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GLOBAL BOUNDEDNESS IN AN INDIRECT CHEMOTAXIS-CONSUMPTION MODEL WITH SIGNAL-DEPENDENT DEGENERATE DIFFUSION

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Abstract. In this article, we consider the consumption chemotaxis system

$$\begin{gathered} u_t = \Delta(uv^\alpha) + au - bu^\gamma, \quad (x,t) \in \Omega \times (0,\infty), \\ v_t = \Delta v - uvw, \quad (x,t) \in \Omega \times (0,\infty), \\ w_t = -\delta w + u, \quad (x,t) \in \Omega \times (0,\infty), \end{gathered}$$

on a smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$ with homogeneous Neumann boundary conditions, where $a > 0$, $b > 0$, $\gamma \geq 2$, and $\delta > 0$. We shown that for sufficiently regular initial data, the associated initial-boundary value problem possesses global bounded classical solutions.

1. Introduction

In 1971, Keller and Segel [16] proposed the model

$$\begin{gathered} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + f(u), \quad x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, \quad x \in \Omega,\ t > 0, \end{gathered} \tag{1.1}$$

to describe the movement of bacteria toward oxygen, and at the same time, oxygen is consumed by the bacteria, where $u = u(x,t)$ denotes the density of the bacteria, where $v = v(x,t)$ represents the oxygen concentration, and $\chi \in R$ represents the chemotactic sensitivity coefficient. In the above system and its various variants, we know that chemotaxis is the directed movement of cells or organisms in response to a concentration gradient of a chemical stimulus, and it plays a crucial role in a wide variety of biological processes. For $f(u) = 0$, Zhang and Li [54] proved that (1.1) admits a global classical solution (u,v) and that this solution converges exponentially to $(\bar{u},0)$ as $t \to \infty$ if either $N \leq 2$ or

$$\chi \leq \frac{1}{6(N+1)\|v_0\|_{L^\infty(\Omega)}}, \quad N \geq 3,$$

where $\bar{u}_0 := \frac{1}{|\Omega|}\int_\Omega u_0$. The global weak solution in a three-dimensional domain has been studied by Tao and Winkler [31]. In [31, 54], and Baghaei and Khelghati in [3] improved the result and showed that system (1.1) has a globally bounded classical

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solution under the condition $\|v_0\|_{L^\infty(\Omega)} \leq \frac{\pi}{\chi\sqrt{2(N+1)}}$. When we replace uv by $uf(v)$, where $f \in C^1([0,\infty))$ is nonnegative with $f(0)=0$, the global generalized solution was proved by Winkler [43] in any dimension. In the case of $N \geq 1$, $b > C_1(N)\|\chi v_0\|_{L^\infty(\Omega)}^{\frac{1}{n}} + C_2(N)\|\chi v_0\|_{L^\infty(\Omega)}^{2N}$, a global bounded classical solution of the system (1.1) was obtained by Lankeit and Wang [23] in 2017. In addition, the variant of (1.1) has been studied in [44, 45, 46, 47].

In regard to (1.1), the utilization of chemotaxis signal by cells may be more complex in realistic situations. The signal may originate from external substances produced indirectly, or consist of several signals produced by different mechanisms, as noted in [33]. In particular, a chemotaxis system with an indirect consumption of signals has been taken into consideration in [11]:

$$
\begin{gathered}
u_t = \Delta u - \nabla\cdot(u\nabla v) + f(u), \quad x \in \Omega, t > 0,\\
v_t = \Delta v - vw, \quad x \in \Omega, t > 0,\\
w_t = -\delta w + u, \quad x \in \Omega, t > 0.
\end{gathered} \tag{1.2}
$$

When $f(u) \equiv 0$, assuming that $n \leq 2$ or $n \geq 3$ with $\|v_0\|_{L^\infty(\Omega)} \leq \frac{1}{3n}$, Fuest [11] proved that (1.2) admits a globally bounded classical solution, which converges to the constant steady state $(\bar{u}_0, 0, \bar{u}_0/\delta)$ as time goes to infinity. When $f(u) = \mu u(1-u)$, $\mu > 0$, if μ is suitably large, Li et al. [21] proved that (1.2) has a globally bounded classical solution. Many of the results related to the qualitative analysis of indirect signal mechanisms can be found in [5, 12, 19].

Recently, the research interest of scholars has gradually shifted to chemotaxis systems with signal-dependent movements [4, 25]. For example, the following Keller-Segel production models with signal-dependent motility

$$
\begin{gathered}
u_t = \Delta(\gamma(v)u) + f(u), \quad x \in \Omega, t > 0,\\
v_t = \Delta v - v + u, \quad x \in \Omega, t > 0.
\end{gathered} \tag{1.3}
$$

If $f(u)=0$ and $k_\gamma \leq \gamma(s) \leq K_\gamma$ for all $s \geq 0$, where $k_\gamma, K_\gamma > 0$, in the case of two dimensions, Tao and Winkler [34] proved that (1.3) has global bounded classical solutions; However, (1.3) has global weak solutions in the high-dimensional case. In particular, under the conditions $\gamma(s) = c_0/s^k (c_0, k > 0)$, if c_0 is small enough, the existence of global classical solutions has been investigated in [53]. If $\gamma(s) = s^{-\alpha}$ with $\alpha > 0$, scholars have also obtained some results on the global existence of classical solutions [1, 8, 9, 15, 40]. If $\gamma(s) = e^{-s}$ for all $s \geq 0$, [7, 14] show a phenomenon of critical mass for (1.3) in the two-dimensional case. Further results on (1.3) can be found in [2, 10, 48]. If $f(u) = \mu u(1-u)$, $\mu > 0$ and $\gamma(s)$ satisfies $\gamma(s) > 0$, $\gamma'(s) < 0$ and $\lim_{s\to+\infty} \gamma'(s)/\gamma(s)$ exists, in the two-dimensional settings, Jin et al. [13] proved the existence of a global classical solution to (1.3). Similar results in higher dimensions were proved by [22, 41]. For some other results on (1.3), see [6, 26, 27, 28].

On the other hand, if the signal is degraded rather than produced by the cells, the consumption of chemotaxis with signal-dependent motility has also been taken into account.

$$
\begin{gathered}
u_t = \Delta(\gamma(v)u) + f(u), \quad x \in \Omega,\ t > 0,\\
v_t = \Delta v - uv, \quad x \in \Omega,\ t > 0.
\end{gathered} \tag{1.4}
$$

If $f(u)=0$, γ is strictly positive on $[0,\infty)$, the existence of global bounded classical solutions has been shown by Li and Zhao [18], provided that $\|v_0\|_{L^\infty(\Omega)}$ is sufficiently small. In the case of $n=2$, the smallness assumption of $\|v_0\|_{L^\infty(\Omega)}$ was removed by Li and Winkler [20], who showed that (1.4) has global classical bounded solutions. When $n \geq 1$, (1.4) possesses global very weak solutions for some weaker regularity properties on γ. If $\gamma(s) = s^{-\alpha}, \alpha > 0$, Tao and Winkler [35] proved that there are global weak solutions to (1.4) provided that $2 \leq n \leq 5$ and $\alpha > \frac{n-2}{6-n}$. If $\gamma(s) = s^\alpha, \alpha > 0$ for all $s \geq 0$, there are some other results in [50, 52, 51].

