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BEHAVIOR NEAR THE EXTINCTION TIME FOR SYSTEMS OF DIFFERENTIAL EQUATIONS WITH SUBLINEAR DISSIPATION TERMS

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Abstract. This article focuses on the behavior near the extinction time of solutions to systems of ordinary differential equations with a sublinear dissipation term. Suppose the dissipation term is a product of a linear mapping A and a positively homogeneous scalar function H of a negative degree $-\alpha$. Then any solution with an extinction time T_* behaves like $(T_* - t)^{1/\alpha}\xi_*$ as time $t \to T_*^-$, where ξ_* is an eigenvector of A. The result allows the higher order terms to be general and the nonlinear function H to take very complicated forms. As a demonstration, our theoretical study is applied to an inhomogeneous population model.

1. Introduction

This article continues our investigations of exact asymptotic behaviors of solutions of systems of nonlinear ordinary differential equations (ODE), see [12, 13, 26, 27], and the Navier–Stokes equations (NSE), see, e.g., [10, 28, 30, 31]. However, in contrast to the cited work above, the current paper studies the asymptotic behavior near a finite extinction time, instead of time infinity, for equations with sublinear dissipation terms, instead of superlinear in [27], or linear in the others. The phenomena of extinction are widely studied in mathematics, biology and physics, see e.g. [14, 15, 16, 17, 32, 36, 37]. In biology, the extinction time reflects the time when the species' populations go extinct, while, for fast diffusive fluid flows in porous media, it indicates the time when the pressure or density vanishes everywhere. For the precise description of the solutions near a finite extinction time, the mathematical results in [14, 16, 36] are only for very specific equations. Instead, our aim is to establish the results for general systems of ODEs. It turns out that we can achieve this by modifying and improving some techniques by Foias and Saut in [18] for the NSE. For that reason, we review [18] and the related literature here. The NSE with potential body forces can be written in the following functional form, which holds in a certain weak sense in a suitable functional space,

$$u' + Au + B(u,u) = 0, \tag{1.1}$$

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where A is the (linear) Stokes operator which has positive eigenvalues and $B(\cdot,\cdot)$ is a bilinear form. It is proved in [18] that any nontrivial solution $u(t)$ of (1.1) has the following asymptotic behavior

$$e^{\Lambda t}u(t) \to \xi_* \quad \text{in any } C^m\text{-norm as } t \to \infty, \tag{1.2}$$

where Λ is an eigenvalue of A and ξ_* is an eigenfunction of A associated with Λ. For finer asymptotic behaviors than (1.2), Foias and Saut develop a theory of asymptotic expansions for the solutions of the NSE (1.1) in [19, 20].

On the one hand, the result (1.2) and its proof are extended to abstract differential inequalities in [22, 21]. On the other hand, the asymptotic expansion theory is developed further for both ODE and partial differential equations (PDE). See [13, 27, 29, 31, 35, 38] for systems without forcing functions, [10, 11, 12, 26, 28, 30] for systems with forcing functions, and [25] for the Lagrangian trajectories for viscous incompressible fluids. Both techniques from [18] and [19] are combined in [13] to deal with nonsmooth ODE systems. Originally, the asymptotic expansions can be obtained independently from the limit (1.2). In [13], however, they are obtained only after the first asymptotic approximation (1.2) is established.

In the original NSE in [18] as well as the systems in the extended work cited above, except for [27], the ODE or PDE have linear dissipation terms. On contrary, the author recently studied in [27] the following ODE system in $\mathbb{R}^n$

$$y' = -H(y)Ay + G(t,y), \tag{1.3}$$

where A is a constant $n\times n$ matrix with positive eigenvalues, H is a positive function, and G represents a higher order term. In [27], H is additionally assumed to be a positively homogeneous function of a positive degree α. Note that the dissipation term $H(y)Ay$ in (1.3) is nonlinear compared with, say, the linear dissipation term Au in (1.1), and the higher order term G is not required to be bilinear like $B(u,u)$. It is proved that any nonzero, decaying solution of (1.3) behaves like $t^{-1/\alpha}\xi_*$ as $t \to \infty$, where ξ_* is an eigenvector of A.

The current paper considers the opposite scenario when the function H in (1.3) has a negative degree $-\alpha$. In this case, many solutions start out with nonzero values and then become zero at a finite time. Such time is called the *extinction time*. Our goal is to describe the behavior of these solutions near this extinction time. The main result can be briefly described as follows. Under appropriate assumptions, any solution $y(t)$ of (1.3) with the extinction time T_* behaves exactly like $(T_*-t)^{1/\alpha}\xi_*$ as time $t \to T_*^-$, where ξ_* is an eigenvector of A. It is worth mentioning that the existence of the extinction time is guaranteed under the small nonzero initial data condition, see Theorem 2.5 below. Our proof will make use and adapt the techniques from [18, 27]. In particular, the recent perturbation method in [27] will be utilized. This method is needed to deal with the nonlinear dissipation in our problem. It will be implemented successfully in this paper for the study of the asymptotic behavior near the finite extinction time, instead of at time infinity as in [27]. The obtained result, in addition to its merits for ODE, also gives hints to a type of results that may be expected for general nonlinear PDE of the similar structure.

This article is organized as follows. Section 2 contains the main results. While the condition for the matrix A is the natural Assumption 2.1, the more technical conditions for the function H are specified in Assumption 2.8. A key requirement of H, namely, property (HC) is introduced in Definition 2.6. Theorem 2.5 states that under appropriate conditions on A, H and G, for any sufficiently small nonzero initial condition, there exists a solution of (1.3) that will become zero at finite time. Solutions with a finite extinction time of this type are the objects of our investigation in this paper. The asymptotic behavior of the solutions to a more general equation (2.11) near the extinction time is established in Theorem 2.9. Its counterpart for equation (1.3) is Theorem 2.10. The proof of Theorem 2.5 is given in Section 3. Section 4 prepares for the proof of Theorem 2.9. Preliminary estimates for the solutions are obtained in Lemma 4.1. Although they provide only a rough description of $y(t)$, the upper and lower bounds with the same rate $1/\alpha$ obtained in (4.1) are important in our further analysis. Section 5 proves Theorem 2.9 for a special case in the form of equation (5.2). This will also serve as the basis for the perturbation argument for the general case in Section 7. In Section 6, we obtain essential properties of the solutions of (2.11) when the matrix A is symmetric. In particular, we establish an eigenvalue Λ as the limit of the quotient $\lambda(t)$, see (6.1), in Proposition 6.1, and a unit vector v_* as the limit of $y(t)/|y(t)|$ in Propositions 6.2 and 6.3. In Section 7, we give proof to Theorem 2.9 first. It combines all the previous preparations with the perturbation method mentioned earlier, see equation (7.2). This equation is a reduction of equation (2.11) of $y(t)$ to the simple form (5.2), but for the projection $R_\Lambda y(t)$ and with a frozen coefficient $\Lambda H(v_*)$. The proof of Theorem 2.10 is then quickly provided. Section 8 contains some examples for the function H in subsection 8.1, and an application to an inhomogeneous population model in subsection 8.2.

Notation. Throughout this article, $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ is the spatial dimension. For any vector $x \in \mathbb{R}^n$, we denote by $|x|$ its Euclidean norm. For an $n \times n$ real matrix $A = (a_{ij})_{1\le i,j\le n}$, its Euclidean norm is

$$\|A\| = \Big(\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2\Big)^{1/2}.$$

The unit sphere in $\mathbb{R}^n$ is $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$.

2. Main results

Assumption 2.1. Matrix A is a (real) diagonalizable $n \times n$ matrix with positive eigenvalues.

Under this assumption, the matrix A has n positive eigenvalues (counting their multiplicities) $\Lambda_1 \le \Lambda_2 \le \Lambda_3 \le \dots \le \Lambda_n$, and there exists an invertible $n \times n$ (real) matrix S such that

$$A = S^{-1} A_0 S, \quad \text{where } A_0 = \operatorname{diag}[\Lambda_1, \Lambda_2, \dots, \Lambda_n]. \tag{2.1}$$

In the case A is symmetric, the matrix S is orthogonal, i.e., $S^{-1} = S^{\mathrm{T}}$, and

$$\Lambda_1 |x|^2 \le x \cdot Ax \le \Lambda_n |x|^2 \quad \text{for all } x \in \mathbb{R}^n. \tag{2.2}$$

More specifically, the distinct eigenvalues of A are denoted by λ_j, with $1 \le j \le d$ for some integer $d \in [1, n]$, and are arranged to be (strictly) increasing in j, i.e.,

$$0 < \lambda_1 = \Lambda_1 < \lambda_2 < \dots < \lambda_d = \Lambda_n.$$

The spectrum of A is $\sigma(A) = \{\Lambda_k : 1 \le k \le n\} = \{\lambda_j : 1 \le j \le d\}$.

For $1 \le k, \ell \le n$, let $E_{k\ell}$ be the elementary $n \times n$ matrix $(\delta_{ki}\delta_{\ell j})_{1\le i,j\le n}$, where δ_{ki} and $\delta_{\ell j}$ are the Kronecker delta symbols. For $\Lambda \in \sigma(A)$, define

$$\widehat{R}_\Lambda = \sum_{1\le i\le n, \Lambda_i=\Lambda} E_{ii} \text{ and } R_\Lambda = S^{-1}\widehat{R}_\Lambda S.$$

Then one immediately has

$$I_n = \sum_{j=1}^{d} R_{\lambda_j}, \quad R_{\lambda_i}R_{\lambda_j} = \delta_{ij}R_{\lambda_j}, \quad AR_{\lambda_j} = R_{\lambda_j}A = \lambda_j R_{\lambda_j}. \tag{2.3}$$

Thanks to (2.3), each R_Λ is a projection, and $R_\Lambda(\mathbb{R}^n)$ is the eigenspace of A associated with the eigenvalue Λ.

In the case A is symmetric, R_Λ is the orthogonal projection from $\mathbb{R}^n$ to the eigenspace of A associated with Λ, and, hence,

$$|R_\Lambda x| \le |x| \text{ for all } x \in \mathbb{R}^n. \tag{2.4}$$

The function H will be assumed to have some type of homogeneity which is specified in the next definition.

Definition 2.2. Let X and Y be two (real) linear spaces, and $\beta < 0$ be a given number. A function $F : X \setminus \{0\} \to Y$ is positively homogeneous of degree β if

$$F(tx) = t^\beta F(x) \text{ for any } x \in X \setminus \{0\} \text{ and } t > 0.$$

We define $\mathcal{H}_\beta(X, Y)$ to be the set of functions from $X \setminus \{0\}$ to Y that are positively homogeneous of degree β.

If $F \in \mathcal{H}_\beta(X, Y)$ and F is not the zero function, then the degree β is unique.

Assumption 2.3. The function H is in $\mathcal{H}_{-\alpha}(\mathbb{R}^n, \mathbb{R})$ for some $\alpha > 0$, and in $C(\mathbb{R}^n \setminus \{0\}, (0, \infty))$.

For a function H in Assumption 2.3, it is positive and continuous on $\mathbb{S}^{n-1}$. Hence, we have

$$0 < c_1 = \min_{|x|=1} H(x) \le \max_{|x|=1} H(x) = c_2 < \infty. \tag{2.5}$$

By writing $H(x) = |x|^{-\alpha} H(x/|x|)$ for any $x \in \mathbb{R}^n \setminus \{0\}$ and using (2.5), we derive

$$c_1|x|^{-\alpha} \le H(x) \le c_2|x|^{-\alpha} \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}. \tag{2.6}$$

Regarding the function G in equation (1.3), we have the following assumption.

Assumption 2.4. Let t_0 be any given number in $[0, \infty)$. We assume that the function $G(t, x)$ is continuous on $[t_0, \infty) \times (\mathbb{R}^n \setminus \{0\})$, and there exist positive numbers c_*, r_*, δ such that

$$|G(t, x)| \le c_*|x|^{1-\alpha+\delta} \text{ for all } t \ge t_0, \text{ and all } x \in \mathbb{R}^n \text{ with } 0 < |x| \le r_*. \tag{2.7}$$

In our first theorem below, we show that the extinction time exists for, at least, certain small solutions of (1.3).

Theorem 2.5. *Under Assumptions 2.1, 2.3 and 2.4, there exists a number $r_0 > 0$ such that for any $y_0 \in \mathbb{R}^n \setminus \{0\}$ with $|y_0| \le r_0$, there are a number $T_* > t_0$ and a function $y \in C^1([t_0, T_*), \mathbb{R}^n)$ such that $y(t_0) = y_0$,*

$$y(t) \ne 0 \quad \text{for all } t \in [t_0, T_*), \tag{2.8}$$

$$\lim_{t\to T_*^-} y(t) = 0, \tag{2.9}$$

and $y(t)$ satisfies equation (1.3) *for all $t \in (t_0, T_*)$. In other words, $y(t)$ is a solution of* (1.3) *on (t_0, T_*) with the initial data y_0 at time t_0 and has the extinction time T_*.*

In fact, $y(t)$ is any solution with the maximal interval $[t_0, T_{\max})$ in the existence theorem 3.1 below, and $T_* = T_{\max}$. Moreover, if additional conditions are imposed to guarantee the uniqueness of solutions to the initial value problem for equation (1.3), then, in the above Theorem 2.5, *any* solution with sufficiently small nonzero initial data must have a finite extinction time. The proof of Theorem 2.5 is given in Section 3.

