\Big| \leq \varepsilon \int_{\mathbb{R}^N} v_n^2 dx + C_\varepsilon \int_{\mathbb{R}^N} |v_n|^p dx,$$

since $\{v_n\} \subset E$ is bound and the arbitrariness of $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(x, v_n)\, dx = 0.$$

When $l = 0$, we have

$$\liminf_{n\to\infty} \Psi(v_n) = \liminf_{n\to\infty} \Big(\frac{1}{2}\|v_n\|^2 - \int_{\mathbb{R}^N} F(x, v_n)\, dx\Big) = \frac{1}{2}. \tag{2.18}$$

To obtain a contradiction, we need the inequality

$$\Psi(tu) \leq \Psi(u), \quad \text{for } t \geq 0 \text{ and } u \in \mathcal{N}. \tag{2.19}$$

Indeed, let $u \in \mathcal{N}$ and define the function

$$\xi(t) := \frac{t^2}{2} f(x,u)u - F(x,tu), \quad t \geq 0,$$

and for any $t > 0$

$$\xi'(t) = tf(x,u)u - f(x,tu)u = tu^2\Big(\frac{f(x,u)}{u} - \frac{f(x,tu)}{tu}\Big).$$

By (A7), for every $t > 0$, we know $\xi(t) \leq \xi(1)$. Then, after integration on $\mathbb{R}^N$ and using that $\langle \Psi'(u), u\rangle = 0$, we have

$$\begin{aligned} \Psi(tu) &= \Psi(tu) - \frac{t^2}{2}\langle \Psi'(u), u\rangle \\ &= \int_{\mathbb{R}^N} \Big(\frac{t^2}{2} f(x,u)u - F(x,tu)\Big)\, dx \\ &\leq \int_{\mathbb{R}^N} \Big(\frac{1}{2} f(x,u)u - F(x,u)\Big)\, dx \\ &= \Psi(u). \end{aligned}$$

This completes the proof of (2.19). On the other hand, taking $t=\frac{1}{\|u_n\|}$ in (2.19), we have

$$\Psi(v_n)=\Psi(tu_n)\le\Psi(u_n)=l+o_n(1)=o_n(1),$$

which contradicts (2.18).

When $l>0$,

$$\liminf_{n\to\infty}\Psi(v_n)=\liminf_{n\to\infty}\Big(\frac{1}{2}\|v_n\|^2-\int_{\mathbb{R}^N}F(x,v_n)\,dx\Big)=2l,$$

and taking $t=\frac{2\sqrt{l}}{\|u_n\|}$ in (2.19) we have

$$\Psi(v_n)\le\Psi(u_n)=l+o_n(1),$$

getting the same contradiction, thus the Claim holds.

By the above claim we infer a contradiction in both cases: when $\{y_n\}$ is bounded or unbounded. This will complete the proof.

Case 1. $\{y_n\}$ is bounded. Then there is $\widetilde{r}>0$ such that $\{y_n\}\subset B_{\widetilde{r}}$. By (2.17), we have

$$\int_{B_r(y_n)}v_n^2dx>\frac{d}{2}.$$

Thus we can choose $\widehat{r}>r+\widetilde{r}$ with $B_r(y_n)\subset B_{\widehat{r}}$ and

$$\int_{B_{\widehat{r}}}v_n^2dx>\frac{d}{2}.$$

Since $\{v_n\}$ is bounded in E, there exists a subsequence still denoted by $\{v_n\}$ such that $v_n\rightharpoonup v$ in E, $v_n\to v$ in $L^p(\mathbb{R}^N)$ for $2\le p<2_s^*$, and $v_n(x)\to v(x)$ a.e. on $\mathbb{R}^N$. In particular, we have

$$\int_{B_{\widehat{r}}}v_n^2dx\to\int_{B_{\widehat{r}}}v^2dx\quad\text{and}\quad\int_{B_{\widehat{r}}}v^2dx\ge\frac{d}{2}>0,$$

which implying that $v\not\equiv0$. Thus there exists a set $\Omega\subset B_{\widehat{r}}$ with the meas$(\Omega)>0$ such that $v(x)\neq0$ for every $x\in\Omega$. Hence for a fixed $x\in\Omega$ and the constant $t>0$, $v_n(x)=\frac{tu_n(x)}{\|u_n\|}\neq0$ when n large enough which implied that $u_n(x)\neq0$. As a consequence of $\|u_n\|\to\infty$, $|u_n(x)|\to\infty$. So $|u_n(x)|\to\infty$ for every $x\in\Omega$. Since

$$\Psi(u_n)=\int_{\mathbb{R}^N}\Big(\frac{1}{2}f(x,u_n)u_n-F(x,u_n)\Big)\,dx\ge\int_{\Omega}\Big(\frac{1}{2}f(x,u_n)u_n-F(x,u_n)\Big)\,dx,$$

by (A9) and Fatou's lemma, we have

$$\liminf_{n\to\infty}\Psi(u_n)\ge\int_{\Omega}\liminf_{n\to\infty}\Big(\frac{1}{2}f(x,u_n)u_n-F(x,u_n)\Big)\,dx=\infty,$$

which imply $\Psi(u_n)\to\infty$, contradicting that $\Psi(u_n)\to l\in\mathbb{R}$.

