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STRUCTURE AND STABILITY OF GLOBAL ATTRACTORS FOR A CAHN-HILLIARD TUMOR GROWTH MODEL WITH CHEMOTAXIS

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Abstract. In this article, we analyze the long-time dynamics of a Cahn-Hilliard tumor growth model, focusing on the geometric structure and stability of its global attractors. Using a Łojasiewicz-Simon type inequality, we first prove that every full trajectory in the global attractor converges to a single stationary point as $t \to \infty$ and to another stationary point as $t \to -\infty$. As a result, we show that the global attractor is the union of the unstable manifolds emanating from the stationary points. We also examine the rate of convergence to these stationary points and provide specific polynomial and exponential rates under certain conditions. Additionally, we demonstrate that the global attractors of the corresponding tumor growth model exhibit upper-semicontinuity with respect to small perturbations of the chemotaxis parameter. Finally, by restricting chemotaxis within a certain interval, we establish the lower-semicontinuity of the global attractors for this model.

1. Introduction

Tumor growth and its associated dynamics have long been a subject of significant interest in the fields of biology, medicine and mathematical modeling. One class of mathematical models that has proven particularly useful in this context is the diffuse interface models. The key mathematical equation used in diffuse interface models for tumor growth is the Cahn-Hilliard equation,

$$\phi_t - \Delta(-\Delta\phi + f(\phi)) = 0. \tag{1.1}$$

Cahn-Hilliard equations were originally developed to describe the phase separation processes in binary materials (see [2]), but they have found applications in wide range of biological systems, including tumor growth.

In this context, the following Cahn-Hilliard system was introduced in [18] to describe tumor growth as a continuum-mixture process.

$$\phi_t = \operatorname{div}(m(\phi)\nabla\mu) + p(\phi)(\chi_\sigma\sigma + \chi_\phi(1-\phi) - \mu) \quad \text{in } (0,T)\times\Omega, \tag{1.2}$$

$$\mu = -\Delta\phi + \Psi'(\phi) - \chi_\phi\sigma \quad \text{in} \quad (0,T)\times\Omega, \tag{1.3}$$

$$\sigma_t = \operatorname{div}(n(\phi)(\chi_\sigma\nabla\sigma - \chi_\phi\nabla\phi)) - p(\phi)(\chi_\sigma\sigma + \chi_\phi(1-\phi) - \mu) \quad \text{in } (0,T)\times\Omega, \tag{1.4}$$

$$\partial_\nu\mu = \partial_\nu\phi = \partial_\nu\sigma = 0 \quad \text{on } (0,T)\times\Gamma. \tag{1.5}$$

Here, $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary Γ and ∂_ν in (1.5) stands for the normal derivative where ν is the outer unit normal to Γ.

Equation (1.2) is a Cahn-Hilliard equation where the order parameter ϕ ranges between -1 and 1, representing the tumorous and healthy phases respectively, with μ denoting the chemical potential for ϕ. Additionally, equation (1.4) is a reaction-diffusion equation where σ represents the chemical concentration acting as a nutrient for the tumor. The terms $m(\phi)$ and $n(\phi)$ represent positive mobilities indicating the diffusivity of the binary mixture and the chemicals. Moreover, Ψ is a potential function characterized by two minima at ± 1. Lastly, $\chi_\phi \geq 0$ stands for the constant representing chemotaxis and active transport, while $\chi_\sigma \geq 0$ denotes the chemical mobility.

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To the best of our knowledge, system (1.2)–(1.5) was first analyzed in the mathematical sense in [10] in the case $\chi_\phi = 0$ and $m(\phi) = n(\phi) = \chi_\sigma = 1$, where the authors proved the well-posedness of the problem and the existence of a global attractor. Subsequently, problem (1.2)–(1.5) with an additional viscosity regularization term was considered in [8, 9]. In [8], the well-posedness of strong solutions was established, and the long-time behavior of the corresponding dynamical system was studied using the concept of the ω-limit set. In [9], some results from [8] were extended to scenarios where the viscosity parameters are independent of each other. Additionally, we refer to [19], where the formal matched asymptotic limit of a quasi-static variant of (1.2)–(1.5) was explored.

Among other valuable contributions in the literature related to similar tumor growth models, we can cite [11, 12, 13, 14]. In [14], a Cahn-Hilliard–Darcy model was introduced for tumor growth with chemotaxis and active transport. In [11] also a Cahn-Hilliard–Darcy system was investigated, and the existence of global weak solutions was established in both two- and three-dimensional cases. In [13], the following tumor growth model was analyzed,

$$
\begin{gathered}
\phi_t = \operatorname{div}(m(\phi)\nabla\mu) + (\lambda_p\sigma - \lambda_a)h(\phi) \quad \text{in } \Omega\times(0,T),\\
\mu = A\Psi'(\phi) - B\Delta\phi - \chi_\phi\sigma \quad \text{in} \quad \Omega\times(0,T),\\
\kappa\sigma_t = \operatorname{div}(n(\phi)(\chi_\sigma\nabla\sigma - \chi_\phi\nabla\phi)) - \lambda_c\sigma h(\phi) \quad \text{in } \Omega\times(0,T),\\
\nabla\mu\cdot\nu = \nabla\phi\cdot\nu = 0, \quad n(\phi)\chi_\sigma\nabla\sigma\cdot\nu = K(\sigma_\infty - \sigma) \quad \text{on } \Gamma\times(0,T).
\end{gathered}
\tag{1.6}
$$

Here, $\kappa = 1$, A, B and K are positive constants. The parameters λ_p, λ_a and λ_c are nonnegative constants representing, respectively, the proliferation rate, the apoptosis rate of the tumor cells, and the nutrient consumption rate. The function $h(\phi)$ is an interpolation function, and σ_∞ denotes the nutrient amount on the boundary. Additionally, $m(\phi)$, $n(\phi)$, χ_ϕ and χ_σ represent the same quantities as in system (1.2)–(1.5). In that paper, well-posedness of the model (1.6) and its quasi-static version ($\kappa = 0$) was established for regular potentials with quadratic growth. On the other hand, in [12], the well-posedness of problem (1.6) with Dirichlet boundary conditions was obtained for regular potentials with higher polynomial growth and even for the singular potentials.

Global attractors are essential tools for analyzing the long-time behavior of infinite-dimensional dynamical systems. Since they are compact, invariant sets that attract all bounded trajectories, understanding their structure and stability is key to gaining deeper insight into the overall dynamics of the system. Some studies have advanced our understanding of global attractors in systems with hyperbolic stationary points, where the number of stationary points is finite [3, 4]. These systems can be described using gradient-like semigroups, where all trajectories move between stationary points, and there are no homoclinic orbits. However, the set of stationary points is generally infinite and those results do not directly apply.

To overcome this limitation, [1] shows a novel approach based on the Łojasiewicz-Simon inequality. In systems where the set of stationary points is infinite, their technique allows one to show that all full trajectories in the global attractor originate from a stationary point as $t \to -\infty$ and converge to another as $t \to \infty$. As a result, the attractor can be described as the union of the unstable manifolds of all stationary points.

This article introduces new results concerning the structure and the stability of the global attractors for problem (1.2)–(1.5), in the case $\chi_\phi > 0$, differing from the approach in [10] and related works. Some foundational results concerning the existence of a global attractor and its characterization as an unstable manifold emanating from the set of stationary points were previously established in [15], in the case $\chi_\phi > 0$. In this paper, we aim to present new findings regarding the geometric structure and stability of the global attractors for problem (1.2)–(1.5). The main contributions of this paper can be summarized as follows.

• Inspired by [1], and using a Łojasiewicz-Simon type inequality, we prove that every full trajectory within the global attractor converges to a single stationary point as $t \to \infty$ and to another stationary point as $t \to -\infty$. We then show that the global attractor is equal to a union of the unstable manifolds emanating from the stationary points, even when the set of stationary points is infinite (see Remark 3.3).

• We obtain the rate of convergence to equilibrium for the full trajectories in the global attractor, in the space $(H^1(\Omega))^* \times (H^1(\Omega))^*$ (see Proposition 4.1). Then, by imposing suitable restrictions on the chemotaxis, we obtain that the every full trajectory in the global attractor converges exponentially to equilibrium (Corollary 4.2).

• In the case where the Łojasiwicz-Simon exponent $0 < \theta < \frac{1}{2}$, we prove that the full trajectories in the global attractor maintain the same convergence rate, established in $(H^1(\Omega))^* \times (H^1(\Omega))^*$ also in the space $\mathcal{Z}_M$. This result is not straightforward and requires careful analysis and detailed estimates beyond standard arguments.

• By considering the chemotaxis and active transport parameter χ_ϕ as a perturbation parameter, we obtain a family of global attractors. We first prove that this family of global attractors is uppersemicontinuous as $\chi_\phi \to 0$. Moreover, restricting the chemotaxis on a suitable interval, we obtain the lower semicontinuity of the global attractors as $\chi_\phi \to 0$.

This article is organized as follows. In Section 2, we introduce the problem and present several foundational results from [15]. In Section 3, we conduct a detailed analysis of the geometric properties of the global attractor, along with convergence of trajectories to equilibrium. Section 4 is dedicated to examining the rate of convergence to equilibrium. Finally, in Section 5, we investigate the upper and lower-semicontinuity properties of the global attractors.

2. Setting of the problem and the previous results

In the rest of this paper, to simplify the notation, we will denote chemotaxis and active transport by χ instead of χ_ϕ. Furthermore, since the choice of χ_σ does not affect the subsequent mathematical analysis, we will assume $\chi_\sigma = 1$. Additionally, we consider the mobilities to be constant, i.e. $m(\phi) = n(\phi) = 1$. Hence, we reformulate the problem (1.2)–(1.5) as follows.

$$\phi_t^\chi = \Delta\mu^\chi + p(\phi^\chi)(\sigma^\chi + \chi(1-\phi^\chi) - \mu^\chi), \quad \text{in } (0,T)\times\Omega, \tag{2.1}$$

$$\mu^\chi = -\Delta\phi^\chi + \Psi'(\phi^\chi) - \chi\sigma^\chi, \quad \text{in } (0,T)\times\Omega, \tag{2.2}$$

$$\sigma_t^\chi = \Delta\sigma^\chi - \chi\Delta\phi^\chi - p(\phi^\chi)(\sigma^\chi + \chi(1-\phi^\chi) - \mu^\chi), \quad \text{in } (0,T)\times\Omega, \tag{2.3}$$

$$\partial_\nu\mu^\chi = \partial_\nu\phi^\chi = \partial_\nu\sigma^\chi = 0, \quad \text{on } (0,T)\times\Gamma, \tag{2.4}$$

where $\chi \geq 0$.

We consider this problem with the following assumptions, which are identical to those in [15].

(A1) The potential $\Psi \in C^2(\mathbb{R})$ can be written as

$$\Psi(s) = \Psi_0(s) + \lambda(s) \tag{2.5}$$

where $\Psi_0 \in C^2(\mathbb{R})$ and $\lambda \in C^2(\mathbb{R})$ satisfies $|\lambda''(s)| \leq \alpha$, for all $s \in \mathbb{R}$, and for some constant $\alpha \geq 0$. Moreover, we assume that

$$c_1(1+|s|^{\rho-2}) \leq \Psi_0''(s) \leq c_2(1+|s|^{\rho-2}), \tag{2.6}$$

$$\Psi(s) \geq R_1|s|^2 - R_2 \tag{2.7}$$

for all $s \in \mathbb{R}$, with c_1, c_2, $R_1 > 2\chi^2$, $R_2 \in \mathbb{R}$ and with $\rho \in [2,6)$.

(A2) The interpolation function $p \in C^{0,1}_{loc}(\mathbb{R})$ satisfies

$$p > 0 \quad \text{and} \quad |p'(s)| \leq c_4(1+|s|^{q-1}) \tag{2.8}$$

for all $s \in \mathbb{R}$, with $c_4 > 0$ and with $q \in [1,4]$.

We also assume the following in the proof of the Łojasiewicz-Simon type inequality.

(A3) The potential $\Psi \in C^\infty(\mathbb{R})$ is an analytic function.

Remark 2.1. Assumption (2.7) is only needed if $\rho = 2$.

Notation. We now introduce some notation that will be used throughout the paper. For a Banach space X, we denote its norm by $\|\cdot\|_X$, its dual space by X^*, and the duality pairing between X^* and X by $\langle\langle \cdot, \cdot \rangle\rangle$. Additionally, we will use $\langle \cdot, \cdot \rangle_X$ for inner product in X. The inner product in $L^2(\Omega)$ will be written as $\langle \cdot, \cdot \rangle$. For every $u \in (H^1(\Omega))^*$, we define the mean value as:

$$\langle u \rangle := \frac{1}{|\Omega|} \langle\langle u, 1 \rangle\rangle \quad \text{for every } u \in (H^1(\Omega))^*.$$

Moreover, we introduce the operator $A : H^1(\Omega) \to (H^1(\Omega))^*$ such that

$$Au = -\Delta u + u \quad \text{and} \quad D(A) = \{\phi \in H^2(\Omega) : \partial_\nu \phi = 0 \text{ on } \Gamma\}.$$

It is worth noting that the restriction of A on $D(A)$ is an isomorphism from $D(A)$ onto $L^2(\Omega)$, i.e. $D(A) = A^{-1}(L^2(\Omega))$. In addition, the following identities hold:

$$\begin{gathered}\langle\langle Au, A^{-1}v^* \rangle\rangle = \langle\langle v^*, u \rangle\rangle \quad \text{for every } u \in H^1(\Omega), v^* \in (H^1(\Omega))^*, \\ \langle\langle u^*, A^{-1}v^* \rangle\rangle = \langle u^*, v^* \rangle_{(H^1(\Omega))^*} \quad \text{for every } u^*, v^* \in (H^1(\Omega))^*,\end{gathered}$$

where $\langle \cdot, \cdot \rangle_{(H^1(\Omega))^*}$ is the dual inner product in $(H^1(\Omega))^*$ corresponding to the usual inner product in $H^1(\Omega)$. Domain of the inverse operator A^{-1} is defined as $D(A^{-1}) = (D(A))^*$ (see [22]). Also, we have

$$\langle\langle v^*, u \rangle\rangle = \int_\Omega v^* u \quad \text{if } v^* \in L^2(\Omega),$$

and

$$\frac{d}{dt} \|v^*\|^2_{(H^1(\Omega))^*} = 2\langle\langle \partial_t v^*, A^{-1}v^* \rangle\rangle \quad \text{for every } v^* \in H^1(0, T; (H^1(\Omega))^*).$$

We now define the weak and strong solutions to the problem (2.1)–(2.4).

