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COMPLICATED ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO DOUBLY NONLINEAR DIFFUSION EQUATION IN UNBOUNDED SPACES

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Abstract. In this article, we study the complicated asymptotic behavior of doubly nonlinear diffusion equations in unbounded spaces. We find a workspace in which is unbounded and can exhibit complicated asymptotic behavior of the solution to the Cauchy problem. To overcome the difficulties caused by the nonlinearity of the equations and the unbounded solutions, we establish propagation estimates, growth estimates, and Weighted L^1-L^∞ estimates for the solutions.

1. Introduction

In this article, we study the complicated asymptotic behavior of solutions to the Cauchy problem for doubly nonlinear diffusion equations,

$$u_t - \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m) = 0 \quad \text{in } S = \mathbb{R}^N \times (0,\infty), \tag{1.1}$$

$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where $m > 0$, $p > 1$, $m(p-1) > 1$ and the nonnegative initial values satisfy

$$u_0 \in Y_\sigma(\mathbb{R}^N) = \{\varphi \in C(\mathbb{R}^N) : \lim_{|x|\to\infty} (1+|x|^2)^{-\sigma/2}\varphi(x) = 0\}.$$

In recent decades, many authors have shown a great interest in studying a number of evolution equations, especially concerning the complicated asymptotic behavior of the solutions[5, 2, 6, 18, 19, 3, 10]. Among these studies, we mention an important work by Kamin et al. [5, 6], for the equation (1.1) with $p = 2$. They showed that the solution of the porous medium equation converges uniformly to the Barenblatt solution of the equation for porous media with the same mass as u_0 if the initial value $u_0 \in L^1_+(\mathbb{R}^N) = \{\varphi \in L^1(\mathbb{R}^N; \varphi(x) \geq 0)\}$. Subsequently, it was shown by Zhao and Yuan [20] that for the doubly nonlinear diffusion equation (1.1), the solution of the multidimensional Cauchy-problem (1.1) converges to the Barenblatt solution of the porous media equation of the same mass as u_0 if the initial value $u_0 \in L^1_+(\mathbb{R}^N) = \{\varphi \in L^1(\mathbb{R}^N; \varphi(x) \geq 0)\}$.

For the solutions of certain evolution equations, the occurrence of complicated asymptotic behavior primarily depends on the workspace to which the initial data belongs, see [9, 8, 17, 1]. Note that in the aforementioned work, the problem concerning the complicated asymptotic behavior of the solutions is only considered in some bounded spaces. Based on the existence theory of doubly nonlinear diffusion equations, it is concluded that the solutions to the problems (1.1)-(1.2) are global, even if the initial data belongs to some unbounded spaces [21, 12, 4, 11]. Thus, the complicated asymptotic behavior of solutions to the doubly nonlinear diffusion equation may also appear in unbounded spaces.

Inspired by this, we focus here on the complicated asymptotic behavior of problems (1.1)-(1.2) in the unbounded spaces $Y_\sigma(\mathbb{R}^N)$ with $\frac{p}{p-1} \leq \sigma < \frac{p}{m(p-1)-1}$. Next, we consider that the solution $u(x,t)$ of problem (1.1)-(1.2) with initial data $u_0 \in Y_\sigma(\mathbb{R}^N)$ may be unbounded solutions. It is worth noting that the doubly nonlinear diffusion equation has nonlinearity and degeneracy not

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found in the heat equation [13, 14]. To overcome these difficulties, we establish the propagation speed estimation and the growth estimation in a given weighted spaces, and extend the weighted $L^1 - L^\infty$ estimates of Zhao and Xu [21] to the unbounded spaces $Y_\sigma(\mathbb{R}^N)$ and apply them to obtain the existence of a weak solution of problem (1.1)-(1.2). Finally, we investigate the asymptotic behavior of solutions for the problem (1.1)-(1.2) and give the fact that the spaces $Y_\sigma(\mathbb{R}^N)$ with $\frac{p}{p-1} \le \sigma < \frac{p}{m(p-1)-1}$ can provide the work spaces where complexity causing the asymptotic behavior of solutions.

Specifically, using the properties of solutions in unbounded spaces, we can obtain that there exists a function $\phi \in Y_\sigma(\mathbb{R}^N)$ such that the set of ω-limits set $\omega_\sigma^{\mu,\beta}(u_0) = Y_\sigma^+(\mathbb{R}^N)$ for some μ, β, where

$$\omega_\sigma^{\mu,\beta}(u_0) = \{f \in Y_\sigma(\mathbb{R}^N) : \exists t_n \to \infty \text{ s.t. } t_n^{\mu/2} u(t_n^\beta x, t_n) \xrightarrow{t_n \to \infty} f \quad \text{in } Y_\sigma(\mathbb{R}^N)\},$$
$$Y_\sigma^+(\mathbb{R}^N) = \{\varphi \in Y_\sigma(\mathbb{R}^N) : \varphi \ge 0 \text{ and } \varphi(0) = 0\},$$

and $u(x,t)$ is the solution of problem (1.1)-(1.2) with the initial data $u_0(x) = \phi(x)$. So the unbounded spaces $Y_\sigma(\mathbb{R}^N)$ can provide the work spaces where complexity occurs in the asymptotic behavior of solution of the Cauchy-problem (1.1)-(1.2), according to Vazquez and Zuazua [15].

The main points of the rest of this paper are as follows. In the next section, we introduce some essential definitions and properties to provide theoretical support. subsequently, we present the propagation estimates, growth estimates, and weighted L^1-L^∞ estimates for the solutions of problem (1.1)-(1.2) with initial data $u_0 \in Y_\sigma(\mathbb{R}^N)$. Finally, we study the complicated asymptotic behavior of the solution of problem (1.1)-(1.2) in the unbounded spaces $Y_\sigma(\mathbb{R}^N)$ given by $\frac{p}{p-1} \le \sigma < \frac{p}{m(p-1)-1}$.

2. Preliminaries

Definition 2.1. For $r > 0$, $f \in L^1_{\rm loc}(\mathbb{R}^N)$, let

$$|||f|||_r = \sup_{R \ge r} R^{-\frac{N[m(p-1)-1]+p}{m(p-1)-1}} \int_{x \le R} |f(x)| \mathrm{d}x.$$

The space X is defined as

$$X = \{f \in L^1_{\rm loc}(\mathbb{R}^N); |||f|||_1 < \infty\}$$

with the norm $|||\cdot|||_1$. Thus X is a Banach space. For any norm $|||\cdot|||_r$ ($r > 0$) is an equivalent norm, taking $f \in X$, we will define it as follows

$$\ell(f) = \lim_{|r| \to \infty} |||f|||_r.$$

Taking a subspace $X_0(\mathbb{R}^N)$ of the function space X, for the sake of convenience, simply denoted as

$$X_0 = \{\varphi \in X;\ \ell(f) = 0\}.$$

If the initial value $u_0 \in X_0$, then the existence and uniqueness of a global weak solution to problem (1.1)-(1.2) is proved in [21, 12, 4, 11]. To prove our result, this solution also needs to satisfy the following proposition as support.

Proposition 2.2 ([11])**.** *If $u_0 \in X_0$, the Cauchy problem for the doubly nonlinear diffusion equation* (1.1)-(1.2) *generates a continuous bounded semigroup in X_0, $S(t):\ u_0 \to u(x,t)$, i.e.*

$$u(x,t) = S(t)u_0 \in C([0,\infty); X_0).$$

Note that the semigroup is bounded in $L^p(\mathbb{R}^N)$ for all $p \ge 1$.

Definition 2.3. Let $0 \le \sigma < \infty, \rho_\sigma(x) = (1 + |x|^2)^{-\sigma/2}$, the weighted space $L^p(\rho_\sigma)$ was defined as

$$L^p(\rho_\sigma) = \{\varphi \in L^1_{\rm loc}(\mathbb{R}^N) : \varphi\rho_\sigma \in L^p(\mathbb{R}^N)\}$$

and $\|\varphi\|_{L^p(\rho_\sigma)} = \|\varphi\rho_\sigma\|_{L^p(\mathbb{R}^N)}$. Also the space $Y_\sigma(\mathbb{R}^N)$ is defined as

$$Y_\sigma(\mathbb{R}^N) = \{\varphi(x) \in C(\mathbb{R}^N) : \lim_{|x| \to \infty} \varphi(x)\rho_\sigma(x) = 0\}$$

with the norm $\|\varphi\|_{Y_\sigma(\mathbb{R}^N)} = \|\varphi\rho_\sigma\|_{L^\infty(\mathbb{R}^N)}$.