Case 2. $\{y_n\}$ is unbounded. We set a new sequence $\widetilde{v}_n(x):=v_n(x-y_n)$ and $\|\widetilde{v}_n\|=\|v_n\|$ is a constant. Thus, up to a subsequence $\widetilde{v}_n\rightharpoonup\widetilde{v}$ in E, $\widetilde{v}_n\to\widetilde{v}$ in $L^p(\mathbb{R}^N)$ for $2\le p<2_s^*$, and $\widetilde{v}_n(x)\to\widetilde{v}(x)$ a.e. on $\mathbb{R}^N$. From (2.17),

$$\liminf_{n\to\infty}\int_{B_r(y_n)}v_n^2dx=\liminf_{n\to\infty}\int_{B_r}\widetilde{v}_n^2dx\ge d$$

and hence

$$\int_{B_r}\widetilde{v}^2dx\ge d>0,$$

implying that $\widetilde{v} \not\equiv 0$. Therefore, there exists a subset $\Lambda \subset B_r$ with positive measure such that $\widetilde{v} \not\equiv 0$ for every $x \in \Lambda$. Similar to Case 1, $|u_n(x+y_n)| \to \infty$ for every $x \in \Lambda$ as $n \to \infty$. So

$$\begin{aligned}
\Psi(u_n) &= \int_{\mathbb{R}^N} \Big(\frac{1}{2} f(x,u_n)u_n - F(x,u_n)\Big)\,dx \\
&\geq \int_{B_r(y_n)} \Big(\frac{1}{2} f(x,u_n)u_n - F(x,u_n)\Big)\,dx \\
&= \int_{B_r} \Big(\frac{1}{2} f(x,u_n(x+y_n))u_n(x+y_n) - F(x,u_n(x+y_n)\Big)\,dx \\
&\geq \int_{\Lambda} \Big(\frac{1}{2} f(x,u_n(x+y_n))u_n(x+y_n) - F(x,u_n(x+y_n))\Big)\,dx.
\end{aligned}$$

Thus, we have the same contradiction with Case 1. □

Lemma 2.7. *If* (A3)–(A4) *hold, then* Ψ *satisfies Palais-Smale condition at any level* $c > 0$.

Proof. Let $\{u_n\} \subset E$ be a $(PS)_c$ sequence of Ψ, that is $\Psi(u_n) \to c$, and $\Psi'(u_n) \to 0$. By Lemma 2.6, we know that $\{u_n\}$ is bounded in E. So we can assume that up to a subsequence, there exists a $u \in E$ such that

$$u_n \rightharpoonup u \quad \text{in } E,$$
$$u_n \to u \quad \text{in } L^p(\mathbb{R}^N) \text{ for } p \in [2, 2_s^*).$$

Observe that

$$\|u_n - u\|^2 = \langle \Psi'(u_n) - \Psi'(u), u_n - u\rangle + \int_{\mathbb{R}^N} \left(f(x,u_n) - f(x,u)\right)(u_n - u)\,dx.$$

It follows from the Hölder inequality and Lemma 2.3 that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^N} (f(x,u_n) - f(x,u))(u_n - u)\,dx\Big| \\
&\leq \int_{\mathbb{R}^N} (|f(x,u_n)| + |f(x,u)|)|u_n - u|\,dx \\
&\leq \int_{\mathbb{R}^N} (\varepsilon|u_n| + \varepsilon|u| + C_\varepsilon|u_n|^{p-1} + C_\varepsilon|u|^{p-1})|u_n - u|\,dx \\
&\leq 4\varepsilon(|u_n|_2^2 + |u|_2^2)|u_n - u|_2 + C_\varepsilon(|u_n|_p^{p-1} + |u|_p^{p-1})|u_n - u|_p.
\end{aligned}$$

Thus we have verified that $u_n \to u$ in E, that is Ψ satisfies $(PS)_c$ condition. □

3. Proof of Theorem 1.1

We define an operator $A : E \to E$ as

$$Au := (-\mathcal{L}_K u + Vu)^{-1} \circ h(u), \quad u \in E,$$

where $h(u) := f(x,u)$. When $u \in E$ fixed, we consider the functional

$$J(v) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(x) - v(y)|^2 K(x-y)\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^N} V(x)v^2 dx - \int_{\mathbb{R}^N} F(x,u)\,dx.$$

It is easy to prove that $J \in C^1(E,\mathbb{R})$ and coercive, bounded below and strictly convex in E. Therefore, by [23, Theorem 1.1], $J(v)$ admits a unique global minimizer