Definition 2.2 ([15]). Let $(\phi_0, \sigma_0) \in H^1(\Omega) \times L^2(\Omega)$ and $T \in (0, \infty)$ be given. A pair (ϕ^χ, σ^χ), satisfying the properties

$$\begin{gathered}(\phi^\chi, \sigma^\chi) \in L^\infty(0, T; H^1(\Omega) \times L^2(\Omega)) \cap L^2(0, T; H^2(\Omega) \times H^1(\Omega)), \\ (\phi^\chi_t, \sigma^\chi_t) \in L^r(0, T; D(A^{-1}) \times D(A^{-1})), \\ \mu^\chi := -\Delta\phi^\chi + \Psi'(\phi^\chi) - \chi\sigma^\chi \in L^2(0, T; H^1(\Omega)), \\ (\phi^\chi(0), \sigma^\chi(0)) = (\phi_0, \sigma_0),\end{gathered}$$

for some $r > 1$, is called a weak solution of problem (2.1)–(2.4) on $[0, T] \times \Omega$, if

$$\begin{gathered}\langle\langle \phi^\chi_t, \eta \rangle\rangle + \langle \nabla\mu^\chi, \nabla\eta \rangle = \langle p(\phi^\chi)(\sigma^\chi + \chi(1 - \phi^\chi) - \mu^\chi), \eta \rangle \\ \langle\langle \sigma^\chi_t, \xi \rangle\rangle + \langle (\nabla\sigma^\chi - \chi\nabla\phi^\chi), \nabla\xi \rangle = -\langle p(\phi^\chi)(\sigma^\chi + \chi(1 - \phi^\chi) - \mu^\chi), \xi \rangle\end{gathered} \tag{2.9}$$

holds on $(0, T) \times \Omega$, for every η, $\xi \in D(A)$.

If the pair (ϕ^χ, σ^χ) also satisfies the properties

$$\begin{gathered}\phi^\chi \in L^\infty(0, T; H^3(\Omega)), \quad \phi^\chi_t \in L^2(0, T; H^1(\Omega)), \\ \sigma^\chi \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)), \quad \sigma^\chi_t \in L^2(0, T; L^2(\Omega)), \\ \mu^\chi \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^3(\Omega)),\end{gathered}$$

then it is called a strong solution of problem (2.1)–(2.4) on $[0, T] \times \Omega$.

The following results on the well posedness of solutions was obtained in [15].

Theorem 2.3 ([15]). *Let conditions* (A1), (A2) *hold. Then for every* $(\phi_0, \sigma_0) \in H^1(\Omega) \times L^2(\Omega)$ *and for every* $T > 0$, *problem* (2.1)–(2.4) *has a weak solution such that*

$$\begin{gathered}\phi^\chi \in L^2(0, T; H^3(\Omega)), \quad \Psi(\phi^\chi) \in L^\infty(0, T; L^1(\Omega)), \\ \nabla N_{\sigma^\chi} \in L^2(0, T; L^2(\Omega)), \quad \sqrt{p(\phi^\chi)}(N_{\sigma^\chi} - \mu^\chi) \in L^2(0, T; L^2(\Omega)), \\ \phi^\chi_t \in L^2(0, T; (H^1(\Omega))^*), \quad \sigma^\chi_t \in L^2(0, T; (H^1(\Omega))^*).\end{gathered}$$

Moreover, the following energy identity holds for the weak solutions,

$$\mathcal{E}_\chi(\phi^\chi(t), \sigma^\chi(t)) + \int_0^t \int_\Omega (|\nabla\mu^\chi|^2 + |\nabla N_{\sigma^\chi}|^2 + p(\phi^\chi)(N_{\sigma^\chi} - \mu^\chi)^2)\, dx\, d\tau = \mathcal{E}_\chi(\phi_0, \sigma_0) \tag{2.10}$$

where

$$\mathcal{E}_\chi(\phi,\sigma) = \frac{1}{2}\|\nabla\phi\|^2_{L^2(\Omega)} + \int \Psi(\phi) + \frac{1}{2}\|\sigma\|^2_{L^2(\Omega)} + \chi\int \sigma(1-\phi)$$

and $N^\chi_\sigma = \sigma^\chi + \chi(1-\phi^\chi)$. *Furthermore, for every initial data* $(\phi_0,\sigma_0) \in H^3(\Omega)\times H^1(\Omega)$ *with* $\partial_\nu\phi_0 = 0$ *on* Γ *and for every* $T>0$, *problem* (2.1)–(2.4) *has a strong solution.*

Theorem 2.4 ([15])**.** *Let conditions* (A1), (A2) *hold. Then, for every initial data* $(\phi_0,\sigma_0)\in H^1(\Omega)\times L^2(\Omega)$ *and for every* $T>0$, *the weak solution of problem* (2.1)–(2.4) *specified by Theorem 2.3 is unique. Moreover, if* $(\phi^\chi_i,\sigma^\chi_i)$, $i=1,2$ *are weak solutions of problem* (2.1)–(2.4) *with initial data* $(\phi_{0i},\sigma_{0i})\in H^1(\Omega)\times L^2(\Omega)$, *respectively, then*

$$\begin{aligned}
&\|\phi^\chi_2(t)-\phi^\chi_1(t)\|_{(H^1(\Omega))^*} + \|\sigma^\chi_2(t)-\sigma^\chi_1(t)\|_{(H^1(\Omega))^*} + \|\phi^\chi_2(t)-\phi^\chi_1(t)\|_{L^2(0,t;H^1(\Omega))}\\
&+ \|\sigma^\chi_2(t)-\sigma^\chi_1(t)\|_{L^2(0,t;L^2(\Omega))}\\
&\le \Lambda(t)\Big(\|\phi^\chi_{02}-\phi^\chi_{01}\|_{(H^1(\Omega))^*} + \|\sigma^\chi_{02}-\sigma^\chi_{01}\|_{(H^1(\Omega))^*}\Big)
\end{aligned}$$

where Λ *is a continuous positive function which depends on the norms of the initial data and* Ψ, p, Ω *and* T.

As a result of Theorems 2.3 and 2.4, problem (2.1)–(2.4) generates a weakly continuous semigroup $\{S^\chi(t)\}_{t\ge0}$ in $H^1(\Omega)\times L^2(\Omega)$, according to the formula $S^\chi(t)(\phi_0,\sigma_0) = (\phi^\chi(t),\sigma^\chi(t))$. Here, $(\phi^\chi(t),\sigma^\chi(t))$ denotes the weak solution determined by Theorem 2.3.

Exploiting (2.4), it is easy to see that problem (2.1)–(2.4) satisfies the total mass conservation as

$$\int_\Omega(\phi^\chi(t)+\sigma^\chi(t))\,dx = \int_\Omega(\phi_0+\sigma_0)\,dx \quad \forall t\ge0.$$

Hence, we need to introduce the following subspaces:

$$\mathcal{Z}_M := \big\{(\phi,\sigma)\in H^1(\Omega)\times L^2(\Omega) : \int_\Omega(\phi+\sigma)\,dx = |\Omega|M\big\}, \tag{2.11}$$

$$\mathcal{Z}^r_M := \big\{(\phi,\sigma)\in H^{2r+1}(\Omega)\times H^r(\Omega) : \int_\Omega(\phi+\sigma)\,dx = |\Omega|M\big\},\quad r>0, \tag{2.12}$$

which are equipped with the usual norms of $H^1(\Omega)\times L^2(\Omega)$ and $H^{2r+1}(\Omega)\times H^r(\Omega)$, respectively. Thus, restricting the phase space of the problem to $\mathcal{Z}_M$, we obtain the dynamical system $(\mathcal{Z}_M, S^\chi_M(t))$, where $\{S^\chi_M(t)\}_{t\ge0}$ denotes the restriction of $\{S^\chi(t)\}_{t\ge0}$ on $\mathcal{Z}_M$. Moreover, we will denote the weak solution of problem (2.1)–(2.4) in $\mathcal{Z}_M$ by $S^\chi_M(t)(\phi_0,\sigma_0) = (\phi^\chi_M(t),\sigma^\chi_M(t))$.

Now, let us define the set of stationary points,

$$\mathcal{N}^\chi_M = \{(\phi,\sigma)\in\mathcal{Z}_M : S^\chi_M(t)(\phi,\sigma) = (\phi,\sigma), \forall t\ge0\},$$

for problem (2.1)–(2.4) in $\mathcal{Z}_M$. The set of stationary points $\mathcal{N}^\chi_M$, as indicated in [15], is a nonempty, bounded subset of $\mathcal{Z}_M$, consisting of solutions to the stationary problem

$$\begin{gathered}
-\Delta\phi^\chi + \Psi'(\phi^\chi) - \chi\sigma^\chi = \mu^\chi_0,\\
\sigma^\chi + \chi(1-\phi^\chi) = \mu^\chi_0,\\
\in_\Omega (\phi^\chi+\sigma^\chi)\,dx = |\Omega|M,
\end{gathered} \tag{2.13}$$

where

$$\mu^\chi_0 = \frac{1}{|\Omega|}\int_\Omega \Psi'(\phi^\chi)\,dx - \chi|\Omega|\int_\Omega \sigma^\chi\,dx.$$

For convenience of the reader, we state some results from [15], which are used in some steps of this paper.

Proposition 2.5 (Asymptotic compactness, [15, Lemma 5.1])**.** *Assume that* (A1), (A2) *are satisfied and* B *is a bounded subset of* $H^1(\Omega)\times L^2(\Omega)$. *Then, every sequence of the form* $\{S^\chi(t_k)(\phi_k,\sigma_k)\}_{k=1}^\infty$, *where* $\{(\phi_k,\sigma_k)\}_{k=1}^\infty\subset B$, $t_k\to\infty$ *has a convergent subsequence in* $H^1(\Omega)\times L^2(\Omega)$.

Proposition 2.6 (Gradient Property, [15, Lemma 5.2])**.** *Under conditions* (A1), (A2), *the dynamical system* $(H^1(\Omega)\times L^2(\Omega), S^\chi(t))$ *is a gradient system, i.e. the energy functional* $\mathcal{E}_\chi$ *is a strict Lyapunov function on the whole phase space* $H^1(\Omega)\times L^2(\Omega)$.

Proposition 2.7 ([15, Lemma 5.3]). *Assume that conditions* (A1), (A2) *are satisfied. Then, energy functional $\mathcal{E}_\chi(\phi,\sigma)$ has at least one minimizer $(\phi_*,\sigma_*)\in Z_M$ such that*

$$\mathcal{E}_\chi(\phi_*,\sigma_*)=\inf_{(\phi,\sigma)\in\mathcal{Z}_M}\mathcal{E}_\chi(\phi,\sigma).$$

Proposition 2.8 ([15, Lemma 5.4]). *Let conditions* (A1), (A2) *hold and (ϕ_*,σ_*) be a minimizer of $\mathcal{E}_\chi(\phi,\sigma)$ in $\mathcal{Z}_M$. Then $(\phi_*,\sigma_*)\in H^2(\Omega)\times H^2(\Omega)$ is the strong solution of the problem* (2.13).

Proposition 2.9 ([15, Lemma 5.5]). *Assume that* (A1), (A2) *are satisfied. Then the set of stationary points $\mathcal{N}_M^\chi$ is nonempty and bounded in $\mathcal{Z}_M$.*

Before stating the main result of [15], we recall the following definitions.

Definition 2.10. Let $\{S(t)\}_{t\geq 0}$ be a semigroup on a metric space (X,d). A set $\mathcal{A}\subset X$ is called a global attractor for the semigroup $\{S(t)\}_{t\geq 0}$, if

- $\mathcal{A}$ is a compact set.
- $\mathcal{A}$ is invariant, i.e. $S(t)\mathcal{A}=\mathcal{A}$, $\forall t\geq 0$.
- $\lim_{t\to\infty}\operatorname{dist}_X(S(t)B,\mathcal{A})=0$, for each bounded set $B\subset X$,

where $\operatorname{dist}(\cdot,\cdot)$ is the Hausdorff semidistance defined as

$$\operatorname{dist}_X(A,B)=\sup_{a\in A}\inf_{b\in B}d(a,b).$$

Definition 2.11. Let $\mathcal{N}$ be the set of stationary points of the dynamical system $(X,S(t))$. We define the unstable manifold $\mathcal{M}^u(\mathcal{N})$ emanating from the set $\mathcal{N}$ as a set of all $y\in X$ such that there exists a full trajectory $\gamma=\{u(t):t\in\mathbb{R}\}$ with the properties

$$u(0)=y\quad\text{and}\quad\lim_{t\to-\infty}\operatorname{dist}_X(u(t),\mathcal{N})=0.$$

The following theorem is the main result of [15] regarding the existence and the regularity of the global attractor.

Theorem 2.12 ([15, Theorem 5.9, Theorem 5.10]). *Assume that* (A1), (A2) *are satisfied. Then the semigroup $\{S_M^\chi(t)\}_{t\geq 0}$ generated by the weak solutions of the problem* (2.1)–(2.4) *possesses a global attractor $\mathcal{A}_M^\chi$ in $\mathcal{Z}_M$, and $\mathcal{A}_M^\chi=\mathcal{M}^u(\mathcal{N}_M^\chi)$. Moreover, the global attractor $\mathcal{A}_M^\chi$ is bounded in $\mathcal{Z}_M^1$.*

3. Further geometric properties of the global attractor

Recalling the definition of the unstable manifold and exploiting the result [7, Theorem 7.5.6], we observe that the global attractor $\mathcal{A}_M^\chi$ consists of full trajectories $\gamma_M^\chi=\{(\phi_M^\chi(t),\sigma_M^\chi(t)):t\in\mathbb{R}\}$ such that

$$\lim_{t\to\infty}\operatorname{dist}_{\mathcal{Z}_M}((\phi_M^\chi(t),\sigma_M^\chi(t)),\mathcal{N}_M^\chi)=0\quad\text{and}\quad\lim_{t\to-\infty}\operatorname{dist}_{\mathcal{Z}_M}((\phi_M^\chi(t),\sigma_M^\chi(t)),\mathcal{N}_M^\chi)=0.\tag{3.1}$$

In this section, we aim to improve (3.1) by proving the following result.