For the behavior of a solution near an extinction time, the function H is required to have an extra property.

Definition 2.6. Let E be a nonempty subset of $\mathbb{R}^n$ and F be a function from E to $\mathbb{R}$. We say F has property (HC) on E if, for any $x_0 \in E$, there exist numbers $r, C, \gamma > 0$ such that

$$|F(x) - F(x_0)| \le C|x - x_0|^\gamma \tag{2.10}$$

for each $x \in E$ with $|x - x_0| < r$.

The following are elementary properties of the functions in Definitions 2.2 and 2.6.

Lemma 2.7. *Let $F \in \mathcal{H}_{-\alpha}(\mathbb{R}^n, \mathbb{R})$ for some $\alpha > 0$.*

- (i) *If $F > 0$ on $\mathbb{S}^{n-1}$, then $F > 0$ on $\mathbb{R}^n \setminus \{0\}$.*
- (ii) *If F is continuous on $\mathbb{S}^{n-1}$, then it is continuous on $\mathbb{R}^n \setminus \{0\}$.*

  Assume F has property (HC) on $\mathbb{S}^{n-1}$ in (iii)–(v) *below.*
- (iii) *Then F has property (HC) on $\mathbb{R}^n \setminus \{0\}$.*
- (iv) *If φ is a function from $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{R}^n \setminus \{0\}$ that has property (HC) on $\mathbb{R}^n \setminus \{0\}$. Then $F \circ \varphi$ has property (HC) on $\mathbb{R}^n \setminus \{0\}$.*
- (v) *If K is an invertible $n \times n$ matrix, then the function $x \in \mathbb{R}^n \setminus \{0\} \mapsto F(Kx)$ has property (HC) on $\mathbb{R}^n \setminus \{0\}$.*

The proof of the above lemma is given in the Appendix.

Assumption 2.8. The function H belongs to $\mathcal{H}_{-\alpha}(\mathbb{R}^n, \mathbb{R})$ for some $\alpha > 0$, has property (HC) on the unit sphere $\mathbb{S}^{n-1}$, and $H > 0$ on $\mathbb{S}^{n-1}$.

If the function H satisfies Assumption 2.8, it is obvious that H is continuous on $\mathbb{S}^{n-1}$. Therefore, thanks to parts (i) and (ii) of Lemma 2.7, it is continuous and positive on $\mathbb{R}^n \setminus \{0\}$. Consequently, H satisfies the conditions in Assumption 2.3. Some examples for the function H will be given in subsection 8.1.

The next result deals with a more general equation than (1.3), namely, equation (2.11) below.

Theorem 2.9 (Main Theorem I)**.** *Let Assumptions 2.1 and 2.8 hold. Let $t_0, T_* \in \mathbb{R}$ be two given numbers with $T_* > t_0 \ge 0$. Assume $y \in C^1([t_0, T_*), \mathbb{R}^n)$ satisfies* (2.8), (2.9) *and*

$$y' = -H(y)Ay + f(t) \text{ for all } t \in (t_0, T_*), \tag{2.11}$$

where f is a continuous function from $[t_0, T_*)$ to $\mathbb{R}^n$ such that

$$|f(t)| \le M|y(t)|^{1-\alpha+\delta}, \text{ for all } t \in [t_0, T_*) \text{ and some constants } M, \delta > 0. \tag{2.12}$$

Then there exist an eigenvalue Λ *of* A *and an eigenvector* ξ_* *of* A *associated with* Λ *such that*

$$|y(t) - (T_* - t)^{1/\alpha}\xi_*| = \mathcal{O}((T_* - t)^{1/\alpha+\varepsilon}) \text{ as } t \to T_*^- \text{ for some } \varepsilon > 0. \tag{2.13}$$

More specifically,

$$|(I_n - R_\Lambda)y(t)| = \mathcal{O}((T_* - t)^{1/\alpha+\varepsilon}) \text{ as } t \to T_*^- \text{ for some } \varepsilon > 0, \tag{2.14}$$

$$|R_\Lambda y(t) - (T_* - t)^{1/\alpha}\xi_*| = \mathcal{O}((T_* - t)^{1/\alpha+\varepsilon}) \text{ as } t \to T_*^- \text{ for some } \varepsilon > 0, \tag{2.15}$$

$$\alpha\Lambda H(\xi_*) = 1. \tag{2.16}$$

With a solution $y(t)$ as in Theorem 2.9, we define, for the sake of convenience,

$$y(T_*) = 0, \text{ and have } y \in C([t_0, T_*], \mathbb{R}^n). \tag{2.17}$$

From Theorem 2.9, we can derive a corresponding result for equation (1.3).

Theorem 2.10 (Main Theorem II)**.** *Let Assumptions 2.1, 2.4 (with a number* $t_0 \geq 0$*) and 2.8 hold. Given a number* $T_* > t_0$*. Assume* $y \in C^1([t_0, T_*), \mathbb{R}^n)$ *has properties* (2.8), (2.9)*, and satisfies equation* (1.3) *for all* $t \in (t_0, T_*)$*. Then there exist an eigenvalue* Λ *and an associated eigenvector* ξ_* *of* A *such that* (2.13)*–*(2.16) *hold true.*

The proofs of Theorems 2.9 and 2.10 are given in Section 7.

3. Existence of the extinction time

We prove Theorem 2.5 in this section. First, we present a standard existence theorem for the solutions.

Theorem 3.1. *Let Assumptions 2.1, 2.3 hold and assume* G *is a continuous function from* $[t_0, \infty) \times \mathbb{R}^n \setminus \{0\}$ *to* $\mathbb{R}^n$*, for some number* $t_0 \in \mathbb{R}$*. Let* $y_0 \in \mathbb{R}^n \setminus \{0\}$*. Then there exist an interval* $[t_0, T_{\max})$*, with* $t_0 < T_{\max} \leq \infty$*, and a function* $y \in C^1([t_0, T_{\max}), \mathbb{R}^n \setminus \{0\})$ *such that*

$$y(t) \text{ that satisfies (1.3) on } (t_0, T_{\max}),\ y(t_0) = y_0, \tag{3.1}$$

and either

(a) $T_{\max} = \infty$*, or*
(b) $T_{\max} < \infty$*, and for any* $\varepsilon > 0$*,* y *cannot be extended to a function of class* $C^1([t_0, T_{\max} + \varepsilon), \mathbb{R}^n \setminus \{0\})$ *that satisties* (1.3) *on the interval* $(t_0, T_{\max} + \varepsilon)$*.*

Moreover, in the case (b)*, it holds, for any compact set* $U \subset \mathbb{R}^n \setminus \{0\}$*, that*

$$y(t) \notin U \text{ when } t \in [t_0, T_{\max}) \text{ is near } T_{\max}. \tag{3.2}$$

Proof. By Peano's Existence Theorem [24, Chapter II, Theorems 2.1, page 10] and the continuity of the function $G(t, x)$, there exist a number $\delta > 0$ and a function $y \in C^1([t_0, t_0 + \delta], \mathbb{R}^n \setminus \{0\})$ such $y(t)$ that satisfies equation (1.3) on $[t_0, t_0 + \delta]$ and $y(t_0) = y_0$. By the Extension Theorem, see [24, Chapter II, Theorem 3.1, page 12] or [23, Chapter I, Theorem 2.1, page 17], applied to this solution y and the open set

$$D = \{(t, x) : t > t_0, x \in \mathbb{R}^n \setminus \{0\}\} \subset \mathbb{R}^{n+1}, \tag{3.3}$$

the current solution y can be extended to a solution $y \in C^1([t_0, T_{\max}), \mathbb{R}^n \setminus \{0\})$, for some $t_0 + \delta \leq T_{\max} \leq \infty$, that has the properties (3.1), with either (a) or (b), and, additionally, in the case (b), one has, for any compact set $V \subset D$,

$$(t, y(t)) \notin V \text{ when } t \in [t_0, T_{\max}) \text{ is near } T_{\max}. \tag{3.4}$$

Now, consider case (b) and a compact set $U \subset \mathbb{R}^n \setminus \{0\}$. Let $V = [T_1, T_2] \times U$ with $T_1 = (t_0 + T_{\max})/2$ and $T_2 = T_{\max} + 1$. If $t \in [t_0, T_{\max})$ is sufficiently close to $T_{\max}$, then $t \in [T_1, T_2]$. Therefore, the desired statement (3.2) follows from (3.4). □

Note that such a solution $y(t)$ in Theorem 3.1 may not be unique. Next we prove Theorem 2.5.

Proof of Theorem 2.5. Let $y(t)$ be a solution of (1.3) as in Theorem 3.1 with the maximal interval $[t_0, T_{\max})$. Then

$$y(t) \neq 0 \text{ for all } t \in [t_0, T_{\max}). \tag{3.5}$$

On $(t_0, T_{\max})$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(|y|^\alpha) = \alpha |y|^{\alpha-2} y' \cdot y = \alpha \left(-|y|^{\alpha-2} H(y)(Ay) \cdot y + |y|^{\alpha-2} G(t,y) \cdot y \right). \tag{3.6}$$

Step 1. Consider A is symmetric first. Take $r_0 > 0$ such that

$$2r_0 \le r_* \text{ and } c_*(2r_0)^\delta \le a_0 := c_1 \Lambda_1 / 2.$$

For $t > t_0$ sufficiently close to t_0, we have $|y(t)| < 2r_0$. Let $[t_0, T)$ be the maximal interval in $[t_0, T_{\max})$ on which $|y(t)| < 2r_0$.

Suppose $T < T_{\max}$. On the one hand, it must hold that

$$|y(T)| = 2r_0. \tag{3.7}$$

On the other hand, combining (3.6) with (2.2), (2.6) and (2.7), we have, for $t \in (t_0, T)$, that

$$\frac{\mathrm{d}}{\mathrm{d}t}(|y|^\alpha) \le \alpha(-c_1\Lambda_1 + c_*|y|^\delta) \le \alpha(-c_1\Lambda_1 + c_*(2r_0)^\delta) \le -\alpha a_0 < 0. \tag{3.8}$$

Thus, $|y(T)|^\alpha \le |y_0|^\alpha$, which implies $|y(T)| \le |y_0| \le r_0$. This contradicts (3.7). Therefore, $T = T_{\max}$. For $t \in (t_0, T_{\max})$, integrating (3.8) from t_0 to t gives

$$|y(t)|^\alpha \le |y_0|^\alpha - \alpha a_0 (t - t_0) \text{ for all } t \in [t_0, T_{\max}). \tag{3.9}$$

Step 2. Consider the general matrix A. Using the equivalence (2.1), we set

$$z(t) = Sy(t) \text{ and } z_0 = z(t_0) = Sy_0. \tag{3.10}$$

Then

$$z' = -\widetilde{H}(z) A_0 z + \widetilde{G}(t,z) \text{ for } t \in (t_0, T_{\max}), \tag{3.11}$$

where

$$\widetilde{H}(z) = H(S^{-1}z), \quad \widetilde{G}(t,z) = SG(t, S^{-1}z) \text{ for } t \in [t_0, T_{\max}) \text{ and } z \in \mathbb{R}^n \setminus \{0\}.$$

Note that

$$\|S^{-1}\|^{-1} \cdot |x| \le |Sx| \le \|S\| \cdot |x| \text{ for all } x \in \mathbb{R}^n. \tag{3.12}$$

Clearly, $\widetilde{H}$ satisfies the same condition as H in Assumption 2.3. Moreover, $\widetilde{G}(t,z)$ is continuous on $[t_0, \infty) \times (\mathbb{R}^n \setminus \{0\})$. For $t \in [t_0, T_{\max})$ and $0 < |z| \le r_* / \|S^{-1}\|$, we have $0 < |S^{-1}z| \le r_*$, and then, by (2.7) and (3.12),

$$|\widetilde{G}(t,z)| \le \|S\| \cdot c_* |S^{-1}z|^{1-\alpha+\delta} \le c_* \|S\| \cdot \begin{cases} \|S^{-1}\|^{1-\alpha+\delta} |z|^{1-\alpha+\delta}, & \text{if } 1-\alpha+\delta \ge 0, \\ \|S\|^{-(1-\alpha+\delta)} |z|^{1-\alpha+\delta}, & \text{otherwise.} \end{cases}$$

We apply the calculations in Step 1 to the solution $z(t)$ of (3.11). When $|y_0|>0$ is sufficiently small, we have $|z_0|>0$ is sufficiently small, and hence, similar to estimate (3.9),

$$|z(t)|^\alpha \le |z_0|^\alpha - \alpha\widetilde{a}_0(t-t_0),\ \text{for all } t\in[t_0,T_{\max}) \text{ and some constant } \widetilde{a}_0>0.$$

Therefore,

$$|y(t)|^\alpha \le \|S^{-1}\|^\alpha |z(t)|^\alpha \le \|S^{-1}\|^\alpha (|Sy_0|^\alpha - \alpha\widetilde{a}_0(t-t_0)) \text{ for all } t\in[t_0,T_{\max}). \tag{3.13}$$

Step 3. If $T_{\max}=\infty$, then (3.13) implies that $|y(t)|^\alpha<0$ for $t>t_0+|Sy_0|^\alpha/(\alpha\widetilde{a}_0)$, which is an obvious contradiction. Therefore, $T_{\max}<\infty$. As a consequence of (3.13),

$$|y(t)|\le R_0 \text{ on } [t_0,T_{\max}), \text{ where } R_0=\|S^{-1}\|\cdot|Sy_0|>0. \tag{3.14}$$

Step 4. Let $T_*=T_{\max}$. Then (3.5) implies (2.8). For any $\varepsilon>0$, let $U=\{x\in\mathbb{R}^n : \varepsilon\le|x|\le 2R_0\}$ in (3.2). Taking into account (3.14), one must have $|y(t)|<\varepsilon$ when $t\in[t_0,T_*)$ is near T_*. This proves the zero limit in (2.9). □

We remark that $y(t)$ may be zero for t larger than the above $T_{\max}$. However, this is excluded from our consideration of the set D in (3.3). The reason is our sole focus on the finite extinction time and the solution before that time.