$v = Au$, and $v = Au$ is the unique solution to the equation

$$-\mathcal{L}_K v + V(x)v = f(x,u), \quad \forall u \in E$$

That is,

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(v(x)-v(y))(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy + \int_{\mathbb{R}^N} V(x)v\varphi\,dx \\
&= \int_{\mathbb{R}^N} f(x,u)\varphi\,dx, \quad \forall \varphi \in E.
\end{aligned} \tag{3.1}$$

Lemma 3.1. *If* (A3), (A4) *hold, then the operator A satisfies:*

(1) *A is continuous and maps bounded sets into bounded sets.*
(2) $\langle \Psi'(u), u - Au\rangle = \|u - Au\|^2$
(3) $\|\Psi'(u)\| \le \|u - Au\|$

Proof. (1) Let $\{u_n\} \subset E$ such that $u_n \to u$ in E. Let $v_n := Au_n$ and $v := Au$. Then (3.1) implies that

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(v_n(x)-v_n(y)\right)\left(\varphi(x)-\varphi(y)\right)K(x-y)\,dx\,dy + \int_{\mathbb{R}^N} V(x)v_n\varphi\,dx \\
&= \int_{\mathbb{R}^N} f(x,u_n)\varphi(x)\,dx, \quad \varphi \in E,
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(v(x)-v(y))(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy + \int_{\mathbb{R}^N} V(x)v\varphi\,dx \\
&= \int_{\mathbb{R}^N} f(x,u)\varphi(x)\,dx, \varphi \in E.
\end{aligned} \tag{3.3}$$

In view of the Hölder inequality, Theorem 2.1 and (3.2), (3.3), we have

$$\begin{aligned}
&\|v_n - v\|^2 \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(v_n(x)-v_n(y)-v(x)+v(y)\right)^2 K(x-y)\,dx\,dy + \int_{\mathbb{R}^N} V(x)|v_n - v|^2 dx \\
&= \int_{\mathbb{R}^N} f(x,u_n)v_n\,dx + \int_{\mathbb{R}^N} f(x,u)v dx - \int_{\mathbb{R}^N} f(x,u_n)v\,dx - \int_{\mathbb{R}^N} f(x,u)v_n\,dx \\
&= \int_{\mathbb{R}^N}\left(f(x,u_n)-f(x,u)\right)(v_n - v)dx \\
&\le \Big(\int_{\mathbb{R}^N}|v_n - v|^{2_s^*}dx\Big)^{\frac{1}{2_s^*}}\Big(\int_{\mathbb{R}^N}|f(x,u_n)-f(x,u)|^{\frac{2_s^*}{2_s^*-1}}dx\Big)^{\frac{2_s^*-1}{2_s^*}} \\
&\le C\|v_n - v\|\Big(\int_{\mathbb{R}^N}|f(x,u_n)-f(x,u)|^{\frac{2_s^*}{2_s^*-1}}dx\Big)^{\frac{2_s^*-1}{2_s^*}}.
\end{aligned}$$

So,

$$\|v_n - v\| \le C\Big(\int_{\mathbb{R}^N}|f(x,u_n)-f(x,u)|^{\frac{2_s^*}{2_s^*-1}}dx\Big)^{\frac{2_s^*-1}{2_s^*}}.$$

It follows from the Lebesgue dominated convergence theorem that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}|f(x,u_n)-f(x,u)|^{\frac{2_s^*}{2_s^*-1}}dx = 0,$$

hence,

$$\|v_n - v\| \to 0, \text{ as } n \to +\infty,$$

which implies that A is continuous on E. Next, we prove the boundedness of A. In (3.1), we taking $\varphi = Au \in E$ and combining Lemma 2.3 and the Hölder inequality, we know

$$
\begin{aligned}
&\|Au\|^2\\
&= \int_{\mathbb{R}^N} f(x,u)Audx\\
&\le C\Big(\int_{\mathbb{R}^N} |Au||u|dx + \int_{\mathbb{R}^N} |u|^{p-1}|Au|dx\Big)\\
&\le C\Big(\int_{\mathbb{R}^N} |u|^2dx\Big)^{1/2}\Big(\int_{\mathbb{R}^N} |Au|^2dx\Big)^{1/2} + C\Big(\int_{\mathbb{R}^N} |u|^pdx\Big)^{\frac{p-1}{p}}\Big(\int_{\mathbb{R}^N} |Au|^pdx\Big)^{1/p}\\
&\le C\|Au\|\left(\|u\| + \|u\|^{p-1}\right).
\end{aligned}
$$

Therefore, $\|Au\| \le C\left(\|u\| + \|u\|^{p-1}\right)$, which implies that A maps bounded sets into bounded sets.