Theorem 3.1. *Let assumptions* (A1)–(A3) *hold. Then for any full trajectory $\gamma_M^\chi=\{(\phi_M^\chi(t),\sigma_M^\chi(t)):t\in\mathbb{R}\}$ in the global attractor $\mathcal{A}_M^\chi$ there exist $(\phi_*,\sigma_*),(\phi_{**},\sigma_{**})\in\mathcal{N}_M^\chi$ such that*

$$\lim_{t\to\infty}\operatorname{dist}_{\mathcal{Z}_M^r}((\phi_M^\chi(t),\sigma_M^\chi(t)),(\phi_*,\sigma_*))=0.\tag{3.2}$$

and

$$\lim_{t\to-\infty}\operatorname{dist}_{\mathcal{Z}_M^r}((\phi_M^\chi(t),\sigma_M^\chi(t)),(\phi_{**},\sigma_{**})=0,\tag{3.3}$$

where $r\in[0,1)$.

As a consequence of Theorem 3.1, we obtain the following geometric property of the global attractor.

Corollary 3.2. *Under assumptions* (A1)–(A3), *the global attractor $\mathcal{A}_M^\chi$ equals to the union of the unstable manifolds of its stationary points, i.e.,*

$$\mathcal{A}_M^\chi=\cup_{(\phi_*,\sigma_*)\in\mathcal{N}_M^\chi}\mathcal{M}^u((\phi_*,\sigma_*)).\tag{3.4}$$

Remark 3.3. We already know that when the set of stationary points $\mathcal{N}_M^\chi$ is finite, the equality (3.4) holds (see for example [7, p.361]). In this section, we demonstrate that the global attractor $\mathcal{A}_M^\chi$ of the problem (2.1)–(2.4) retains the property (3.4), even if the set of stationary points $\mathcal{N}_M^\chi$ is infinite.

Remark 3.4. In our framework, the dynamical system associated with problem (2.1)–(2.4) is a gradient system, i.e. the energy functional $E_\chi(\phi_M^\chi, \sigma_M^\chi)$ serves as a strict Lyapunov functional (see Proposition 2.6). Hence, the system admits no heteroclinic cycles. The present work goes beyond establishing the existence of a Lyapunov structure; it also aims to characterize the geometric structure of the global attractor. From Theorem 3.1 we obtain that $\mathcal{A}_M^\chi$ consists of full trajectories connecting distinct equilibrium points, which correspond to heteroclinic orbits.

It is important to emphasize that this geometric structure of the attractor is not a direct consequence of the gradient property itself. Systems whose attractors possessing the properties (3.2) and (3.3) are often referred to as gradient-like systems. Although many gradient systems become gradient-like under additional analytical tools, such as the existence of Łojasiewicz-Simon-type inequalities, not every gradient system is necessarily gradient-like (see [1]).

To prove Theorem 3.1, we first present some properties of the limit sets. Subsequently, we will derive an appropriate uniform Łojasiewicz-Simon type inequality. Finally, by using these establishments, we prove Theorem 3.1.

3.1. **Properties of limit sets.** Let us start with definitions of the limit sets.

Definition 3.5. For any $(\phi_0, \sigma_0) \in \mathcal{Z}_M$ the ω-limit set of (ϕ_0, σ_0) is defined by

$$\begin{aligned}\omega_M^\chi((\phi_0, \sigma_0)) := \big\{&(\phi_*, \sigma_*) \in \mathcal{A}_M^\chi : \exists\{(\phi_M^\chi(t_k), \sigma_M^\chi(t_k))\}_{k=1}^\infty, \text{ such that} \\ &(\phi_M^\chi(0), \sigma_M^\chi(0)) = (\phi_0, \sigma_0), t_k \nearrow \infty \text{ and } (\phi_M^\chi(t_k), \sigma_M^\chi(t_k)) \to (\phi_*, \sigma_*) \text{ strongly in } \mathcal{Z}_M\big\}.\end{aligned}$$

Moreover, for any $(\phi_0, \sigma_0) \in \mathcal{A}_M^\chi$ the α-limit set of (ϕ_0, σ_0) can be defined as follows:

$$\begin{aligned}\alpha_M^\chi((\phi_0, \sigma_0)) := \big\{&(\phi_{**}, \sigma_{**}) \in \mathcal{A}_M^\chi : \exists\{(\phi_M^\chi(t_k), \sigma_M^\chi(t_k))\}_{k=1}^\infty, \text{ such that} \\ &(\phi_M^\chi(0), \sigma_M^\chi(0)) = (\phi_0, \sigma_0), t_k \searrow -\infty \text{ and } (\phi_M^\chi(t_k), \sigma_M^\chi(t_k)) \to (\phi_{**}, \sigma_{**}) \text{ strongly in } \mathcal{Z}_M\big\}.\end{aligned}$$

On account of the above definitions, we deduce the following lemma.

Lemma 3.6. *Let $(\phi_0, \sigma_0) \in \mathcal{A}_M^\chi$ and $(\phi_M^\chi(t), \sigma_M^\chi(t))$ be the full trajectory passing through (ϕ_0, σ_0). Then, the sets $\omega_M^\chi((\phi_0, \sigma_0))$ and $\alpha_M^\chi((\phi_0, \sigma_0))$ are nonempty, compact, invariant subsets of $\mathcal{Z}_M$ and there exist constants $\mathcal{E}_M^{\chi,\infty^+}$ and $\mathcal{E}_M^{\chi,\infty^-}$ such that*

$$\begin{aligned}\mathcal{E}_\chi(\phi_*, \sigma_*) &= \lim_{t\to\infty} \mathcal{E}_\chi(\phi_M^\chi(t), \sigma_M^\chi(t)) = \mathcal{E}_M^{\chi,\infty^+} \quad \forall(\phi_*, \sigma_*) \in \omega_M^\chi((\phi_0, \sigma_0)), \\ \mathcal{E}_\chi(\phi_{**}, \sigma_{**}) &= \lim_{t\to-\infty} \mathcal{E}_\chi(\phi_M^\chi(t), \sigma_M^\chi(t)) = \mathcal{E}_M^{\chi,\infty^-} \quad \forall(\phi_{**}, \sigma_{**}) \in \alpha_M^\chi((\phi_0, \sigma_0)).\end{aligned} \tag{3.5}$$

Moreover, $\omega_M^\chi((\phi_0, \sigma_0)) \subset \mathcal{N}_M^\chi$ and $\alpha_M^\chi((\phi_0, \sigma_0)) \subset \mathcal{N}_M^\chi$.

Proof. By using the asymptotic compactness property stated in Proposition 2.5, one can see that for any $(\phi_0, \sigma_0) \in \mathcal{A}_M^\chi$ the sets $\omega_M^\chi((\phi_0, \sigma_0))$ and $\alpha_M^\chi((\phi_0, \sigma_0))$ are nonempty, compact and invariant subset of $\mathcal{Z}_M$ (see for example [7, p. 339]).

On the other hand, since $\mathcal{E}_\chi(\phi_M^\chi(t), \sigma_M^\chi(t))$ is a non-increasing functional which is bounded from above and below (cf. Proposition 2.6, Proposition 2.7), we infer that there exist constants $\mathcal{E}_M^{\chi,\infty^+}$ and $\mathcal{E}_M^{\chi,\infty^-}$ such that

$$\begin{aligned}\lim_{t\to\infty} \mathcal{E}_\chi(\phi_M^\chi(t), \sigma_M^\chi(t)) &= \mathcal{E}_M^{\chi,\infty^+}, \\ \lim_{t\to-\infty} \mathcal{E}_\chi(\phi_M^\chi(t), \sigma_M^\chi(t)) &= \mathcal{E}_M^{\chi,\infty^-}.\end{aligned} \tag{3.6}$$

Moreover, for any $(\phi_*, \sigma_*) \in \omega_M^\chi((\phi_0, \sigma_0))$ there exists $t_n \to \infty$ (as $n \to \infty$) such that

$$\lim_{n\to\infty} (\phi_M^\chi(t_n), \sigma_M^\chi(t_n)) = (\phi_*, \sigma_*). \tag{3.7}$$

Similarly, for any $(\phi_{**},\sigma_{**}) \in \alpha_M^\chi((\phi_0,\sigma_0))$ there exists $\tilde{t}_n \to -\infty$ (as $n\to\infty$) such that

$$\lim_{n\to\infty}(\phi_M^\chi(\tilde{t}_n),\sigma_M^\chi(\tilde{t}_n)) = (\phi_{**},\sigma_{**}). \tag{3.8}$$

From (3.6)-(3.8) we obtain that (3.5) is satisfied. Now, using the invariance property of $\omega_M^\chi((\phi_0,\sigma_0))$ and $\alpha_M^\chi((\phi_0,\sigma_0))$, from (3.6)-(3.8) it can be deduced that

$$\begin{aligned} \mathcal{E}_\chi(S_M^\chi(t)(\phi_*,\sigma_*)) &= \mathcal{E}_M^{\chi,\infty^+} \quad \forall(\phi_*,\sigma_*) \in \omega_M^\chi((\phi_0,\sigma_0)), \quad \forall t\ge 0,\\ \mathcal{E}_\chi(S_M^\chi(t)(\phi_{**},\sigma_{**})) &= \mathcal{E}_M^{\chi,\infty^-} \quad \forall(\phi_{**},\sigma_{**}) \in \alpha_M^\chi((\phi_0,\sigma_0)), \quad \forall t\ge 0, \end{aligned} \tag{3.9}$$

which yields

$$\begin{aligned} \mathcal{E}_\chi(S_M^\chi(t)(\phi_*,\sigma_*)) &= \mathcal{E}_\chi((\phi_*,\sigma_*)) \quad \forall(\phi_*,\sigma_*) \in \omega_M^\chi((\phi_0,\sigma_0)), \quad \forall t\ge 0,\\ \mathcal{E}_\chi(S_M^\chi(t)(\phi_{**},\sigma_{**})) &= \mathcal{E}_\chi((\phi_{**},\sigma_{**})) \quad \forall(\phi_{**},\sigma_{**}) \in \alpha_M^\chi((\phi_0,\sigma_0)), \quad \forall t\ge 0. \end{aligned}$$

Hence, using that $\mathcal{E}_\chi(\phi_M^\chi(t),\sigma_M^\chi(t))$ is a strict Lyapunov functional (see Proposition 2.6), we have $(\phi_*,\sigma_*),(\phi_{**},\sigma_{**}) \in \mathcal{N}_M^\chi$, which completes the proof. □

3.2. **Łojasiewicz-Simon type inequality.** In the proof of Theorem 3.1, Łojasiewicz-Simon type inequality assumes a pivotal position. Its derivation primarily stems from the theoretical framework developed in Chill [5] and Chill et al. [6]. Let us now proceed with the definition of the pertinent spaces, which are obtained by setting $M=0$ in (2.11) and (2.12):

$$\begin{aligned} \mathcal{Z}_0 &:= \big\{(\phi,\sigma)\in H^1(\Omega)\times L^2(\Omega): \int_\Omega(\phi+\sigma)\,dx=0\big\},\\ \mathcal{Z}_0^r &:= \big\{(\phi,\sigma)\in H^{2r+1}(\Omega)\times H^r(\Omega): \int_\Omega(\phi+\sigma)\,dx=0\big\}, \quad r>0. \end{aligned}$$

These spaces are Hilbert spaces equipped with usual norms of $H^1(\Omega)\times L^2(\Omega)$ and $H^{2r+1}(\Omega)\times H^r(\Omega)$, respectively. Furthermore, introducing the Hilbert space

$$\mathcal{H}_0 := \big\{(\phi,\sigma)\in L^2(\Omega)\times L^2(\Omega): \int_\Omega(\phi+\sigma)\,dx=0\big\}, \tag{3.10}$$

we obtain the Hilbert triple $\mathcal{Z}_0 \hookrightarrow \mathcal{H}_0 = \mathcal{H}_0^* \hookrightarrow \mathcal{Z}_0^*$.

It can be inferred from Theorem 2.12 that the every full trajectory $\gamma_M^\chi = \{(\phi_M^\chi(t),\sigma_M^\chi(t)) : t\in\mathbb{R}\}$ in the global attractor $\mathcal{A}_M^\chi$, and thus the set of stationary points $\mathcal{N}_M^\chi$, is bounded in $\mathcal{Z}_M^r$, $r\in[0,1]$.

Now, for any solution pair $(\phi_M^\chi,\sigma_M^\chi)$ of the problem (2.1)–(2.4) in the phase space $\mathcal{Z}_M$, let us define the shifted pair

$$(\overline{\phi_M^\chi},\overline{\sigma_M^\chi}) := (\phi_M^\chi - c_1, \sigma_M^\chi - c_2)$$

where $c_1 = \frac{(1+\chi)M}{1+2\chi}$ and $c_2 = \frac{\chi M}{1+2\chi}$. Since

$$c_1 + c_2 = M \quad \text{and} \quad \chi c_1 - (1+\chi)c_2 = 0, \tag{3.11}$$

the shifted pair $(\overline{\phi_M^\chi},\overline{\sigma_M^\chi})$ is in $\mathcal{Z}_0$ and solves problem (2.1)–(2.4) written for the functions

$$\Psi_M := \Psi(\cdot + \frac{M}{1+\chi}) \quad \text{and} \quad p_M := p(\cdot + \frac{M}{1+\chi})$$

instead of Ψ and p. Hence, without lose of generality we can study on the spaces $\mathcal{Z}_0^r$, $r\in[0,1]$.
Notation. From this point onward, for simplicity of notation, we will denote the solution pair in $\mathcal{Z}_0$ as (ϕ^χ,σ^χ) instead of $(\phi_0^\chi,\sigma_0^\chi)$, and the full trajectory in $\mathcal{Z}_0^1$ as $\gamma^\chi = \{(\phi^\chi(t),\sigma^\chi(t)) : t\in\mathbb{R}\}$ instead of $\gamma_0^\chi = \{(\phi_0^\chi(t),\sigma_0^\chi(t)) : t\in\mathbb{R}\}$. Additionally, we will refer to the set of stationary points in $\mathcal{Z}_0^1$ as $\mathcal{N}^\chi$ instead of $\mathcal{N}_0^\chi$.