4. Preliminary estimates

In this section, we prepare for the proof of Theorem 2.9 by obtaining preliminary estimates for the solution $y(t)$ of equation (2.11). They even hold under a weaker condition than Assumption 2.8.

Lemma 4.1. *Let Assumptions 2.1 and 2.3 hold. Given numbers $T_*>t_0\ge0$, let $y\in C^1([t_0,T_*),\mathbb{R}^n)$ satisfy* (2.8), (2.9), (2.11)–(2.12). *Then there are positive constants C_1 and C_2 such that*

$$C_1(T_*-t)^{1/\alpha}\le|y(t)|\le C_2(T_*-t)^{1/\alpha}\quad \text{for all } t\in[t_0,T_*]. \tag{4.1}$$

Proof. We prove (4.1) for the case the matrix A is symmetric first and then for A not symmetric.

Case 1. Consider A is symmetric. For $t\in(t_0,T_*)$, we calculate, similarly to (3.6),

$$\frac{\mathrm{d}}{\mathrm{d}t}(|y|^\alpha)=\alpha\left(-|y|^{\alpha-2}H(y)(Ay)\cdot y+|y|^{\alpha-2}f(t)\cdot y\right). \tag{4.2}$$

Utilizing (2.2), (2.6), and (2.12), one has

$$\alpha(-c_2\Lambda_n-M|y|^\delta)\le\frac{\mathrm{d}}{\mathrm{d}t}(|y|^\alpha)\le\alpha(-c_1\Lambda_1+M|y|^\delta).$$

Let $a_1=c_1\Lambda_1/2$ and $a_2=c_2\Lambda_n+1$. Let $r_0>0$ be such that $Mr_0^\delta=\min\{1,c_1\Lambda_1/2\}$.

Thanks to (2.9), there is $T\in(t_0,T_*)$ such that $|y(t)|\le r_0$ on (T,T_*). Hence,

$$-\alpha a_2\le\frac{\mathrm{d}}{\mathrm{d}t}(|y|^\alpha)\le-\alpha a_1 \text{ on } (T,T_*). \tag{4.3}$$

For $t\in[T,T_*)$, integrating (4.3) from t to $t'\in(t,T_*)$, passing to the limit $t'\to T_*^-$, and using (2.9), we obtain

$$\alpha a_1(T_*-t)\le|y(t)|^\alpha\le\alpha a_2(T_*-t) \text{ for all } t\in[T,T_*]. \tag{4.4}$$

Above, (4.4) holds for $t = T_*$ thanks to (2.17). Note also that

$$0 < a_3 := \min_{t\in[t_0,T]} (T_* - t)^{-1/\alpha}|y(t)| \le a_4 := \max_{t\in[t_0,T]} (T_* - t)^{-1/\alpha}|y(t)| < \infty. \quad (4.5)$$

Combining (4.4) with (4.5), we obtain the desired estimates in (4.1) with

$$C_1 = \min\{(\alpha a_1)^{1/\alpha}, a_3\} \text{ and } C_2 = \max\{(\alpha a_2)^{1/\alpha}, a_4\}.$$

Case 2. Consider A is not symmetric. Let $A = S^{-1}A_0S$ as in (2.1). Same as (3.10), we set $z(t) = Sy(t)$ for $t \in [t_0, T_*]$. Then z belongs to $C^1([t_0, T_*), \mathbb{R}^n \setminus \{0\}) \cap C([t_0, T_*], \mathbb{R}^n)$, $z(t) \neq 0$ for all $t \in [t_0, T_*)$, $z(T_*) = 0$, and

$$z' = -\widetilde{H}(z)A_0z + \widetilde{f}(t) \text{ for } t \in (t_0, T_*), \quad (4.6)$$

where

$$\widetilde{H}(z) = H(S^{-1}z) \text{ for } z \in \mathbb{R}^n \setminus \{0\}, \text{ and } \widetilde{f}(t) = Sf(t) \text{ for } t \in [t_0, T_*). \quad (4.7)$$

Thanks to Assumption 2.3, we can verify that $\widetilde{H}$ belongs to $\mathcal{H}_{-\alpha}(\mathbb{R}^n, \mathbb{R})$ and is positive and continuous on $\mathbb{R}^n \setminus \{0\}$. Moreover, it is clear that the function $\widetilde{f}$ is continuous on $[t_0, T_*)$. Thanks to (2.12) and (3.12), it satisfies, for $t \in [t_0, T_*)$,

$$|\widetilde{f}(t)| \le \|S\| \cdot |f(t)| \le \|S\|M|y(t)|^{1-\alpha+\delta} \le \widetilde{M}|z(t)|^{1-\alpha+\delta}, \quad (4.8)$$

where

$$\widetilde{M} = \begin{cases} M\|S\| \cdot \|S^{-1}\|^{1-\alpha+\delta}, & \text{if } 1 - \alpha + \delta \ge 0, \\ M\|S\| \cdot \|S\|^{-(1-\alpha+\delta)} = M\|S\|^{\alpha-\delta}, & \text{otherwise.} \end{cases}$$

Therefore, we can apply the result in Case 1 to the solution $z(t)$ and equation (4.6). Then there exist two positive constants C_1' and C_2' such that

$$C_1'(T_* - t)^{1/\alpha} \le |z(t)| \le C_2'(T_* - t)^{1/\alpha} \quad \text{for all } t \in [t_0, T_*]. \quad (4.9)$$

Combining (4.9) with the relations in (3.12), we obtain the estimates in (4.1) for $y(t)$. □

The following are two immediate consequences of Lemma 4.1.

(a) By (2.6) and (4.1), we have, for all $t \in [t_0, T_*)$,

$$C_3(T_* - t)^{-1} \le H(y(t)) \le C_4(T_* - t)^{-1}, \quad (4.10)$$

where $C_3 = c_1C_2^{-\alpha}$ and $C_4 = c_2C_1^{-\alpha}$.

(b) We also observe from (2.12) and (4.1) that, for all $t \in [t_0, T_*)$,

$$|f(t)| \le M(T_* - t)^{1/\alpha-1+\delta/\alpha} \cdot \begin{cases} C_2^{1-\alpha+\delta}, & \text{if } 1 - \alpha + \delta \ge 0, \\ C_1^{1-\alpha+\delta}, & \text{otherwise.} \end{cases} \quad (4.11)$$

5. Proof for a special case

Let a be an arbitrarily positive number. Consider equation (2.11) in the case

$$A = I_n \text{ and } H(x) = a|x|^{-\alpha}, \quad (5.1)$$

that is, equation (2.11) becomes

$$y' = -a|y|^{-\alpha}y + f(t) \text{ for } t \in (t_0, T_*). \quad (5.2)$$

Theorem 2.9 for this particular case is simply the following.

Theorem 5.1. *Given numbers $T_* > t_0 \geq 0$. Let $y \in C^1([t_0,T_*),\mathbb{R}^n)$ and $f \in C([t_0,T_*),\mathbb{R}^n)$ satisfy (2.8), (2.9), (2.12) and (5.2). Then there exists a vector $\xi_* \in \mathbb{R}^n$ such that*

$$|\xi_*| = (\alpha a)^{1/\alpha}, \tag{5.3}$$

and, as $t \to T_*^-$,

$$|y(t) - (T_* - t)^{1/\alpha}\xi_*| = \mathcal{O}((T_* - t)^{1/\alpha+\varepsilon}) \quad \text{for some } \varepsilon > 0. \tag{5.4}$$

Proof. Regarding the function f that satisfies (2.12), we observe, for any number $\delta' \in (0,\delta)$, that

$$|f(t)| \leq M'|y(t)|^{1-\alpha+\delta'} \text{ for all } t \in [t_0,T_*), \tag{5.5}$$

where

$$M' = M \max_{t\in[t_0,T_*]} |y(t)|^{\delta-\delta'} \in (0,\infty). \tag{5.6}$$

Note that we used (2.17) in (5.6). Because of property (5.5), we can assume that $\delta < \alpha$ in (2.12).

With the matrix A and function H in (5.1), they certainly satisfy Assumptions 2.1 and 2.3. Then Lemma 4.1 applies and the estimates from above and below for $|y(t)|$ in (4.1), and estimate (4.11) for $|f(t)|$ hold.

For $t \in (t_0,T_*)$, from (4.2) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(|y|^\alpha) = -\alpha a + \alpha|y|^{\alpha-2}f(t)\cdot y. \tag{5.7}$$

Integrating equation (5.7) from t to $t' \in (t,T_*)$, passing to the limit $t' \to T_*^-$, and using (2.9) give

$$|y(t)|^\alpha = \alpha a(T_* - t) + g(t), \text{ where } g(t) = -\alpha\int_t^{T_*} |y(\tau)|^{\alpha-2}f(\tau)\cdot y(\tau)\mathrm{d}\tau. \tag{5.8}$$

Hence, for all $t \in [t_0,T_*)$, one has $\alpha a(T_* - t) + g(t) > 0$.

Using the Cauchy-Schwarz inequality, (2.12) and the upper bound of $|y(t)|$ in (4.1), we estimate

$$\begin{aligned}|g(t)| &\leq \alpha\int_t^{T_*} |y(\tau)|^{\alpha-1}|f(\tau)|\mathrm{d}\tau \leq \alpha M\int_t^{T_*} |y(\tau)|^\delta \mathrm{d}\tau\\ &\leq \alpha M C_2^\delta \int_t^{T_*} (T_* - \tau)^{\delta/\alpha}\mathrm{d}\tau.\end{aligned}$$

We obtain

$$|g(t)| \leq C_3(T_* - t)^{1+\delta/\alpha} \text{ for all } t \in [t_0,T_*), \text{ where } C_3 = \frac{\alpha M C_2^\delta}{1+\delta/\alpha}. \tag{5.9}$$

We consider equation (5.2) as a linear equation of y with time-dependent coefficient $-a|y(t)|^{-\alpha}$ and forcing function $f(t)$. By the variation of constants formula, we solve for $y(t)$ explicitly as

$$y(t) = e^{-J(t)}\Big(y_0 + \int_{t_0}^t e^{J(\tau)}f(\tau)\mathrm{d}\tau\Big) \quad \text{for } t \in [t_0,T_*),$$

where

$$J(t) = a\int_{t_0}^t |y(\tau)|^{-\alpha}\mathrm{d}\tau. \tag{5.10}$$

Using (5.8) in (5.10), we rewrite $J(t)$ as

$$J(t) = \int_{t_0}^{t} \frac{a}{a\alpha(T_* - \tau) + g(\tau)} \mathrm{d}\tau = J_1(t) + J_2(t),$$

where

$$J_1(t) = \int_{t_0}^{t} \frac{1}{\alpha(T_* - \tau)} \mathrm{d}\tau \text{ and } J_2(t) = \int_{t_0}^{t} h(\tau)\mathrm{d}\tau,$$

with

$$h(\tau) = \frac{-g(\tau)}{\alpha(T_* - \tau)(a\alpha(T_* - \tau) + g(\tau))}.$$

Clearly,

$$J_1(t) = -\frac{1}{\alpha}\ln(T_* - t) + \frac{1}{\alpha}\ln(T_* - t_0).$$

Therefore,

$$y(t) = \frac{(T_* - t)^{1/\alpha}}{(T_* - t_0)^{1/\alpha}} e^{-J_2(t)} \Big(y_0 + (T_* - t_0)^{1/\alpha} \int_{t_0}^{t} \frac{e^{J_2(\tau)}}{(T_* - \tau)^{1/\alpha}} f(\tau)\mathrm{d}\tau\Big). \tag{5.11}$$

Consider the integrand $h(\tau)$ of $J_2(t)$. Taking into account the estimate of $|g(\tau)|$ in (5.9), we assert that, as $\tau \to T_*^-$,

$$|h(\tau)| = \mathcal{O}(|g(\tau)|(T_* - \tau)^{-2}) = \mathcal{O}((T_* - \tau)^{-1+\delta/\alpha}). \tag{5.12}$$

Thus,

$$\lim_{t \to T_*^-} J_2(t) = \int_{t_0}^{T_*} h(\tau)\mathrm{d}\tau = J_* \in \mathbb{R}, \tag{5.13}$$

$$J_2(t) = J_* - h_1(t), \text{ where } h_1(t) = \int_{t}^{T_*} h(\tau)\mathrm{d}\tau \in \mathbb{R}. \tag{5.14}$$