This section verifies that both $Y_\sigma(\mathbb{R}^N)$ and $L^p(\rho_\sigma)$ are Banach spaces.

Remark 2.4. From the above definitions, we can see that $Y_\sigma(\mathbb{R}^N) \subset L^\infty(\rho_\sigma)$ for every $\sigma \geq 0$, and $Y_\sigma(\mathbb{R}^N) \subset X_0$ for $\frac{p}{p-1} \leq \sigma < \frac{p}{m(p-1)-1}$, and $Y_0(\mathbb{R}^N) = C_0(\mathbb{R}^N)$, and $L^p(\rho_0) = L^p(\mathbb{R}^N)$ for $\sigma = 0$.

Proposition 2.5 ([12, 21]). *For any $u_0 \in X$, there exists a constant $T = T(u_0)$ and a weak solution $u(x,t)$ to problems (1.1)-(1.2) in $Q_T = \mathbb{R}^N \times [0,T)$, where*

$$T(u_0) = \begin{cases} C\ell(u_0)^{1-m(p-1)} & \text{if } \ell(u_0) > 0, \\ \infty & \text{if } \ell(u_0) = 0. \end{cases}$$

Furthermore, if $u_0 \in C(\mathbb{R}^N)$, then $u(x,t) \in C(\mathbb{R}^N \times [0,T))$.

Definition 2.6. A nonnegative function $u(x,t)$ is called a solution of (1.1)-(1.2) if u satisfies

(i) $u \in C(S_T)$, $u_m \in l^p_{\text{loc}}(0,T; W^{1,p}_{\text{loc}}(R^N))$, u_t is a regular measure on S_T of locally bounded variation;
(ii) $\int_{S_T} [u(x,t)\varphi_t(x,t) - |Du^m|^{p-2} Du^m \cdot D\varphi] \mathrm{d}x\mathrm{d}\tau = 0$, $\forall \varphi \in C_0^1(S_T)$;
(iii) $\lim\limits_{t\to 0} \int_{|x|<R} |u(x,t) - u_0(x)| \mathrm{d}x = 0, \forall R > 0$,

where $S_T = \mathbb{R}^N \times (0,T)$ and $0 \leq T < \infty$. As a way of measuring the growth of a function $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ as $|x| \to \infty$, we let

$$|||f|||_{r_\rho} = \sup_{\rho \geq r} \rho^{-\frac{\lambda}{m(p-1)-1}} \int_{B_\rho} |f(x)|\rho_\sigma \mathrm{d}x,$$

where $r > 0$, $\lambda = N[m(p-1)-1] + p$, $B_\rho = \{x \in \mathbb{R}^N : |x| < \rho\}$.

Definition 2.7. Let $\frac{p}{p-1} \leq \sigma < \frac{p}{m(p-1)-1}$, μ, $\beta > 0$, and $u_0 \in Y_\sigma(\mathbb{R}^N)$. Then the ω-limit is as follows

$$\omega_\sigma^{\mu,\beta}(u_0) = \{f \in Y_\sigma(\mathbb{R}^N) : \exists t_n \to \infty \text{ s.t. } D^{\mu,\beta}_{\sqrt{t_n}}[S(t_n)u_0] \xrightarrow{t_n\to\infty} f \text{ in } Y_\sigma(\mathbb{R}^N)\}.$$

For $\varphi \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $\lambda > 0$ there are $D^{\mu,\beta}_\lambda \varphi(x) = \lambda^\mu \varphi(\lambda^{2\beta} x)$, $S(t)$ is the semigroup operator given in Proposition2.2.

In [3, 16], the following exchange relation between semigroup operators and dilation operators is obtained,

$$D^{\mu,\beta}_\lambda [S(\lambda^2 t)u_0] = S(\lambda^{2-2p\beta-\mu[m(p-1)-1]} t)[D^{\mu,\beta}_\lambda u_0].$$

For $\lambda > 0$, the proof will not be repeated here.

Definition 2.8. Let $d(x) = \sup\{R : u_0(y) = 0 \text{ a.e. in } B_R(x)\}$ be the distance from x to the support point of u_0. We define

$$\Omega(t) = \{x \in \mathbb{R}^N : u(x,t) > 0\}.$$

We also define the ρ-neighborhood of $\Omega(t)$ as

$$\Omega_\rho(t) = \{x \in \mathbb{R}^N : d(x, \Omega(t)) < \rho\},$$

where $d(x, \Omega(t))$ is the distance from x to $\Omega(t)$.

3. Some estimates

In this section, we provide estimates for the solution of problem (1.1)-(1.2) using initial data $u_0 \in L^1(\rho_\sigma)$. To accomplish this, we need to use the following lemma given in [16], with particular emphasis in [21].

Lemma 3.1. *Given a non-negative weak solution $u(x,t)$ of problem (1.1)-(1.2). For every $x \in \mathbb{R}^N$, if*

$$B(x) = \sup_{R \geq 0} R^{-\frac{N[m(p-1)-1]+p}{m(p-1)-1}} \int_{B_R(x)} u_0(y)\mathrm{d}y < \infty,$$

where $B_R(x) = \{y : |x-y| < R\}$, for all $0 < t \leq C(p,N)B(x)^{-[m(p-1)-1]}$, then $u(x,t) = 0$.

Next, we prove the propagation estimates of solutions to problem (1.1)–(1.2) when the initial value u_0 belongs to $L^1(\rho_\sigma)$ and $L^\infty(\rho_\sigma)$.

Theorem 3.2 (Propagation estimates). *Let $\frac{p}{p-1} \leq \sigma < \frac{p}{m(p-1)-1}$ and assume that $u_0 \in L^1(\rho_\sigma)$. Then for all $0 \leq t_1 \leq t_2 < \infty$,*

$$\Omega(t_2) \subset \Omega_{\rho(t_2-t_1)}(t_1),$$

where

$$\begin{aligned}\rho(t_2-t_1) = C \max\big\{&\|u_0\|_{L^1(\rho_\sigma)}^{\frac{m(p-1)-1}{N[m(p-1)-1]+p}} (t_2-t_1)^{\frac{1}{N[m(p-1)-1]+p}},\\ &\|u_0\|_{L^1(\rho_\sigma)}^{-\frac{m(p-1)-1}{p+(N-\sigma)[m(p-1)-1]}} (t_2-t_1)^{\frac{1}{p+(N-\sigma)[m(p-1)-1]}}\big\}.\end{aligned}$$

Proof. We consider only the case of $t_1 = 0$, assuming that $x_0 \in \mathbb{R}^N$ and $B_R(x_0) > 0$, if $R < d(x_0)$, then

$$\int_{B_R(x_0)} u_0(y)\mathrm{d}y = 0.$$

If $R \geq d(x_0)$, then

$$\begin{aligned}
&R^{-\frac{N[m(p-1)-1]+p}{m(p-1)-1}} \int_{B_R(x_0)} u_0(y)\mathrm{d}y\\
&= R^{-\frac{N[m(p-1)-1]+p}{m(p-1)-1}} \int_{B_R(x_0)} u_0(y)\rho_\sigma(x)(1+|x|^2)^{\sigma/2}\mathrm{d}y\\
&\leq R^{-\frac{N[m(p-1)-1]+p}{m(p-1)-1}} (1+R^2)^{\sigma/2} \int_{B_R(x_0)} u_0(y)\rho_\sigma(x)\mathrm{d}y\\
&\leq R^{-\frac{N[m(p-1)-1]+p}{m(p-1)-1}} (1+R^2)^{\sigma/2} \|u_0\|_{L^1(\rho_\sigma)}\\
&\leq 2^{\sigma/2}\|u_0\|_{L^1(\rho_\sigma)} \max\{R^{-\frac{N[m(p-1)-1]+p}{m(p-1)-1}}, R^{-\frac{(N-\sigma)[m(p-1)-1]+p}{m(p-1)-1}}\}\\
&\leq 2^{\sigma/2}\|u_0\|_{L^1(\rho_\sigma)} \max\{d(x_0)^{-\frac{N[m(p-1)-1]+p}{m(p-1)-1}}, d(x_0)^{-\frac{(N-\sigma)[m(p-1)-1]+p}{m(p-1)-1}}\}.
\end{aligned} \tag{3.1}$$