We firstly state the following auxiliary lemma for the energy functional $\mathcal{E}_\chi$, which can be proved by using similar assertions in [6, Lemma 6.2] and arguing as in the proof of Proposition 2.8.

Lemma 3.7. *The energy functional $\mathcal{E}_\chi$ is twice continuously Frechet differentiable. For every $(\phi,\sigma),(\eta,\xi)\in\mathcal{Z}_0$, it holds*

$$\langle\langle \mathcal{E}_\chi'(\phi,\sigma),(\eta,\xi)\rangle\rangle_{\mathcal{Z}_0}=\int_\Omega(\nabla\phi\nabla\eta+(\Psi'(\phi)-\chi\sigma)\eta+(\sigma+\chi(1-\phi))\xi)\,dx. \tag{3.12}$$

Moreover, for every $(\phi,\sigma),(\eta_1,\xi_1),(\eta_2,\xi_2)\in\mathcal{Z}_0$, we have

$$\langle \mathcal{E}_\chi''(\phi,\sigma)(\eta_1,\xi_1),(\eta_2,\xi_2)\rangle_{\mathcal{Z}_0}=\int_\Omega(\nabla\eta_1\nabla\eta_2+\Psi''(\phi)\eta_1\eta_2-\chi\xi_2\eta_1+\xi_1\xi_2-\chi\xi_1\eta_2)\,dx. \tag{3.13}$$

Now, we will prove a suitable version of the Łojasiewicz-Simon type inequality which can be proved by adapting the abstract result [5, Corollary 3.11]. At this point, it worths to mention that every critical point of the energy functional $\mathcal{E}_\chi$ in $\mathcal{Z}_0$ is a stationary point and also bounded in $\mathcal{Z}_0^1$ (see Proposition (2.8)). The proof of the inequality will be done by arguing as in [6, Proposition 6.6]. We will state the proof for the convenience of the reader.

Lemma 3.8. *Suppose that* (A3) *is satisfied and $(\phi_*,\sigma_*)\in\mathcal{Z}_0^1$ is a critical point of the functional $\mathcal{E}_\chi$. Then, there exist constants $\theta\in(0,\frac12]$ and $C,\beta>0$ depending on (ϕ_*,σ_*) such that*

$$|\mathcal{E}_\chi(\phi,\sigma)-\mathcal{E}_\chi(\phi_*,\sigma_*)|^{1-\theta}\le C\|\mathcal{E}_\chi'(\phi,\sigma)\|_{\mathcal{Z}_0^*} \tag{3.14}$$

for all $(\phi,\sigma)\in\mathcal{Z}_0$ such that $\|(\phi,\sigma)-(\phi_*,\sigma_*)\|_{\mathcal{Z}_0}<\beta$.

Proof. Firstly, Sobolev embedding theorem yields that $\mathcal{Z}_0^1\subset L^\infty(\Omega)\times L^2(\Omega)$. Hence, the restriction of $\mathcal{E}_\chi'$ to $\mathcal{Z}_0^1$ is an analytic function with values in $\mathcal{H}_0$ (see e.g. [5, Corollary 4.6]). On the other hand, the associated bilinear form of the linearization $\mathcal{E}_\chi''(\phi,\sigma)$ is continuous, symmetric, elliptic operator. Hence, applying Lax-Milgram theorem, $\mathcal{E}_\chi''(\phi,\sigma)$ has a nonempty resolvent set. Additionally, since the embeddings $\mathcal{Z}_0\hookrightarrow\mathcal{Z}_0^*$ and $\mathcal{Z}_0^1\hookrightarrow\mathcal{H}_0$ are compact, $\mathcal{E}_\chi''(\phi,\sigma)$ and $\mathcal{E}_\chi''(\phi,\sigma)|_{\mathcal{Z}_0^1}$ have compact resolvents on $\mathcal{Z}_0^*$ and $\mathcal{H}_0$, respectively. Then, from the Fredholm alternative the kernel $\ker\mathcal{E}_\chi''(\phi,\sigma)$ is finite dimensional and the ranges $\operatorname{Rg}\mathcal{E}_\chi''(\phi,\sigma)$ and $\operatorname{Rg}\mathcal{E}_\chi''(\phi,\sigma)|_{\mathcal{Z}_0^1}$ are closed in $\mathcal{Z}_0^*$ and $\mathcal{H}_0$. Moreover, $\mathcal{Z}_0^*$ (resp. $\mathcal{H}_0$) is a direct orthogonal sum of $\ker\mathcal{E}_\chi''(\phi,\sigma)$ and $\operatorname{Rg}\mathcal{E}_\chi''(\phi,\sigma)$ (resp. $\operatorname{Rg}\mathcal{E}_\chi''(\phi,\sigma)|_{\mathcal{Z}_0^1}$). Hence we can define a continuous orthogonal projection $P:\mathcal{Z}_0\to\mathcal{Z}_0$ with $\operatorname{Rg}P=\ker\mathcal{E}_\chi''(\phi,\sigma)$. Consequently, we can complete the proof by applying the results in [5, Corollary 3.11] with choice of $X=\mathcal{Z}_0^1$, $V=\mathcal{Z}_0$, $Y=\mathcal{H}_0$, $W=\mathcal{Z}_0^*$. □

Now, we will prove the uniform version of the Łojasiewicz-Simon type inequality, which annihilates the dependence of the constants on the choice of the critical points, for the limit sets.

Lemma 3.9. *Assume that* (A1)–(A3) *are satisfied.*

(i) *For each $(\phi_0,\sigma_0)\in\mathcal{Z}_0$, there exists an open neighborhood $\mathcal{U}\subset\mathcal{Z}_0^r, r<1$ of $\omega_0^\chi((\phi_0,\sigma_0))$ and the constants $\theta\in(0,\frac12]$ and $C>0$ such that*

$$|\mathcal{E}_\chi(\phi,\sigma)-\mathcal{E}_0^{\chi,\infty^+}|^{1-\theta}\le C\|\mathcal{E}_\chi'(\phi,\sigma)\|_{\mathcal{Z}_0^*} \tag{3.15}$$

for all $(\phi,\sigma)\in\mathcal{U}$, where $\mathcal{E}_0^{\chi,\infty^+}$ is already determined in Lemma 3.6.

(ii) *Moreover, for any $(\phi_0,\sigma_0)\in\mathcal{A}_0^\chi\subset\mathcal{Z}_0^1$, there exists an open neighborhood $\mathcal{V}\subset\mathcal{Z}_0^r, r<1$ of $\alpha_0^\chi((\phi_0,\sigma_0))$ and the constants $\tilde\theta\in(0,\frac12]$ and $\tilde C>0$ such that*

$$|\mathcal{E}_\chi(\phi,\sigma)-\mathcal{E}_0^{\chi,\infty^-}|^{1-\tilde\theta}\le \tilde C\|\mathcal{E}_\chi'(\phi,\sigma)\|_{\mathcal{Z}_0^*} \tag{3.16}$$

for all $(\phi,\sigma)\in\mathcal{V}$, where $\mathcal{E}_0^{\chi,\infty^-}$ is already determined in Lemma 3.6.

Proof. (i) As a result of Lemma 3.8, for every $(\phi_{0j},\sigma_{0j})\in\omega_0^\chi((\phi_0,\sigma_0))$ there exist constants $\theta_j\in(0,\frac12]$ and $C_j,\beta_j>0$ such that

$$|\mathcal{E}_\chi(\phi,\sigma)-\mathcal{E}_\chi(\phi_{0j},\sigma_{0j})|^{1-\theta_j}\le C_j\|\mathcal{E}_\chi'(\phi,\sigma)\|_{\mathcal{Z}_0^*}$$

for all $(\phi,\sigma)\in\mathcal{Z}_0$ such that $\|(\phi,\sigma)-(\phi_{0j},\sigma_{0j})\|_{\mathcal{Z}_0}<\beta_j$.

Furthermore, since $\omega_0^\chi((\phi_0,\sigma_0))$ is bounded in $\mathcal{Z}_0^1$, it is compact in $\mathcal{Z}_0^r$ for all $r<1$. Therefore, it can be covered by a finitely many balls in $\mathcal{Z}_0^r$. Therefore, taking $\mathcal{U}$ as the union of these balls, we achieve the existence of uniform constants $\theta \in (0, \frac{1}{2}]$ and $C > 0$ such that (3.15) holds for all $(\phi, \sigma) \in \mathcal{U}$.

(ii) We can establish (3.16), by using the compactness of $\alpha_0^\chi((\phi_0, \sigma_0))$ in $\mathcal{Z}_0^r$ for all $r < 1$ and arguing as in the proof of i. $\square$

Now, we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1. As previously discussed, it is sufficient to consider the case $M = 0$. Assume that $(\phi_0, \sigma_0) \in \mathcal{A}_0^\chi \subset \mathcal{Z}_0^1$ and that $\{(\phi^\chi(t), \sigma^\chi(t)) : t \in \mathbb{R}\}$ denotes the corresponding full trajectory passing through the point (ϕ_0, σ_0). To prove the theorem, it is enough to show that $\omega_0^\chi((\phi_0, \sigma_0))$ and $\alpha_0^\chi((\phi_0, \sigma_0))$ consist of a single point. From Lemma 3.6, it follows that the energy functional $\mathcal{E}_\chi(.,.)$ is constant on $\omega_0^\chi((\phi_0, \sigma_0))$ and $\alpha_0^\chi((\phi_0, \sigma_0))$. Furthermore, the following holds.

$$\mathcal{E}_0^{\chi,\infty^+} := \lim_{t\to\infty} \mathcal{E}_\chi(\phi^\chi(t), \sigma^\chi(t)) = E(\phi_*, \sigma_*), \quad \text{for all } (\phi_*, \sigma_*) \in \omega_0^\chi((\phi_0, \sigma_0)), \tag{3.17}$$

$$\mathcal{E}_0^{\chi,\infty^-} := \lim_{t\to-\infty} \mathcal{E}_\chi(\phi^\chi(t), \sigma^\chi(t)) = E(\phi_{**}, \sigma_{**}), \quad \text{for all } (\phi_{**}, \sigma_{**}) \in \alpha_0^\chi((\phi_0, \sigma_0)). \tag{3.18}$$

Moreover, we have proved in Lemma 3.9 that there exist open neighborhoods $\mathcal{U}$ of $\omega_0^\chi((\phi_0, \sigma_0))$ and $\mathcal{V}$ of $\alpha_0^\chi((\phi_0, \sigma_0))$ such that the inequalities (3.15) and (3.16) are satisfied on $\mathcal{U}$ and $\mathcal{V}$, respectively. Besides, the definition of the limit sets (Definition 3.5) yields that there exists $T_0 > 0$ such that

$$(\phi^\chi(t), \sigma^\chi(t)) \in \mathcal{U} \subset \mathcal{Z}_0^r, \quad r < 1, \ \forall t \geq T_0, \tag{3.19}$$

$$(\phi^\chi(-t), \sigma^\chi(-t)) \in \mathcal{V} \subset \mathcal{Z}_0^r, \quad r < 1, \ \forall t \geq T_0. \tag{3.20}$$

Thus, we obtain from Lemma 3.9 that

$$|\mathcal{E}_\chi(\phi^\chi(t), \sigma^\chi(t)) - \mathcal{E}_0^{\chi,\infty^+}|^{1-\theta} \leq C\|\mathcal{E}'_\chi(\phi^\chi(t), \sigma^\chi(t))\|_{\mathcal{Z}_0^*} \quad \forall t \geq T_0, \tag{3.21}$$

$$|\mathcal{E}_\chi(\phi^\chi(-t), \sigma^\chi(-t)) - \mathcal{E}_0^{\chi,\infty^-}|^{1-\tilde{\theta}} \leq \tilde{C}\|\mathcal{E}'_\chi(\phi^\chi(-t), \sigma^\chi(-t))\|_{\mathcal{Z}_0^*} \quad \forall t \geq T_0. \tag{3.22}$$

We separate the following part of the proof into two steps. In the first step, we present a detailed proof of forward convergence to equilibrium, namely (3.2) stated in the theorem. In the second step, we will establish the backward convergence (3.3), mainly referring to the method used in the first step.

Step 1: Proof of (3.2). First we assume that there exists a time t_* such that $\mathcal{E}_\chi(\phi^\chi(t_*), \sigma^\chi(t_*)) = \mathcal{E}_0^{\chi,\infty^+}$. Since the energy functional $\mathcal{E}_\chi(\phi^\chi(t), \sigma^\chi(t))$ is a strict Lyapunov functional, there exists a unique limit $(\phi_\infty^\chi, \sigma_\infty^\chi) \in \mathcal{N}^\chi$ such that

$$(\phi_0, \sigma_0) = (\phi^\chi(t), \sigma^\chi(t)) = (\phi^\chi(t_*), \sigma^\chi(t_*)) = (\phi_\infty^\chi, \sigma_\infty^\chi).$$

Hence, the proof of (3.2) is complete in this case.