When $f(u) = au - bu^l$, $a > 0$, $b > 0$, Wang [37] proved that if one of the following 3 conditions is satisfied: (i) $n \leq 2$ and $l > 1$, (ii) $n \geq 3$ and $l > 2$, (iii) $n \geq 3$, $l = 2$ and b is sufficiently large; then there exists a bounded classical solution in (1.4), while in the case of $n \geq 3$ and $l \in (1,2]$, (1.4) admits at least one global weak solution which becomes smooth after some waiting time. If $\gamma(s) = s^\alpha, \alpha \geq 1$, Wang established a global classical solution in [38]. Conversely, (1.4) admits at least one global weak solution in the case $\alpha > 0$ if γ has rather mild regularities. Moreover, if γ is suitably smooth with $\alpha > 1$, then the above weak solutions eventually become smooth. Some scholars have also studied the system (1.4) in other situations, we refer the reader to [24, 29, 39].

Recently, Wang [36] studied the chemotaxis system

$$
\begin{gathered}
u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \eta(u - u^m), \quad x \in \Omega,\ t > 0, \\
v_t = \Delta v - u^\theta v w, \quad x \in \Omega,\ t > 0, \\
0 = \Delta w - w + u^\alpha, \quad x \in \Omega,\ t > 0
\end{gathered} \tag{1.5}
$$

and obtained bounded classical solutions of the corresponding initial value problem.

With the above work as a motivation, in this article we consider the $\varphi(v) = v^\alpha$ for $\alpha > 0$, and the system

$$
\begin{gathered}
u_t = \Delta(u v^\alpha) + au - bu^\gamma, \quad x \in \Omega,\ t > 0, \\
v_t = \Delta v - uvw, \quad x \in \Omega,\ t > 0, \\
w_t = -\delta w + u, \quad x \in \Omega,\ t > 0, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0, \\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x) \quad x \in \Omega,
\end{gathered} \tag{1.6}
$$

where $\Omega \subset \mathbb{R}^n$ is a smooth and bounded domain, $\frac{\partial}{\partial \nu}$ denotes the derivative with respect the outer normal of $\partial\Omega$, and $\alpha > 0, a > 0, b > 0, \gamma \geq 2$. The initial data satisfy

$$
\begin{gathered}
u_0 \in C^0(\Omega) \text{ with } u_0 \geq 0 \text{ in } \Omega, \\
v_0,\ w_0 \in W^{1,\infty}(\Omega) \text{ with } v_0 > 0,\ w_0 \geq 0 \text{ in } \Omega.
\end{gathered} \tag{1.7}
$$

Our main goal is to study the initial boundary value problem of (1.6) and consider its global bounded solutions in the classical sense. This is stated in the following theorem.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded domain with smooth boundary, the parameters $a, b, \delta > 0$ and $\alpha \geq 1$. Suppose that the initial data satisfy* (1.7), *if on of the following 3 cases holds: (i) $\gamma > 2$, (ii) $n \leq 3$ and $\gamma = 2$, (iii) $n \geq 4$, $\gamma = 2$ and*

$$
b > \Big(\frac{n-2}{2}\Big)^{\frac{n+2}{n}} (n-1)^{-2/n} (n + \sqrt{n})^{\frac{4}{n}} \alpha^{\frac{2(n+2)}{n}} \|v_0\|_{L^\infty(\Omega)}^{\frac{\alpha(n+2)-2}{n}}
$$

$$+\frac{2^{2n+5}(n+\sqrt{n}+1)^n(n-2+\sqrt{n})^{\frac{n+2}{2}}}{\delta^{\frac{n+2}{2}}(n+2)(n-1)^{\frac{n}{2}}}\|v_0\|_{L^\infty(\Omega)};$$

then (1.6) *has a global bounded classical solution* (u,v,w) *with*

$$u\in C^0(\bar\Omega\times[0,\infty))\cap C^{2,1}(\bar\Omega\times(0,\infty)),$$
$$v\in\cap_{\theta>n}C^0([0,\infty);W^{1,\theta}(\Omega))\cap C^{2,1}(\bar\Omega\times(0,\infty)),$$
$$w\in C^0(\bar\Omega\times[0,\infty))\cap C^{0,1}(\bar\Omega\times(0,\infty)).$$

Remark 1.2. When $n\ge 2$, reference [36] shows the existence of classical solutions for system (1.5). For (1.5), if the first and third equations are replaced by $u_t=\Delta(uv^\alpha)+au-bu^\gamma$, $w_t=-\delta w+u$ and $\theta=1$ respectively, this paper considers the global boundedness for an indirect chemotaxis-consumption model with signal-dependent degenerate diffusion.

2. Preliminaries

We first present a criterion for existence and extensibility of local solutions.

Lemma 2.1. *Let* $\Omega\subset\mathbb{R}^n$ $(n\ge 1)$ *be a bounded domain with smooth boundary, and let* $a,b,\delta,\alpha>0$. *Assume that* (1.7) *holds. Then model* (1.6) *possesses a nonnegative local classical solution* (u,v,w) *with* $u\in C^0(\bar\Omega\times[0,T_{\max}))\cap C^{2,1}(\bar\Omega\times(0,T_{\max}))$, $v\in\cap_{\theta>n}C^0([0,T_{\max});W^{1,\theta}(\Omega))\cap C^{2,1}(\bar\Omega\times(0,T_{\max}))$ *and* $w\in C^0(\bar\Omega\times[0,T_{\max}))\cap C^{0,1}(\bar\Omega\times(0,T_{\max}))$. *Moreover, if* $T_{\max}<\infty$, *then*

$$\limsup_{t\nearrow T_{\max}}\|u(\cdot,t)\|_{L^\infty(\Omega)}=\infty.$$

Based on the established parabolic theory, the proof of the above lemma is similar to [13]. We omit it here.

Lemma 2.2. *Let the hypotheses of* (1.7) *hold. Then there exists* $C>0$ *such that*

$$\|u(\cdot,t)\|_{L^1(\Omega)}\le C\quad\text{for all } t\in(0,T_{\max}),\tag{2.1}$$
$$0\le v\le\|v_0\|_{L^\infty(\Omega)}\quad\text{in } \Omega\times(0,T_{\max}),\tag{2.2}$$
$$\int_t^{t+\epsilon}\int_\Omega u^\gamma\le C\quad\text{for all } t\in(0,T_{\max}-\epsilon),\tag{2.3}$$

where $\epsilon=\min\{1,\frac12 T_{\max}\}$.