So, from the above two equations it follows that

$$\begin{aligned}
B(x_0) &= \sup_{R \geq d(x_0)} R^{-\frac{N[m(p-1)-1]+p}{m(p-1)-1}} \int_{B_R(x_0)} u_0(y)\mathrm{d}y\\
&\leq C\|u_0\|_{L^1(\rho_\sigma)} \max\{d(x_0)^{-\frac{N[m(p-1)-1]+p}{m(p-1)-1}}, d(x_0)^{-\frac{(N-\sigma)[m(p-1)-1]+p}{m(p-1)-1}}\}.
\end{aligned} \tag{3.2}$$

By Lemma 3.1 $u(x_0,t) = 0$, for

$$0 < t \leq C\|u_0\|_{L^1(\rho_\sigma)}^{-[m(p-1)-1]} \min\{d(x_0)^{N[m(p-1)-1]+p}, d(x_0)^{(N-\sigma)[m(p-1)-1]+p}\}.$$

Then $\Omega(t) \subset \Omega_{\rho(t)}(0)$, where

$$\rho(t) = C\max\big\{\|u_0\|_{L^1(\rho_\sigma)}^{\frac{m(p-1)-1}{N[m(p-1)-1]+p}} t^{\frac{1}{N[m(p-1)-1]+p}}, \|u_0\|_{L^1(\rho_\sigma)}^{-\frac{m(p-1)-1}{p+(N-\sigma)[m(p-1)-1]}} t^{\frac{1}{p+(N-\sigma)[m(p-1)-1]}}\big\}.$$

The proof is complete. □

Theorem 3.3 (Propagation estimates). *Let $\frac{p}{p-1} \leq \sigma < \frac{p}{m(p-1)-1}$ and assume that*

$$u_0 \in L^\infty(\rho_\sigma). \tag{3.3}$$

Then for any $0 \leq t_1 \leq t_2 < \infty$, we have $\Omega(t_2) \subset \Omega_{\rho(t_2-t_1)}(t_1)$. Note that $\Omega_{\rho(t_2-t_1)}(t_1)$ is a ρ-neighborhood of $\Omega(t_2)$ and

$$\rho((t_2-t_1)) = C\max\big\{\|u_0\|_{L^\infty(\rho_\sigma)}^{\frac{m(p-1)-1}{p}} (t_2-t_1)^{1/p}, \|u_0\|_{L^\infty(\rho_\sigma)}^{\frac{-[m(p-1)-1]}{p-N[m(p-1)-1]}} (t_2-t_1)^{\frac{1}{p-N[m(p-1)-1]}}\big\}.$$

Proof. As above, we only consider the case $t_1 = 0$. Suppose that $x_0 \in \mathbb{R}^N$ and $B_R(x_0) > 0$. Then, for $R < d(x_0)$, it holds

$$\int_{B_R(x_0)} u_0(y)dy = 0.$$

As for $R \geq d(x_0)$, using Definition 2.3, it can be estimated as

$$\begin{aligned}
&R^{-\frac{N[m(p-1)-1]+p}{m(p-1)-1}} \int_{B_R(x_0)} u_0(y)\mathrm{d}y \\
&= R^{-\frac{N[m(p-1)-1]+p}{m(p-1)-1}} \int_{B_R(x_0)} u_0(y)\rho_\sigma(x)(1+|x|^2)^{\sigma/2}\mathrm{d}y \\
&\leq R^{-\frac{N[m(p-1)-1]+p}{m(p-1)-1}} \|u_0\|_{L^\infty(\rho_\sigma)}(1+R^2)^{\sigma/2} \int_{B_R(x_0)} \mathrm{d}y \\
&\leq R^{-\frac{p}{m(p-1)-1}} \|u_0\|_{L^\infty(\rho_\sigma)}(1+R^2)^{\sigma/2} \\
&\leq 2^{\sigma/2}\|u_0\|_{L^\infty(\rho_\sigma)} \max\{R^{-\frac{p}{m(p-1)-1}}, R^{-\frac{p}{m(p-1)-1}+\sigma}\} \\
&\leq 2^{\sigma/2}\|u_0\|_{L^\infty(\rho_\sigma)} \max\{d(x_0)^{-\frac{p}{m(p-1)-1}}, d(x_0)^{-\frac{p}{m(p-1)-1}+\sigma}\};
\end{aligned} \tag{3.4}$$

therefore,

$$\begin{aligned}
B(x_0) &= \sup\big\{R \geq d(x_0) R^{-\frac{N[m(p-1)-1]+p}{m(p-1)-1}} \int_{B_R(x_0)} u_0(y)\mathrm{d}y\big\} \\
&\leq C\|u_0\|_{L^\infty(\rho_\sigma)} \max\big\{d(x_0)^{-\frac{p}{m(p-1)-1}}, d(x_0)^{-\frac{p}{m(p-1)-1}+\sigma}\big\}.
\end{aligned} \tag{3.5}$$

Using Lemma3.1 it follows that $u(x_0,t) = 0$, for every

$$0 < t \leq C\|u_0\|_{L^\infty(\rho_\sigma)} \min\big\{d(x_0)^{-\frac{p}{m(p-1)-1}}, d(x_0)^{-\frac{p}{m(p-1)-1}+\sigma}\big\}.$$

Hence $\Omega(t) \subset \Omega_{\rho(t)}(0)$, where

$$\rho(t) = C\max\big\{\|u_0\|_{L^\infty(\rho_\sigma)}^{\frac{m(p-1)-1}{p}} t^{1/p}, \|u_0\|_{L^\infty(\rho_\sigma)}^{\frac{-[m(p-1)-1]}{p-N[m(p-1)-1]}} t^{\frac{1}{p-N[m(p-1)-1]}}\big\}.$$

We have thus completed the proof. □

In the following theorem, we examine the properties of the solutions to problem (1.1)-(1.2) when the initial value satisfies $u_0 \in L^\infty(\rho_\sigma)$.

Theorem 3.4 (Growth estimates). *Let $\frac{p}{p-1} \leq \sigma < \frac{p}{m(p-1)-1}$, if $0 \leq u_0 \in L^\infty(\rho_\sigma)$. Then there exists a constant C such that*

$$0 \leq S(t)u_0(x) \leq C((1+t)^{\frac{2}{p-\sigma[m(p-1)-1]}} + |x|^2)^{\sigma/2},$$

thereby,

$$\|S(t)u_0(x)\|_{L^\infty(\rho_\sigma)} \leq C(1+t)^{\frac{\sigma}{p-\sigma[m(p-1)-1]}}.$$

Furthermore, if $0 \leq u_0 \in Y_\sigma(\mathbb{R}^N)$, for any $t \geq 0$, we have

$$S(t)u_0(x) \in Y_\sigma(\mathbb{R}^N).$$

Proof. We consider the problem

$$\begin{gathered}
u_t - \Delta u^m = 0 \quad \text{in } S, \\
u(x,0) = v_0(x) = M|x|^\sigma \quad \text{in } \mathbb{R}^N.
\end{gathered} \tag{3.6}$$

For $\lambda > 0$, let

$$\lambda_1 = \lambda^{\frac{p-\sigma[m(p-1)-1]}{2p}}, \quad \mu = \frac{2\sigma}{p-\sigma[m(p-1)-1]}, \quad \beta = \frac{1}{p-\sigma[m(p-1)-1]}$$

and satisfying the equation

$$2 - \mu[m(p-1)-1] - 2p\beta = 0.$$

Applying the exchange relation given in Definition 2.7, it follows that

$$\begin{aligned}\lambda^{-\frac{\sigma}{p}}[S(\lambda^{1-\frac{\sigma[m(p-1)-1]}{p}}t)v_0](\lambda^{1/p}x) &= \lambda_1^{\mu}[S(\lambda_1^2 t)v_0](\lambda_1^{2\beta}x)\\ &= S(t)[\lambda_1^{\mu}v_0(\lambda_1^{2\beta}\cdot)](x)\\ &= S(t)v_0(x).\end{aligned} \tag{3.7}$$

In the above equation, setting $t=1,\ s=\lambda^{\frac{p-\sigma[m(p-1)-1]}{p}}$ and $g(x)=S(1)v_0(x)$, we obtain

$$S(s)v_0(x) = s^{\frac{p}{p-\sigma[m(p-1)-1]}}g(s^{-\frac{1}{p-\sigma[m(p-1)-1]}}x). \tag{3.8}$$