Now, we assume that $\mathcal{E}_\chi(\phi^\chi(t), \sigma^\chi(t)) > \mathcal{E}_0^{\chi,\infty^+}$ for all $t \geq 0$. Integrating by parts on (3.12), we obtain that

$$\langle \mathcal{E}'_\chi(\phi^\chi, \sigma^\chi), (\eta, \xi)\rangle_{\mathcal{Z}_0} = \int_\Omega \mu^\chi \eta \, dx + \int_\Omega N_\sigma^\chi \xi \, dx \tag{3.23}$$

for every $(\eta, \xi) \in \mathcal{Z}_0$. Moreover, since $(\eta, \xi) \in \mathcal{Z}_0$, denoting $c := \frac{\langle \mu^\chi\rangle + \langle N_\sigma^\chi\rangle}{2}$ we have

$$\begin{aligned}
&\int_\Omega \mu^\chi \eta \, dx + \int_\Omega N_\sigma^\chi \xi \, dx \\
&= \int_\Omega (\mu^\chi - c)\eta \, dx + \int_\Omega (N_\sigma^\chi - c)\xi \, dx \\
&= \frac{1}{2}\int_\Omega (\mu^\chi - \langle \mu^\chi\rangle + \mu^\chi - \langle N_\sigma^\chi\rangle)\eta \, dx + \frac{1}{2}\int_\Omega (N_\sigma^\chi - \langle N_\sigma^\chi\rangle + N_\sigma^\chi - \langle \mu^\chi\rangle)\xi \, dx \\
&\leq (\|\mu^\chi - \langle \mu^\chi\rangle\|_{L^2(\Omega)} + \|\mu^\chi - N_\sigma^\chi\|_{L^2(\Omega)} + \|N_\sigma^\chi - \langle N_\sigma^\chi\rangle\|_{L^2(\Omega)})(\|\eta\|_{L^2(\Omega)} + \|\xi\|_{L^2(\Omega)}).
\end{aligned}$$

Considering the previous inequality in (3.23), we have

$$\|\mathcal{E}'_\chi(\phi^\chi, \sigma^\chi)\|_{\mathcal{Z}_0^*} \leq C(\|\mu^\chi - \langle \mu^\chi\rangle\|_{L^2(\Omega)} + \|\mu^\chi - N_\sigma^\chi\|_{L^2(\Omega)} + \|N_\sigma^\chi - \langle N_\sigma^\chi\rangle\|_{L^2(\Omega)}). \tag{3.24}$$

Furthermore, recalling the embedding $H^2(\Omega) \hookrightarrow C(\overline{\Omega})$ and (A2), we obtain

$$\|\mu^\chi - N_\sigma^\chi\|_{L^2(\Omega)} \leq \max_{x\in\overline{\Omega}}\Big(\frac{1}{\sqrt{p(\phi^\chi)}}\Big)\|\sqrt{p(\phi^\chi)}(\mu^\chi - N_\sigma^\chi)\|_{L^2(\Omega)} \leq C\|\sqrt{p(\phi^\chi)}(\mu^\chi - N_\sigma^\chi)\|_{L^2(\Omega)}.$$

Therefore, applying the Poincare-Wirtinger inequality and considering the last estimate in (3.24), we deduce that

$$\begin{aligned}
&\|\mathcal{E}'_\chi(\phi^\chi(t), \sigma^\chi(t))\|_{\mathcal{Z}_0^*} \\
&\leq C\Big(\|\nabla\mu^\chi(t)\|_{L^2(\Omega)} + \|\nabla N_\sigma^\chi(t)\|_{L^2(\Omega)} + \|\sqrt{p(\phi^\chi)}(\mu^\chi(t) - N_\sigma^\chi(t))\|_{L^2(\Omega)}\Big),
\end{aligned} \tag{3.25}$$

for all $t \in \mathbb{R}$. Considering (3.25) to estimate the right-hand side of (3.21), we have

$$|\mathcal{E}_\chi(\phi^\chi(t), \sigma^\chi(t)) - \mathcal{E}_0^{\chi,\infty^+}|^{1-\theta} \leq C\Upsilon_\chi(t) \quad \forall t \geq T_0, \tag{3.26}$$

where

$$\Upsilon_\chi(t) = \Big(\|\nabla\mu^\chi(t)\|_{L^2(\Omega)} + \|\nabla N_\sigma^\chi(t)\|_{L^2(\Omega)} + \|\sqrt{p(\phi^\chi(t))}(\mu^\chi(t) - N_\sigma^\chi(t))\|_{L^2(\Omega)}\Big)^{1/2}.$$

On the other hand, rewriting the energy identity (2.10) in the differential form, we infer that

$$\frac{d}{dt}(\mathcal{E}_\chi(\phi^\chi(t), \sigma^\chi(t))) = -(\Upsilon_\chi(t))^2 \quad \forall t \in \mathbb{R}. \tag{3.27}$$

Recalling that $\mathcal{E}_\chi(\phi^\chi(t), \sigma^\chi(t)) - \mathcal{E}_0^{\chi,\infty^+} > 0$ and using (3.27) in (3.26), we obtain

$$\frac{d}{dt}\Big(\mathcal{E}_\chi(\phi^\chi(t), \sigma^\chi(t)) - \mathcal{E}_0^{\chi,\infty^+}\Big) + C\Big(\mathcal{E}_\chi(\phi^\chi(t), \sigma^\chi(t)) - \mathcal{E}_0^{\chi,\infty^+}\Big)^{2(1-\theta)} \leq 0 \quad \forall t \geq T_0. \tag{3.28}$$

Now, let us examine the ordinary differential inequality

$$\frac{d}{dt}y(t) + C\,(y(t))^{2(1-\theta)} \leq 0, \quad y(t) > 0 \quad \forall t \geq T_0. \tag{3.29}$$

In the case $\theta = \mathbf{\frac{1}{2}}$,

$$\frac{d}{dt}y(t) + C\,(y(t)) \leq 0, \quad y(t) > 0 \quad \forall t \geq T_0,$$

this yields

$$y(t) \leq Ke^{-C(t-T_0)}, \quad \forall t > T_0,$$

for some constant K depending on T_0.

In the case $\mathbf{0} < \theta < \mathbf{\frac{1}{2}}$, defining $v(t) := (y(t))^{2\theta-1}$, we obtain

$$v'(t) + (2\theta - 1)\,C \geq 0 \quad \forall t \geq T_0.$$

Integrating from T_0 to t, we infer that

$$v(t) \geq v(T_0) + (1-2\theta)\,C(t-T_0) > (1-2\theta)C(t-T_0), \quad \forall t > T_0,$$

since $v(T_0) > 0$. From the last estimate, it follows that

$$y(t) \leq \tilde{K}(t-T_0)^{\frac{-1}{1-2\theta}}, \quad \forall t > T_0,$$

for some constant $\tilde{K}$ depending on the exponent θ and T_0.

In the following, we will denote by C, K and $\tilde{K}$ the generic constants depending on θ and T_0. Since the inequality in (3.28) can be written in the form of (3.29), we deduce that

$$\begin{gathered}
\Big(\mathcal{E}_\chi(\phi^\chi(t), \sigma^\chi(t)) - \mathcal{E}_0^{\chi,\infty^+}\Big) \leq Ke^{-C(t-T_0)} \quad \text{if } \theta = \frac{1}{2}, \\
\Big(\mathcal{E}_\chi(\phi^\chi(t), \sigma^\chi(t)) - \mathcal{E}_0^{\chi,\infty^+}\Big) \leq \tilde{K}(t-T_0)^{\frac{-1}{1-2\theta}} \quad \text{if } \theta \in (0, \frac{1}{2}).
\end{gathered} \tag{3.30}$$

Recalling the weak formulation formula (2.9), we obtain

$$\int_t^T \big\|(\phi_t^\chi(s), \sigma_t^\chi(s))\big\|^2_{(H^1(\Omega))^*\times(H^1(\Omega))^*}\,ds \leq \int_t^T (\Upsilon_\chi(s))^2\,ds \quad \forall T > t \geq T_0. \tag{3.31}$$

Next, integrating (3.27) from t to T, we have

$$\int_t^T (\Upsilon_\chi(s))^2\,ds = \mathcal{E}_\chi(\phi^\chi(t),\sigma^\chi(t)) - \mathcal{E}_\chi(\phi^\chi(T),\sigma^\chi(T)) \quad \forall T > t \geq T_0.$$

Considering the last equality in (3.31), we infer that

$$\int_t^T \big\|(\phi_t^\chi(s),\sigma_t^\chi(s))\big\|^2_{(H^1(\Omega))^*\times(H^1(\Omega))^*}\,ds \leq \mathcal{E}_\chi(\phi^\chi(t),\sigma^\chi(t)) - \mathcal{E}_\chi(\phi^\chi(T),\sigma^\chi(T)) \quad \forall T > t \geq T_0.$$

Passing to limit as $T \to \infty$ in the previous estimate, and using (3.30), we deduce that

$$\begin{gathered}\int_t^\infty \big\|(\phi_t^\chi(s),\sigma_t^\chi(s))\big\|^2_{(H^1(\Omega))^*\times(H^1(\Omega))^*}\,ds \leq K e^{-C(t-T_0)} \quad \forall t \geq T_0,\\ \int_t^\infty \big\|(\phi_t^\chi(s),\sigma_t^\chi(s))^2_{(H^1(\Omega))^*\times(H^1(\Omega))^*}\,ds \leq \tilde{K}(t-T_0)^{\frac{-1}{1-2\theta}} \quad \forall t \geq T_0.\end{gathered}$$

Then, arguing as in [17, Lemma 3.2, Lemma 3.3], it follows from the last estimate that

$$\begin{gathered}\int_t^\infty \big\|(\phi_t^\chi(s),\sigma_t^\chi(s))\big\|_{(H^1(\Omega))^*\times(H^1(\Omega))^*}\,ds \leq K e^{-C(t-T_0)} \quad \text{if } \theta = \frac{1}{2},\\ \int_t^\infty \big\|(\phi_t^\chi(s),\sigma_t^\chi(s))\big\|_{(H^1(\Omega))^*\times(H^1(\Omega))^*}\,ds \leq \tilde{K}(t-T_0)^{\frac{-\theta}{1-2\theta}} \quad \text{if } \theta \in (0,\frac{1}{2}),\end{gathered} \tag{3.32}$$

for all $t \geq T_0$.

Furthermore, we have

$$\big\|(\phi^\chi(t),\sigma^\chi(t)) - (\phi^\chi(T),\sigma^\chi(T))\big\|_{(H^1(\Omega))^*\times(H^1(\Omega))^*} \leq \int_t^T \big\|(\phi_t^\chi(s),\sigma_t^\chi(s))\big\|_{(H^1(\Omega))^*\times(H^1(\Omega))^*}\,ds.$$

Hence, with the help of (3.32) and using Cauchy criterion for the existence of limits, we obtain that there exists $(\phi_*,\sigma_*) \in \omega_0^\chi((\phi_0,\sigma_0))$ such that

$$\lim_{t\to\infty} \big\|(\phi^\chi(t),\sigma^\chi(t) - (\phi_*,\sigma_*))\big\|_{(H^1(\Omega))^*\times(H^1(\Omega))^*} = 0. \tag{3.33}$$

Furthermore, recalling that the global attractor is bounded in $\mathcal{Z}_0^1$ and using interpolation between the spaces $(H^1(\Omega))^* \times (H^1(\Omega))^*$ and $\mathcal{Z}_0^1$, we can complete the proof of (3.2).

Step 2. By using the analogous observations (3.18), (3.20), (3.22) and following the similar steps demonstrated in Step 1, we can prove the backward convergence to equilibrium (3.3). Consequently, proof is complete. □

4. Rate of convergence to equilibrium

From the proof of Theorem 3.1, we can deduce the rate of convergence in the space $(H^1(\Omega))^* \times (H^1(\Omega))^*$.

Proposition 4.1. *Assume that* (A1)–(A3) *hold, and θ, $\tilde{\theta}$ are the Lojasiewicz exponents fixed in Lemma 3.9. Let $T_0 > 0$ be the time fixed in the proof of Theorem 3.1. Then for any full trajectory $\gamma_M^\chi = \{(\phi_M^\chi(t),\sigma_M^\chi(t)) : t \in \mathbb{R}\}$ in the global attractor $\mathcal{A}_M^\chi$, there exist $(\phi_*,\sigma_*) \in \mathcal{N}_M^\chi$ and positive constants $C, K_1, \tilde{K}_1$ such that*

$$\begin{gathered}\big\|(\phi_M^\chi(t),\sigma_M^\chi(t) - (\phi_*,\sigma_*))\big\|_{(H^1(\Omega))^*\times(H^1(\Omega))^*} \leq K_1\,e^{-C(t-T_0)}, \quad \text{if } \theta = \frac{1}{2},\\ \big\|(\phi_M^\chi(t),\sigma_M^\chi(t) - (\phi_*,\sigma_*))\big\|_{(H^1(\Omega))^*\times(H^1(\Omega))^*} \leq \tilde{K}_1\,(t-T_0)^{\frac{-\theta}{1-2\theta}} \quad \text{if } 0 < \theta < \frac{1}{2},\end{gathered} \tag{4.1}$$

for all $t \geq T_0$.

Moreover, there exist $(\phi_{**},\sigma_{**}) \in \mathcal{N}_M^\chi$ and positive constants $\tilde{C}, K_2, \tilde{K}_2$ such that

$$\begin{gathered}\big\|(\phi_M^\chi(-t),\sigma_M^\chi(-t) - (\phi_{**},\sigma_{**}))\big\|_{(H^1(\Omega))^*\times(H^1(\Omega))^*} \leq K_2\,e^{-\tilde{C}(t-T_0)}, \quad \text{if } \tilde{\theta} = \frac{1}{2},\\ \big\|(\phi_M^\chi(-t),\sigma_M^\chi(-t) - (\phi_{**},\sigma_{**}))\big\|_{(H^1(\Omega))^*\times(H^1(\Omega))^*} \leq \tilde{K}_2\,(t-T_0)^{\frac{-\tilde{\theta}}{1-2\tilde{\theta}}} \quad \text{if } 0 < \tilde{\theta} < \frac{1}{2},\end{gathered} \tag{4.2}$$

for all $t \geq T_0$.