From estimate (5.12), it follows that

$$|h_1(t)| = \mathcal{O}((T_* - t)^{\delta/\alpha}) \text{ as } t \to T_*^-. \tag{5.15}$$

Regarding the integral in formula (5.11), we have, thanks to estimate (4.11) of $|f(t)|$, that

$$\frac{|f(t)|}{(T_* - t)^{1/\alpha}} = \mathcal{O}((T_* - t)^{-1+\delta/\alpha}) \text{ as } t \to T_*^-. \tag{5.16}$$

Hence,

$$\lim_{t \to T_*^-} \int_{t_0}^{t} \frac{e^{J_2(\tau)}}{(T_* - \tau)^{1/\alpha}} f(\tau)\mathrm{d}\tau = \int_{t_0}^{T_*} \frac{e^{J_2(\tau)}}{(T_* - \tau)^{1/\alpha}} f(\tau)\mathrm{d}\tau = \eta_* \in \mathbb{R}^n,$$

and

$$\int_{t_0}^{t} \frac{e^{J_2(\tau)}}{(T_* - \tau)^{1/\alpha}} f(\tau)\mathrm{d}\tau = \eta_* - \eta(t), \tag{5.17}$$

where

$$\eta(t) = \int_{t}^{T_*} \frac{e^{J_2(\tau)}}{(T_* - \tau)^{1/\alpha}} f(\tau)\mathrm{d}\tau \in \mathbb{R}^n.$$

It follows from (5.13) and (5.16) that

$$|\eta(t)| = \mathcal{O}((T_* - t)^{\delta/\alpha}) \text{ as } t \to T_*^-. \tag{5.18}$$

Combining (5.11), (5.14) and (5.17) gives

$$y(t) = \frac{(T_* - t)^{1/\alpha}}{(T_* - t_0)^{1/\alpha}} e^{-J_* + h_1(t)} \Big(y_0 + (T_* - t_0)^{1/\alpha}(\eta_* - \eta(t)) \Big) \text{ for } t \in [t_0, T_*).$$

Then

$$\begin{aligned} & y(t) - (T_* - t)^{1/\alpha} e^{-J_*} \Big(\frac{y_0}{(T_* - t_0)^{1/\alpha}} + \eta_* \Big) \\ &= (T_* - t)^{1/\alpha} e^{-J_*} (e^{h_1(t)} - 1) \Big(\frac{y_0}{(T_* - t_0)^{1/\alpha}} + \eta_* \Big) - (T_* - t)^{1/\alpha} e^{-J_* + h_1(t)} \eta(t). \end{aligned}$$

Let $\xi_* = e^{-J_*}((T_* - t_0)^{-1/\alpha} y_0 + \eta_*) \in \mathbb{R}^n$. This expression and properties (5.15), (5.18) imply, as $t \to T_*^-$,

$$\begin{aligned} |y(t) - (T_* - t)^{1/\alpha} \xi_*| &= \mathcal{O}\Big((T_* - t)^{1/\alpha} (|e^{h_1(t)} - 1| + |\eta(t)|) \Big) \\ &= \mathcal{O}\Big((T_* - t)^{1/\alpha} (|h_1(t)| + |\eta(t)|) \Big), \end{aligned}$$

thus,

$$|y(t) - (T_* - t)^{1/\alpha} \xi_*| = \mathcal{O}((T_* - t)^{1/\alpha + \delta/\alpha}). \tag{5.19}$$

Therefore, we obtain the desired estimate (5.4). Because of the lower bound of $|y(t)|$ in (4.1), the vector ξ_* in (5.4) must be nonzeo.

Now we prove property (5.3). By the triangle inequality and (5.19), one has

$$\big| (T_* - t)^{-1/\alpha} |y(t)| - |\xi_*| \big| = \mathcal{O}((T_* - t)^{\delta/\alpha}). \tag{5.20}$$

From (5.8),

$$(T_* - t)^{-1/\alpha} |y(t)| = \Big(a\alpha + \frac{g(t)}{T_* - t} \Big)^{1/\alpha}. \tag{5.21}$$

Taking into account estimate (5.9) of $|g(t)|$, from (5.21) we have that as $t \to T_*^-$,

$$\big| (T_* - t)^{-1/\alpha} |y(t)| - (a\alpha)^{1/\alpha} \big| = \mathcal{O}\Big(\frac{|g(t)|}{T_* - t} \Big) = \mathcal{O}((T_* - t)^{\delta/\alpha}). \tag{5.22}$$

From the two asymptotic estimates (5.20) and (5.22), one must have $|\xi_*| = (a\alpha)^{1/\alpha}$, which proves (5.3). The proof is complete. □

Remark 5.2. In the case dimension $n = 1$, Theorem 5.1 already proves Theorem 2.9 for any positive constant A and positive function $H \in \mathcal{H}_{-\alpha}(\mathbb{R}, \mathbb{R})$. We justify this fact below.

Let $y(t)$ be the solution of (2.11) as in Theorem 2.9. With $H \in \mathcal{H}_{-\alpha}(\mathbb{R}, \mathbb{R})$, we have

$$H(x) = \begin{cases} |x|^{-\alpha} H(1), & \text{for } x > 0, \\ |x|^{-\alpha} H(-1), & \text{for } x < 0. \end{cases}$$

In general, $H(1) \neq H(-1)$, hence it appears that we do not have equation (5.2) yet. However, for our continuous solution $y(t) \neq 0$ on $[t_0, T_*)$, we must have either $y(t) > 0$ on $[t_0, T_*)$ or $y(t) < 0$ on $[t_0, T_*)$. Therefore, $y(t)$, in fact, satisfies (5.2) for all $t \in (t_0, T_*)$, with $a = AH(1)$ or $a = AH(-1)$. Then Theorem 5.1 applies. (As a side note, because $\mathbb{S}^0 = \{-1, 1\}$, property (HC) on $\mathbb{S}^0$ is automatically satisfied.)

6. Solutions when the matrix A is symmetric

In this section, we assume Assumptions 2.1 and 2.3 hold and, additionally, the matrix A is symmetric. Let numbers $T_* > t_0 \geq 0$ be given, and let function $y \in C^1([t_0, T_*), \mathbb{R}^n)$ satisfy (2.8), (2.9), (2.11)–(2.12). For $t \in [t_0, T_*)$, define

$$\lambda(t) = \frac{y(t) \cdot Ay(t)}{|y(t)|^2} \text{ and } v(t) = \frac{y(t)}{|y(t)|}. \tag{6.1}$$

(The quotient $\lambda(t)$ in (6.1) imitates the Dirichlet quotient for the heat equations when A is the negative Laplacian.) Then $\lambda \in C^1([t_0, T_*), \mathbb{R})$ and $v \in C^1([t_0, T_*), \mathbb{R}^n)$. Moreover, one has, $|v(t)| = 1$ and, thanks to (2.2),

$$\Lambda_1 \leq \lambda(t) \leq \Lambda_n \leq \|A\| \text{ for all } t \in [t_0, T_*). \tag{6.2}$$

Proposition 6.1. *One has*

$$\lim_{t \to T_*^-} \lambda(t) = \Lambda \in \sigma(A).$$

Proof. For $t \in (t_0, T_*)$, we have

$$\lambda'(t) = \frac{2}{|y|^2} y' \cdot Ay - \frac{2(y \cdot Ay)}{|y|^4} y' \cdot y = \frac{2}{|y|^2} y' \cdot (Ay - \lambda(t)y). \tag{6.3}$$

By equation (2.11), we write y' as

$$y' = -H(y)(Ay - \lambda(t)y) - \lambda(t)H(y)y + f(t),$$

and use it in (6.3) to obtain

$$\lambda'(t) = -\frac{2H(y)}{|y|^2}|Ay - \lambda(t)y|^2 - \frac{2\lambda(t)H(y)}{|y|^2} y \cdot (Ay - \lambda(t)y) + h(t),$$

where

$$h(t) = \frac{2}{|y(t)|^2} f(t) \cdot (Ay(t) - \lambda(t)y(t)).$$

Because $y(t) \cdot (Ay(t) - \lambda(t)y(t)) = 0$, it follows that

$$\lambda'(t) = -2H(y)|Av - \lambda(t)v|^2 + h(t). \tag{6.4}$$

Using (2.12), (6.2), the fact $|v(t)| = 1$, and then (4.1), we estimate

$$|h(t)| \leq 4M\|A\| \cdot |y(t)|^{-\alpha+\delta} \leq C_5(T_* - t)^{-1+\delta/\alpha} \text{ for all } t \in [t_0, T_*), \tag{6.5}$$

where C_5 is $4M\|A\|C_2^{-\alpha+\delta}$ if $\delta \geq \alpha$, or $4M\|A\|C_1^{-\alpha+\delta}$ otherwise.

For $t, t' \in [t_0, T_*)$ with $t' > t$, integrating equation (6.4) from t to t' gives

$$\lambda(t') - \lambda(t) + 2\int_t^{t'} H(y(\tau))|Av(\tau) - \lambda(\tau)v(\tau)|^2 \mathrm{d}\tau = \int_t^{t'} h(\tau)\mathrm{d}\tau. \tag{6.6}$$

Thanks to (6.5), the last integral can be estimated as

$$\Big|\int_t^{t'} h(\tau)\mathrm{d}\tau\Big| \leq \frac{\alpha C_5}{\delta}(T_* - t)^{\delta/\alpha}. \tag{6.7}$$

By taking the limit superior of (6.6), as $t' \to T_*^-$, we derive

$$\limsup_{t' \to T_*^-} \lambda(t') \leq \lambda(t) + \frac{\alpha C_5}{\delta}(T_* - t)^{\delta/\alpha} < \infty. \tag{6.8}$$

Then taking the limit inferior of (6.8), as $t \to T_*^-$, yields

$$\limsup_{t' \to T_*^-} \lambda(t') \le \liminf_{t \to T_*^-} \lambda(t).$$

This and (6.2) imply

$$\lim_{t \to T_*^-} \lambda(t) = \Lambda \in [\Lambda_1, \Lambda_n]. \tag{6.9}$$

It remains to prove that Λ is an eigenvalue of A. Using properties (6.7) and (6.9) in (6.6) and by the Cauchy criterion, as $t, t' \to T_*^-$, we obtain

$$\int_{t_0}^{T_*} H(y(\tau))|Av(\tau) - \lambda(\tau)v(\tau)|^2 \mathrm{d}\tau < \infty. \tag{6.10}$$

We claim that

$$\forall \varepsilon \in (0, T_* - t_0), \exists t \in [T_* - \varepsilon, T_*) : |Av(t) - \lambda(t)v(t)| < \varepsilon. \tag{6.11}$$

Indeed, suppose the claim (6.11) is not true, then

$$\exists \varepsilon_0 \in (0, T_* - t_0), \forall t \in [T_* - \varepsilon_0, T_*) : |Av(t) - \lambda(t)v(t)| \ge \varepsilon_0. \tag{6.12}$$

Combining (6.12) with property (4.10), we have

$$\int_{T_*-\varepsilon_0}^{T_*} H(y(\tau))|Av(\tau) - \lambda(\tau)v(\tau)|^2 \mathrm{d}\tau \ge \int_{T_*-\varepsilon_0}^{T_*} C_3(T_* - \tau)^{-1}\varepsilon_0^2 \mathrm{d}\tau = \infty,$$

which contradicts (6.10). Hence, the claim (6.11) is true.

Thanks to (6.11), there exists a sequence $(t_j)_{j=1}^\infty \subset [t_0, T_*)$ such that

$$\lim_{j \to \infty} t_j = T_* \text{ and } \lim_{j \to \infty} |Av(t_j) - \lambda(t_j)v(t_j)| = 0. \tag{6.13}$$

The first equation in (6.13) and (6.9) imply $\lambda(t_j) \to \Lambda$ as $j \to \infty$. Because $v(t_j) \in \mathbb{S}^{n-1}$ for all j, we can extract a subsequence $(v(t_{j_k}))_{k=1}^\infty$, such that $v(t_{j_k}) \to \bar{v} \in \mathbb{S}^{n-1}$ as $k \to \infty$. Combining these limits with the second equation in (6.13) written with $j = j_k$ and $k \to \infty$ yields $A\bar{v} = \Lambda\bar{v}$. Therefore, Λ is an eigenvalue of A. □

From here to the end of this section, Λ is the eigenvalue in Proposition 6.1.