Indeed, it follows from regularity theory that for $t>0$,

$$0 \le S(t)v_0 \in C^{\infty}((0,\infty)\times\mathbb{R}^N)\cap C([0,\infty)\times\mathbb{R}^N),$$

for $|x|=1$, taking the limit in both sides of the above equation simultaneously with respect to $s\to 0$ as follows that

$$S(s)v_0(x)\to v_0(x)=M|x|^{\sigma}=M,$$

and taking

$$y = s^{-\frac{1}{p-\sigma[m(p-1)-1]}}x;$$

note that $|y|\to\infty$ as $s\to 0$. Then upon

$$|y|^{-\sigma}g(y)-M\to 0 \quad \text{as} \quad |y|\to\infty, \tag{3.9}$$

there exists a non-negative constant C such that

$$g(x)\le C(1+|x|^2)^{\sigma/2}.$$

We deduce from (3.8) that

$$0\le S(s)v_0(x)\le C(s^{\frac{2}{p-\sigma[m(p-1)-1]}}+|x|^2)^{\sigma/2}.$$

This implies

$$S(t)g(x)=S(t)[S(1)v_0](x)=S(t+1)v_0(x)\le C((1+t)^{\frac{2}{p-\sigma[m(p-1)-1]}}+|x|^2)^{\sigma/2}.$$

Putting $\varphi(x)=M(1+|x|^2)^{\sigma/2}$, we have

$$S(t)\varphi(x)\le C((1+t)^{\frac{2}{p-\sigma[m(p-1)-1]}}+|x|^2)^{\sigma/2}.$$

If we let $M=\|u_0\|_{L^{\infty}(\rho_\sigma)}$ in question (3.6). we have

$$S(t)u_0(x)\le C((1+t)^{\frac{2}{p-\sigma[m(p-1)-1]}}+|x|^2)^{\sigma/2}.$$

Next we prove the second part of the theory, owing to

$$0\le u_0\in Y_\sigma(\mathbb{R}^N)\subset L^{\infty}(\rho_\sigma).$$

For every $t>0$, $R>0$, let

$$R(t)=R+1+C\max\{\|u_0\|_{L^{\infty}(\rho_\sigma)}^{\frac{m(p-1)-1}{p}}t^{1/p}, \|u_0\|_{L^{\infty}(\rho_\sigma)}^{\frac{-[m(p-1)-1]}{p-N[m(p-1)-1]}}t^{\frac{1}{p-N[m(p-1)-1]}}\},$$

and taking $\chi_{R+1}(x)$ to be a cut-off function defined on B_{R+1} associated with B_R, then $\chi_{R+1}(x)\in C_0^{\infty}(\mathbb{R}^N)$, $0\le\chi_{R+1}(x)\le 1$, such that

$$\chi_{R+1}(x)=\begin{cases}1 & \text{for } x\in B_R,\\ 0 & \text{for } x\notin B_{R+1}.\end{cases}$$

We can deduce from Lemma3.1 that

$$\operatorname{supp}[S(t)(\chi_{R+1}u_0)]\subset\{x\in\mathbb{R}^N : |x|\le R(t)\}.$$

It means that for $t>0$, $R>0$,, the value of $S(t)u_0(x)$ in $\mathbb{R}^N\setminus B_{R(t)}$ depends only on the initial value $u_0(x)$ in $\mathbb{R}^N\setminus B_{R(t)}$, in other words, if $|x|>R(t)$, then

$$S(t)[(1-\chi_{R+1})u_0](x)=S(t)u0(x). \tag{3.10}$$

For any $\varepsilon > 0$, and the following assumptions are satisfied $0 \le u_0 \in Y_\sigma(\mathbb{R}^N)$, then there exists a constant $R_1 > 1 > 0$, It follows that for $|x| > R_1$,

$$(1+|x|^2)^{-\sigma/2} u_0(x) < \frac{\varepsilon}{2}.$$

At this point lettin $R = R_1$ and we obtain

$$(1-\chi_{R+1})u_0 < \varepsilon |x|^\sigma.$$

Let $M = \varepsilon$ in equation (3.6), and then use equation (3.9) and the Comparison Principle to show that there exists a constant R_2 and the following inequality holds when $|x| > R_2$,

$$S(1)[(1-\chi_{R+1})u_0](x) \le g(x) \le 2\varepsilon |x|^\sigma. \tag{3.11}$$

If $|x| t^{-\frac{1}{p-\sigma[m(p-1)-1]}} > R_2$, then

$$S(t)[(1-\chi_{R+1})u_0](x) \le t^{\frac{\sigma}{p-\sigma[m(p-1)-1]}} g(t^{-\frac{1}{p-\sigma[m(p-1)-1]}} x) \le 2\varepsilon |x|^\sigma. \tag{3.12}$$

Then combining (3.10) with the above equation, if

$$|x| > \max\{R_1(t), t^{\frac{1}{p-\sigma[m(p-1)-1]}} R_2\},$$

then, according to (3.11),

$$S(t)u_0(x) \le 2\varepsilon |x|^\sigma. \tag{3.13}$$

Clearly, using the first part of the theorem, for $\forall t > 0$,

$$S(t)u_0(x) \in L^\infty(\rho_\sigma) \quad \text{and} \quad S(t)u_0(x) \in C(\mathbb{R}^N).$$

By (3.13), it follows that for $\forall t > 0$,

$$S(t)u_0(x) \in Y_\sigma(\mathbb{R}^N). \tag{3.14}$$

The proof of all statements of Theorem3.4 is complet. $\square$

The ideas of the proof of the following theorem come from [21, 4].

Theorem 3.5 (Weighted L^1-L^∞ estimates). *Let $\frac{p}{p-1} \le \sigma < \frac{p}{m(p-1)-1}$, if $u_0 \in L^1(\rho_\sigma)$, then for all $t > 0$,*

$$u(t) = S(t)u_0 \in L^\infty(\rho_\sigma). \tag{3.15}$$

Moreover, if $0 < t \le \gamma \|u_0\|_{L^1(\rho_\sigma)}^{-[m(p-1)-1]}$, then

$$\|u(\cdot,t)\|_{L^\infty(\rho_\sigma)} \le \gamma t^{-N/\lambda} \|u_0\|_{L^1(\rho_\sigma)}^{p/\lambda}. \tag{3.16}$$

Proof. We consider the Cauchy problem

$$u_t - \operatorname{div}(|\nabla u^m|^{p-2} \nabla u^m) = 0 \quad \text{in } S_T, \tag{3.17}$$

$$u(\cdot,0) = u_0(\cdot) \quad \text{in } \mathbb{R}^N, \tag{3.18}$$

where $u_0(x) \in L^1(\rho_\sigma)$, by the results of [20], equations (3.17) and (3.18) have a nonnegative solution u satisfying

$$u \in C([0,T] : L^1(\mathbb{R}^N) \cap L^\infty(S_T) \cap C(S_T), u^m \cap L^p_{\rm loc}(0,T : W^{1,p}(\mathbb{R}^N)).$$

Such regularity will suffice to justify the calculations to follow. We will denote by γ a generic positive constant depending only on N, m, p. The proof of Theorem3.5 requires several steps. Let $T > 0$, $\rho > 0$ be fixed and consider the sequences

$$T_j = \frac{T}{2} - \frac{T}{2^{j+2}}, \quad \rho_j = \rho + \frac{\rho_j}{2}, \quad \bar\rho_j = \frac{1}{2}(\rho_j + \rho_{j+1}) = \rho + \frac{3\rho}{2^{j+2}}.$$

Set $B_j = B_{\rho_j}$, $\bar B_j = B_{\bar\rho_j}$, $Q_j = B_j \times (T_j, T)$, $\bar Q_j = \bar B_j \times (T_{j+1}, T)$, $k_j = k - \frac{k}{2^{j+1}}$, $k > 0$, and let $\xi_j(x,t)$ be a piecewise smooth cut-off function in Q_j satisfying

$$\xi_j(x,t) = 1, \text{ for } (x,t) \in \bar Q_j, \quad |D\xi_j| \le \frac{2^{j+2}}{\rho}, \text{ for } 0 \le \xi_{jt} \le \frac{2^{j+2}}{T},$$

Finally, we define

$$\varphi(t)=\varphi_r(t)=\sup_{\tau\in(0,t)}\tau^{N/\lambda}\sup_{\rho>r}\frac{\|u(\cdot,\tau)\|_{L^\infty(\rho_\sigma),B_\rho}}{\rho^{\frac{p}{m(p-1)-1}}}, \tag{3.19}$$

where $r>0$, $\lambda=N[m(p-1)-1]+p$, and

$$\psi(t)=\sup_{\tau\in(0,t)}|||u(\cdot,\tau)|||_{r_\rho}.$$

Since $u_0(x)\in L^1(\rho_\sigma)$, $u\in L^\infty(S_T)$ and $\varphi(t)$, $\psi(t)$ are well defined.