Proof. Integrating the first equation in (1.6) yields

$$\frac{d}{dt}\int_\Omega u=a\int_\Omega u-b\int_\Omega u^\gamma\le a\int_\Omega u-b|\Omega|^{1-\gamma}\Big(\int_\Omega u\Big)^l\quad\text{for all } t\in(0,T_{\max}),\tag{2.4}$$

upon an ODE comparison argument implies (2.1). By the non-negativity of u, v and w, using the maximum principle for the second equation of system (1.6) leads to (2.2). By integrating (2.4), we can easily obtain (2.3). □

Lemma 2.3 ([49, Lemma 3.4])**.** *Assume* $p\ge 2$ *and* $\phi\in C^2(\bar\Omega)$ *is positive with* $\frac{\partial\phi}{\partial\nu}$ *on* $\partial\Omega$. *Then*

$$\int_\Omega\phi^{-p-1}|\nabla\phi|^{p+2}\le(p+\sqrt{n})^2\int_\Omega\phi^{-p+3}|\nabla\phi|^{p-2}|D^2\ln\phi|^2,$$
$$\int_\Omega\phi^{-p+1}|\nabla\phi|^{p-2}|D^2\phi|^2\le(p+\sqrt{n}+1)^2\int_\Omega\phi^{-p+3}|\nabla\phi|^{p-2}|D^2\ln\phi|^2.$$

Lemma 2.4 ([32]). *Let $T \in (0,\infty]$, $t_0 \in (0,T)$, $\lambda > 0$, $k > 0$. Assume that $y : [0,T) \to [0,\infty)$ is absolutely continuous and that*

$$y'(t) + \lambda y(t) \le h(t) \quad \textit{for all } t \in (0,T)$$

with the nonnegative function $h \in L^l_{loc}([0,T))$ fulfilling

$$\int_t^{t+t_0} h(s)\,ds \le k \quad \textit{for all } t \in (0, T-t_0).$$

Then

$$y(t) \le \max\Big\{y(0)+k, 2k + \frac{k}{\lambda t_0}\Big\} \quad \textit{for all } t \in (0,T).$$

Lemma 2.5 ([49, Lemma 3.5]). *Suppose $p \ge 2$ and $\mu > 0$. Then there exists $C = C(p,\mu) > 0$ such that for every positive $\phi \in C^2(\bar\Omega)$ with $\frac{\partial\phi}{\partial\nu} = 0$ on $\partial\Omega$ we have*

$$\begin{aligned} &\int_{\partial\Omega} \phi^{-p+1}|\nabla\phi|^{p-2}\frac{\partial|\nabla\phi|^2}{\partial\nu} \\ &\le \mu\int_\Omega \phi^{-p-1}|\nabla\phi|^{p+2} + \mu\int_\Omega \phi^{-p+1}|\nabla\phi|^{p-2}|D^2\phi|^2 + C\int_\Omega \phi. \end{aligned}$$

Lemma 2.6. *Assume that the initial value satisfies (1.7) and $q \in [1,\gamma]$. Then there exists $C > 0$ such that*

$$\int_\Omega w^q(\cdot,t) \le C \quad \textit{for all } t \in (0,T_{\max}). \tag{2.5}$$

Proof. Using w^{q-1} to test the third equation of (1.6) and integrating gives

$$\begin{aligned} \frac{1}{q}\frac{d}{dt}\int_\Omega w^q &= -\delta\int_\Omega w^q + \int_\Omega uw^{q-1} \\ &\le -\delta\int_\Omega w^q + \frac{\delta}{2}\int_\Omega w^q + C\int_\Omega u^q \end{aligned} \tag{2.6}$$

for all $t \in (0,T_{\max})$ which implies (2.5) with (2.3) and Lemma 2.4. □

Lemma 2.7. *Let $T \in (0,T_{\max})$. For some $p > \frac{n}{2}$, there exists $C_1 > 0$ such that*

$$\|w(\cdot,t)\|_{L^p(\Omega)} \le C_1 \quad \textit{for all } t \in (0,T_{\max}). \tag{2.7}$$

Then there exists $C_2(T) > 0$ such that

$$v(x,t) \ge C_2(T) \quad \text{for all } t \in (0,T). \tag{2.8}$$

Proof. Let $z(x,t) := \ln\frac{1}{v(x,t)}$. Then, using the second equation in (1.6), we see that

$$\begin{aligned} &z_t = \Delta z - |\nabla z|^2 + uw, \quad x \in \Omega,\ t > 0, \\ &\frac{\partial z}{\partial\nu} = 0, \quad x \in \partial\Omega,\ t > 0, \\ &z(x,0) = z_0(x) = \ln\frac{1}{v_0(x)}, \quad x \in \Omega. \end{aligned} \tag{2.9}$$

We use the variation-of-constants formula for the first equation of (2.9) to obtain

$$z(\cdot,t) \le e^{t\Delta}z_0 + \int_0^t e^{(t-s)\Delta}uw\,ds \quad \text{for all } t \in (0,T).$$

Then, by (2.7), (2.18) and the smoothing properties of the Neumann heat semigroup $(e^{t\Delta})_{t\geq 0}$ defined on Ω [42, Lemma 1.3], we can find constants $c_1, c_2 > 0$ such that

$$\begin{aligned}
&\|z(\cdot,t)\|_{L^\infty(\Omega)} \\
&\leq \|e^{t\Delta}z_0\|_{L^\infty(\Omega)} + \int_0^t \|e^{(t-s)\Delta}u(\cdot,s)w(\cdot,s)\|_{L^\infty(\Omega)}\,ds \\
&\leq \|z_0\|_{L^\infty(\Omega)} + c_1\int_0^t \{1+(t-s)^{-\frac{n}{2p}}\}\|u(\cdot,s)w(\cdot,s)\|_{L^p(\Omega)}\,ds \\
&\leq \|z_0\|_{L^\infty(\Omega)} + c_1\int_0^t \{1+(t-s)^{-\frac{n}{2p}}\}\|w(\cdot,s)\|_{L^p(\Omega)}\|u(\cdot,s)\|_{L^\infty(\Omega)}\,ds \\
&\leq \|z_0\|_{L^\infty(\Omega)} + c_2\int_0^T (1+\sigma^{-\frac{n}{2p}})\,d\sigma \quad \text{for all } t\in(0,T).
\end{aligned} \tag{2.10}$$

From the definition of z, in view of $p > \frac{n}{2}$ and using (2.10), we can easily derive (2.8). ☐

Lemma 2.8. *Let the initial value satisfy* (1.7). *If there exists $C > 0$ and $q \geq 1$ such that $q > \frac{n}{2}$ and*

$$\|u(\cdot,t)\|_{L^q(\Omega)} \leq C \quad \text{for all } t\in(0,T). \tag{2.11}$$

Then for all $T \in (0, T_{\max})$, there exists $C(T) > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} < C(T) \quad \text{for all } t\in(0,T_{\max}). \tag{2.12}$$

Proof. Testing the third equation in (1.6) with qw^{q-1} and applying Young's inequality to find a positive constant c_1 such that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega w^q &= -q\delta\int_\Omega w^q + q\int_\Omega uw^{q-1} \\
&\leq -\frac{q\delta}{2}\int_\Omega w^q + c_1\int_\Omega u^q \quad \text{for all } t\in(0,T_{\max}).
\end{aligned} \tag{2.13}$$

From (2.11) and (2.13), it follows that for some $c_2 > 0$,

$$\int_\Omega w^q \leq c_2 \quad \text{for all } t\in(0,T_{\max}). \tag{2.14}$$

From $q > n/2$, we have $\frac{nq}{(n-q)_+} > n$. So, we can pick $\Theta > \max\{1, \frac{n}{2}\}$ such that $\frac{nq}{(n-q)_+} > 2\Theta > n$. By applying (2.14) and [17, Lemma 1.2], we conclude that there exists a positive constant c_3 such that $\|\nabla v(\cdot,t)\|_{L^{2\Theta}(\Omega)} \leq c_3$ for all $t\in(0,T_{\max})$. Moreover, for arbitrary $T\in(0,T_{\max})$, (2.14) and Lemma 2.7 imply $v > c_4(T)$ for some $c_4(T) > 0$ in $\Omega\times(0,T)$.