(2) Taking $\varphi = u - Au \in E$ into (3.1), we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (Au(x) - Au(y))(u(x) - Au(x) - u(y) + Au(y))K(x-y)\,dx\,dy\\
&+ \int_{\mathbb{R}^N} V(x)Au(u - Au)\,dx\\
&= \int_{\mathbb{R}^N} f(x,u)(u - Au)\,dx.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\langle \Psi'(u), u - Au\rangle\\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (u(x) - u(y))(u(x) - Au(x) - u(y) + Au(y))K(x-y)\,dx\,dy\\
&\quad + \int_{\mathbb{R}^N} V(x)u(u - Au)\,dx - \int_{\mathbb{R}^N} f(x,u)(u - Au)\,dx\\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (u(x) - Au(x) - u(y) + Au(y))^2K(x-y)\,dx\,dy\\
&\quad + \int_{\mathbb{R}^N} V(x)(u - Au)^2dx\\
&= \|u - Au\|^2.
\end{aligned}
$$

(3) By the Hölder inequality, for any $\varphi \in E$, we obtain

$$
\begin{aligned}
&|\langle \Psi'(u), \varphi\rangle|\\
&= \Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (u(x) - u(y))(\varphi(x) - \varphi(y))K(x-y)\,dx\,dy\\
&\quad + \int_{\mathbb{R}^N} V(x)u\varphi\,dx - \int_{\mathbb{R}^N} f(x,u)\varphi\,dx\Big|\\
&\le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |(u(x) - Au(x) - u(y) + Au(y))(\varphi(x) - \varphi(y))K(x-y)|\,dx\,dy\\
&\quad + \int_{\mathbb{R}^N} |V(x)(u - Au)\varphi|\,dx
\end{aligned}
$$

$$
\begin{aligned}
&\le \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u(x)-Au(x)-u(y)+Au(y)|^2K(x-y)\,dx\,dy\Big)^{1/2}\\
&\quad\times\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\varphi(x)-\varphi(y)|^2K(x-y)\,dx\,dy\Big)^{1/2}\\
&\quad+\Big(\int_{\mathbb{R}^N}V(x)|u-Au|^2dx\Big)^{1/2}\Big(\int_{\mathbb{R}^N}V(x)\varphi^2dx\Big)^{1/2}\\
&\le \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u(x)-Au(x)-u(y)+Au(y)|^2K(x-y)\,dx\,dy\\
&\quad+\int_{\mathbb{R}^N}V(x)|u-Au|^2dx\Big)^{1/2}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\varphi(x)-\varphi(y)|^2K(x-y)\,dx\,dy\\
&\quad+\int_{\mathbb{R}^N}V(x)\varphi^2dx\Big)^{1/2}\\
&=\|u-Au\|\|\varphi\|,
\end{aligned}
$$

which implies that $\|\Psi'(u)\|\le\|u-Au\|$. □

As in [6], we consider the convex cones $E^+:=\{u\in X^s: u\ge 0\}$ and $E^-:=\{u\in X^s: u\le 0\}$. For an arbitrary $\varepsilon>0$, we define

$$
D_\varepsilon^+:=\{u\in X^s:\operatorname{dist}(u,E^+)<\varepsilon\},\quad D_\varepsilon^-:=\{u\in X^s:\operatorname{dist}(u,E^-)<\varepsilon\},
$$

where $\operatorname{dist}(u,E^\pm)=\inf_{v\in E^\pm}\|v-u\|$.

Lemma 3.2. *If (A3)–(A8) hold, then there exists $\varepsilon_0>0$ such that for $0<\varepsilon<\varepsilon_0$, $A(\partial D_\varepsilon^+)\subset D_\varepsilon^+$, $A(\partial D_\varepsilon^-)\subset D_\varepsilon^-$.*

Proof. Taking $\varphi=v^+$ in (3.1), by the Hölder inequality and Lemma 2.3, we obtain that

$$
\begin{aligned}
\|v^+\|^2&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\big(v^+(x)-v^+(y)\big)^2K(x-y)\,dx\,dy+\int_{\mathbb{R}^N}V(x)\left|v^+\right|^2dx\\
&\le\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\big(v^+(x)-v^+(y)\big)^2K(x-y)\,dx\,dy+\int_{\mathbb{R}^N}V(x)vv^+dx+\big(v^+,v^-\big)\\
&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(v(x)-v(y))\big(v^+(x)-v^+(y)\big)K(x-y)\,dx\,dy+\int_{\mathbb{R}^N}V(x)vv^+dx\\
&=\int_{\mathbb{R}^N}f(x,u)v^+dx\\
&\le\int_{\mathbb{R}^N}f\left(x,u^+\right)v^+dx\\
&\le\int_{\mathbb{R}^N}\Big(\varepsilon\left|u^+\right|+C_\varepsilon\left|u^+\right|^{p-1}\Big)v^+dx\\
&\le\varepsilon\left|u^+\right|_2\left|v^+\right|_2+C_\varepsilon\left|u^+\right|_p^{p-1}\left|v^+\right|_p.
\end{aligned}
$$