Lemma 3.6. *There exists a constant $\gamma(N,m,p)$, such that for any $t>0$,*

$$\|u(\cdot,t)\|_{L^\infty(\rho_\sigma),B_\rho}\le\gamma[K(t)]^{\frac{N+p}{\gamma}}\Big(\int_{\frac{t}{4}}^t\int_{B2_\rho}(u\rho_\sigma)^{mp}\mathrm{d}x\mathrm{d}\tau\Big)^{p/\beta}, \tag{3.20}$$

where $\beta=N(m(p-1)-1)+mp^2$, and

$$K(t)=t^{\frac{-N(m(p-1)-1)}{\lambda}}\varphi^{m(p-1)-1}(t)+t^{-1}. \tag{3.21}$$

Proof. Let $\chi_j(x)$ be a segmented smooth cut-off function in $\bar{B}_j$ that satisfies the following conditions: $x\in B_{j+1}$, $\chi_j(x)=1$, and $|D\chi_j(x)|\le 2^{j+2}\rho^{-1}$. Set

$$w_j=(u\rho_\sigma-k_j)_+^{\frac{m(2p-1)-1}{p}}. \tag{3.22}$$

Recall the the Gagliardo-Nirenberg inequality, for $t\in(0,T)$,

$$\|w_j\chi_j\|_{q,\bar{B}_j(t)}\le\gamma\|D(w_j\chi_j)\|^\eta_{p,\bar{B}_j(t)}\|w_j\chi_j\|^{1-\eta}_{s,\bar{B}_j(t)}, \tag{3.23}$$

where

$$\eta=\Big(\frac{1}{s}-\frac{1}{q}\Big)\Big(\frac{1}{N}-\frac{1}{p}+\frac{1}{s}\Big)^{-1}$$

and $\gamma=\gamma(N,m,p)$. Choose $\eta=\frac{p}{q}$, $s=\frac{mp^2}{m(2p-1)-1}$, and $q=p(1+\frac{s}{N})$. An application of q-powers to the integration of (T_{j+1},T) results in the following:

$$\begin{aligned}
&\iint_{Q_{j+1}}w_{j+1}^{p(1+\frac{s}{N})}\mathrm{d}x\mathrm{d}\tau\\
&\le\iint_{\bar{Q}_j}(w_{j+1}\chi_j)^q\mathrm{d}x\mathrm{d}\tau\\
&\le\gamma\Big(\iint_{\bar{Q}_j}|Dw_{j+1}|^q\mathrm{d}x\mathrm{d}\tau+\frac{2^{jp}}{\rho^p}\iint_{\bar{Q}_j}w_{j+1}^p\mathrm{d}x\mathrm{d}\tau\Big)\Big(\sup_{T_{j+1}\le t\le T}\int_{\bar{B}_j(t)}w_{j+1}^s\mathrm{d}x\Big)^{p/N}.
\end{aligned}\tag{3.24}$$

Next, we estimate the right-hand side of the aforementioned equation. We use ξ_j as defined in (3.24). We multiply (3.17) by the test function

$$\varphi(x,t)=\xi_j^p(u\rho_\sigma-k_{j+1})_+^{mp-1}=\max\{0,\xi_j^p(u\rho_\sigma-k_{j+1})^{mp-1}\} \tag{3.25}$$

and integrate over Q_j to obtain

$$\begin{aligned}
&\sup_{T_{j+1}\le t\le T}\int_{\bar{B}_j(t)}w_{j+1}^s\mathrm{d}x+\iint_{\bar{Q}_j}|Dw_{j+1}|^q\mathrm{d}x\mathrm{d}\tau\\
&\le\frac{2^{jp}\gamma}{\rho^p}\iint_{Q_j\cap\{u\rho_\sigma\ge k_{j+1}\}}(u\rho_\sigma)^{(m-1)(p-1)}(u\rho_\sigma-k_j)^{mp+p-2}\mathrm{d}x\mathrm{d}\tau\\
&\quad+\frac{\gamma 2^j}{T}\iint_{Q_j}(u\rho_\sigma-k_j)_+^{mp}\mathrm{d}x\mathrm{d}\tau.
\end{aligned}\tag{3.26}$$

For $p>2$, we have

$$(u\rho_\sigma)^{(m-1)(p-1)}(u\rho_\sigma-k_j)_+^{p-2}\rho^{-p}\le\gamma t^{\frac{-N(m(p-1)-1)}{\lambda}}\varphi^{m(p-1)-1}(t).$$

For $p<2$, if $\frac{u\rho_\sigma}{2}\ge k_j$ such that

$$(u\rho_\sigma)^{(m-1)(p-1)}(u\rho_\sigma-k_j)^{p-2}\le(u\rho_\sigma)^{(m-1)(p-1)}(\frac{u\rho_\sigma}{2})^{p-2}\le C(u\rho_\sigma)^{m(p-1)-1},$$

and if $k_{j+1} \leq u\rho_\sigma \leq 2k_j$ such that

$$(u\rho_\sigma)^{(m-1)(p-1)}(u\rho_\sigma - k_j)^{p-2} \leq (u\rho_\sigma)^{(m-1)(p-1)}(k_{j+1} - k_j)^{p-2} \leq C2^{(2-p)j}(u\rho_\sigma)^{m(p-1)-1}.$$

Under the above conditions,

$$(u\rho_\sigma)^{(m-1)(p-1)}(u\rho_\sigma - k_j)^{p-2}\rho^{-p} \leq \gamma t^{\frac{-N(m(p-1)-1)}{\lambda}}\varphi^{m(p-1)-1}(t) \quad \text{in } \{u\rho_\sigma \geq k_{j+1}\}.$$

Thus, (3.26) can be written as

$$\sup_{T_{j+1}\leq t\leq T}\int_{\bar{B}_j(t)} w_{j+1}^s \mathrm{d}x + \iint_{\bar{Q}_j} |Dw_{j+1}|^q \mathrm{d}x\mathrm{d}\tau \leq \gamma 2^{j(p+1)}K(T)\iint_{Q_j} w_j^s \mathrm{d}x\mathrm{d}\tau\,, \tag{3.27}$$

Again, by the above definition of w_j, it follow that

$$\rho^{-p}\iint_{\bar{Q}_j} w_{j+1}^p \mathrm{d}x\mathrm{d}\tau \leq \gamma K(T)\iint_{Q_j} w_j^s \mathrm{d}x\mathrm{d}\tau\,. \tag{3.28}$$

By applying equation (3.27) to equation (3.24) and estimating the right-hand side of (3.27), the following result is obtained:

$$\iint_{Q_{j+1}} w_{j+1}^q \mathrm{d}x\mathrm{d}\tau \leq \gamma\Big(2^{(p+1)j}K(T)\Big)^{\frac{N+p}{N}}\Big(\iint_{Q_j} w_j^s \mathrm{d}x\mathrm{d}\tau\Big)^{\frac{N+p}{N}}. \tag{3.29}$$