For all $p > 1$, multiplying the first equation of (1.6) by pu^{p-1} and employing Young's inequality, we can arrive that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u^p + \frac{c_4^\alpha(T)p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \int_\Omega u^p \\
&\leq \frac{\alpha^2 c_5(T)p(p-1)}{2}\int_\Omega u^p|\nabla v|^2 + (ap+1)\int_\Omega u^p
\end{aligned} \tag{2.15}$$

for all $t \in (0, T_{\max})$, where $c_5(T) = \max\{\|v_0\|^{\alpha-2}_{L^\infty(\Omega)}, c_4^{\alpha-2}(T)\}$. It is clear from the previously mentioned range of Θ that $\frac{2\Theta}{\Theta-1} < \frac{2n}{(n-2)_+}$, using an Ehrling type inequality and Lemma 2.2, one can find $c_7 = c_7(p) > 0$ such that

$$\begin{aligned} c_5 \int_\Omega u^p |\nabla v|^2 &\le c_5 \|u^{\frac{p}{2}}\|^2_{L^{\frac{2\Theta}{\Theta-1}}(\Omega)} \|\nabla v\|^2_{L^{2\Theta}(\Omega)} \\ &\le c_5 c_3^2 \|u^{\frac{p}{2}}\|^2_{L^{\frac{2\Theta}{\Theta-1}}(\Omega)} \\ &\le c_4 \int_\Omega |\nabla u^{\frac{p}{2}}|^2 + c_7 \end{aligned} \tag{2.16}$$

for all $t \in (0, T_{\max})$. By combining (2.15) and (2.16) and applying an ODE argument, we conclude that there exists a constant $c_8 = c_8(p, T) > 0$ such that $\|u(\cdot, t)\|_{L^p(\Omega)} \le c_8 \quad$ for all $t \in (0, T_{\max})$. Using [17, Lemma 1.2] again, we can find that $c_9 > 0$ such that

$$\|\nabla v\|_{L^\infty(\Omega)} \le c_9 \quad \text{for all } t \in (0, T_{\max}). \tag{2.17}$$

In accordance with [30, Lemma A.1], we can find a positive constant $c_{10}(T)$ such that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \le c_{10}(T) \quad \text{for all } t \in (0, T_{\max}). \tag{2.18}$$

Applying the variation-of-constants formula for w, we have

$$w(\cdot, t) = e^{-\delta t} w_0 + \int_0^t e^{-\delta(t-s)} u(\cdot, s)\, ds,$$

which implies from (2.18) that there exists $c_{11}(T) > 0$ such that

$$\|w(\cdot, t)\|_{L^\infty(\Omega)} \le c_{11}(T) \quad \text{for all } t \in (0, T_{\max}). \tag{2.19}$$

Thus, by (2.17)–(2.19) and the extensibility criterion from Lemma 2.1, the proof is complete. □

3. Existence of global solutions

In this section, we will provide a proof of the main theorem.

Lemma 3.1. *Assume that the initial value satisfies* (1.7). *If $\gamma > 1$, then for all $p > 1$, there exist $C_1, C_2 > 0$ such that*

$$\begin{aligned} \frac{d}{dt} \int_\Omega u^p + \frac{p(p-1)}{2} \int_\Omega u^{p-2} v^\alpha |\nabla u|^2 &\le \frac{\alpha^2 p(p-1)}{2} \int_\Omega u^p v^{\alpha-2} |\nabla v|^2 + ap \int_\Omega u^p \\ &\quad - bp \int_\Omega u^{p+\gamma-1} \quad \textit{for all } t \in (0, T_{\max}) \end{aligned} \tag{3.1}$$

and

$$\frac{d}{dt} \int_\Omega w^{p+1} + \frac{(p+1)\delta}{2} \int_\Omega w^{p+1} \le (\frac{2}{\delta})^p \int_\Omega u^{p+1} \quad \textit{for all } t \in (0, T_{\max}), \tag{3.2}$$

where

$$K = \frac{\max_{0 \le s \le \|v_0\|_{L^\infty(\Omega)}} |\phi'(s)|}{\sqrt{\min_{0 \le s \le \|v_0\|_{L^\infty(\Omega)}} \phi(s)}}.$$

Proof. Multiplying the first equation of (1.6) by pu^{p-1} and applying Young's inequality, we obtain

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega u^p &= -p(p-1)\int_\Omega u^{p-2}v^\alpha|\nabla v|^2 - \alpha p(p-1)\int_\Omega u^{p-1}v^{\alpha-1}\nabla u\cdot\nabla v \\
&\quad + ap\int_\Omega u^p - bp\int_\Omega u^{p+\gamma-1} \\
&\le -\frac{p(p-1)}{2}\int_\Omega u^{p-2}v^\alpha|\nabla v|^2 + \frac{\alpha^2 p(p-1)}{2}\int_\Omega u^p v^{\alpha-2}|\nabla v|^2 \\
&\quad + ap\int_\Omega u^p - bp\int_\Omega u^{p+\gamma-1} \quad \text{for all } t\in(0,T_{\max}),
\end{aligned}
\tag{3.3}
$$

which results in (3.1).

The third equation in the system (1.6) is tested using $(p+1)w^p$, then applying Young's inequality to this resultant ensures that (3.2) is true. □

Lemma 3.2. *Under the assumptions of* (1.7)*, for all* $p\ge 2$*, we have*

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega v^{-p+1}|\nabla v|^p + p(p-1)\int_\Omega v^{-p+3}|\nabla v|^{p-2}|D^2\ln v|^2 \\
&\le \frac{p}{2}\int_{\partial\Omega} v^{-p+1}|\nabla v|^{p-2}\cdot\frac{\partial|\nabla v|^2}{\partial v} + p(p-2+\sqrt{n})\int_\Omega wv^{-p+2}|\nabla v|^{p-2}|D^2 v|
\end{aligned}
\tag{3.4}
$$

for all $t\in(0,T_{\max})$.