Let $u\in E$ and $v=Au$, by the Theorem 2.1, for any $p\in[2,2_s^*]$, there exists $C_p>0$ such that

$$
|u^\pm|_p=\inf_{v\in E^\mp}|v-u|_p\le C_p\inf_{v\in E^\mp}\|v-u\|=C_p\operatorname{dist}(u,E^\mp). \tag{3.4}
$$

It is easy to know that $\operatorname{dist}(v,E^-)\le\|v^+\|$, combining with (3.4) we know there exists $C>0$ such that

$$
\operatorname{dist}(v,E^-)\|v^+\|\le\|v^+\|^2
$$

$$\le \varepsilon|u^+|_2|v^+|_2 + C_\varepsilon|u^+|_p^{p-1}|v^+|_p$$
$$\le C\left(\varepsilon C_\varepsilon \operatorname{dist}(u,E^-) + C_\varepsilon C_p^{p-1}(\operatorname{dist}(u,E^-))^{p-1}\right)\|v^+\|.$$

Therefore,

$$\operatorname{dist}(Au,E^-) \le C(\varepsilon \operatorname{dist}(u,E^-) + C_\varepsilon(\operatorname{dist}(u,E^-))^{p-1}),$$

where $C>0$ is different from the previous line. Then there exists $\varepsilon_0>0$ such that, for all $\varepsilon\in(0,\varepsilon_0)$ and $u\in\partial D_\varepsilon^-$

$$\operatorname{dist}(Au,E^-) < \varepsilon.$$

In particular, we have $A\left(\partial D_\varepsilon^-\right)\subset D_\varepsilon^-$. $A\left(\partial D_\varepsilon^+\right)\subset D_\varepsilon^+$ can be proved analogously. □

By the Lemma 3.1, we only know that A is continuous. Next, we construct a locally Lipschitz continuous operator B which inherits the properties of A. Similar to [7, Lemma 2.1], we have the following lemma.

Lemma 3.3. *There is a locally Lipschitz continuous odd operator $B: E\backslash\mathcal{K}\to E$ satisfies the following properties:*

1. $B\left(\partial D_\varepsilon^-\right)\subset D_\varepsilon^-$, $B\left(\partial D_\varepsilon^+\right)\subset D_\varepsilon^+$;
2. $\frac12\|u-Bu\|\le\|u-Au\|\le 2\|u-Bu\|$;
3. $\langle\Psi'(u),u-Bu\rangle\ge\frac12\|u-Au\|^2$;
4. $\|\Psi'(u)\|\le 2\|u-Bu\|$;

where $\mathcal{K}=\{u\in E\mid\Psi'(u)=0\}$.

By a similar arguments as [15, Lemma 3.4], we have the following Lemma.

Lemma 3.4. *Suppose that N is a symmetric closed neighborhood of $\mathcal{K}_c:=\{u\in E\mid\Psi'(u)=0\ ,\ \Psi(u)=c\}$. Then there exists $\varepsilon_1>0$ such that for $0<\varepsilon<\varepsilon'<\varepsilon_1$, and a continuous map $\sigma:[0,1]\times E\to E$ satisfying:*

1. $\sigma(0,u)=u,\forall u\in E$.
2. $\sigma(t,u)=u,\forall t\in[0,1],\Psi(u)\notin[c-\varepsilon',c+\varepsilon']$.
3. $\sigma(t,-u)=-\sigma(t,u),\forall(t,u)\in[0,1]\times E$.
4. $\sigma\left(1,\Psi^{c+\varepsilon}\backslash N\right)\subset\Psi^{c-\varepsilon}$.
5. $\sigma(t,\overline{D_\varepsilon^+})\subset D_\varepsilon^+,\sigma(t,\overline{D_\varepsilon^-})\subset D_\varepsilon^-$.

In particular, if N is a symmetric closed neighborhood of $\mathcal{K}_c\backslash W$, where $W=D_\varepsilon^+\cup D_\varepsilon^-$, then there exists $\varepsilon_1>0$ such that for $0<\varepsilon<\varepsilon_1$ there will be a continuous map $\eta:E\to E$ such that

6. $\eta(-u)=-\eta(u),\forall u\in E$.
7. $\eta|_{\Psi^{c-2\varepsilon}}=id$.
8. $\eta\left(\Psi^{c+\varepsilon}\backslash(N\cup W)\right)\subset\Psi^{c-\varepsilon}$.
9. $\eta(\overline{D_\varepsilon^+})\subset D_\varepsilon^+,\eta(\overline{D_\varepsilon^-})\subset D_\varepsilon^-$.