By Hölder inequality,

$$\iint_{Q_j}(u\rho_\sigma - k_{j+1})_+^{mp}\mathrm{d}x\mathrm{d}\tau \leq \Big(\iint_{Q_{j+1}} w_{j+1}^q \mathrm{d}x\mathrm{d}\tau\Big)^{s/q}|A_{j+1}|^{1-\frac{s}{q}}, \tag{3.30}$$

where $A_j = \{(x,t) \in Q_j : u(x,t)\rho_\sigma > k_j\}$, $\quad j = 0,1,2,\dots$ and $|A_j| = \operatorname{meas} A_j$. Thus

$$\begin{aligned}\iint_{Q_j}(u\rho_\sigma - k_{j+1})_+^{mp}\mathrm{d}x\mathrm{d}\tau &\geq \iint_{Q_j\cap[u\rho_\sigma > k_{n+1}]}(u\rho_\sigma - k_{j+1})^{mp}\mathrm{d}x\mathrm{d}\tau \\ &\geq |k_{j+1} - k_j|^p|A_{j+1}| \\ &\geq \frac{k^{mp}}{2^{mp(j+2)}}|A_{j+1}|,\end{aligned} \tag{3.31}$$

we deduce from (3.29) and (3.30) that

$$\iint_{Q_j}(u\rho_\sigma - k_{j+1})_+^{mp}\mathrm{d}x\mathrm{d}\tau \leq \gamma b^j[K(T)]^{\frac{(N+p)s}{Nq}}k^{-mp(1-\frac{s}{q})}\Big(\iint_{Q_j}(u\rho_\sigma - k_j)_+^{mp}\mathrm{d}x\mathrm{d}\tau\Big)^{1+\frac{ps}{Nq}}, \tag{3.32}$$

it is worth noting that $b = 2^{mp+[(1+\frac{p}{N})(p+1)-mp]}\frac{s}{q}$. By using [7, Lemma 5.6] it is known that

$$\iint_{Q_j}(u\rho_\sigma - k_j)_+^{mp}\mathrm{d}x\mathrm{d}\tau \to \iint_{Q_\infty}(u\rho_\sigma - k)_+^{mp}\mathrm{d}x\mathrm{d}\tau = 0,$$

provided

$$\iint_{Q_0}(u\rho_\sigma)^{mp}\mathrm{d}x\mathrm{d}\tau \leq \gamma[K(T)]^{\frac{N+p}{p}}k^{mN(\frac{q}{s}-1)}.$$

In this scenario, if the value of k is selected and satisfies the requisite conditions

$$k = \gamma[K(T)]^{\frac{N+p}{\beta}}\Big(\iint_{Q_0}(u\rho_\sigma)^{mp}\mathrm{d}x\mathrm{d}\tau\Big)^{p/\beta}, \quad \beta = N(m(p-1)-1) + mp^2, \tag{3.33}$$

then $\sup_{Q_\infty} u\rho_\sigma \leq k$. Since $T > 0$ is arbitrary, the lemma follows. $\square$

Lemma 3.7. *For all $t > 0$ it holds*

$$\varphi(t) \leq \gamma\int_0^t \tau^{-\frac{N[m(p-1)-1]}{\lambda}}[\varphi(\tau)]^{m(p-1)}d\tau + \gamma\psi(t)^{p/\lambda}, \quad \lambda = N(m(p-1)-1) + p. \tag{3.34}$$

Proof. Multiply both sides of (3.20) by $\rho^{\frac{p}{m(p-1)-1}}$ and $\tau^{\frac{N}{\lambda}}$, $\tau \in (\frac{t}{2}, t)$ simultaneously. It is necessary to recall the definition of $K(t)$ as presented in equation (3.21), we have that for all $\tau \in (\frac{t}{2}, t)$, and all $t > 0$,

$$
\begin{aligned}
&\tau^{\frac{N}{\lambda}}\|u(\cdot,\tau)\|_{L^\infty(\rho_\sigma),B_\rho}\rho^{-\frac{p}{m(p-1)-1}} \\
&\le \gamma\tau^{\frac{N}{\lambda}-\frac{N(N+p)[m(p-1)-1]}{\lambda\beta}}[\varphi(\tau)]^{\frac{(N+p)[m(p-1)-1]}{\beta}}\Big(\int_{\frac{t}{4}}^{t}\int_{B_{2\rho}}\rho^{-\frac{\beta}{m(p-1)-1}}(u\rho_\sigma)^{mp}\mathrm{d}x\mathrm{d}\tau\Big)^{p/\beta} \\
&\quad+\gamma\tau^{\frac{N}{\lambda}-\frac{N+p}{\lambda\beta}}\Big(\int_{\frac{t}{4}}^{t}\int_{B_{2\rho}}\rho^{-\frac{\beta}{m(p-1)-1}}(u\rho_\sigma)^{mp}\mathrm{d}x\mathrm{d}\tau\Big)^{p/\beta} \\
&= E_1 + E_2.
\end{aligned}
\tag{3.35}
$$

Subsequently, the values of E_1 and E_2 are estimated:

$$
\begin{aligned}
E_1 &\le \gamma_1[\varphi(\tau)]^{1-\frac{p}{\beta}}\Big[\int_{\frac{t}{4}}^{t}\tau^{-\frac{N[m(p-1)-1]}{\lambda}}\Big(\tau^{\frac{N}{\lambda}}\frac{\|u(\cdot,\tau)\|_{L^\infty(\rho_\sigma),B_{2\rho}}}{(2\rho)^{\frac{p}{m(p-1)-1}}}\Big)^{mp}\mathrm{d}\tau\Big]^{p/\beta} \\
&\le \gamma_1[\varphi(t)]^{1-\frac{p}{\beta}}\Big(\int_0^t\tau^{-\frac{N[m(p-1)-1]}{\lambda}}[\varphi(\tau)]^{m(p-1)}\,\mathrm{d}\tau\Big)^{p/\beta} \\
&\le \frac{1}{4}\varphi(t)+\gamma_2\int_0^t\tau^{-\frac{N[m(p-1)-1]}{\lambda}}[\varphi(\tau)]^{m(p-1)}\,\mathrm{d}\tau,
\end{aligned}
\tag{3.36}
$$

and

$$
\begin{aligned}
E_2 &\le \gamma\Big(\frac{1}{t}\int_{\frac{t}{4}}^{t}\tau^{-\frac{N[m(p-1)-1]}{\lambda}}\Big(\frac{\|u(\cdot,\tau)\|_{L^\infty(\rho_\sigma),B_{2\rho}}}{(2\rho)^{\frac{p}{m(p-1)-1}}}\Big)^{mp-1}\frac{\int_{B_{2\rho}}u(x,\tau)\rho_\sigma\mathrm{d}x}{(2\rho)^{\frac{\lambda}{m(p-1)-1}}}\,\mathrm{d}\tau\Big)^{p/\beta} \\
&\le \gamma[\varphi(t)]^{\frac{p(mp-1)}{\beta}}\Big(\frac{1}{t}\int_0^t|||u(\cdot,\tau)|||_{r_\rho}\mathrm{d}\tau\Big)^{p/\beta} \\
&\le \gamma[\varphi(t)]^{\frac{p(mp-1)}{\beta}}[\psi(t)]^{p/\beta} \\
&\le \frac{1}{4}\varphi(t)+\gamma_3[\psi(t)]^{p/\lambda}.
\end{aligned}
\tag{3.37}
$$

The above conclusions are all based on Cauchy inequality scaling estimates. Now, we carry these estimates in (3.35) and take the supremum first over all $p \ge r > 0$ and then over all $\tau \in (0, t)$, for all $t > 0$. Recalling the definition (3.19) of $\varphi(t)$, the lemma follows. □

Lemma 3.8. *Let us consider a piecewise smooth cutoff function, designated as $\chi(x)$, within the context of $B_{2\rho}$, such that in B_ρ satisfying $\chi = 1$ and $|D\chi| \le \frac{2}{\rho}$. Let $\rho > r > 0$, for $\forall t > 0$ such that*

$$
\begin{aligned}
&\int_0^t\int_{B_{2\rho}}|Du^m|^{p-1}\chi^{p-1}\rho_\sigma\,\mathrm{d}x\,\mathrm{d}\tau \\
&\le \gamma\rho^{1+\frac{\lambda}{m(p-1)-1}}\Big\{\int_0^t\tau^{\frac{p+1}{\lambda}}[\varphi(\tau)]^{\frac{[m(p-1)-1](p+1)}{p}}\psi(\tau)\,\mathrm{d}\tau \\
&\quad+\int_0^t\tau^{\frac{1}{\lambda}-1}[\varphi(\tau)]^{\frac{m(p-1)-1}{p}}\psi(\tau)\,\mathrm{d}\tau\Big\}^{\frac{p-1}{p}}\Big(\int_0^t\tau^{\frac{1}{\lambda}-1}[\varphi(\tau)]^{\frac{m(p-1)-1}{p}}\psi(\tau)\,\mathrm{d}\tau\Big)^{\frac{1}{p}}.
\end{aligned}
\tag{3.38}
$$