Proposition 6.2. *There is $\varepsilon > 0$ such that*

$$|(I_n - R_\Lambda)v(t)| = \mathcal{O}((T_* - t)^\varepsilon) \text{ as } t \to T_*^-. \tag{6.14}$$

Proof. If $\sigma(A) = \{\Lambda\}$, then $R_\Lambda = \mathrm{Id}$ and (6.14) is true. Consider the case $\sigma(A) \ne \{\Lambda\}$. We calculate

$$v' = \frac{1}{|y|}y' - \frac{1}{|y|^3}(y' \cdot y)y = -\frac{H(y)}{|y|}Ay + \frac{1}{|y|}f(t) + \frac{H(y)(Ay)\cdot y}{|y|^3}y - \frac{f(t)\cdot y}{|y|^3}y.$$

We define the function $g : [t_0, T_*) \to \mathbb{R}^n$ by

$$g(t) = \frac{1}{|y(t)|}f(t) - \frac{f(t)\cdot y(t)}{|y(t)|^3}y(t).$$

Then we have

$$v' = -H(y)(Av - \lambda(t)v) + g(t) \text{ for all } t \in (t_0, T_*). \tag{6.15}$$

Using property (2.12) of $f(t)$, one can estimate

$$|g(t)| \le 2M|y(t)|^{-\alpha+\delta} \text{ for all } t \in [t_0, T_*). \tag{6.16}$$

Let $\lambda_j \in \sigma(A)\backslash\{\Lambda\}$. Applying R_{λ_j} to equation (6.15) and taking the dot product with $R_{\lambda_j}v$ yield

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|R_{\lambda_j}v|^2 = -H(y)(\lambda_j - \lambda(t))|R_{\lambda_j}v|^2 + R_{\lambda_j}g(t)\cdot R_{\lambda_j}v. \tag{6.17}$$

Set

$$\mu = \min\{|\lambda_j - \Lambda| : 1 \le j \le d, \lambda_j \ne \Lambda\} > 0. \tag{6.18}$$

Applying Cauchy–Schwarz's inequality, inequality (2.4) to $|R_{\lambda_j}g(t)|$, estimate (6.16) for $|g(t)|$, and then Cauchy's inequality, we have

$$|R_{\lambda_j}g(t)\cdot R_{\lambda_j}v| \le 2M|y|^{-\alpha+\delta}|R_{\lambda_j}v| \le \frac{\mu}{4}H(y)|R_{\lambda_j}v|^2 + \frac{4M^2|y|^{-2\alpha+2\delta}}{\mu H(y)}.$$

Using the first inequality of (2.6) to estimate the last $H(y)$ gives

$$|R_{\lambda_j}g(t)\cdot R_{\lambda_j}v| \le \frac{\mu}{4}H(y)|R_{\lambda_j}v|^2 + \frac{4M^2}{\mu c_1}|y|^{-\alpha+2\delta}.$$

Utilizing the estimates in (4.1) for the norm $|y(t)|$, we obtain, for $t \in [t_0, T_*)$,

$$|R_{\lambda_j}g(t)\cdot R_{\lambda_j}v| \le \frac{\mu}{4}H(y)|R_{\lambda_j}v|^2 + \frac{C_6}{2}(T_*-t)^{-1+2\delta/\alpha}, \tag{6.19}$$

where

$$C_6 = \frac{8M^2}{\mu c_1}\cdot \begin{cases} C_2^{-\alpha+2\delta}, & \text{if } \delta \ge \alpha/2,\\ C_1^{-\alpha+2\delta}, & \text{otherwise.}\end{cases}$$

Below, $T \in (t_0, T_*)$ is fixed and can be taken sufficiently close to T_* such that

$$|\lambda(t) - \Lambda| \le \frac{\mu}{4} \text{ for all } t \in [T, T_*). \tag{6.20}$$

Case $\lambda_j > \Lambda$. In this case, combining (6.17) and (6.19) yields, for $t \in (t_0, T_*)$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|R_{\lambda_j}v|^2 \le -(\lambda_j - \lambda(t) - \frac{\mu}{4})H(y)|R_{\lambda_j}v|^2 + \frac{C_6}{2}(T_*-t)^{-1+2\delta/\alpha}.$$

By definition (6.18) of μ and the choice (6.20), one has, for all $t \in [T, T_*)$,

$$\lambda_j - \lambda(t) - \frac{\mu}{4} = (\lambda_j - \Lambda) + (\Lambda - \lambda(t)) - \frac{\mu}{4} \ge \mu - \frac{\mu}{4} - \frac{\mu}{4} = \frac{\mu}{2}. \tag{6.21}$$

Thus, for $t \in [T, T_*)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}|R_{\lambda_j}v|^2 \le -\mu H(y)|R_{\lambda_j}v|^2 + C_6(T_*-t)^{-1+2\delta/\alpha}. \tag{6.22}$$

Let t and $\bar t$ be any numbers in $[T, T_*)$ with $t > \bar t$. It follows from (6.22) that

$$\begin{aligned} &|R_{\lambda_j}v(t)|^2\\ &\le e^{-\mu\int_{\bar t}^t H(y(\tau))\mathrm{d}\tau}|R_{\lambda_j}v(\bar t)|^2 + C_6\int_{\bar t}^t e^{-\mu\int_\tau^t H(y(s))\mathrm{d}s}(T_*-\tau)^{-1+2\delta/\alpha}\mathrm{d}\tau. \end{aligned} \tag{6.23}$$

With C_3 being the positive constant in (4.10), we fix a number $\theta > 0$ such that

$$\theta \le C_3 \text{ and } \theta\mu < 2\delta/\alpha.$$

Then

$$H(y(t)) \ge \theta(T_*-t)^{-1} \text{ for all } t \in [t_0, T_*). \tag{6.24}$$

Utilizing this estimate in (6.23) gives

$$|R_{\lambda_j}v(t)|^2$$

$$\begin{aligned}
&\le e^{-\theta\mu\int_{\bar t}^t (T_*-\tau)^{-1}\mathrm{d}\tau}|R_{\lambda_j}v(\bar t)|^2 + C_6\int_{\bar t}^t e^{-\theta\mu\int_\tau^t (T_*-s)^{-1}\mathrm{d}s}(T_*-\tau)^{-1+2\delta/\alpha}\mathrm{d}\tau\\
&= \frac{(T_*-t)^{\theta\mu}}{(T_*-\bar t)^{\theta\mu}}|R_{\lambda_j}v(\bar t)|^2 + C_6(T_*-t)^{\theta\mu}\int_{\bar t}^t (T_*-\tau)^{-1+2\delta/\alpha-\theta\mu}\mathrm{d}\tau\\
&= \frac{(T_*-t)^{\theta\mu}}{(T_*-\bar t)^{\theta\mu}}|R_{\lambda_j}v(\bar t)|^2 + \frac{C_6(T_*-t)^{\theta\mu}}{2\delta/\alpha-\theta\mu}\Big((T_*-\bar t)^{2\delta/\alpha-\theta\mu}-(T_*-t)^{2\delta/\alpha-\theta\mu}\Big).
\end{aligned}$$

Therefore,

$$|R_{\lambda_j}v(t)|^2 \le \Big(\frac{|R_{\lambda_j}v(\bar t)|^2}{(T_*-\bar t)^{\theta\mu}} + \frac{C_6(T_*-\bar t)^{2\delta/\alpha-\theta\mu}}{2\delta/\alpha-\theta\mu}\Big)(T_*-t)^{\theta\mu}. \tag{6.25}$$

Having $\bar t = T$ in (6.25), we obtain

$$|R_{\lambda_j}v(t)| = \mathcal{O}((T_*-t)^{\theta\mu/2}) \text{ as } t\to T_*^-. \tag{6.26}$$

Case $\lambda_j < \Lambda$. Using (6.19) to have a lower bound for the last term in (6.17), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|R_{\lambda_j}v|^2 \ge (\lambda(t)-\lambda_j-\frac{\mu}{4})H(y)|R_{\lambda_j}v|^2 - \frac{C_6}{2}(T_*-t)^{-1+2\delta/\alpha}.$$

Same as (6.21), one has, for $t\in[T,T_*)$,

$$\lambda(t)-\lambda_j-\frac{\mu}{4} = (\lambda(t)-\Lambda)+(\Lambda-\lambda_j)-\frac{\mu}{4} \ge -\frac{\mu}{4}+\mu-\frac{\mu}{4} = \frac{\mu}{2}.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}|R_{\lambda_j}v|^2 \ge \mu H(y)|R_{\lambda_j}v|^2 - C_6(T_*-t)^{-1+2\delta/\alpha}.$$

Then, for any $t,\bar t\in[T,T_*)$ with $t>\bar t$, one has

$$\begin{aligned}
&e^{-\mu\int_{\bar t}^t H(y(\tau))\mathrm{d}\tau}|R_{\lambda_j}v(t)|^2 - |R_{\lambda_j}v(\bar t)|^2\\
&\ge -C_6\int_{\bar t}^t e^{-\mu\int_{\bar t}^\tau H(y(s))\mathrm{d}s}(T_*-\tau)^{-1+2\delta/\alpha}\mathrm{d}\tau.
\end{aligned} \tag{6.27}$$

Note from (6.24) that $\int_{\bar t}^{T_*} H(y(\tau))\mathrm{d}\tau = \infty$, and from (2.4) that $|R_{\lambda_j}v(t)| \le |v(t)| = 1$. Then

$$\lim_{t\to T_*^-} e^{-\mu\int_{\bar t}^t H(y(\tau))\mathrm{d}\tau}|R_{\lambda_j}v(t)|^2 = 0.$$

Letting $t\to T_*^-$ in (6.27) and using (6.24) yield

$$\begin{aligned}
|R_{\lambda_j}v(\bar t)|^2 &\le C_6\int_{\bar t}^{T_*} e^{-\mu\int_{\bar t}^\tau H(y(s))\mathrm{d}s}(T_*-\tau)^{-1+2\delta/\alpha}\mathrm{d}\tau\\
&\le C_6\int_{\bar t}^{T_*}\frac{(T_*-\tau)^{\theta\mu}}{(T_*-\bar t)^{\theta\mu}}(T_*-\tau)^{-1+2\delta/\alpha}\mathrm{d}\tau = \frac{C_6}{\theta\mu+2\delta/\alpha}(T_*-\bar t)^{2\delta/\alpha}.
\end{aligned}$$

Therefore,

$$|R_{\lambda_j}v(\bar t)| = \mathcal{O}((T_*-\bar t)^{\delta/\alpha}) \quad \text{as } \bar t\to T_*^-. \tag{6.28}$$

We estimate $|(I_n-R_\Lambda)v(t)|$ now. We have

$$|(I_n-R_\Lambda)v(t)| = \Big|\sum_{1\le j\le d,\lambda_j\ne\Lambda} R_{\lambda_j}v(t)\Big| \le \sum_{1\le j\le d,\lambda_j\ne\Lambda}|R_{\lambda_j}v(t)|. \tag{6.29}$$

In the last sum in (6.29), we estimate $|R_{\lambda_j} v(t)|$ for all $\lambda_j > \Lambda$ by (6.26), and estimate $|R_{\lambda_j} v(t)|$ for all $\lambda_j < \Lambda$ by (6.28). This results in the desired estimate (6.14) for $|(I_n - R_\Lambda)v|$, with $\varepsilon = \min\{\theta\mu/2, \delta/\alpha\} = \theta\mu/2$. □

We derive from Proposition 6.2 more specific estimates for $y(t)$. Let $\varepsilon > 0$ be as in Proposition 6.2. On the one hand, we have

$$|(I_n - R_\Lambda)y(t)| = |y(t)| \cdot |(I_n - R_\Lambda)v(t)|.$$

Together with (4.1) and (6.14), it yields

$$|(I_n - R_\Lambda)y(t)| = \mathcal{O}((T_* - t)^{1/\alpha+\varepsilon}) \text{ as } t \to T_*^-. \tag{6.30}$$

On the other hand, by the triangle inequality and (4.1), one has

$$\begin{aligned}
|R_\Lambda y(t)| &\le |y(t)| + |(I_n - R_\Lambda)y(t)| \le C_2(T_* - t)^{1/\alpha} + |(I_n - R_\Lambda)y(t)|,\\
|R_\Lambda y(t)| &\ge |y(t)| - |(I_n - R_\Lambda)y(t)| \ge C_1(T_* - t)^{1/\alpha} - |(I_n - R_\Lambda)y(t)|.
\end{aligned}$$

Combining these inequalities with estimate (6.30) for $|(I_n - R_\Lambda)y(t)|$, we deduce that there exist numbers $T_0 \in [t_0, T_*)$ and $C_7, C_8 > 0$ such that

$$C_7(T_* - t)^{1/\alpha} \le |R_\Lambda y(t)| \le C_8(T_* - t)^{1/\alpha} \quad \text{for all } t \in [T_0, T_*). \tag{6.31}$$

Proposition 6.3. *There exists a unit vector $v_* \in \mathbb{R}^n$ such that*

$$|R_\Lambda v(t) - v_*| = \mathcal{O}((T_* - t)^{\varepsilon}) \text{ as } t \to T_*^- \text{ for some } \varepsilon > 0. \tag{6.32}$$

Proof. Let $\varepsilon_0 > 0$ be such that (6.14) holds for $\varepsilon = \varepsilon_0$. Then one has

$$\big|1 - |R_\Lambda v(t)|\big| = \big||v(t)| - |R_\Lambda v(t)|\big| \le |v(t) - R_\Lambda v(t)| = \mathcal{O}((T_* - t)^{\varepsilon_0}). \tag{6.33}$$

Let T_0 be as in (6.31). Recall that C_4 is the positive constant in (4.10). We fix a number $\varepsilon_1 > 0$ such that

$$C_4\varepsilon_1 < \delta/\alpha. \tag{6.34}$$

Thanks to Proposition 6.1, there is $T \in [T_0, T_*)$ such that

$$|\lambda(t) - \Lambda| \le \varepsilon_1 \text{ for all } t \in [T, T_*).$$

Note from (6.31) that $R_\Lambda v(t) \ne 0$ for all $t \in [T, T_*)$. Applying R_Λ to equation (6.15) yields, for $t \in (t_0, T_*)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} R_\Lambda v = -H(y)(\Lambda - \lambda(t))R_\Lambda v + R_\Lambda g(t). \tag{6.35}$$