Proof of Theorem 1.1. Let $\lambda_i, i=1,2,\dots$ be the ith eigenvalue of (2.5) and e_i be the eigenfunction corresponding to λ_i, $X_j=\operatorname{span}\{e_1,e_2,\cdots,e_j\}$. Firstly, we define

$$M:=\{u\in E\mid\frac14\|u\|^2>\int_{\mathbb{R}^N}F(x,u)\,dx\}\cup B_\rho,$$

where $\rho>0$ such that

$$\{u\in E\mid\frac14\|u\|^2=\int_{\mathbb{R}^N}F(x,u)\,dx\}\cap\partial B_\rho\neq\emptyset.$$

By Lemma 2.3, the interpolation inequality and the definition of λ_i, which is defined in Proposition 2.2, for all $u \in \partial M \cap X_{j-1}^{\perp}$, there exists different constants $C > 0$ such that

$$\begin{aligned} \int_{\mathbb{R}^N} F(x,u)\,dx &\le \int_{\mathbb{R}^N} \left(\varepsilon|u|^2 + C_\varepsilon |u|^p\right)\,dx \\ &\le C\int_{\mathbb{R}^N} |u|^p dx \\ &\le C\Big(\int_{\mathbb{R}^N} |u|^2 dx\Big)^{p\theta/2}\Big(\int_{\mathbb{R}^N} |u|^{2_s^*} dx\Big)^{\frac{p(1-\theta)}{2_s^*}} \\ &\le C\lambda_j^{-\frac{p\theta}{2}} \|u\|^{p\theta}\|u\|^{p(1-\theta)} \\ &= C\lambda_j^{-\frac{p\theta}{2}}\Big(\int_{\mathbb{R}^N} F(x,u)\,dx\Big)^{p/2}, \end{aligned}$$

where $\theta \in (0,1)$ satisfying $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{2_s^*}$, thus

$$\int_{\mathbb{R}^N} F(x,u)\,dx \ge C\lambda_j^{\frac{p\theta}{p-2}}. \tag{3.5}$$

By (3.5), for any $u \in \partial M \cap X_{j-1}^{\perp}$, we have

$$\Psi(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u)\,dx = \int_{\mathbb{R}^N} F(x,u)\,dx \ge C\lambda_j^{\frac{p\theta}{p-2}}.$$

Sincep $\frac{p\theta}{p-2} > 0$, we obtain

$$\inf_{u\in\partial M\cap X_{j-1}^{\perp}} \Psi(u) \ge C\lambda_j^{\frac{p\theta}{p-2}} \to +\infty \quad \text{as } j \to +\infty.$$

By similar arguments to those in Lemma 2.4, we can choose R_j large enough such that $\Psi(u) < 0$ for $u \in X_j \setminus B_{R_j}$. Like in [18], we define

$$c_j = \inf_{D\in\Gamma_j} \sup_{u\in D\setminus W} \Psi(u),$$

where

$$\begin{aligned} \Gamma_j = \big\{ &H\left(X_{j+1}\cap B_{R_{j+1}}\right) : H \in C\left(X_{j+1}\cap B_{R_{j+1}}, E\right), H \text{ is odd and} \\ &H\big|_{X_{j+1}\cap\partial B_{R_{j+1}}} = id\big\}. \end{aligned}$$

Next, we assert that

$$(D\backslash W) \cap X_{j-1}^{\perp} \cap \partial M \ne \emptyset, \forall D \in \Gamma_j, j \ge 2.$$

In fact, by the definition of Γ_j, when $D = H\left(X_{j+1}\cap B_{R_{j+1}}\right)$, we know $H \in C\left(X_{j+1}\cap B_{R_{j+1}}, X\right)$, H is odd and $H|_{\partial B_{R_{j+1}}\cap X_{j+1}} = id$. Let $\hat{\text{O}} = \{u \in X_{j+1}\cap B_{R_{j+1}} \mid H(u) \in intM\}$ and O be the connected component of $\hat{O}$ containing 0. Clearly O is a bounded symmetric neighborhood of 0 in X_{j+1} and $O \cap X_{j+1} \cap \partial B_{R_{j+1}} = \emptyset$. By Borsuk's theorem [30],

$$\gamma(\partial O) = j+1 \text{ and } H(\partial O) \subset \partial M,$$

where $\gamma(\partial O)$ denote the genus of ∂O, readers can learn more about the properties of genus from [26, 30]. Now, we define $I : W \cap \partial M \to \mathbb{R}$ by

$$I(u) = \int_{\mathbb{R}^N} F\left(x,u^+\right)\,dx - \int_{\mathbb{R}^N} F(x,u^-)\,dx. \tag{3.6}$$

It is easy to see that I is an odd continuous map and $0 \notin I(W \cap \partial M)$. Indeed, if $0 \in I(W \cap \partial M)$, then there exists $u \in W \cap \partial M$ such that $\int_{\mathbb{R}^N} F(x,u^+)\,dx = \int_{\mathbb{R}^N} F(x,u^-)\,dx$. When $u \in W$, we know

$$\int_{\mathbb{R}^N} F(x,u^+)\,dx = \int_{\mathbb{R}^N} F(x,u^-)\,dx \le C\varepsilon.$$