Proof. The initial requirement is that u be strictly positive. This can be made strict by substituting $u + \varepsilon$ for u. By Hölder inequality,

$$
\begin{aligned}
\int_0^t\int_{B_{2\rho}}|Du^m|^{p-1}\chi^{p-1}\rho_\sigma\mathrm{d}x\mathrm{d}\tau &\le \int_0^t\int_{B_{2\rho}}\Big(\tau^{\frac{m(p-1)-1}{p^2}}|Du^m|^{p-1}(u\rho_\sigma)^{\frac{(h-m)(p-1)}{p}}\chi^{p-1}\rho_\sigma^{\frac{m(p-1)}{p}}\Big) \\
&\quad\times\Big(\tau^{-\frac{m(p-1)-1}{p^2}}(u\rho_\sigma)^{-\frac{(h-m)(p-1)}{p}}\rho_\sigma^{\frac{p-m(p-1)}{p}}\Big)\mathrm{d}x\mathrm{d}\tau \\
&\le \Big(\int_0^t\int_{B_{2\rho}}\tau^{1/p}|Du^m|^{p-1}(u\rho_\sigma)^{h-m}\chi^p\rho_\sigma^m\mathrm{d}x\mathrm{d}\tau\Big)^{\frac{p-1}{p}}
\end{aligned}
$$

$$
\begin{aligned}
&\times \Big(\int_0^t\int_{B_{2\rho}} \tau^{-\frac{p-1}{p}} (u\rho_\sigma)^{-(p-1)(h-m)} \rho_\sigma^{p-m(p-1)} \mathrm{d}x\mathrm{d}\tau\Big)^{1/p} \\
&\le \Big(\int_0^t\int_{B_{2\rho}} \tau^{1/p}|Du^m|^{p-1}(u\rho_\sigma)^{h-m}\chi^p\rho_\sigma^m \mathrm{d}x\mathrm{d}\tau\Big)^{\frac{p-1}{p}} \\
&\times \Big(\int_0^t\int_{B_{2\rho}} \tau^{-\frac{p-1}{p}} (u\rho_\sigma)^{-(p-1)(h-m)} \mathrm{d}x\mathrm{d}\tau\Big)^{1/p} \\
&= [F_1(t)]^{\frac{p-1}{p}}[F_2(t)]^{1/p},
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^t\int_{B_{2\rho}} |Du^m|^{p-1}\chi^{p-1}\rho_\sigma \mathrm{d}x\mathrm{d}\tau &\le \int_0^t\int_{B_{2\rho}} \Big(\tau^{\frac{m(p-1)-1}{p^2}}|Du^m|^{p-1}(u\rho_\sigma)^{\frac{(h-m)(p-1)}{p}}\chi^{p-1}\rho_\sigma^{\frac{m(p-1)}{p}}\Big) \\
&\quad \times \Big(\tau^{-\frac{m(p-1)-1}{p^2}}(u\rho_\sigma)^{-\frac{(h-m)(p-1)}{p}}\rho_\sigma^{\frac{p-m(p-1)}{p}}\Big)\mathrm{d}x\mathrm{d}\tau \\
&\le \Big(\int_0^t\int_{B_{2\rho}} \tau^{1/p}|Du^m|^{p-1}(u\rho_\sigma)^{h-m}\chi^p\rho_\sigma^m \mathrm{d}x\mathrm{d}\tau\Big)^{\frac{p-1}{p}} \\
&\quad \times \Big(\int_0^t\int_{B_{2\rho}} \tau^{-\frac{p-1}{p}} (u\rho_\sigma)^{-(p-1)(h-m)} \rho_\sigma^{p-m(p-1)} \mathrm{d}x\mathrm{d}\tau\Big)^{1/p} \\
&\le \Big(\int_0^t\int_{B_{2\rho}} \tau^{1/p}|Du^m|^{p-1}(u\rho_\sigma)^{h-m}\chi^p\rho_\sigma^m \mathrm{d}x\mathrm{d}\tau\Big)^{\frac{p-1}{p}} \\
&\quad \times \Big(\int_0^t\int_{B_{2\rho}} \tau^{-\frac{p-1}{p}} (u\rho_\sigma)^{-(p-1)(h-m)} \mathrm{d}x\mathrm{d}\tau\Big)^{1/p} \\
&= [F_1(t)]^{\frac{p-1}{p}}[F_2(t)]^{1/p},
\end{aligned}
$$

where $h = \big(m(p-1)-1\big)/p$, for estimating $J_1(t)$, we take $\varphi = \tau^{1/p}(u\rho_\sigma)^h\chi^p$ in the weak formulation of (3.17) and obtain

$$
\begin{aligned}
F_1(t) &\le \frac{\gamma}{\rho^p}\int_0^t\int_{B_{2\rho}} \tau^{1/p}(u\rho_\sigma)^{h+m(p-1)}\mathrm{d}x\mathrm{d}\tau + \gamma\int_0^t\int_{B_{2\rho}} \tau^{\frac{1}{p}-1}(u\rho_\sigma)^{h+1}\mathrm{d}x\mathrm{d}\tau \\
&= V_1 + V_2.
\end{aligned} \tag{3.39}
$$

Estimating V_1, V_2, separately, we have

$$
\begin{gathered}
V_1 \le \gamma\rho^{1+\frac{\lambda}{m(p-1)-1}} \int_0^t \tau^{\frac{p+1}{\lambda}}[\varphi(t)]^{\frac{(p+1)[m(p-1)-1]}{p}}\psi(t)\mathrm{d}\tau, \\
V_2 \le \gamma\rho^{1+\frac{\lambda}{m(p-1)-1}} \int_0^t \tau^{\frac{1}{\lambda}-1}[\varphi(t)]^{\frac{m(p-1)-1}{p}}\psi(t)\mathrm{d}\tau.
\end{gathered} \tag{3.40}
$$

Note that $F_2(t) = V_2$, and combing (3.39)-(3.40) the conclusion follows. □

Lemma 3.9. *It holds*

$$
\begin{aligned}
\psi(t) \le \gamma|||u_0|||_{r_\rho} &+ \gamma\int_0^t \tau^{\frac{p+1}{\lambda}-1}[\varphi(\tau)]^{\frac{(p+1)[m(p-1)-1]}{p}}\psi(\tau)\mathrm{d}\tau \\
&+ \int_0^t \tau^{\frac{1}{\lambda}-1}[\varphi(\tau)]^{\frac{m(p-1)-1}{p}}\psi(\tau)\mathrm{d}\tau.
\end{aligned} \tag{3.41}
$$

Proof. It is necessary to take the cutoff function $\chi(x)$ introduced in Lemma 3.8. By simultaneously multiplying both sides of Equation (3.20) by $\varphi(x) = \chi^p(x)\rho_\sigma$ and utilizing distributional integrals and standard calculations, we obtain

$$
\int_{B_\rho} u(x,t)\rho_\sigma \mathrm{d}x \le \int_{B_{2\rho}} u_0\rho_\sigma\mathrm{d}x + \frac{\gamma}{\rho}\int_0^t\int_{B_{2\rho}} |Du^m|^{p-1}\chi^{p-1}\rho_\sigma\mathrm{d}x\mathrm{d}\tau, \tag{3.42}
$$

valid for all $t > 0$ and all $p > r > 0$. Multiply both sides by $\rho^{-\frac{\lambda}{m(p-1)-1}}$, estimate the last term by Lemma 3.8, and take the supremum over all $p > r > 0$. $\square$

Lemma 3.10. *For t positive, let $\varphi(t)$ and $\psi(t)$ be two continuous, non-decreasing functions satisfying the following conditions:*

$$\varphi(t) \le \gamma \int_0^t \tau^{-\frac{N[m(p-1)-1]}{\lambda}} [\varphi(\tau)]^{m(p-1)} d\tau + \gamma[\psi(t)]^{p/\lambda}, \tag{3.43}$$

$$\begin{aligned} \psi(t) \le\ & \gamma\rho^{-\frac{\lambda}{m(p-)-1}} \|u(\cdot,\tau)\|_{L^1(\rho_\sigma),B_\rho} \\ & + \gamma \int_0^t \tau^{\frac{p+1}{\lambda}-1} [\varphi(\tau)]^{\frac{(p+1)[m(p-1)-1]}{p}} \psi(\tau) \mathrm{d}\tau \\ & + \gamma \int_0^t \tau^{\frac{1}{\lambda}-1} [\varphi(\tau)]^{\frac{m(p-1)-1}{p}} \psi(\tau) \mathrm{d}\tau. \end{aligned} \tag{3.44}$$

Furthermore, there exist constants γ_0, γ_1, γ_2, which are dependent on the variables γ, m, p and N, and which satisfy

$$\varphi(t) \le \gamma_1 |||u_0|||_{r_\rho}^{p/\lambda}, \tag{3.45}$$

$$\psi(t) \le \gamma_2 |||u_0|||_{r_\rho}, \tag{3.46}$$

for all $0 < t < \gamma_0 |||u_0|||_{r_\rho}^{-[m(p-1)-1]}$.