Proof. Applying integration by parts to the second equation in (1.6) and using the well-known equation $2\nabla v\cdot\nabla\Delta v = \Delta|\nabla v|^2 - 2|D^2 v|^2$, we find that

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega v^{1-p}|\nabla v|^p \\
&= p\int_\Omega v^{1-p}|\nabla v|^{p-2}\nabla v\cdot\nabla(\Delta v - uvw) - (p-1)\int_\Omega v^{-p}|\nabla v|^p(\Delta v - uvw) \\
&= \frac{p}{2}\int_\Omega v^{1-p}|\nabla v|^{p-2}(\Delta|\nabla v|^2 - 2|D^2 v|^2) - p\int_\Omega v^{1-p}|\nabla v|^{p-2}\nabla v\cdot\nabla(uvw) \\
&- (p-1)\int_\Omega v^{-p}|\nabla v|^p\Delta v + (p-1)\int_\Omega wv^{1-p}|\nabla v|^p \\
&= p(p-1)\int_\Omega v^{-p}|\nabla v|^{p-2}\nabla v\cdot\nabla|\nabla v|^2 - p\int_\Omega v^{1-p}|\nabla v|^{p-2}|D^2 v|^2 \\
&\quad - \frac{p(p-2)}{4}\int_\Omega v^{1-p}|\nabla v|^{p-4}|\nabla|\nabla v|^2|^2 - p(p-1)\int_\Omega v^{-p-1}|\nabla v|^{p+2} \\
&\quad + \frac{p}{2}\int_{\partial\Omega} v^{1-p}|\nabla v|^{p-2}\cdot\frac{\partial|\nabla v|^2}{\partial\nu} + \frac{p(p-2)}{2}\int_\Omega wv^{-p+2}|\nabla v|^{p-4}\nabla v\cdot\nabla|\nabla v|^2 \\
&\quad + p\int_\Omega wv^{-p+2}|\nabla v|^{p-2}\Delta v - (p-1)^2\int_\Omega wv^{1-p}|\nabla v|^p.
\end{aligned}
\tag{3.5}
$$

The pointwise identity [49, Lemma 3.2] and $\nabla|\nabla v|^2 = 2D^2 v\cdot\nabla v$ imply that

$$\begin{aligned} &p(p-1)\int_\Omega v^{-p}|\nabla v|^{p-2}\nabla v\cdot\nabla|\nabla v|^2 - p\int_\Omega v^{1-p}|\nabla v|^{p-2}|D^2v|^2\\ &-\frac{p(p-2)}{4}\int_\Omega v^{1-p}|\nabla v|^{p-4}|\nabla|\nabla v|^2|^2 - p(p-1)\int_\Omega v^{-p-1}|\nabla v|^{p+2}\\ &=-p(p-1)\int_\Omega v^{-p+3}|\nabla v|^{p-2}(\frac{1}{v^4}|\nabla v|^4-\frac{1}{v^3}\nabla v\cdot\nabla|\nabla v|^2+\frac{1}{v^2}|D^2v|^2)\\ &=-p(p-1)\int_\Omega v^{-p+3}|\nabla v|^{p-2}|D^2\ln v|^2. \end{aligned}\tag{3.6}$$

Using the well-known inequality $|\Delta v|\le\sqrt{n}|D^2v|$, the sixth and seventh summands on the right hand side of (3.5) can be estimated as

$$\begin{aligned} &\frac{p(p-2)}{2}\int_\Omega wv^{-p+2}|\nabla v|^{p-4}\nabla v\cdot\nabla|\nabla v|^2 + p\int_\Omega wv^{-p+2}|\nabla v|^{p-2}\Delta v\\ &\le p(p-2+\sqrt{n})\int_\Omega wv^{-p+2}|\nabla v|^{p-2}|D^2v|. \end{aligned}\tag{3.7}$$

Substituting (3.6) and (3.7) into (3.5) yields the result of this lemma. □

Lemma 3.3. *Let $\gamma>1$ and $p>1$. If $q\ge 2$ satisfies $q<2(p+\gamma-2)$. Then there exists a constant $C>0$ such that*

$$\int_\Omega u^{\frac{q+2}{2}} \le \frac{(q+2)\delta}{8k_1}\cdot(\frac{\delta}{2})^{q/2}\cdot\frac{bp}{4}\int_\Omega u^{p+\gamma-1}+C \quad \textit{for all } t\in(0,T_{\max}),\tag{3.8}$$

where k_1 is a positive constant.

Proof. From $q<2(p+\gamma-2)$, we have $\frac{q+2}{2}<p+\gamma-1$. Thus, we derive from $\frac{q+2}{2}+(p+\gamma-2-\frac{q}{2})=p+\gamma-1$ that

$$p+\gamma-1-\frac{q+2}{2}=p+\gamma-2-\frac{q}{2}>0.$$

Applying Young's inequality to $\frac{8k_1}{(q+2)\delta}(\frac{2}{\delta})^{\frac{q}{2}}\int_\Omega u^{\frac{q+2}{2}}$, it is easy to see that for some $c>0$,

$$\frac{8k_1}{(q+2)\delta}(\frac{2}{\delta})^{q/2}\int_\Omega u^{\frac{q+2}{2}}\le\frac{bp}{4}\int_\Omega u^{p+\gamma-1}+c \text{ for all } t\in(0,T_{\max}).\tag{3.9}$$

The proof is complete. □

Lemma 3.4. *Let $\alpha\ge1$, $\gamma>1$ and $p>2$ satisfying $q>\frac{2p}{\gamma-1}$. Then one can find a positive constant C such that*

$$\begin{aligned} &\frac{p(p-1)}{2}\Big(\frac{q(q-1)}{2p(p-1)(q+\sqrt{n})^2}\Big)^{-2/q}\alpha^{\frac{2(q+2)}{q}}\|v_0\|_{L^\infty(\Omega)}^{\frac{\alpha(q+2)-2}{q}}\int_\Omega u^{\frac{p(q+2)}{q}}\\ &\le\frac{bp}{4}\int_\Omega u^{p+\gamma-1}+C \end{aligned}\tag{3.10}$$

for all $t\in(0,T_{\max})$.

Proof. From $q>\frac{2p}{\gamma-1}$, we have $q(\gamma-1)-2p>0$, then

$$p+\gamma-1-\frac{p(q+2)}{q}=\frac{(\gamma-1)q-2p}{q}>0.$$

Thus, we apply Young's inequality to

$$\frac{p(p-1)}{2}\Big(\frac{q(q-1)}{2p(p-1)(q+\sqrt{n})^2}\Big)^{-2/q}\alpha^{\frac{2(q+2)}{q}}\|v_0\|_{L^\infty(\Omega)}^{\frac{\alpha(q+2)-2}{q}}\int_\Omega u^{\frac{p(q+2)}{q}}$$

we show that there exists $c>0$ such that

$$\begin{aligned}
&\frac{p(p-1)}{2}\Big(\frac{q(q-1)}{2p(p-1)(q+\sqrt{n})^2}\Big)^{-2/q}\alpha^{\frac{2(q+2)}{q}}\|v_0\|_{L^\infty(\Omega)}^{\frac{\alpha(q+2)-2}{q}}\int_\Omega u^{\frac{p(q+2)}{q}}\\
&\le \frac{bp}{4}\int_\Omega u^{p+\gamma-1}+c \quad \text{for all } t\in(0,T_{\max}).
\end{aligned}$$

Therefore, we obtain the desired result. □

Lemma 3.5. *Assume* (1.7) *holds,* $\alpha\ge 1$, $\gamma>2$. *Then, for all* $p>1$ *and* $q\ge 2$, *one can find a positive constant* C *such that*

$$\|u(\cdot,t)\|_{L^p(\Omega)}\le C \quad \textit{for all } t\in(0,T_{\max}). \tag{3.11}$$