Proof. As discussed previously, it is sufficient to prove the case $M = 0$. We have already obtained in Theorem 3.1 that such equilibrium points exist. We will only prove (4.1), since (4.2) can be established in the same manner. First, recall the inequality

$$\big\|(\phi^\chi(t), \sigma^\chi(t)) - (\phi^\chi(T), \sigma^\chi(T))\big\|_{(H^1(\Omega))^* \times (H^1(\Omega))^*} \le \int_t^T \big\|(\phi_t^\chi(s), \sigma_t^\chi(s))\big\|_{(H^1(\Omega))^* \times (H^1(\Omega))^*}\, ds,$$

established in the proof of Theorem 3.1. Passing to limit as $T \to \infty$ in this inequality and considering (3.32) proved in Theorem 3.1, we complete the proof of (4.1). $\square$

Now, let λ_2 denote the second eigenvalue of Laplace operator with Neumann boundary condition. By imposing an additional assumption on the chemotaxis and active transport term χ, we obtain exponential convergence to equilibrium.

Corollary 4.2. *Let* (A1)–(A3) *hold, and let $T_0 > 0$ be the time fixed in the proof of Theorem 3.1. Suppose $\chi^2 < \min(\lambda_2 + c_1 - \alpha, 1))$. Then, for any full trajectory $\gamma_M^\chi = \{(\phi_M^\chi(t), \sigma_M^\chi(t)) : t \in \mathbb{R}\}$ in the global attractor $\mathcal{A}_M^\chi$ there exist $(\phi_*, \sigma_*) \in \mathcal{N}_M^\chi$ and positive constants C, K_1 such that*

$$\big\|(\phi_M^\chi(t), \sigma_M^\chi(t) - (\phi_*, \sigma_*))\big\|_{(H^1(\Omega))^* \times (H^1(\Omega))^*} \le K_1\, e^{-C(t-T_0)},$$

for all $t \ge T_0$.

Proof. As mentioned previously, we consider the case $M = 0$. To prove the corollary, it is sufficient to prove that Łojasiewicz-Simon inequality (3.14) is satisfied with the exponent $\theta = \frac{1}{2}$. To this end, we will examine the kernel of the linearization $\mathcal{E}_\chi''(\phi, \sigma)$. If $(\eta, \xi) \in \ker \mathcal{E}_\chi''(\phi, \sigma)$, then it solves the problem

$$-\Delta\eta + \Psi''(\phi) - \chi\xi = 0, \tag{4.3}$$

$$\xi - \chi\eta = 0. \tag{4.4}$$

Testing (4.3) with η and (4.4) with ξ, we obtain

$$\|\nabla\eta\|_{L^2(\Omega)}^2 + \int_\Omega \Psi''(\phi)|\eta 1|^2\, dx - 2\chi \int_\Omega \eta\xi\, dx + \|\xi\|_{L^2(\Omega)}^2 = 0,$$

which yields

$$(\lambda_2 + c_1 - \alpha - \chi^2)\|\eta\|^2 + (1 - \chi^2)\|\xi\|^2 = 0.$$

Recalling the assumptions on the parameter χ, we infer that $(\eta, \xi) = (0, 0)$, which means that $\ker\, \mathcal{E}_\chi''(\phi, \sigma) = \{(0, 0)\}$. Hence, using [5, Corollary 3.12], we deduce that the Łojasiewicz-Simon inequality (3.14) is satisfied with the exponent $\theta = \frac{1}{2}$. $\square$

Remark 4.3. The stationary point $(\phi_*, \sigma_*) \in \mathcal{N}_M^\chi$ is called hyperbolic if the linearization $\mathcal{E}_\chi''(\phi_*, \sigma_*)$ is invertible, i.e., $\ker \mathcal{E}_\chi''(\phi_*, \sigma_*) = \{(0, 0)\}$. Therefore, under the assumptions of the Corollary 4.2, all stationary points are hyperbolic.

4.1. **Rate of convergence in $\mathcal{Z}_M$ for $0 < \theta < 1/2$.** In this section, we will demonstrate that when $0 < \theta < 1/2$, the full trajectories in the global attractor maintain the same convergence rate as established in $(H^1(\Omega))^* \times (H^1(\Omega))^*$ (see Proposition 4.1) within the space $\mathcal{Z}_M$.

First, note that $(-\Delta)^{-1}$ denotes the inverse of the minus Laplace operator associated with Neumann boundary conditions, acting on functions with zero spatial average. Namely, the norm $\|(-\Delta)^{-1/2} \cdot \|_{L^2(\Omega)}$ is a norm on $\{v \in (H^1(\Omega))^* : \langle v \rangle = 0\}$, which is equivalent to the usual norm of $(H^1(\Omega))^*$. Furthermore, $\|(-\Delta)^{1/2} \cdot \|_{L^2(\Omega)} = \|\nabla \cdot \|_{L^2(\Omega)}$ is equivalent to the usual norm on $H^1(\Omega)$.

In the following theorem, we use the ideas in [21].

Theorem 4.4. *Assume* (A1)–(A3) *hold, and $\theta, \tilde{\theta} \in (0, \frac{1}{2})$ are the Łojasiewicz exponents fixed in Lemma 3.9. Let $T_0 > 0$ be the time fixed in the proof of Theorem 3.1. Then, for any full trajectory $\gamma_M^\chi = \{(\phi_M^\chi(t), \sigma_M^\chi(t)) : t \in \mathbb{R}\}$ in the global attractor $\mathcal{A}_M^\chi$, there exist $(\phi_*, \sigma_*) \in \mathcal{N}_M^\chi$ and $\kappa > 0$ such that*

$$\big\|(\phi_M^\chi(t), \sigma_M^\chi(t) - (\phi_*, \sigma_*))\big\|_{\mathcal{Z}_M} \le \kappa\, (t - T_0)^{\frac{-\theta}{1-2\theta}}, \tag{4.5}$$

for all $t \geq T_0$. Moreover, there exist $(\phi_{**}, \sigma_{**}) \in \mathcal{N}_M^\chi$ and $\tilde{\kappa} > 0$ such that

$$\left\|(\phi_M^\chi(-t), \sigma_M^\chi(-t) - (\phi_{**}, \sigma_{**}))\right\|_{\mathcal{Z}_M} \leq \tilde{\kappa}\,(t - T_0)^{\frac{-\tilde{\theta}}{1-2\tilde{\theta}}}, \tag{4.6}$$

for all $t \geq T_0$.

Proof. As previously discussed, it is sufficient to consider the case $M = 0$. We will only prove (4.5) and do not present the proof of (4.6), as it can be established following the same steps.

From Proposition 4.1, we know that for any full trajectory $\gamma^\chi = \{(\phi^\chi(t), \sigma^\chi(t)) : t \in \mathbb{R}\}$ in the global attractor $\mathcal{A}^\chi$, there exist $(\phi_*, \sigma_*) \in \mathcal{N}^\chi$ such that (4.1) is satisfied.

Defining $\bar{\phi^\chi} := \phi^\chi - \phi_*$, $\bar{\sigma^\chi} := \sigma^\chi - \sigma_*$ and $\bar{\mu^\chi} := \mu - \mu_0$, we obtain from (2.9) that

$$\langle \bar{\phi^\chi}_t, \eta\rangle + \langle \nabla\bar{\mu^\chi}, \nabla\eta\rangle = \langle p(\phi^\chi)(\bar{\sigma^\chi} - \chi\bar{\phi^\chi} - \bar{\mu^\chi}), \eta\rangle, \tag{4.7}$$

$$\bar{\mu^\chi} = -\Delta\bar{\phi^\chi} + \Psi'(\phi^\chi) - \Psi'(\phi_*) - \chi\bar{\sigma^\chi}, \tag{4.8}$$

$$\langle \bar{\sigma^\chi}_t, \xi\rangle + \langle \nabla\bar{\sigma^\chi}, \nabla\xi\rangle = \chi\langle \nabla\bar{\phi^\chi}, \nabla\xi\rangle - \langle p(\phi^\chi)(\bar{\sigma^\chi} - \chi\bar{\phi^\chi} - \bar{\mu^\chi}), \eta\rangle, \tag{4.9}$$

for all $\eta, \xi \in H^1(\Omega)$. Choosing $\eta = (-\Delta)^{-1}\bar{\phi^\chi}$ in (4.7) and $\xi = (-\Delta)^{-1}\bar{\sigma^\chi}$ in (4.9), and summing the obtained identities, we infer that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\bar{\phi^\chi}\|^2_{(H^1(\Omega))^*} + \|\bar{\sigma^\chi}\|^2_{(H^1(\Omega))^*}) + \|\bar{\phi^\chi}\|^2_{H^1(\Omega)} + \|\bar{\sigma^\chi}\|^2_{L^2(\Omega)} + \langle \Psi'(\phi^\chi) - \Psi'(\phi_*), \bar{\phi^\chi}\rangle \\
&= 2\chi\int \bar{\sigma^\chi}\bar{\phi^\chi}\,dx + \langle p(\phi^\chi)(\bar{\sigma^\chi} - \chi\bar{\phi^\chi} - \bar{\mu^\chi}), (-\Delta)^{-1}\bar{\phi^\chi}\rangle \\
&\quad - \langle p(\phi^\chi)(\bar{\sigma^\chi} - \chi\bar{\phi^\chi} - \bar{\mu^\chi}), (-\Delta)^{-1}\bar{\sigma^\chi}\rangle.
\end{aligned} \tag{4.10}$$

Now, we will evaluate the terms on the right-hand side of (4.10). First, let us observe that for any $\eta \in H^1(\Omega)$, since $\phi^\chi \in H^3(\Omega) \subset L^\infty$, it holds

$$|\langle p(\phi^\chi)(\bar{\sigma^\chi} - \chi\bar{\phi^\chi} - \bar{\mu^\chi}), \eta\rangle| \leq \|\bar{\sigma^\chi} - \chi\bar{\phi^\chi} - \bar{\mu^\chi}\|_{L^2(\Omega)}\|\eta\|_{L^2(\Omega)} \leq C\|\eta\|_{H^1(\Omega)}.$$

Then

$$\begin{aligned}
&|\langle p(\phi^\chi)(\bar{\sigma^\chi} - \chi\bar{\phi^\chi} - \bar{\mu^\chi}), (-\Delta)^{-1}\bar{\phi^\chi}\rangle| + |\langle p(\phi^\chi)(\bar{\sigma^\chi} - \chi\bar{\phi^\chi} - \bar{\mu^\chi}), (-\Delta)^{-1}\bar{\sigma^\chi}\rangle| \\
&\leq \|p(\phi^\chi)(\bar{\sigma^\chi} - \chi\bar{\phi^\chi} - \bar{\mu^\chi})\|_{(H^1(\Omega))^*}(\|\bar{\phi^\chi}\|_{(H^1(\Omega))^*} + \|\bar{\sigma^\chi}\|_{(H^1(\Omega))^*}) \\
&\leq C(\|\bar{\phi^\chi}\|_{(H^1(\Omega))^*} + \|\bar{\sigma^\chi}\|_{(H^1(\Omega))^*}).
\end{aligned} \tag{4.11}$$

On the other hand, for the first term on the right-hand side of (4.10), we have

$$2\chi\Big|\int_\Omega \bar{\sigma^\chi}\bar{\phi^\chi}\,dx\Big| \leq 4\chi^2\|\bar{\sigma^\chi}\|^2_{(H^1(\Omega))^*} + \frac{1}{4}\|\bar{\phi^\chi}\|^2_{H^1(\Omega)}. \tag{4.12}$$

Moreover, since Ψ_0' is monotone and Λ' is Lipschitz continuous, for the nonlinear term we obtain

$$\begin{aligned}
&(\Psi'(\phi^\chi) - \Psi'(\phi_*), \bar{\phi^\chi}) = (\Psi_0'(\phi^\chi) - \Psi_0'(\phi_*), \bar{\phi^\chi}) + (\Lambda'(\phi^\chi) - \Lambda'(\phi_*), \bar{\phi^\chi}) \\
&\geq -\alpha\|\bar{\phi^\chi}\|^2_{L^2} \geq -\frac{1}{4}\|\bar{\phi^\chi}\|^2_{H^1(\Omega)} - c\|\bar{\phi^\chi}\|^2_{(H^1(\Omega))^*},
\end{aligned} \tag{4.13}$$

for some $c > 0$.

In the following, C denotes a generic constant, which may vary from line to line and even within the same line. Now, considering the estimates (4.11), (4.12) and (4.13) in (4.10), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\bar{\phi^\chi}\|^2_{(H^1(\Omega))^*} + \|\bar{\sigma^\chi}\|^2_{(H^1(\Omega))^*}\Big) + \|\bar{\phi^\chi}\|^2_{H^1(\Omega)} + \|\bar{\sigma^\chi}\|^2_{L^2(\Omega)} \\
&\leq C(\|\bar{\phi^\chi}\|_{(H^1(\Omega))^*} + \|\bar{\sigma^\chi}\|_{(H^1(\Omega))^*}).
\end{aligned} \tag{4.14}$$

Next, testing (4.7) with $\bar{\mu^\chi}$ and (4.9) with $\bar{\sigma^\chi} - \chi\bar{\phi^\chi}$, we infer that

$$\begin{aligned}
&\frac{d}{dt}\Big(\frac{1}{2}\|\bar{\phi^\chi}\|^2_{H^1(\Omega)} + \frac{1}{2}\|\bar{\sigma^\chi}\|^2_{L^2(\Omega)} + \chi\int \bar{\sigma^\chi}\bar{\phi^\chi}\,dx\Big) \\
&+ \frac{d}{dt}\Big(\int_\Omega \Psi(\phi^\chi)\,dx - \int_\Omega \Psi(\phi_*)\,dx + \int_\Omega \Psi'(\phi_*)\phi_*\,dx - \int_\Omega \Psi'(\phi_*)\phi^\chi\,dx\Big) \\
&+ \int_\Omega \Big(|\nabla\bar{\mu^\chi}|^2 + |\nabla\bar{\sigma^\chi} - \chi\nabla\bar{\phi^\chi}|^2 + p(\phi^\chi)(\bar{\sigma^\chi} - \chi\bar{\phi^\chi} - \mu^\chi)^2\Big)\,dx = 0.
\end{aligned} \tag{4.15}$$