But when $u \in \partial M$, there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} F(x,u)\,dx \ge C > 0,$$

which is a contradiction when ε is small enough. As a consequence, $\gamma(\partial M \cap W) = 1$. Thus, $\gamma((I(\partial O) \setminus W) \cap \partial M) \ge j + 1 - 1 = j$, which is contradict to $\operatorname{codim}\left(X_{j-1}^{\perp}\right) = j - 1 < j$. So $H(\partial O) \setminus W \cap \partial M \cap X_{j-1}^{\perp} \neq \emptyset$, since $H(\partial O) \setminus W \subset D \setminus W$, then the claim holds. To complete the proof, we only need to prove $\mathcal{K}_{c_j} \setminus W \neq \emptyset, j \ge 2$. Otherwise, it follows from Lemma 3.4 that there exists $\varepsilon > 0$ and an odd continuous map $\eta : E \to E$ such that

$$\begin{gathered}\eta|_{\Psi^{c_j-2\varepsilon}} = id,\\ \eta(\Psi^{c_j+\varepsilon} \setminus W) \subset \Psi^{c_j-\varepsilon},\\ \eta(\overline{D_\varepsilon^{\pm}}) \subset D_\varepsilon^{\pm}.\end{gathered}$$

Hence, regarding the ε mentioned above, there exists $D_0 \in \Gamma_j$ such that

$$\sup_{u \in D_0 \setminus W} \Psi(u) < c_j + \varepsilon,$$

that is $D_0 \backslash W \subset \Psi^{c_j+\varepsilon}$. Let $U := \eta(D_0)$, it is easy to verify that $U \in \Gamma_j$ and $c_j \le \sup_{u \in U \backslash W} \Psi(u)$. Note that

$$U \backslash W = \eta(D_0) \backslash W \subset (\eta(D_0 \backslash W) \cup \eta(W)) \backslash W \subset \eta(D_0 \backslash W) \backslash W \subset \eta(\Psi^{c_j+\varepsilon} \backslash W) \subset \Psi^{c_j-\varepsilon}.$$

Thus

$$c_j \le \sup_{u \in U \backslash W} \Psi(u) \le c_j - \varepsilon,$$

which is a contradiction. The proof is complete. □

Acknowledgments. This work was supported by the NSFC 12261107, and by the Yunnan key Laboratory of Modern Analytical Mathematics and Applications.

References

[1] A. Ambrosetti, P. H. Rabinowitz; Dual variational methods in critical point theory and applications, *J. Funct. Anal.*, **14** (1973), 349–381.

[2] T. Bartsch, Z.-Q. Wang; Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^N$, *Comm. Partial Differential Equations*, **20** (1995), 1725–1741.

[3] T. Bartsch, Z.-Q. Wang; On the existence of sign changing solutions for semilinear Dirichlet problems, *Topol. Methods Nonlinear Anal.*, **7** (1996), 115–131.

[4] T. Bartsch; Critical point theory on partially ordered Hilbert spaces, *J. Funct. Anal.*, **186** (2001), 117–152.

[5] T. Bartsch, A. Pankov, Z.-Q. Wang; Nonlinear Schrödinger equations with steep potential well, *Commun. Contemp. Math.*, **3** (2001), 549–569.

[6] T. Bartsch, Z. Liu, T. Weth; Sign changing solutions of superlinear Schrödinger equations, *Comm. Partial Differential Equations*, **29** (2004), 25–42.

[7] T. Bartsch, Z. Liu, T. Weth; Nodal solutions of a p-Laplacian equation, *Proc. London Math. Soc.(3)*, **91** (2005), 129–152.