Proof. It follows from Lemms 3.1 and 3.2 that

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_\Omega u^p+\int_\Omega v^{-q+1}|\nabla v|^q\Big)+\int_\Omega u^p+\int_\Omega v^{-q+1}|\nabla v|^q\\
&+q(q-1)\int_\Omega v^{-q+3}|\nabla v|^{q-2}|D^2\ln v|^2\\
&\le \frac{\alpha^2p(p-1)}{2}\int_\Omega u^pv^{\alpha-2}|\nabla v|^2+(ap+1)\int_\Omega u^p-bp\int_\Omega u^{p+l-1}\\
&\quad+q(q-2+\sqrt{n})\int_\Omega wv^{-q+2}|\nabla v|^{q-2}|D^2v|\\
&\quad+\frac{q}{2}\int_{\partial\Omega}v^{-q+1}|\nabla v|^{q-2}\cdot\frac{\partial|\nabla v|^2}{\partial\nu}+\int_\Omega v^{-q+1}|\nabla v|^q.
\end{aligned}\tag{3.12}$$

Using Young's inequality and Lemma 2.3, we see that for any $\zeta>0$,

$$\begin{aligned}
&\frac{\alpha^2p(p-1)}{2}\int_\Omega u^pv^{\alpha-2}|\nabla v|^2\\
&\le \frac{p(p-1)}{2}\zeta^{\frac{q+2}{2}}\int_\Omega v^{-q-1}|\nabla v|^{q+2}\\
&\quad+\frac{p(p-1)}{2}\zeta^{-\frac{q+2}{q}}\alpha^{\frac{2(q+2)}{q}}\int_\Omega u^{\frac{p(q+2)}{q}}v^{\frac{\alpha(q+2)-2}{q}}\\
&\le \frac{p(p-1)(q+\sqrt{n})^2}{2}\zeta^{\frac{q+2}{2}}\int_\Omega v^{-q+3}|\nabla v|^{q-2}|D^2\ln v|^2\\
&\quad+\frac{p(p-1)}{2}\zeta^{-\frac{q+2}{q}}\alpha^{\frac{2(q+2)}{q}}\|v_0\|_{L^\infty(\Omega)}^{\frac{\alpha(q+2)-2}{q}}\int_\Omega u^{\frac{p(q+2)}{q}}.
\end{aligned}\tag{3.13}$$

By selecting

$$\zeta=\big(\frac{q(q-1)}{2p(p-1)(q+\sqrt{n})^2}\big)^{\frac{2}{q+2}}$$

and applying (3.13), we estimate that

$$\begin{aligned}&\frac{\alpha^2 p(p-1)}{2}\int_\Omega u^p v^{\alpha-2}|\nabla v|^2\\&\le \frac{q(q-1)}{4}\int_\Omega v^{-q+3}|\nabla v|^{q-2}|D^2\ln v|^2\\&\quad+\frac{p(p-1)}{2}\Big(\frac{q(q-1)}{2p(p-1)(q+\sqrt{n})^2}\Big)^{-2/q}\alpha^{\frac{2(q+2)}{q}}\|v_0\|_{L^\infty(\Omega)}^{\frac{\alpha(q+2)-2}{q}}\int_\Omega u^{\frac{p(q+2)}{q}}.\end{aligned}\tag{3.14}$$

For any $\eta_1,\ \eta_2>0$, we can see from Young's inequality that

$$\begin{aligned}&q(q-2+\sqrt{n})\int_\Omega wv^{-q+2}|\nabla v|^{q-2}|D^2v|\\&\le \eta_1\int_\Omega v^{-q+1}|\nabla v|^{q-2}|D^2v|^2+\eta_1^{-1}q^2(q-2+\sqrt{n})^2\int_\Omega w^2v^{-q+3}|\nabla v|^{q-2}\\&\le \eta_1(q+\sqrt{n}+1)^2\int_\Omega v^{-q+3}|\nabla v|^{q-2}|D^2\ln v|^2\\&\quad+\eta_1^{-1}q^2(q-2+\sqrt{n})^2\eta_2^{\frac{q+2}{q-2}}\int_\Omega v^{-q-1}|\nabla v|^{q+2}\\&\quad+\eta_1^{-1}q^2(q-2+\sqrt{n})^2\eta_2^{-\frac{q+2}{4}}\int_\Omega w^{\frac{q+2}{2}}v\\&\le (q+\sqrt{n}+1)^2\big\{\eta_1+\eta_1^{-1}q^2(q-2+\sqrt{n})^2\eta_2^{\frac{q+2}{q-2}}\big\}\int_\Omega v^{-q+3}|\nabla v|^{q-2}|D^2\ln v|^2\\&\quad+\eta_1^{-1}q^2(q-2+\sqrt{n})^2\eta_2^{-\frac{q+2}{4}}\|v_0\|_{L^\infty(\Omega)}\int_\Omega w^{\frac{q+2}{2}}.\end{aligned}\tag{3.15}$$

If we let

$$\eta_1=\frac{q(q-1)}{8(q+\sqrt{n}+1)^2},\eta_2=\Big(\frac{(q-1)^2}{64(q+\sqrt{n}+1)^4(q-2+\sqrt{n})^2}\Big)^{\frac{q-2}{q+2}},$$

then (3.15) can be simplified as

$$\begin{aligned}&q(q-2+\sqrt{n})\int_\Omega wv^{-q+2}|\nabla v|^{q-2}|D^2v|\\&\le \frac{q(q-1)}{4}\int_\Omega v^{-q+3}|\nabla v|^{q-2}|D^2\ln v|^2\\&\quad+\frac{8^{q/2}q(q+\sqrt{n}+1)^q(q-2+\sqrt{n})^{\frac{q+2}{2}}\|v_0\|_{L^\infty(\Omega)}}{(q-1)^{q/2}}\int_\Omega w^{\frac{q+2}{2}}.\end{aligned}\tag{3.16}$$

Substituting (3.14) and (3.16) into (3.12) yields

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_\Omega u^p+\int_\Omega v^{-q+1}|\nabla v|^q\Big)+\int_\Omega u^p+\int_\Omega v^{-q+1}|\nabla v|^q\\
&+\frac{q(q-1)}{2}\int_\Omega v^{-q+3}|\nabla v|^{q-2}|D^2\ln v|^2\\
&\le \frac{p(p-1)}{2}\big(\frac{q(q-1)}{2p(p-1)(q+\sqrt{n})^2}\big)^{-2/q}\alpha^{\frac{2(q+2)}{q}}\|v_0\|_{L^\infty(\Omega)}^{\frac{\alpha(q+2)-2}{q}}\int_\Omega u^{\frac{p(q+2)}{q}}\\
&+k_1\int_\Omega w^{\frac{q+2}{2}}+\frac{q}{2}\int_{\partial\Omega} v^{-q+1}|\nabla v|^{q-2}\cdot\frac{\partial|\nabla v|^2}{\partial\nu}\\
&+\int_\Omega v^{-q+1}|\nabla v|^q+(ap+1)\int_\Omega u^p-bp\int_\Omega u^{p+\gamma-1},
\end{aligned}\tag{3.17}$$

where

$$k_1=\frac{8^{q/2}q(q+\sqrt{n}+1)^q(q-2+\sqrt{n})^{\frac{q+2}{2}}\|v_0\|_{L^\infty(\Omega)}}{(q-1)^{q/2}}.$$