Then, for $t \in [T, T_*)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}|R_\Lambda v| = \frac{1}{|R_\Lambda v|}\Big(\frac{\mathrm{d}}{\mathrm{d}t}R_\Lambda v\Big) \cdot R_\Lambda v = -H(y)(\Lambda - \lambda(t))|R_\Lambda v| + g_1(t), \tag{6.36}$$

where

$$g_1(t) = \frac{R_\Lambda g(t) \cdot R_\Lambda v(t)}{|R_\Lambda v(t)|}.$$

Solving for solution $|R_\Lambda v(t)|$ by the variation of constants formula from the differential equation (6.36) gives, for $\bar{t}, t \in [T, T_*)$ with $t > \bar{t}$,

$$|R_\Lambda v(t)| = e^{-\int_{\bar t}^t H(y(\tau))(\Lambda - \lambda(\tau))\mathrm{d}\tau}\Big(|R_\Lambda v(\bar t)| + \int_{\bar t}^t e^{\int_{\bar t}^\tau H(y(s))(\Lambda - \lambda(s))\mathrm{d}s} g_1(\tau)\mathrm{d}\tau\Big).$$

It yields

$$\begin{aligned}
&\int_{\bar t}^t H(y(\tau))(\Lambda-\lambda(\tau))\mathrm{d}\tau\\
&= \ln\Big(|R_\Lambda v(\bar t)| + \int_{\bar t}^t e^{\int_{\bar t}^\tau H(y(s))(\Lambda-\lambda(s))\mathrm{d}s} g_1(\tau)\mathrm{d}\tau\Big) - \ln|R_\Lambda v(t)|.
\end{aligned} \tag{6.37}$$

We have from (2.4), (4.1) and (6.16) that

$$|g_1(t)|\le |R_\Lambda g(t)|\le |g(t)|\le C_9(T_*-t)^{-1+\delta/\alpha} \text{ for all } t\in[T,T_*), \tag{6.38}$$

where C_9 is $2MC_2^{-\alpha+\delta}$ if $\delta\ge\alpha$, and is $2MC_1^{-\alpha+\delta}$ otherwise. By (6.38) and (4.10), we have, for $\tau\in[\bar t,T_*)$,

$$\begin{aligned}
e^{\int_{\bar t}^\tau H(y(s))(\Lambda-\lambda(s))\mathrm{d}s}|g_1(\tau)| &\le e^{\int_{\bar t}^\tau C_4\varepsilon_1(T_*-s)^{-1}\mathrm{d}s} C_9(T_*-\tau)^{-1+\delta/\alpha}\\
&= C_9(T_*-\bar t)^{C_4\varepsilon_1}(T_*-\tau)^{-1+\delta/\alpha-C_4\varepsilon_1}.
\end{aligned}$$

Thanks to this and (6.34),

$$\begin{aligned}
\lim_{t\to T_*^-}\int_{\bar t}^t e^{\int_{\bar t}^\tau H(y(s))(\Lambda-\lambda(s))\mathrm{d}s} g_1(\tau)\mathrm{d}\tau &= \int_{\bar t}^{T_*} e^{\int_{\bar t}^\tau H(y(s))(\Lambda-\lambda(s))\mathrm{d}s} g_1(\tau)\mathrm{d}\tau\\
&= \eta(\bar t)\in\mathbb{R}.
\end{aligned} \tag{6.39}$$

Note that

$$|\eta(\bar t)|\le C_9(T_*-\bar t)^{C_4\varepsilon_1}\int_{\bar t}^{T_*}(T_*-\tau)^{-1+\delta/\alpha-C_4\varepsilon_1}\mathrm{d}\tau = \frac{C_9}{\delta/\alpha - C_4\varepsilon_1}(T_*-\bar t)^{\delta/\alpha}. \tag{6.40}$$

Passing to the limit as $t\to T_*^-$ in (6.37), and using that $|R_\Lambda v(t)|\to 1$, thanks to (6.33), we have

$$\int_{\bar t}^{T_*} H(y(\tau))(\Lambda-\lambda(\tau))\mathrm{d}\tau = \ln(|R_\Lambda v(\bar t)| + \eta(\bar t))\in\mathbb{R}. \tag{6.41}$$

By (6.41), we can define, for $t\in[T,T_*)$,

$$h(t) = \int_t^{T_*} H(y(\tau))(\Lambda-\lambda(\tau))\mathrm{d}\tau\in\mathbb{R}.$$

We rewrite (6.41) for $\bar t = t$ as

$$h(t) = \ln(|R_\Lambda v(t)| + \eta(t)) = \ln(1 + (|R_\Lambda v(t)|-1) + \eta(t)).$$

With this expression and properties (6.33) and (6.40), we have, as $t\to T_*^-$,

$$\begin{aligned}
|h(t)| &= \mathcal{O}(\big||R_\Lambda v(t)|-1\big| + |\eta(t)|)\\
&= \mathcal{O}((T_*-t)^{\varepsilon_0} + (T_*-t)^{\delta/\alpha}) = \mathcal{O}((T_*-\bar t)^{\varepsilon_2}),
\end{aligned} \tag{6.42}$$

where $\varepsilon_2 = \min\{\varepsilon_0,\delta/\alpha\}$.

Solving for $R_\Lambda v(t)$ from (6.35) by the variation of constants formula, one has

$$R_\Lambda v(t) = e^{-\int_{\bar t}^t H(y(\tau))(\Lambda-\lambda(\tau))\mathrm{d}\tau}\Big(R_\Lambda v(\bar t) + \int_{\bar t}^t e^{\int_{\bar t}^\tau H(y(s))(\Lambda-\lambda(s))\mathrm{d}s} R_\Lambda g(\tau)\mathrm{d}\tau\Big). \tag{6.43}$$

Using the same arguments as those from (6.38) to (6.40) with $R_\Lambda g(\tau)$ replacing $g_1(\tau)$, we obtain, similar to (6.39) and (6.40), that

$$\lim_{t\to T_*^-}\int_{\bar t}^t e^{\int_{\bar t}^\tau H(y(s))(\Lambda-\lambda(s))\mathrm{d}s} R_\Lambda g(\tau)\mathrm{d}\tau = \int_{\bar t}^{T_*} e^{\int_{\bar t}^\tau H(y(s))(\Lambda-\lambda(s))\mathrm{d}s} R_\Lambda g(\tau)\mathrm{d}\tau$$

$$= X(\bar t) \in \mathbb{R}^n$$

for all $\bar t \in [T, T_*)$, and

$$|X(\bar t)| = \mathcal{O}((T_* - \bar t)^{\varepsilon_2}) \text{ as } \bar t \to T_*^-. \tag{6.44}$$

Taking $t \to T_*^-$ in (6.43) gives

$$\lim_{t\to T_*^-} R_\Lambda v(t) = v_* := e^{-h(\bar t)}(R_\Lambda v(\bar t) + X(\bar t)) \in \mathbb{R}^n.$$

Note that

$$X(t) = \int_t^{T_*} e^{h(t)-h(\tau)} R_\Lambda g(\tau) \mathrm{d}\tau.$$

Using $h(t)$, $X(t)$ and v_*, we rewrite (6.43) as

$$R_\Lambda v(t) = e^{h(t)-h(\bar t)}\Big(R_\Lambda v(\bar t) + X(\bar t) - \int_t^{T_*} e^{h(\bar t)-h(\tau)} R_\Lambda g(\tau)\mathrm{d}\tau\Big) = e^{h(t)} v_* - X(t).$$

Thus,

$$|R_\Lambda v(t) - v_*| \le |e^{h(t)} - 1| \cdot |v_*| + |X(t)|.$$

Using (6.42) and (6.44), we deduce, as $t \to T_*^-$,

$$|R_\Lambda v(t) - v_*| = \mathcal{O}(|h(t))| + |X(t)|) = \mathcal{O}((T_* - t)^{\varepsilon_2}).$$

Therefore, we obtain the desired estimate (6.32). Because of (6.32) and (6.33), we have $|v_*| = 1$. The proof is complete. □

Some immediate consequences of (6.14) and (6.32) are

$$\lim_{t\to T_*^-} R_\Lambda v(t) = \lim_{t\to T_*^-} v(t) = v_*, \tag{6.45}$$

$$|v(t) - v_*| = \mathcal{O}((T_* - t)^{\varepsilon}) \text{ as } t \to T_*^- \text{ for some } \varepsilon > 0. \tag{6.46}$$

7. Proofs of the main theorems

We prove Theorems 2.9 and 2.10 in this section.

Proof of Theorem 2.9. Since H satisfies Assumption 2.8, it follows that, thanks to Lemma 2.7, H is positive, continuous on $\mathbb{R}^n \setminus \{0\}$, and has property (HC) on $\mathbb{R}^n \setminus \{0\}$.

Case 1. Consider the case A is symmetric first. We use the same notation as in Section 6. The estimate (2.14) already comes from (6.30). We prove (2.15) next. Applying R_Λ to equation (2.11), we have

$$(R_\Lambda y)' = -\Lambda H(y) R_\Lambda y + R_\Lambda f(t). \tag{7.1}$$

Let v_* be the unit vector in Proposition 6.3, and $\varepsilon_0 > 0$ be such that (6.14), (6.30), (6.32), and (6.46) hold for $\varepsilon = \varepsilon_0$. We rewrite $H(y)$ on the right-hand side of (7.1) as

$$H(y) = |y|^{-\alpha} H(v) = |R_\Lambda y|^{-\alpha} H(v_*) + g_0(t),$$

where

$$\begin{aligned} g_0(t) &= |y(t)|^{-\alpha}(H(v(t)) - H(v_*)) + (|y(t)|^{-\alpha} - |R_\Lambda y(t)|^{-\alpha}) H(v_*) \\ &= |y(t)|^{-\alpha}\big\{H(v(t)) - H(v_*) + (1 - |R_\Lambda v(t)|^{-\alpha}) H(v_*)\big\}. \end{aligned}$$

Then

$$(R_\Lambda y)' = -\Lambda H(v_*) |R_\Lambda y|^{-\alpha} R_\Lambda y + f_0(t), \tag{7.2}$$

where $f_0(t) = -\Lambda g_0(t) R_\Lambda y(t) + R_\Lambda f(t)$.

We estimate $|g_0(t)|$ first. We combine inequality (2.10) in Definition 2.6 applied to $F = H$, $E = \mathbb{S}^{n-1}$, $x_0 = v_*$ and $x = v(t)$ for t sufficiently close to T_*, with estimate (6.46). Then there exists a number $\gamma > 0$ such that, as $t \to T_*^-$,

$$|H(v(t)) - H(v_*)| = \mathcal{O}(|v(t) - v_*|^\gamma) = \mathcal{O}((T_* - t)^{\gamma\varepsilon_0}). \tag{7.3}$$

It is elementary to see $|s^{-\alpha} - 1| = \mathcal{O}(|s - 1|)$ as $s \to 1$. By taking $s = |R_\Lambda v(t)|$, which goes to 1 as $t \to T_*^-$ thanks to (6.45), and using estimate (6.33), we derive

$$\left|1 - |R_\Lambda v(t)|^{-\alpha}\right| = \mathcal{O}\left(\left|1 - |R_\Lambda v(t)|\right|\right) = \mathcal{O}((T_* - t)^{\varepsilon_0}) \text{ as } t \to T_*^-. \tag{7.4}$$

Combining (7.3), (7.4) with (4.1), we obtain

$$|g_0(t)| = \mathcal{O}(|y(t)|^{-\alpha}((T_* - t)^{\gamma\varepsilon_0} + (T_* - t)^{\varepsilon_0})) = \mathcal{O}((T_* - t)^{-1+\varepsilon_1}) \tag{7.5}$$

as $t \to T_*^-$, where $\varepsilon_1 = \varepsilon_0 \min\{1, \gamma\}$.

We estimate $|f_0(t)|$ now. As $t \to T_*^-$, we have from (4.1) and (4.11) that

$$|R_\Lambda y(t)| = \mathcal{O}((T_* - t)^{1/\alpha}) \text{ and } |R_\Lambda f(t)| = \mathcal{O}((T_* - t)^{1/\alpha - 1 + \delta/\alpha}). \tag{7.6}$$

Combining (7.5) and (7.6) gives, as $t \to T_*^-$,

$$|f_0(t)| = \mathcal{O}((T_* - t)^{-1+\varepsilon_1}(T_* - t)^{1/\alpha} + (T_* - t)^{1/\alpha - 1 + \delta/\alpha}) = \mathcal{O}((T_* - t)^{1/\alpha - 1 + \varepsilon_2/\alpha}),$$

where $\varepsilon_2 = \min\{\varepsilon_1\alpha, \delta\}$. By the virtue of the lower bound of $|R_\Lambda y(t)|$ in (6.31), we actually have

$$|f_0(t)| = \mathcal{O}(|R_\Lambda y(t)|^{1-\alpha+\varepsilon_2}).$$

Fix a number $t_0' \in [t_0, T_*)$ such that $R_\Lambda y(t) \neq 0$ and $|f_0(t)| \leq M_0 |R_\Lambda y(t)|^{1-\alpha+\varepsilon_2}$ for all $t \in [t_0', T_*)$, where M_0 is a positive constant. Of course, one already has $R_\Lambda y(T_*) = 0$.