Using the Newton-Leibniz formula and recalling that $\phi^\chi, \phi_* \in H^3(\Omega) \subset L^\infty(\Omega)$,

$$\begin{aligned}
&\Big| \int_\Omega \Psi(\phi^\chi)\,dx - \int_\Omega \Psi(\phi_*)\,dx + \int_\Omega \Psi'(\phi_*)\phi_*\,dx - \int_\Omega \Psi'(\phi_*)\phi^\chi\,dx \Big| \\
&\le \Big| \int_\Omega \int_0^1 \int_0^1 \Psi''(\tau s\phi + (1-\tau s)\phi_*)(\phi^\chi - \phi_*)^2\,ds\,d\tau\,dx \Big| \\
&\le C\|\bar{\phi^\chi}\|^2_{L^2(\Omega)} \\
&\le \frac{1}{8}\|\bar{\phi^\chi}\|^2_{H^1(\Omega)} + C\|\bar{\phi^\chi}\|^2_{(H^1(\Omega))^*}\,.
\end{aligned} \tag{4.16}$$

Furthermore,

$$\big|\chi \int \bar{\sigma^\chi}\bar{\phi^\chi}\,dx\big| \le C\|\bar{\phi^\chi}\|^2_{L^2(\Omega)} \le \frac{1}{8}\|\bar{\phi^\chi}\|^2_{H^1(\Omega)} + C\|\bar{\sigma^\chi}\|^2_{(H^1(\Omega))^*}. \tag{4.17}$$

Now defining,

$$\begin{aligned}
\Phi(t) := \frac{1}{2}\|\nabla\bar{\phi^\chi}\|^2_{L^2(\Omega)} + \frac{1}{2}\|\bar{\sigma^\chi}\|^2_{L^2(\Omega)} + \chi \int \bar{\sigma^\chi}\bar{\phi^\chi}\,dx \\
+ \int_\Omega \Psi(\phi^\chi)\,dx - \int_\Omega \Psi(\phi_*)\,dx + \int_\Omega \Psi'(\phi_*)\phi_*\,dx \\
- \int_\Omega \Psi'(\phi_*)\phi^\chi\,dx + \frac{1}{2}(\|\bar{\phi^\chi}\|^2_{(H^1(\Omega))^*} + \|\bar{\sigma^\chi}\|^2_{(H^1(\Omega))^*}),
\end{aligned}$$

we deduce from (4.16) and (4.17) that

$$\big|\Phi(t) - C(\|\bar{\phi^\chi}\|^2_{H^1(\Omega)} + \|\bar{\sigma^\chi}\|^2_{L^2(\Omega)})\big| \le \|\bar{\phi^\chi}\|^2_{(H^1(\Omega))^*} + \|\bar{\sigma^\chi}\|^2_{(H^1(\Omega))^*}. \tag{4.18}$$

Therefore, adding (4.14) and (4.15), we obtain

$$\frac{d}{dt}(\Phi(t)) + \beta\,\Phi(t) \le C(\|\bar{\phi^\chi}\|_{(H^1(\Omega))^*} + \|\bar{\sigma^\chi}\|_{(H^1(\Omega))^*}), \tag{4.19}$$

for some $\beta > 0$. Considering (4.1) in (4.19), we have

$$\frac{d}{dt}(\Phi(t)) + \beta\Phi(t) \le C\,(t - T_0)^{\frac{-\theta}{1-2\theta}}.$$

Using that the exponential function grows faster than any polynomial, it follows from the last inequality that

$$\begin{aligned}
\Phi(t) &\le \Phi(T_0)e^{-\beta(t-T_0)} + e^{-\beta t}C\int_{T_0}^t e^{\beta\tau}(\tau - T_0)^{\frac{-2\theta}{1-2\theta}}\,d\tau \\
&\le Ce^{-\beta t} + C\,(t - T_0)^{\frac{-\theta}{1-2\theta}} \\
&\le \kappa_1\,(t - T_0)^{\frac{-\theta}{1-2\theta}},
\end{aligned} \tag{4.20}$$

for some $\kappa_1 > 0$. Then, recalling (4.18), we obtain from (4.1) and (4.20) that

$$\|\bar{\phi^\chi}\|^2_{H^1(\Omega)} + \|\bar{\sigma^\chi}\|^2_{L^2(\Omega)} \le \Phi(t) + \|\bar{\phi^\chi}\|^2_{(H^1(\Omega))^*} + \|\bar{\sigma^\chi}\|^2_{(H^1(\Omega))^*} \le \kappa_2\,(t - T_0)^{\frac{-\theta}{1-2\theta}}, \tag{4.21}$$

for some $\kappa_2 > 0$. Hence, we conclude that (4.5) is satisfied in $\mathcal{Z}_0$, i.e.,

$$\big\|(\phi^\chi(t), \sigma^\chi(t) - (\phi_*, \sigma_*))\big\|_{\mathcal{Z}_0} \le \kappa\,(t - T_0)^{\frac{-\theta}{1-2\theta}}, \tag{4.22}$$

for some $\kappa > 0$. □

5. Stability of the Attractors

In this section, we consider the chemotaxis and active transport parameter χ as a perturbation parameter. Then, from Theorem 2.12, there exists a family of global attractors $\{\mathcal{A}_M^\chi\}_{\chi\ge 0}$ for the family of the semigroups $\{S_M^\chi\}_{\chi\ge 0}$ acting on the phase space Z_M. Since M is fixed and has no effect on the following calculations, we omit M from the notation for the sake of simplicity. In the following part of the paper, we will use the notations $\{\mathcal{A}^\chi\}_{\chi\ge 0}$ and $(\mathcal{Z}_M, S^\chi)$ instead of $\{\mathcal{A}_M^\chi\}_{\chi\ge 0}$ and $(\mathcal{Z}_M, S_M^\chi)$, respectively. Here, it is worth recalling that $\mathcal{A}^0$ and S^0 denote the global attractor and the semigroup, respectively, in the case where chemotaxis and active transport are neglected ($\chi = 0$). We investigate the stability of the family of the global attractors $\{\mathcal{A}^{\chi}\}_{\chi\geq 0}$ as $\chi \to 0$.

Definition 5.1. Let X be a Banach space, and I be a metric space. The family of global attractors $\{\mathcal{A}^{\alpha}\}_{\alpha\in I}$ is called upper semicontinuous at the point $\alpha_0 \in I$ if

$$\lim_{\alpha\to\alpha_0} \operatorname{dist}_X(\mathcal{A}^{\alpha}, \mathcal{A}^{\alpha_0}) = 0,$$

and it is called lower semicontinuous at the point $\alpha_0 \in I$ if

$$\lim_{\alpha\to\alpha_0} \operatorname{dist}_X(\mathcal{A}^{\alpha_0}, \mathcal{A}^{\alpha}) = 0,$$

where $\operatorname{dist}_X(\cdot,\cdot)$ is the Hausdorff semidistance defined in Definition 2.10.

We exploit the following abstract result proven in [20] to obtain the upper semicontinuity of the family of global attractors $\{\mathcal{A}^{\chi}\}_{\chi\geq 0}$ as $\chi \to 0$.

Theorem 5.2. *Suppose that a dynamical system $(X, S^{\alpha}(t))$ possesses a global attractor $\mathcal{A}^{\alpha}$ for every $\alpha \in I$, where I is a complete metric space. Assume that the following conditions hold:*

(1) *There exists a compact set K such that $\underset{\alpha\in I}{\cup} \mathcal{A}^{\alpha} \subset K$,*
(2) *If $\alpha_k \to \alpha_0$, $x_k \in \mathcal{A}^{\alpha_k}$ and $x_k \to x_0$, then $S^{\alpha_k}(t_0)x_k \to S^{\alpha_0}(t_0)x_0$ for some $t_0 > 0$.*

Then, the family of attractors is upper semicontinuous at the point α_0.

We begin by proving the following lemma, which states the continuous dependence of the solutions on the parameter χ.

Lemma 5.3. *Assume that* (A1), (A2) *are satisfied and $\{\chi_n\}_{n\in\mathbb{N}} \in [0, \chi_0]$, $\quad\chi_n \to 0$ as $n \to \infty$. For any initial datum $(\phi_0, \sigma_0) \in \mathcal{Z}_M$, we have*

$$S^{\chi_n}(\cdot)(\phi_0, \sigma_0) \to S^0(\cdot)(\phi_0, \sigma_0) \quad \text{strongly in } C([0, T]; L^2(\Omega) \times L^2(\Omega)).$$

Proof. Let us denote $(\phi_n, \sigma_n) := S^{\chi_n}(t)(\phi_0, \sigma_0)$. Applying the same procedure used in the proof of [15, Theorem 3.1] to the problem

$$\begin{gathered}
\langle\langle \phi_t^{\chi_n}, \eta\rangle\rangle + \langle \nabla\mu^{\chi_n}, \nabla\eta\rangle = \langle p(\phi^{\chi_n})(\sigma^{\chi_n} + \chi_n(1 - \phi^{\chi_n}) - \mu^{\chi_n}), \eta\rangle, \\
\langle \mu^{\chi_n}, \eta\rangle = \langle \nabla\phi^{\chi_n}, \nabla\eta\rangle + \langle \Psi'(\phi^{\chi_n}), \eta\rangle - \langle \chi_n\sigma^{\chi_n}, \eta\rangle, \\
\langle\langle \sigma_t^{\chi_n}, \xi\rangle\rangle + \langle(\nabla\sigma^{\chi_n} - \chi_n\nabla\phi^{\chi_n}), \nabla\xi\rangle = -\langle p(\phi^{\chi_n})(\sigma^{\chi_n} + \chi_n(1 - \phi^{\chi_n}) - \mu^{\chi_n}), \xi\rangle,
\end{gathered} \tag{5.1}$$

we obtain the uniform bounds

$$\begin{gathered}
\|\phi_n\|_{L^{\infty}(0,T;H^1(\Omega))\cap L^2(0,T;H^3(\Omega))} \leq C, \quad \|\phi_{nt}\|_{L^2(0,T;(H^1(\Omega))^*)} \leq C \\
\|\sigma_n\|_{L^{\infty}(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \leq C, \quad \|\sigma_{nt}\|_{L^2(0,T;(H^1(\Omega))^*)} \leq C, \\
\|\mu_n\|_{L^2(0,T;H^1(\Omega))} \leq C,
\end{gathered}$$

where $\mu_n := \mu^{\chi_n}$.

Therefore, from the Banach-Alaoglu theorem it follows that up to a subsequence,

$$\begin{gathered}
(\phi_n, \sigma_n) \to (\hat{\phi}, \hat{\sigma}) \quad \text{weakly-star in } L^{\infty}(0, T; H^1(\Omega) \times L^2(\Omega)), \\
(\phi_{nt}, \sigma_{nt}) \to (\hat{\phi}_t, \hat{\sigma}_t) \quad \text{weakly in } L^2(0, T; (H^1(\Omega))^* \times (H^1(\Omega))^*), \\
(\phi_n, \sigma_n) \to (\hat{\phi}, \hat{\sigma}) \quad \text{weakly in } L^2(0, T; H^3(\Omega) \times H^1(\Omega)), \\
\mu_n \to \hat{\mu} \quad \text{weakly in } L^2(0, T; H^1(\Omega)),
\end{gathered} \tag{5.2}$$

where

$$\begin{gathered}
\hat{\phi} \in L^{\infty}(0, T; H^1(\Omega)) \cap L^2(0, T; H^3(\Omega)), \\
\hat{\sigma} \in L^{\infty}(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \\
\hat{\mu} \in L^2(0, T; H^1(\Omega)), \\
\hat{\phi}_t,\ \hat{\sigma}_t \in L^2(0, T; (H^1(\Omega))^*).
\end{gathered}$$

From (5.2), using the Aubin-Lions lemma, we infer that (up to a subsequence)

$$\phi_n \to \hat{\phi} \quad \text{strongly in } C([0,T]; L^\kappa(\Omega)), \text{ for } 2 \le \kappa < 6. \tag{5.3}$$

Moreover, from (5.2), using the Lions-Magenes lemma, we infer that (up to a subsequence)

$$\sigma_n \to \hat{\sigma} \quad \text{strongly in } C([0,T]; L^2(\Omega)). \tag{5.4}$$

Then, arguing as in [15, Theorem 3.1], we obtain (up to a subsequences)

$$p(\phi_n) \to p(\hat{\phi}), \quad \text{strongly in } L^2(0,T; L^{6/5}(\Omega)) \tag{5.5}$$

and

$$(\sigma^{\chi_n} + \chi_n(1-\phi^{\chi_n}) - \mu^{\chi_n}) \to (\hat{\sigma} - \hat{\mu}), \quad \text{weakly in } L^2(0,T; L^6(\Omega)). \tag{5.6}$$

Hence, we can pass to the limit in the term $\langle p(\phi^{\chi_n})(\sigma^{\chi_n} + \chi_n(1-\phi^{\chi_n}) - \mu^{\chi_n}), \eta\rangle$ with the help of (5.5) and (5.6). Thanks to all of the convergence results established above, we can pass to the limit in the problem (5.1) and deduce that $(\hat{\phi}, \hat{\sigma}, \hat{\mu})$ is a weak solution of the problem (2.1)–(2.4) without chemotaxis and active transport, i.e., with the parameter $\chi = 0$. Then, using the uniqueness of weak solutions to the problem (2.1)–(2.4), we deduce that $(\hat{\phi}, \hat{\sigma}) = S^0(t)(\phi_0, \sigma_0)$. Consequently, every convergent subsequence of $\{(\phi_n, \sigma_n)\}_{n\in\mathbb{N}}$ has the same limit, so we conclude that (5.3) and (5.4) are satisfied by the whole sequence. Hence, the proof is completed. □

Now, we prove that the $\underset{0\le\chi\le\chi_0}{\cup} \mathcal{A}^\chi$ is relatively compact.