Then, we estimate the second and third terms on the right-hand side of (3.17). By (2.2), Lemmas 2.3, 2.5 and Young's inequality, we find that for any $\varrho>0$, there exists a positive constant c_1 such that

$$\begin{aligned}
&\frac{q}{2}\int_{\partial\Omega} v^{-q+1}|\nabla v|^{q-2}\cdot\frac{\partial|\nabla v|^2}{\partial\nu}\\
&\le \varrho\int_\Omega v^{-q-1}|\nabla v|^{q+2}+\varrho\int_\Omega v^{-q+1}|\nabla v|^{q-2}|D^2v|^2+c_1\int_\Omega v\\
&\le 2(q+\sqrt{n}+1)^2\varrho\int_\Omega v^{-q+3}|\nabla v|^{q-2}|D^2\ln v|^2+c_1|\Omega|\|v_0\|_{L^\infty(\Omega)}
\end{aligned}\tag{3.18}$$

and

$$\begin{aligned}
&\int_\Omega v^{-q+1}|\nabla v|^q\\
&\le \varrho^{\frac{q+2}{q}}\int_\Omega v^{-q-1}|\nabla v|^{q+2}+\varrho^{-\frac{q+2}{2}}\int_\Omega v\\
&\le (q+\sqrt{n})^2\varrho^{\frac{q+2}{q}}\int_\Omega v^{-q+3}|\nabla v|^{q-2}|D^2\ln v|^2+\varrho^{-\frac{q+2}{2}}|\Omega|\|v_0\|_{L^\infty(\Omega)}.
\end{aligned}\tag{3.19}$$

Because

$$(ap+1)\int_\Omega u^p-bp\int_\Omega u^{p+\gamma-1}\le-\frac{bp}{2}\int_\Omega u^{p+\gamma-1}+c_2\tag{3.20}$$

for some $c_2>0$, by selecting a suitably small ϱ and substituting (3.18)–(3.20) into (3.17), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega u^p+\int_\Omega v^{-q+1}|\nabla v|^q\Big)+\int_\Omega u^p+\int_\Omega v^{-q+1}|\nabla v|^q\\
&\le \frac{p(p-1)}{2}\big(\frac{q(q-1)}{2p(p-1)(q+\sqrt{n})^2}\big)^{-2/q}\alpha^{\frac{2(q+2)}{q}}\|v_0\|_{L^\infty(\Omega)}^{\frac{\alpha(q+2)-2}{q}}\int_\Omega u^{\frac{p(q+2)}{q}}\\
&+k_1\int_\Omega w^{\frac{q+2}{2}}-\frac{bp}{2}\int_\Omega u^{p+\gamma-1}+C.
\end{aligned}\tag{3.21}$$

Multiplying the third equation of the system (1.6) by $(\frac{q}{2}+1)w^{q/2}$ gives

$$\frac{d}{dt}\int_\Omega w^{\frac{q+2}{2}} + (\frac{q}{2}+1)\delta\int_\Omega w^{\frac{q+2}{2}} \le (\frac{q}{2}+1)\int_\Omega w^{q/2}u. \tag{3.22}$$

With the help of Young's inequality, we have

$$\begin{aligned}(\frac{q}{2}+1)\int_\Omega w^{q/2}u &\le \frac{(q+2)\delta}{4}\int_\Omega w^{\frac{q}{2}+1} + (\frac{2q}{(q+2)\delta})^{q/2}\int_\Omega u^{\frac{q}{2}+1}\\ &\le \frac{(q+2)\delta}{4}\int_\Omega w^{\frac{q}{2}+1} + (\frac{2}{\delta})^{q/2}\int_\Omega u^{\frac{q}{2}+1}.\end{aligned} \tag{3.23}$$

Collecting (3.22) and (3.23), we obtain

$$\frac{d}{dt}\int_\Omega w^{\frac{q+2}{2}} + \frac{(q+2)\delta}{4}\int_\Omega w^{\frac{q+2}{2}} \le (\frac{2}{\delta})^{q/2}\int_\Omega u^{\frac{q+2}{2}}. \tag{3.24}$$

Multiplying $\frac{8k_1}{(q+2)\delta}$ in the both sides of (3.24), we have

$$\frac{8k_1}{(q+2)\delta}\cdot\frac{d}{dt}\int_\Omega w^{\frac{q+2}{2}} + 2k_1\int_\Omega w^{\frac{q+2}{2}} \le \frac{8k_1}{(q+2)\delta}(\frac{2}{\delta})^{q/2}\int_\Omega u^{\frac{q+2}{2}}. \tag{3.25}$$

Inserting (3.25) into (3.21), we see that

$$\begin{aligned}&\frac{d}{dt}\Big(\int_\Omega u^p + \int_\Omega v^{-q+1}|\nabla v|^q + \frac{8k_1}{(q+2)\delta}\int_\Omega w^{\frac{q+2}{2}}\Big)\\ &+\int_\Omega u^p + \int_\Omega v^{-q+1}|\nabla v|^q + k_1\int_\Omega w^{\frac{q+2}{2}}\\ &\le \frac{p(p-1)}{2}(\frac{q(q-1)}{2p(p-1)(q+\sqrt{n})^2})^{-2/q}\alpha^{\frac{2(q+2)}{q}}\|v_0\|_{L^\infty(\Omega)}^{\frac{\alpha(q+2)-2}{q}}\int_\Omega u^{\frac{p(q+2)}{q}}\\ &\quad + \frac{8k_1}{(q+2)\delta}(\frac{2}{\delta})^{q/2}\int_\Omega u^{\frac{q+2}{2}} - \frac{bp}{2}\int_\Omega u^{p+\gamma-1} + C,\end{aligned} \tag{3.26}$$

where k_1 is defined as in (3.17). Because

$$2(p+\gamma-2) - \frac{2p}{\gamma-1} = \frac{2[(p+\gamma-2)(\gamma-1)-p]}{\gamma-1} = \frac{2(\gamma-2)(p+\gamma-1)}{\gamma-1} > 0.$$

Then, by selecting $q \ge 2$, we have

$$\frac{2p}{\gamma-1} < q < 2(p+\gamma-2). \tag{3.27}$$

Moreover, in Lemma 3.3, taking $k = k_1$ and combining Lemma 3.4 with (3.26), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega u^{\mathrm{p}} + \int_\Omega v^{-q+1}|\nabla v|^q + \frac{8\kappa_1}{(q+2)\delta}\int_\Omega w^{\frac{q+2}{2}}\Big)\\ &+\min\big\{1, \frac{(q+2)\delta}{8}\big\}\Big(\int_\Omega u^p + \int_\Omega v^{-q+1}|\nabla v|^q + \frac{8\kappa_1}{(q+2)\delta}\int_\Omega w^{\frac{q+2}{2}}\Big) \le c_0\end{aligned}$$

for some $c_0 > 0$. Thus, using a standard ODE comparison parameter gives (3.11). $\square$