We apply Theorem 5.1 to solution $R_\Lambda y(t)$ of equation (7.2) on the interval $[t_0', T_*)$. Specifically, $R_\Lambda y(t)$ satisfies equation (5.2) on (t_0', T_*) with constant $a = \Lambda H(v_*)$ and $f = f_0$. Then there exists a nonzero vector $\xi_* \in \mathbb{R}^n$ such that

$$|R_\Lambda y(t) - (T_* - t)^{1/\alpha}\xi_*| = \mathcal{O}((T_* - t)^{1/\alpha + \varepsilon_3}) \text{ for some } \varepsilon_3 > 0, \tag{7.7}$$

$$|\xi_*| = (\alpha \Lambda H(v_*))^{1/\alpha}. \tag{7.8}$$

The desired statement (2.15) immediately follows from (7.7).

Because

$$\xi_* = \lim_{t \to T_*^-} (T_* - t)^{-1/\alpha} R_\Lambda y(t),$$

by (2.15), and the fact $\xi_* \neq 0$, we have $\xi_* \in R_\Lambda(\mathbb{R}^n) \setminus \{0\}$. Hence, ξ_* is an eigenvector of A associated with Λ.

Next, we prove (2.13). Writing

$$y(t) - (T_* - t)^{1/\alpha}\xi_* = (I_n - R_\Lambda)y(t) + (R_\Lambda y(t) - (T_* - t)^{1/\alpha}\xi_*),$$

and using the estimate (6.30) with $\varepsilon = \varepsilon_0$, and estimate (7.7) yield

$$|y(t) - (T_* - t)^{1/\alpha}\xi_*| = \mathcal{O}((T_* - t)^{1/\alpha + \varepsilon_0} + (T_* - t)^{1/\alpha + \varepsilon_3}).$$

This implies (2.13) with $\varepsilon = \min\{\varepsilon_0, \varepsilon_3\}$.

Finally, we prove (2.16). Let $w(t) = (T_* - t)^{-1/\alpha} y(t)$ and write $v(t) = w(t)/|w(t)|$. Passing $t \to T_*^-$ and noticing that $v(t) \to v_*$ and $w(t) \to \xi_*$, thanks to (6.45) and (2.13), we obtain

$$v_* = \xi_*/|\xi_*| \tag{7.9}$$

Then it follows from (7.8), the fact H is positively homogeneous of degree $-\alpha$, and relation (7.9) that

$$1 = \alpha\Lambda H(v_*)|\xi_*|^{-\alpha} = \alpha\Lambda H(|\xi_*|v_*) = \alpha\Lambda H(\xi_*).$$

Hence, we obtain (2.16). This completes the proof for the case of symmetric matrix A.

Case 2. Consider the case A is not symmetric. Let A_0 and S be as in (2.1). Same as (3.10), we set $z(t) = Sy(t)$ on $[t_0, T_*]$. Then $z(t)$ satisfies equation (4.6) with $\widetilde{H}$ and $\widetilde{f}$ defined in (4.7). One can verify the following facts.

- $z(t) \neq 0$ for $t \in [t_0, T_*)$ and $z(T_*) = 0$.
- $\widetilde{H} \in \mathcal{H}_{-\alpha}(\mathbb{R}^n)$ and, thanks to parts (i) and (v) of Lemma 2.7, $\widetilde{H} > 0$ on $\mathbb{S}^{n-1}$ and $\widetilde{H}$ has property (HC) on $\mathbb{S}^{n-1}$.
- Thanks to (4.8), $\widetilde{f}(t)$ and $z(t)$ satisfy condition (2.12) with the same numbers α, δ, t_0, T_*, and constant $\widetilde{M}$ in place of M.

We apply the results already established in Case 1 above to the solution $z(t)$ of equation (4.6). Note that A_0 replaces A and $\widehat{R}_{\lambda_j}$ replaces R_{λ_j}. Then there exist an eigenvalue Λ of A_0 and an eigenvector ξ_0 of A_0 associated with Λ such that

$$\begin{gathered} |(I_n - \widehat{R}_\Lambda)z(t)| = \mathcal{O}((T_* - t)^{1/\alpha+\varepsilon}), \\ |\widehat{R}_\Lambda z(t) - (T_* - t)^{1/\alpha}\xi_0| = \mathcal{O}((T_* - t)^{1/\alpha+\varepsilon}) \end{gathered} \tag{7.10}$$

for some number $\varepsilon > 0$, and

$$\alpha\Lambda\widetilde{H}(\xi_0) = 1. \tag{7.11}$$

Let $\xi_* = S^{-1}\xi_0$. Then Λ is an eigenvalue of A and ξ_* is an eigenvector of A associated with Λ. We rewrite (7.10) as

$$\begin{gathered} |S(I_n - R_\Lambda)y(t)| = \mathcal{O}((T_* - t)^{1/\alpha+\varepsilon}), \\ |S(R_\Lambda y(t) - (T_* - t)^{-1/\alpha}\xi_*)| = \mathcal{O}((T_* - t)^{1/\alpha+\varepsilon}), \end{gathered}$$

which imply (2.14) and (2.15). By (2.14), (2.15) and the triangle inequality, we obtain (2.13) in the same way as in Case 1.

Finally, (2.16) follows from (7.11) and the relation $\widetilde{H}(\xi_0) = H(\xi_*)$. The proof of Theorem 2.9 is complete. □

Proof of Theorem 2.10. Set $f(t) = G(t, y(t))$ for $t \in [t_0, T_*)$. Thanks to (2.9), the is a number $t_0' \in [t_0, T_*)$ such that

$$|y(t)| \le r_* \text{ for all } t \in [t_0', T_*).$$

This property and (2.7) imply

$$|f(t)| \le c_*|y(t)|^{1-\alpha+\delta} \text{ for all } t \in [t_0', T_*).$$

Thus, $f(t)$ satisfies condition (2.12) with t_0' replacing t_0. Applying Theorem 2.9 to the interval $[t_0', T_*)$ in place of $[t_0, T_*)$, we obtain the statements of Theorem 2.10. □

8. Examples and applications

8.1. **Examples.** We give some examples for the function H in Assumptions 2.3 and 2.8. For simplicity, we consider the case dimension $n = 2$. One can easily generalize them for any higher dimension n.

(a) For $x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}$, let
$$H_1(x) = (x_1^4 + 5x_2^4)^{-3},$$
$$H_2(x) = \Big[(3|x_1|^{3/2} + |x_2|^{3/2})^{1/3} + (2|x_1|^{5/3} + 7|x_2|^{5/3})^{3/10}\Big]^{-1/8}.$$
Then H_1 is in $\mathcal{H}_{-3/4}(\mathbb{R}^2, \mathbb{R})$ while H_2 is in $\mathcal{H}_{-1/16}(\mathbb{R}^2, \mathbb{R})$. Both functions belong to $C^1(\mathbb{R}^2 \setminus \{0\})$. Hence, they have property (HC) on $\mathbb{S}^1$ with the same power $\gamma = 1$ in (2.10).

(b) Another example is
$$H(x) = \frac{\sqrt{|x_1|} + \sqrt{|x_2|}}{|x|} \text{ for } x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}.$$
Then H belongs to $\mathcal{H}_{-1/2}(\mathbb{R}^2, \mathbb{R})$, is positive on $\mathbb{S}^1$ and has property (HC) on $\mathbb{S}^1$ with the same power $\gamma = 1/2$ in (2.10). Unlike the previous two examples, this function H is not in $C^1(\mathbb{R}^2 \setminus \{0\})$.

In fact, the function H can be very complicated, see similar examples in [27, Example 5.7] and [13, Section 6].

8.2. **Applications.** We give an application to a population model [32] in mathematical biology. Consider an inhomogeneous population composed of individuals with different death rates. This population consists of n clones, each has the size $y_i(t)$, for $i = 1, 2, \dots, n$, at time t with the death rate $k_i > 0$, where the constants k_i are mutually distinct. The model [32, Equations (3.1) and (3.2)] is
$$y_i' = -k_i y_i g(N) \quad \text{for } i = 1, 2, \dots, n, \tag{8.1}$$
where
$$N = N(t) := y_1(t) + y_2(t) + \dots + y_n(t) \text{ is the total population size,} \tag{8.2}$$
and $g(s) : (0, \infty) \to (0, \infty)$ is an appropriate function. Note that the integral form of $N(t)$ in [32, Equation (3.2)] becomes the finite sum in (8.2).

Below, we consider the case
$$g(s) = ks^{-\alpha} \text{ for any } s > 0, \text{ where } k > 0 \text{ and } \alpha > 0 \text{ are some constants.} \tag{8.3}$$
This generalizes the consideration $\alpha = 1$ in [32, Equation (4.2)].

Define the matrix A and function $H(x)$, for $x = (x_1, \dots, x_n) \in \mathbb{R}^n \setminus \{0\}$, as follows
$$A = k \operatorname{diag}[k_1, k_2, \dots, k_n] \text{ and } H(x) = (|x_1| + |x_2| + \dots + |x_n|)^{-\alpha}.$$
We denote $\mathcal{C} = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0 \text{ for } i = 1, 2, \dots, n\}$. With $y = (y_1, \dots, y_n)$, if $y(t) \in \mathcal{C} \setminus \{0\}$ is a solution of the system (8.1), (8.2), (8.3), then it is a solution of system (1.3) with $G \equiv 0$, i.e.,
$$y' = -H(y)Ay. \tag{8.4}$$

Note that the matrix A is symmetric and satisfies Assumption 2.1. The function H clearly belongs to $H_{-\alpha}(\mathbb{R}^n, \mathbb{R})$.

Claim. The function $H(x)$ has property (HC) on $\mathbb{S}^{n-1}$ with the same power $\gamma = \min\{1, \alpha\}$ in (2.10).

Thus, H satisfies Assumption 2.8.

Proof of the Claim. We denote the ℓ^1-norm $|x|_{\ell^1} = |x_1| + |x_2| + \ldots + |x_n|$. Then there exists a constant $c_0 \geq 1$ such that

$$c_0^{-1}|x| \leq |x|_{\ell^1} \leq c_0|x| \text{ for all } x \in \mathbb{R}^n.$$

For $x, y \in \mathbb{S}^{n-1}$, we have both norms $|x|_{\ell^1}$ and $|y|_{\ell^1}$ belong to the interval $[c_0^{-1}, c_0]$, and, hence,

$$|H(x) - H(y)| = \frac{\big||y|_{\ell^1}^\alpha - |x|_{\ell^1}^\alpha\big|}{|x|_{\ell^1}^\alpha |y|_{\ell^1}^\alpha} \leq c_0^{2\alpha}\big||y|_{\ell^1}^\alpha - |x|_{\ell^1}^\alpha\big|.$$

When $0 < \alpha \leq 1$, utilizing the inequality $\big||y|_{\ell^1}^\alpha - |x|_{\ell^1}^\alpha\big| \leq \big||y|_{\ell^1} - |x|_{\ell^1}\big|^\alpha$, and applying the triangle inequality, one obtains

$$|H(x) - H(y)| \leq c_0^{2\alpha}|y - x|_{\ell^1}^\alpha \leq c_0^{3\alpha}|y - x|^\alpha.$$

When $\alpha > 1$, applying the Mean Value Theorem to the function $s \mapsto s^\alpha$ and values $s_1 = |y|_{\ell^1}$, $s_2 = |x|_{\ell^1}$ both belonging to the interval $[c_0^{-1}, c_0]$ yields the existence of a number $C_0 > 0$ depending on c_0 and α such that

$$||y|_{\ell^1}^\alpha - |x|_{\ell^1}^\alpha| \leq C_0\, ||y|_{\ell^1} - |x|_{\ell^1}|\,.$$

Then applying the triangle inequality again, we deduce

$$|H(x) - H(y)| \leq c_0^{2\alpha}C_0\, ||y|_{\ell^1} - |x|_{\ell^1}| \leq c_0^{2\alpha}C_0|y - x|_{\ell^1} \leq c_0^{2\alpha+1}C_0|y - x|.$$

Therefore, the Claim is true. □

Theorem 8.1. *Let $y_0 \in \mathcal{C} \setminus \{0\}$ be sufficiently small, then there exist a number $T_* > 0$ and a function $y \in C^1([0, T_*), \mathcal{C} \setminus \{0\})$ such that $y(t)$ satisfies* (8.1), (8.2), (8.3) *for all $t \in (0, T_*)$, $y(0) = y_0$, and* (2.9) *holds.*

Proof. Applying Theorem 2.5 to equation (8.4) and $t_0 = 0$, we obtain a number $T_* > 0$ and a function $y \in C^1([0, T_*), \mathbb{R}^n \setminus \{0\})$ such that $y(t)$ satisfies equation (8.4) for all $t \in (0, T_*)$, $y(0) = y_0$, and (2.9) holds. Since $y(t) \neq 0$, for all $t \in [0, T_*)$, we have, for each $i = 1, 2, \ldots, n$,

$$y_i(t) = y_i(0)e^{-kk_i\int_0^t H(y(\tau))\mathrm{d}\tau}. \tag{8.5}$$

Together with the fact $y(0) \in \mathcal{C} \setminus \{0\}$, this implies $y(t) \in \mathcal{C} \setminus \{0\}$, and hence $y(t)$ is a solution of (8.1), (8.2), (8.3) for all $t \in (0, T_*)$. □

Let $\{e_i : i = 1, 2, \ldots, n\}$ denote the standard canonical basis of $\mathbb{R}^n$.