Lemma 5.4. *Assume that* (A1), (A2) *are satisfied and* $\chi \in [0, \chi_0]$. *Then, the family of the global attractors* $\{\mathcal{A}^\chi\}_{0\le\chi\le\chi_0}$ *for problem* (2.1)–(2.4) *is relatively compact in* $\mathcal{Z}_M^1$. *Namely, there exists a compact set* $\mathcal{K} \subset \mathcal{Z}_M$ *such that*

$$\cup_{0\le\chi\le\chi_0} \mathcal{A}^\chi \subset \mathcal{K}.$$

Proof. In [15], it was shown that the global attractor $\mathcal{A}^\chi$ is bounded in $\mathcal{Z}_M^1$. Exactly, using the regularization of weak solutions, it can be observed that for all $t > 0$ (cf. the proof of [15, Theorem 3.3]),

$$\int_t^{t+1} \Big(\|\phi^\chi(\tau)\|^2_{H^3(\Omega)} + \|\sigma^\chi(\tau)\|^2_{H^1(\Omega)} \Big) d\tau \le C,$$

where C is independent of χ and depends on χ_0. Hence, by following the same steps in [15, Theorem 5.10], we readily obtain that the family of the global attractors $\{\mathcal{A}^\chi\}_{0\le\chi\le\chi_0}$ is uniformly bounded in $\mathcal{Z}_M^1$. Precisely, there exist $\mathcal{B} \in \mathcal{Z}_M^1$ such that

$$\cup_{0\le\chi\le\chi_0} \mathcal{A}^\chi \subset \mathcal{B}.$$

Hence, by choosing $\mathcal{K} = \overline{\mathcal{B}}^{H^1(\Omega)\times L^2(\Omega)}$, we obtain that $\mathcal{K}$ is bounded in $H^3(\Omega) \times H^1(\Omega)$. Then, by the compact embedding $H^3(\Omega) \times H^1(\Omega) \hookrightarrow H^1(\Omega) \times L^2(\Omega)$, the set $\mathcal{K}$ is compact in $\mathcal{Z}_M$, and the proof is complete. □

Lemma 5.5. *Under the assumptions of Lemma 5.4, for every bounded set* $B \subset \mathcal{Z}_M$, *there exists a time* $T_0(B) > 0$ *such that*

$$\sup_{0\le\chi\le\chi_0} \sup_{(\phi,\sigma)\in B} \|S^\chi(t)(\phi,\sigma)\|_{H^3(\Omega)\times H^1(\Omega)} \le R, \quad \forall t \ge T_0(B),$$

for some constant $R > 0$.

Proof. From Lemma 5.4, we observe that there exists a bounded absorbing set $\mathcal{B}$ in $H^3(\Omega)\times H^1(\Omega)$ such that for all $0 \le \chi \le \chi_0$ and for every bounded set $B \subset \mathcal{Z}_M$, there exists a time $t_0^\chi(B) > 0$ such that

$$S^\chi(t)B \subset \mathcal{B} \quad \text{for all } t \ge t_0^\chi(B).$$

Since for any $t \ge 0$, the state $S^\chi(t)$ depends continuously on the parameter χ (Lemma 5.3), we infer that the time $t_0^\chi(B)$ also depends continuously on χ. Therefore, we deduce that

$$\cup_{0\le\chi\le\chi_0} S^\chi(t)B \subset \mathcal{B} \quad \text{for all } t \ge T_0(B) := \max_{0\le\chi\le\chi_0} t_0^\chi(B).$$

We complete the proof of the lemma by choosing R as the diameter of the absorbing set $\mathcal{B}$. □

Lemma 5.6. *Let the assumptions of Lemma 5.4 hold and let $B \in \mathcal{Z}_M$ be a bounded set. Then, for any $(\phi_1, \sigma_1), (\phi_2, \sigma_2) \in B$, there exists a time $T_0(B) > 0$ such that the semigroup $\{S^\chi(t)\}_{t\geq 0}$, generated by the weak solution of the problem (2.1)–(2.4), satisfies*

$$\begin{aligned}&\|S^\chi(t)(\phi_1,\sigma_1) - S^\chi(t)(\phi_2,\sigma_2)\|_{H^1(\Omega)\times L^2(\Omega)}\\&\leq C\Big(\|\phi_1-\phi_2\|^{1/2}_{H^1(\Omega)} + \|\sigma_1-\sigma_2\|^{1/2}_{L^2(\Omega)}\Big) \quad \forall t \geq T_0(B),\end{aligned}$$

where C is independent of χ.

Proof. Using the interpolation inequality, we have

$$\begin{aligned}&\|S^\chi(t)(\phi_1,\sigma_1) - S^\chi(t)(\phi_2,\sigma_2)\|^2_{H^1(\Omega)\times L^2(\Omega)}\\&\leq \|S^\chi(t)(\phi_1,\sigma_1) - S^\chi(t)(\phi_2,\sigma_2)\|_{H^3(\Omega)\times H^1(\Omega)}\|S^\chi(t)(\phi_1,\sigma_1) - S^\chi(t)(\phi_2,\sigma_2)\|_{(H^1(\Omega))^*\times(H^1(\Omega))^*}.\end{aligned}$$

From Lemma 5.5, there exists a time $T_0(B) > 0$ such that

$$\begin{aligned}&\|S^\chi(t)(\phi_1,\sigma_1) - S^\chi(t)(\phi_2,\sigma_2)\|^2_{H^1(\Omega)\times L^2(\Omega)}\\&\leq 2R\|S^\chi(t)(\phi_1,\sigma_1) - S^\chi(t)(\phi_2,\sigma_2)\|_{(H^1(\Omega))^*\times(H^1(\Omega))^*} \quad \forall t \geq T_0(B).\end{aligned}\tag{5.7}$$

Taking into account Theorem 2.4 in (5.7), we complete the proof. □

Lemma 5.7. *Let the assumptions of Lemma 5.4 hold. Then, If $\chi_n \to 0$, $(\phi_n, \sigma_n) \in \mathcal{A}^{\chi_n}$ and $(\phi_n, \sigma_n) \to (\phi_0, \sigma_0)$, then $S^{\chi_n}(t_0)(\phi_n, \sigma_n) \to S^0(t_0)(\phi_0, \sigma_0)$ for some $t_0 > 0$.*

Proof. Firstly, from Lemma 5.3, we have

$$S^{\chi_n}(t)(\phi_0,\sigma_0) \to S^0(t)(\phi_0,\sigma_0) \quad \text{strongly in } L^2(\Omega)\times L^2(\Omega)\ \forall t \geq 0.\tag{5.8}$$

On the other hand, Lemma 5.5 implies that there exists $T_0 > 0$ such that the sequence $\{S^{\chi_n}(t)(\phi_0, \sigma_0) : \chi_n \in [0, \chi_0],\ t \geq T_0\}$ is uniformly bounded in $H^3(\Omega) \times H^1(\Omega)$. Therefore, (5.8) yields

$$S^{\chi_n}(t)(\phi_0,\sigma_0) \to S^0(t)(\phi_0,\sigma_0) \quad \text{strongly in } H^1(\Omega)\times L^2(\Omega)\ \forall t \geq T_0.\tag{5.9}$$

Finally, applying the triangle inequality and Lemma 5.6, we have

$$\begin{aligned}&\|S^{\chi_n}(t)(\phi_n,\sigma_n) - S^0(t)(\phi_0,\sigma_0)\|_{H^1(\Omega)\times L^2(\Omega)}\\&\leq \|S^{\chi_n}(t)(\phi_n,\sigma_n) - S^{\chi_n}(t)(\phi_0,\sigma_0)\|_{H^1(\Omega)\times L^2(\Omega)}\\&\quad + \|S^{\chi_n}(t)(\phi_0,\sigma_0) - S^0(t)(\phi_0,\sigma_0)\|_{H^1(\Omega)\times L^2(\Omega)}\\&\leq C\Big(\|\phi_n-\phi_0\|^{1/2}_{H^1(\Omega)} + \|\sigma_n-\sigma_0\|^{1/2}_{L^2(\Omega)}\Big)\\&\quad + \|S^{\chi_n}(t)(\phi_0,\sigma_0) - S^0(t)(\phi_0,\sigma_0)\|_{H^1(\Omega)\times L^2(\Omega)} \quad \forall t \geq T_0.\end{aligned}\tag{5.10}$$

Hence, taking into account (5.9) in (5.10) and choosing $t_0 > T_0$, we obtain the desired result of the lemma. □

Now, we are in a position to prove the main result of this section.

Theorem 5.8. *Let the conditions of Lemma 5.4 hold. Then the family of global attractors $\{\mathcal{A}^\chi\}_{0\leq\chi\leq\chi_0}$ is upper-semicontinuous as $\chi \to 0$, i.e.,*

$$\lim_{\chi\to 0} \operatorname{dist}_{\mathcal{Z}_M}(\mathcal{A}^\chi, \mathcal{A}^0) = 0.$$

Proof. Lemmas 5.4 and 5.7 verify that all assumptions of Theorem 5.2 are satisfied. Therefore, the proof is completed by applying Theorem 5.2 introduced in [20]. □

As mentioned in Remark 4.3, if the chemotaxis parameter satisfies $\chi^2 < \min(\lambda_2 + c_1 - \alpha, 1))$, all stationary points in $\mathcal{N}^\chi$ are hyperbolic. Consequently, by applying the well-known result [16, Theorem 3.8.9], we also observe the lower-semicontinuity of the set of global attractors.

Theorem 5.9. *Assume that* (A1)–(A3) *are satisfied, and let $\chi^2 < \min(\lambda_2 + c_1 - \alpha, 1))$. Then, the family of global attractors $\{\mathcal{A}^\chi\}_{0\leq\chi\leq\chi_0}$ is lower-semicontinuous as $\chi \to 0$, i.e.,*

$$\lim_{\chi\to 0} \operatorname{dist}_{\mathcal{Z}_M}(\mathcal{A}^0, \mathcal{A}^\chi) = 0.$$

References

[1] E. R. Aragao-Costa, A. N. Carvalho, P. Marín-Rubio, G. Planas; Gradient-like nonlinear semigroups with infinitely many equilibria and applications to cascade systems, *Topol. Methods Nonlinear Anal*, **42** (2013), no. 2, 345-376.

[2] J. Cahn , J. Hilliard; Free energy of a nonuniform system I. Interfacial free energy, *J. Chem. Phys.*, **2** (1958), 205–245.

[3] A. N. Carvalho, J. A. Langa; An extension of the concept of gradient semigroups which is stable under perturbation. *Journal of Differential Equations*, **246** (2009), no. 7, 2646–2668.

[4] A. N. Carvalho, J. A. Langa, J. C. Robinson; Lower semicontinuity of attractors for non-autonomous dynamical systems. *Ergodic Theory and Dynamical Systems*, **29** (2009), no: 6, 1765-1780.

[5] R. Chill; On the Łojasiewicz–Simon gradient inequality, *J. Funct. Anal.*, **201** (2003), no. 2, 572–601.

[6] R. Chill, E. Făsangová, J. Prüss; Convergence to steady state of solutions of the Cahn-Hilliard and Caginalp equations with dynamic boundary conditions, *Math. Nachr.*, **279** (2006), no. 13-14, 1448–1462.

[7] I. Chueshov, I. Lasiecka; *Von Karman evolution equations: Well-posedness and long-time dynamics*, Springer Monographs in Mathematics. Springer, New York, 2010.

[8] P. Colli, G. Gilardi, D. Hilhorst; On a Cahn-Hilliard type phase field system related to tumor growth, *Discrete Contin. Dyn. Syst.*, **35** (2015), no. 6, 2423–2442.

[9] P. Colli, G. Gilardi, E. Rocca, J. Sprekels; Vanishing viscosities and error estimate for a Cahn–Hilliard type phase field system related to tumor growth, *Nonlinear Anal. Real World Appl.*, **26** (2015), 93–108.

[10] S. Frigeri, M. Grasselli, E. Rocca; On a diffuse interface model of tumour growth, *European J. Appl. Math.*, **26** (2015), no. 2, 215-243.

[11] H. Garcke, K. F. Lam; Global weak solutions and asymptotic limits of a Cahn-Hilliard–Darcy system modelling tumour growth, *AIMS Math.*, **1** (2016), no. 3, 318–360.

[12] H. Garcke, K. F. Lam; Analysis of a Cahn-Hilliard system with non-zero Dirichlet conditions modeling tumor growth with chemotaxis, *Discrete Contin. Dyn. Syst.*, **37** (2017), no. 8, 4277–4308.

[13] H. Garcke, K. F. Lam; Well-posedness of a Cahn-Hilliard system modelling tumour growth with chemotaxis and active transport, *European J. Appl. Math.*, **28** (2017), no. 2, 284-316.

[14] H. Garcke, K. F. Lam, E. Sitka, V. Styles; A Cahn–Hilliard-Darcy model for tumour growth with chemotaxis and active transport, *Math. Models Methods Appl. Sci.*, 26(06):1095–1148, 2016.

[15] H. Garcke, S. Yayla; Long-time dynamics for a Cahn-Hilliard tumor growth model with chemotaxis, *Z. Angew. Math. Phys.*, **71** (2020), no. 4, Paper No. 123.

[16] J. K. Hale. *Asymptotic behavior of dissipative systems*, no. 25. American Mathematical Soc., 2010.

[17] A. Haraux, M. Ali Jendoubi, O. Kavian; Rate of decay to equilibrium in some semilinear parabolic equations, *J. Evol. Equ.*, **3** (2003), no. 3, 463–484.

[18] A. Hawkins-Daarud, K. G. van der Zee, J. T. Oden; Numerical simulation of a thermodynamically consistent four-species tumor growth model, *Int. J. Numer. Methods Biomed. Eng.*, **28** (2012), 3–24.

[19] D. Hilhorst, J. Kampmann, T. N. Nguyen, K. G. van der Zee; Formal asymptotic limit of a diffuse-interface tumor-growth model, *Math. Models Methods Appl. Sci.*, **25** (2015), no. 06, 1011–1043.

[20] L. V. Kapitanskiĭ, I. N. Kostin; Attractors of nonlinear evolution equations and their approximations, *Algebra i Analiz*, **2** (1990), no. 1, 114–140.

[21] K. F. Lam, H. Wu; Convergence to equilibrium for a bulk-surface Allen-Cahn system coupled through a nonlinear robin boundary condition, *Discrete Contin. Dyn. Syst.*, **40** (2020), no. 3, 1847–1878.

[22] A. Miranville; *The Cahn-Hilliard equation: recent advances and applications*, SIAM, 2019.

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