Lemma 3.6. *Let $\gamma = 2$, $n \ge 4$, $p > 1$ and assume that* (1.7) *holds, and b satisfies*

$$b > \mu_1(p,n)\alpha^{\frac{2(p+1)}{p}}\|v_0\|_{L^\infty(\Omega)}^{\frac{\alpha(p+1)-1}{p}} + \mu_2(p,n,\delta)\|v_0\|_{L^\infty(\Omega)}, \tag{3.28}$$

where

$$\mu_1(p,n) = (p-1)^{\frac{p+1}{p}}(2p+\sqrt{n})^{\frac{2}{p}}(2p-1)^{-\frac{1}{p}}, \tag{3.29}$$

$$\mu_2(p,n,\delta) = \frac{2^{4p+4}(2p+\sqrt{n}+1)^{2p}(2p-2+\sqrt{n})^{p+1}}{(p+1)(2p-1)^p\delta^{p+1}}. \tag{3.30}$$

Then one can find $C>0$ such that

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le C \quad \text{for all } t\in(0,T_{\max}). \tag{3.31}$$

Proof. Since $\gamma=2$, from (3.29), (3.30) and setting $q=2p$ in (3.26), we deduce that

$$\begin{aligned} &\frac{d}{dt}\Big(\int_\Omega u^p + \int_\Omega v^{1-2p}|\nabla v|^{2p} + \frac{4k_1}{(p+1)\delta}\int_\Omega w^{p+1}\Big) \\ &+ \int_\Omega u^p + \int_\Omega v^{1-2p}|\nabla v|^{2p} + k_1\int_\Omega w^{p+1} \\ &\le -\frac{p}{2}\big\{b - \mu_1(p,n)\alpha^{\frac{2(p+1)}{p}}\|v_0\|_{L^\infty(\Omega)}^{\frac{\alpha(p+1)-1}{p}} - \mu_2(p,n)\|v_0\|_{L^\infty(\Omega)}\big\}\int_\Omega u^{p+1} + C \end{aligned} \tag{3.32}$$

for all $t\in(0,T_{\max})$. Using (3.28), (3.32) can be reduced to

$$\begin{aligned} &\frac{d}{dt}\Big(\int_\Omega u^p + \int_\Omega v^{1-2p}|\nabla v|^{2p} + \frac{4k_1}{(p+1)\delta}\int_\Omega w^{p+1}\Big) \\ &+ \min\big\{1, \frac{(p+1)\delta}{4}\big\}\Big(\int_\Omega u^p + \int_\Omega v^{1-2p}|\nabla v|^2 p + \frac{4k_1}{(p+1)\delta}\int_\Omega w^{p+1}\Big) \\ &\le C. \end{aligned} \tag{3.33}$$

Thus, a standard ODE comparison argument leads to (3.31). $\square$

Lemma 3.7. *Let $\gamma=2$ and $n=2,3$, Then, for all $T>0$, there exists $C(T)>0$ such that*

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le C \quad \text{for all } t\in(0,T_{\max}). \tag{3.34}$$

Proof. Letting $p=2$ in (3.1), we have

$$\frac{d}{dt}\int_\Omega u^2 + \int_\Omega v^\alpha|\nabla u|^2 + \int_\Omega u^2 \le \alpha^2\int_\Omega v^{\alpha-2}u^2|\nabla v|^2 + (2a+1)\int_\Omega u^2. \tag{3.35}$$

For $\gamma=2$, (2.5) and Lemma 2.7 show that for any $T>0$, there exists a constant $c_1(T)>0$ such that $v\ge c_1(T)$ in $\Omega\times(0,T)$. Thus (3.35) can be rewritten as

$$\frac{d}{dt}\int_\Omega u^2 + c_1^\alpha(T)\int_\Omega|\nabla u|^2 + \int_\Omega u^2 \le \alpha^2 c_2(T)\int_\Omega u^2|\nabla v|^2 + (2a+1)\int_\Omega u^2 \tag{3.36}$$

with $c_2(T) := \max\Big\{c_1^{\alpha-2}(T), \|v_0\|_{L^\infty(\Omega)}^{\alpha-2}\Big\}$. Using $\gamma=2,\ n=2,3$, in conjunction with (2.5) and [17, Lemma 1.2], one can find $c_3>0$ such that

$$\|v(\cdot,t)\|_{W^{1,4}(\Omega)} \le c_3 \quad \text{for all } t\in(0,T_{\max}). \tag{3.37}$$

Using $n = 2,3$, (2.1), (3.37) and applying Hölder's inequality and Ehrling type inequality, we estimate taht for some $c_4(T) > 0$,

$$\begin{aligned} &\alpha^2 c_2(T)\int_\Omega u^2|\nabla v|^2 + (2a+1)\int_\Omega u^2 \\ &\le \alpha^2 c_2(T)\|u\|^2_{L^4(\Omega)}\|\nabla v\|^2_{L^4(\Omega)} + (2a+1)\|u\|^2_{L^2(\Omega)} \\ &\le \alpha^2 c_2(T)c_3^2\|u\|^2_{L^4(\Omega)} + (2a+1)\|u\|^2_{L^2(\Omega)} \\ &\le \frac{c_1^\alpha(T)}{2}\int_\Omega |\nabla u|^2 + c_4(T) \end{aligned} \tag{3.38}$$

for all $t \in (0, T_{\max})$. Collecting (3.38) and (3.36), we estimate

$$\frac{d}{dt}\int_\Omega u^2 + \frac{c_1^\alpha(T)}{2}\int_\Omega |\nabla u|^2 + \int_\Omega u^2 \le c_4(T) \quad \text{for all } t \in (0, T_{\max}). \tag{3.39}$$

We complete the proof of (3.34) by using an ODE argument. □

Lemma 3.8. *Let $\gamma \ge 1$. Then, for all $n \ge 1$, if one of the following 3 conditions is true: (i) $l > 2$; (ii) $l = 2$ and $n \le 3$; (iii) $l = 2, n \ge 4$ and $b > \mu_1(\frac{n}{2}, n)\alpha^{\frac{2(n+2)}{n}}\|v_0\|_{L^\infty(\Omega)}^{\frac{\alpha(n+2)-2}{n}} + \mu_2(\frac{n}{2}, n, \delta)\|v_0\|_{L^\infty(\Omega)}$, where μ_1 μ_2 are defined as in* (3.29) *and* (3.30)*.*
Then for all $T \in (0, T_{\max})$, one can find $C(T) > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} \le C(T) \quad \textit{for all } t \in (0,T).$$

Proof. Let $p \ge 1$ such that $p > \frac{n}{2}$. For the parameter γ, the following three cases are discussed. If $\gamma > 2$, we know that $\|u(\cdot,t)\|_{L^p(\Omega)}$ is bounded from Lemma 3.5. If $\gamma = 2$, from (2.1) and Lemma 3.7, we obtain that $\|u(\cdot,t)\|_{L^1(\Omega)}$ is bounded under the condition $n = 1$ and $\|u(\cdot,t)\|_{L^2(\Omega)}$ is bounded under the condition $n = 2,3$. If $\gamma = 2$ and $n \ge 4$, since $p > \frac{n}{2}$, using Lemma 3.6 with respect to b, we also get that $\|u(\cdot,t)\|_{L^p(\Omega)}$ is bounded for all $t \in (0, T_{\max})$. Finally, we use Lemma 2.8 to complete the proof. □

Now Theorem 1.1 follows directly from Lemmas 2.1 and 3.8.

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