Theorem 8.2. *Let $T_* > 0$ and $y \in C^1([0, T_*), \mathcal{C} \setminus \{0\})$ be such that $y(t)$ satisfies* (8.1), (8.2), (8.3) *for all $t \in (0, T_*)$ and* (2.9) *holds. Then there is an integer $i \in [1, n]$ and a number $\varepsilon > 0$ such that*

$$\Big|y(t) - (\alpha kk_i)^{1/\alpha}(T_* - t)^{1/\alpha}e_i\Big| = \mathcal{O}\Big((T_* - t)^{1/\alpha+\varepsilon}\Big) \text{ as } t \to T_*^-. \tag{8.6}$$

Proof. Since $y(t) \in \mathcal{C}\setminus\{0\}$, the function y in fact is a solution of (8.4) on $(0, T_*)$ with the extinction time T_*. By the virtue of Theorem 2.10 applied to equation (8.4) and $t_0 = 0$, there exist a number $\varepsilon > 0$, an eigenvalue Λ of A and a corresponding eigenvector ξ_*, such that

$$\left|y(t) - (T_* - t)^{1/\alpha}\xi_*\right| = \mathcal{O}\left((T_* - t)^{1/\alpha+\varepsilon}\right) \text{ as } t \to T_*^-. \tag{8.7}$$

Clearly, $\Lambda = kk_i$ for some $1 \le i \le n$, and $\xi_* = Ke_i$ for some number $K \neq 0$. Multiplying estimate (8.7) by $(T_* - t)^{-1/\alpha}$ and using the ith coordinate, one derives

$$K = \lim_{t \to T_*^-} y_i(t)(T_* - t)^{-1/\alpha},$$

which implies $K \ge 0$. Moreover, thanks to (2.16), $|K|^\alpha = \alpha kk_i$. Therefore, $K = (\alpha kk_i)^{1/\alpha}$ and we obtain (8.6) from (8.7). □

Note in this demonstration that the system (8.1) is simpler than (1.3) and (2.11) which were theoretically studied in the previous sections.

Remark 8.3. The following final statements are in order.

(a) There is another totally different approach to the local properties of solutions of ODE based on the Poincaré–Dulac normal form [1, 3, 34]. It has been generalized and developed by many and for so long, see the books [8, 9, 33], recent papers such as [4, 5, 6, 7], and references therein. Our approach is relatively new and only recently used to explore different classes of equations and problems in ODE. For more comparisons between the other approach and ours, see [27, Remark 5.8] and [13, Remark 6.14].

(b) It is an open problem whether the solutions of (1.3) admit an asymptotic expansion similar to those in [18, 13] near the extinction time. Taking some indications from [13], we expect, in the case the answer is affirmative, that our result (2.13) will play an important role in its proof.

9. Appendix

Proof of Lemma 2.7. We denote $E = \mathbb{R}^n \setminus \{0\}$. For any $x \in E$, we can write $F(x) = |x|^{-\alpha}F(x/|x|)$. Hence, part (i) is obvious. For parts (ii)–(iv), the proofs are similar to the proof of [27, Lemma 5.1], and *"the verification of Assumption 5.2 for $\widetilde{H}$"* in the proof of [27, Theorem 5.3]. We present the key arguments here.

Let $x, \xi \in E$. Then

$$\begin{aligned} |F(x) - F(\xi)| &= \left||x|^{-\alpha}F(x/|x|) - |\xi|^{-\alpha}F(\xi/|\xi|)\right| \\ &\le |x|^{-\alpha}\,|F(x/|x|) - F(\xi/|\xi|)| + \left||x|^{-\alpha} - |\xi|^{-\alpha}\right| \cdot \left|F(\xi/|\xi|)\right| \end{aligned} \tag{9.1}$$

Using inequality (9.1) and the fact that functions $x \in E \mapsto x/|x|$ and $x \in E \mapsto |x|^{-\alpha}$ are C^1-functions, we can prove parts (ii) and (iii).

We prove part (iv) now. Suppose F has property (HC) on $\mathbb{S}^{n-1}$. By part (iii), F has property (HC) on E. Clearly, φ is a continuous function on E. Let ξ be any vector in E. Consider $x \in E$ sufficiently close to ξ. As $x \to \xi$, we have $\varphi(x) \to \varphi(\xi)$. Using inequality (2.10) for function F and $x_0 := \varphi(\xi) \in E$, $x := \varphi(x) \in E$ with constant C and power γ, and then inequality (2.10) again for function φ and $x_0 := \xi \in E$, $x \in E$ with constant C' and power γ', we have

$$|F(\varphi(x)) - F(\varphi(\xi))| \le C|\varphi(x) - \varphi(\xi)|^\gamma \le C\,{C'}^\gamma |x - \xi|^{\gamma\gamma'}.$$

Therefore, the function $F \circ \varphi$ has property (HC) on E.

Part (v) is a direct consequence of part (iv) with $\varphi(x) = Kx$. We omit the details. □

References

[1] V. I. Arnol'd; *Geometrical methods in the theory of ordinary differential equations*, second ed., A Series of Comprehensive Studies in Mathematics, vol. 250, Springer-Verlag, New York, 1988. MR 947141 (89h:58049)

[2] Viorel Barbu, Giuseppe Da Prato, Michael Röckner; *Finite time extinction for solutions to fast diffusion stochastic porous media equations*, C. R. Math. Acad. Sci. Paris **347** (2009), no. 1-2, 81–84. MR 2536755

[3] Yuri N. Bibikov; *Local theory of nonlinear analytic ordinary differential equations*, Lecture Notes in Mathematics, vol. 702, Springer-Verlag, Berlin-New York, 1979. MR 547669

[4] A. D. Bruno; *Asymptotic behavior and expansions of solutions of an ordinary differential equation*, Uspekhi Mat. Nauk **59** (2004), no. 3(357), 31–80. MR 2116535

[5] A. D. Bruno; *Power-logarithmic expansions of solutions of a system of ordinary differential equations*, Dokl. Akad. Nauk **419** (2008), no. 3, 298–302. MR 2462086

[6] A. D. Bruno; *Power-exponential expansions of solutions of an ordinary differential equation*, Dokl. Akad. Nauk **444** (2012), no. 2, 137–142. MR 2986098

[7] A. D. Bruno; *On complicated expansions of solutions to ODES*, Comput. Math. Math. Phys. **58** (2018), no. 3, 328–347. MR 3785768

[8] Alexander D. Bruno; *Local methods in nonlinear differential equations*, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1989. MR 993771

[9] Alexander D. Bruno; *Power geometry in algebraic and differential equations*, North-Holland Mathematical Library, vol. 57, North-Holland Publishing Co., Amsterdam, 2000. MR 1773512

[10] Dat Cao, Luan Hoang; *Asymptotic expansions in a general system of decaying functions for solutions of the Navier-Stokes equations*, Ann. Mat. Pura Appl. (4) **199** (2020), no. 3, 1023–1072. MR 4102801

[11] Dat Cao, Luan Hoang; *Long-time asymptotic expansions for Navier-Stokes equations with power-decaying forces*, Proc. Roy. Soc. Edinburgh Sect. A **150** (2020), no. 2, 569–606. MR 4080452

[12] Dat Cao, Luan Hoang; *Asymptotic expansions with exponential, power, and logarithmic functions for non-autonomous nonlinear differential equations*, J. Evol. Equ. **21** (2021), no. 2, 1179–1225. MR 4278393

[13] Dat Cao, Luan Hoang, Thinh Kieu; *Infinite series asymptotic expansions for decaying solutions of dissipative differential equations with non-smooth nonlinearity*, Qual. Theory Dyn. Syst. **20** (2021), no. 3, Paper No. 62, 38 pp. MR 4284023

[14] Beomjun Choi, Robert J. McCann, Christian Seis; *Asymptotics near extinction for nonlinear fast diffusion on a bounded domain*, Arch. Ration. Mech. Anal. **247** (2023), no. 2, Paper No. 16, 48. MR 4553941

[15] Ana D. Davidson, Marcus J. Hamilton, Alison G. Boyer, James H. Brown, Gerardo Ceballos; *Multiple ecological pathways to extinction in mammals*, Proceedings of the National Academy of Sciences **106** (2009), no. 26, 10702–10705.

[16] Manuel del Pino, Mariel Sáez; *Asymptotic description of vanishing in a fast-diffusion equation with absorption*, Differential Integral Equations **15** (2002), no. 8, 1009–1023. MR 1895576

[17] Jesús Ildefonso Díaz, Teresa Pierantozzi, Luis Vázquez; *Finite time extinction for nonlinear fractional evolution equations and related properties*, Electron. J. Differential Equations (2016), Paper No. 239, 13. MR 3547428

[18] C. Foias, J.-C. Saut; *Asymptotic behavior, as $t \to +\infty$, of solutions of Navier-Stokes equations and nonlinear spectral manifolds*, Indiana Univ. Math. J. **33** (1984), no. 3, 459–477. MR MR740960 (85f:35164)

[19] C. Foias, J.-C. Saut; *Linearization and normal form of the Navier-Stokes equations with potential forces*, Ann. Inst. H. Poincaré Anal. Non Linéaire **4** (1987), no. 1, 1–47. MR MR877990 (88d:35158)

[20] C. Foias, J.-C. Saut; *Asymptotic integration of Navier-Stokes equations with potential forces. I*, Indiana Univ. Math. J. **40** (1991), no. 1, 305–320. MR MR1101233 (92g:35175)

[21] Jean-Michel Ghidaglia; *Long time behaviour of solutions of abstract inequalities: applications to thermohydraulic and magnetohydrodynamic equations*, J. Differential Equations **61** (1986), no. 2, 268–294. MR 823404

[22] Jean-Michel Ghidaglia; *Some backward uniqueness results*, Nonlinear Anal. **10** (1986), no. 8, 777–790. MR 851146

[23] Jack K. Hale; *Ordinary differential equations*, second ed., Robert E. Krieger Publishing Co. Inc., Huntington, N.Y., 1980. MR 587488 (82e:34001)

[24] Philip Hartman; *Ordinary differential equations*, John Wiley & Sons, Inc., New York-London-Sydney, 1964. MR 0171038

[25] Luan Hoang; *Asymptotic expansions for the Lagrangian trajectories from solutions of the Navier–Stokes equations*, Comm. Math. Physics **383** (2021), no. 2, 981–995.

[26] Luan Hoang; *Asymptotic expansions about infinity for solutions of nonlinear differential equations with coherently decaying forcing functions*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **XXV** (2024), no. 1, 311–370.

[27] Luan Hoang; *Long-time behaviour of solutions of superlinear systems of differential equations*, Dynamical Systems **39** (2024), no. 1, 79–107.

[28] Luan Hoang; *The Navier–Stokes equations with body forces decaying coherently in time*, J. Math. Anal. Appl. **531** (2024), no. 2, Part 1, 127863 (39 pp).

[29] Luan T. Hoang, Vincent R. Martinez; *Asymptotic expansion in Gevrey spaces for solutions of Navier-Stokes equations*, Asymptot. Anal. **104** (2017), no. 3–4, 167–190.

[30] Luan T. Hoang, Vincent R. Martinez; *Asymptotic expansion for solutions of the Navier-Stokes equations with non-potential body forces*, J. Math. Anal. Appl. **462** (2018), no. 1, 84–113.

[31] Luan T. Hoang, Edriss S. Titi; *Asymptotic expansions in time for rotating incompressible viscous fluids*, Ann. Inst. H. Poincaré Anal. Non Linéaire **38** (2021), no. 1, 109–137. MR 4200479

[32] G P Karev, I Kareva; *Mathematical modeling of extinction of inhomogeneous populations*, Bull Math Biol. **78** (2016), no. 4, 834–858.

[33] Valery V. Kozlov, Stanislav D. Furta; *Asymptotic solutions of strongly nonlinear systems of differential equations*, Springer Monographs in Mathematics, Springer, Heidelberg, 2013, Translated from the 2009 Russian second edition by Lester J. Senechal. MR 3025385

[34] Solomon Lefschetz; *Differential equations: geometric theory*, second ed., Dover Publications, Inc., New York, 1977. MR 0435481

[35] Gheorghe Minea; *Investigation of the Foias-Saut normalization in the finite-dimensional case*, J. Dynam. Differential Equations **10** (1998), no. 1, 189–207. MR MR1607538 (2000b:34058)

[36] Masashi Misawa, Kenta Nakamura, Md Abu Hanif Sarkar; *A finite time extinction profile and optimal decay for a fast diffusive doubly nonlinear equation*, NoDEA Nonlinear Differential Equations Appl. **30** (2023), no. 3, Paper No. 43, 48. MR 4575468

[37] Janik Schüttler, Robert L. Jack, Michael E. Cates; *Effects of phase separation on extinction times in population models*, J. Stat. Mech. Theory Exp. (2024), no. 8, Paper No. 083209, 47. MR 4809006

[38] Y. Shi; *A Foias-Saut type of expansion for dissipative wave equations*, Comm. Partial Differential Equations **25** (2000), no. 11-12, 2287–2331. MR 1789928

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