[8] T. Bartsch, T. Weth; Three nodal solutions of singularly perturbed elliptic equations on domains without topology, *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, **22** (2005), 259–281.
[9] C. Bucur, E. Valdinoci; Nonlocal diffusion and applications, *Lect. Notes Unione Mat. Ital.*, Springer; Unione Matematica Italiana, Bologna, 2016.
[10] A. Castro, J. Cossio, J. Neuberger; A sign-changing solution for a superlinear Dirichlet problem, *Rocky Mountain J. Math.*, **27** (1997), 1041–1053.
[11] X. Chang, Z.-Q. Wang; Nodal and multiple solutions of nonlinear problems involving the fractional Laplacian, *J. Differential Equations*, **256** (2014), 2965–2992.
[12] E. N. Dancer, Y. Du; On sign-changing solutions of certain semilinear elliptic problems, *Appl. Anal.*, **56** (1995), 193–206.
[13] E. Di Nezza, G. Palatucci, E. Valdinoci; Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.*, **136** (2012), 521–573.
[14] G. Gu, Y. Yu, F. Zhao; The least energy sign-changing solution for a nonlocal problem, *J. Math. Phys.*, **58** (2017), 051505.
[15] G. Gu, W. Zhang, F. Zhao; Infinitely many sign-changing solutions for a nonlocal problem, *Ann. Mat. Pura Appl.(4)*, **197** (2018), 1429–1444.
[16] G. Gu, X. Zhang, F. Zhao; A compact embedding result and its applications to a nonlocal Schrödinger equation, *Math. Nachr.*, **297** (2024), 707–715.
[17] L. Jeanjean, K. Tanaka; A positive solution for an asymptotically linear elliptic problem on R^N autonomous at infinity, *ESAIM Control Optim. Calc. Var.*, **7** (2002), 597–614.
[18] S. Li, Z.-Q. Wang; Ljusternik-Schnirelman theory in partially ordered Hilbert spaces, *Trans. Amer. Math. Soc.*, **354** (2002), 3207–3227.
[19] P. L. Lions; The concentration-compactness principle in the calculus of variations. The locally compact case. I, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **1** (1984), 109–145.
[20] P. L. Lions; The concentration-compactness principle in the calculus of variations. The locally compact case. II, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **1** (1984), 223–283.
[21] Z. Liu, J. Su, T. Weth; Compactness results for Schrödinger equations with asymptotically linear terms, *J. Differential Equations*, **231** (2006), 501–512.
[22] L. Maia, O. Miyagaki, S. Soares; A sign-changing solution for an asymptotically linear Schrödinger equation, *Proc. Edinb. Math. Soc. (2)*, **58** (2015), 697–716.
[23] J. Mawhin, M. Willem; Critical point theory and Hamiltonian systems, *Appl. Math. Sci.*, Springer-Verlag, New York, 1989.
[24] G. Molica Bisci, V. Radulescu, R. Servadei; Variational methods for nonlocal fractional problems, *Encyclopedia Math. Appl.*, Cambridge University Press, Cambridge, 2016.
[25] L. Nikolai; Fractional quantum mechanics and Lévy path integrals, *Phys. Lett. A*, **268** (2000), 298-305.
[26] P. H. Rabinowitz; Minimax methods in critical point theory with applications to differential equations, *CBMS Regional Conf. Ser. in Math.*, Conference Board of the Mathematical Sciences, Washington, 1986.
[27] M. Ralf, K. Joseph; The random walk's guide to anomalous diffusion: a fractional dynamics approach, *Phys. Rep.*, **339** (2000), 1-77.
[28] R. Servadei, E. Valdinoci; Mountain pass solutions for non-local elliptic operators, *J. Math. Anal. Appl.*, **389** (2012), 887–898.
[29] R. Servadei, E. Valdinoci; Variational methods for non-local operators of elliptic type, *Discrete Contin. Dyn. Syst.*, **33** (2013), 2105–2137.
[30] M. Struwe; Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems, *Ergeb. Math. Grenzgeb.(3)*, Springer-Verlag, Berlin, 2008.
[31] C. A. Stuart; Self-trapping of an electromagnetic field and bifurcation from the essential spectrum, *Arch. Rational Mech. Anal.*, **113** (1990), 65–96.
[32] C. A. Stuart, H. Zhou; A variational problem related to self-trapping of an electromagnetic field, *Math. Methods Appl. Sci.*, **19** (1996), 1397–1407.
[33] C. A. Stuart, H. Zhou; Applying the mountain pass theorem to an asymptotically linear elliptic equation on $\mathbb{R}^N$, *Comm. Partial Differential Equations*, **24** (1999), 1731–1758.
[34] A. Sumiyoshi, T. Stefan; Anomalous diffusion in view of Einstein's 1905 theory of Brownian motion, *Physica A*, **356** (2005), 403-407.
[35] F. Van Heerden; Multiple solutions for a Schrödinger type equation with an asymptotically linear term, *Nonlinear Anal.*, **55** (2003), 739–758.

[36] F. Van Heerden, Z.-Q. Wang; Schrödinger type equations with asymptotically linear nonlinearities, *Differential Integral Equations*, **16** (2003), 257–280.
[37] Z. Wang, H. Zhou; Radial sign-changing solution for fractional Schrödinger equation, *Discrete Contin. Dyn. Syst.*, **36** (2016), 499–508.
[38] T. Weth; Energy bounds for entire nodal solutions of autonomous superlinear equations, *Calc. Var. Partial Differential Equations*, **27** (2006), 421–437.

Ruowen Qiu
Department of Mathematics, Yunnan Normal University, Kunming, 650221, China
Email address: 1239814486@qq.com

Renqing You
Department of Mathematics, Yunnan Normal University, Kunming, 650221, China
Email address: 1768332868@qq.com

Fukun Zhao (corresponding author)
Department of Mathematics, Yunnan Normal University, Kunming, 650221, China
Email address: fukunzhao@163.com