Proof. It is established that the function $t \to \psi(t)$ is non-decreasing. For all $t^* > 0$, from (3.43) we have

$$\varphi(t) \le \gamma \int_0^t \tau^{-\frac{N(m(p-1)-1)}{\lambda}} [\varphi(\tau)]^{m(p-1)} d\tau + \gamma[\psi(t)]^{p/\lambda}, \tag{3.47}$$

and satisfy $0 < t \le t^*$. For the constant t^* will be chosen later, at this stage is an arbitrary positive value of t. As a consequence of the preceding argument, it follows from (3.47) that $\varphi(t)$ can be displayed using the solution to

$$G'(t) = \gamma t^{-\frac{N[m(p-1)-1]}{\lambda}} G(t)^{p-1},\ G(0) = \gamma\psi(t^*)^{p/\lambda}. \tag{3.48}$$

Solving explicitly

$$\varphi(t) \le G(t) = \gamma\psi(t^*)^{p/\lambda} \{1 - \gamma[m(p-1)-1][t\psi(t^*)^{m(p-1)-1}]^{p/\lambda}\}^{-\frac{1}{m(p-1)}}. \tag{3.49}$$

For any given $t \in [0, t^*]$, it is guaranteed that the value within the brackets is positive. Provided that the bracket is of a positive nature.

In this instance, the aforementioned estimate can be expressed as $t = t^*$. Given that t^* is an arbitrary positive number, it can be inferred that

$$\varphi(t) \le \gamma\psi(t)^{p/\lambda} \{1 - \gamma[m(p-1)-1][t\psi(t)^{m(p-1)-1}]^{p/\lambda}\}^{-\frac{1}{m(p-1)}}, \tag{3.50}$$

for all $t > 0$ for which the bracket is positive. If on t we impose

$$\gamma(t\psi(t)^{m(p-1)-1})^{p/\lambda} \le \frac{2^{-[m(p-1)-1]}}{m(p-1)-1} [2^{m(p-1)-1} - 1]. \tag{3.51}$$

The following statement will be obtained: There exist two constants $\bar{\gamma}_1$ and $\bar{\gamma}_0$, and are dependent only on N, m, and p, such that

$$\varphi(t) \le \bar{\gamma}_1 \psi(t)^{p/\lambda}, \tag{3.52}$$

for all t satisfying

$$[t\psi(t)^{m(p-1)-1}]^{p/\lambda} \le \bar{\gamma}_0. \tag{3.53}$$

Substituting inequality (3.52) into (3.44) yields

$$\psi(t) \le \gamma |||u_0|||_{r_\rho} + \gamma \int_0^t \tau^{\frac{1}{\lambda}-1} \psi(t)^{\frac{m(p-1)-1}{p}+1} \mathrm{d}\tau, \tag{3.54}$$

for all t for which (3.53) holds and for a new constant $\gamma = \gamma(N, m, p)$. It can be reasonably deduced that the solution to $\psi(\cdot)$ is

$$H'(t) = \gamma t^{\frac{1}{\lambda}-1} H(t)^{\frac{m(p-1)-1}{p}+1}, \quad H(0) = \gamma |||u_0|||_{r_\rho}, \tag{3.55}$$

whence

$$\psi(t) \le H(t) = |||u_0|||_{r_\rho} \big[1 - \gamma(t|||u_0|||_{r_\rho}^{m(p-1)-1})^{p/\lambda}\big]^{-\frac{\lambda}{m(p-1)-1}}, \tag{3.56}$$

ensure that the inside of the brackets are positive.

The constant γ being quantitatively determined a prior, we can find a constant $\bar{\bar{\gamma}}_0$ such that

$$\psi(t) \le \gamma_2 |||u_0|||_{r_\rho}, \tag{3.57}$$

as long as t is so small as to satisfy (3.53) and

$$0 < t < \bar{\bar{\gamma}}_0 |||u_0|||_{r_\rho}^{-[m(p-1)-1]}. \tag{3.58}$$

Putting together (3.53), (3.57), and (3.58), we deduce that there exists a constant $\gamma_0 = \gamma_0(N, m, p)$ such that if $0 < t < \gamma_0 |||u_0|||_{r_\rho}^{-[m(p-1)-1]}$, (3.53) and (3.58) holds. □

Now combining (3.52) and (3.57) leads to the following inference.

Corollary 3.11. *There exist constants γ_0, γ_1, γ_2 depending only on N, m, p such that $\rho \ge r > 0$,*

$$\|u(,t)\|_{\infty,B_\rho} \le \gamma_1 t^{-N/\lambda} \rho^{\frac{p}{m(p-1)-1}} |||u_0|||_{r_\rho}^{-[m(p-1)-1]}. \tag{3.59}$$

for all $0 < t < \gamma_0 |||u_0|||_{r_\rho}^{-[m(p-1)-1]}$.

Corollary 3.12. *There exist constants γ, depending only on N, m, p such that*

$$\|u(,t)\|_{L^\infty(\rho_\sigma)} \le \gamma t^{-N/\lambda} \|u_0\|_{L^1(\rho_\sigma)}^{-[m(p-1)-1]}, \tag{3.60}$$

for all $0 < t < \gamma \|u_0\|_{L^1(\rho_\sigma)}^{-[m(p-1)-1]}$. Setting $T = \gamma \|u_0\|_{L^1(\rho_\sigma)}^{-[m(p-1)-1]}$, we duce from (3.60) that

$$S(t)u_0 \in L^\infty(\rho_\sigma). \tag{3.61}$$

Hence, for $T \le t$, by Theorem3.5, we have

$$S(t)u_0 = S(T-t)[S(T)u_0] \in L^\infty(\rho_\sigma), \tag{3.62}$$

and the proof is complete.

Remark 3.13. The $L^1 - L^\infty$estimates for ball B_ρ have been reflected in paper [20]. To obtain our results, we generalize them to the weighted space $L^p(\rho_\sigma)$ and provide the weights L^1-L^∞estimates.

□

4. Complicated asymptotic behavior

To complete the body of this article, we introduce the work spaces $Y_\sigma(\mathbb{R}^N)$ with $\frac{p}{p-1} \le \sigma < \frac{p}{m(p-1)-1}$ and study the complicated asymptotic behavior of the solutions to problems (1.1)-(1.2) in these spaces. We define a closed subset of $Y_\sigma(\mathbb{R}^N)$ as

$$Y_\sigma^+(\mathbb{R}^N) = \{\varphi \in Y_\sigma(\mathbb{R}^N) : \varphi(x) \ge 0 \text{ and } \varphi(0) = 0\}. \tag{4.1}$$

For convenience, let

$$\gamma = N[m(p-1)-1] + p, \quad \eta = \frac{1}{p - \sigma[m(p-1)-1]}.$$

Our main task is to study the complicated asymptotic behavior of the solution of problem (1.1)-(1.2).

Theorem 4.1. *Let*
$$\frac{p}{p-1} \le \sigma < \frac{p}{m(p-1)-1}, \tag{4.2}$$
suppose there exist $\mu > 0$ and $\beta > 0$, and the following inequalities are satisfied
$$\beta > \max(\frac{2-\mu[m(p-1)-1]}{2p}, \eta), \tag{4.3}$$
$$0 < \mu + \frac{2p\beta\sigma}{\gamma} < \frac{2N}{\gamma}. \tag{4.4}$$
Then for any $f \in Y_\sigma^+(\mathbb{R}^N) \cap L^1(\rho_\sigma)$, there exists an initial value $u_0 \in Y_\sigma^+(\mathbb{R}^N)$, and a subsequence $\{t_n\}$ converging to ∞ as $n \to \infty$, such that
$$D^{\mu,\beta}_{\sqrt{t_n}}[S(t_n)u_0] \xrightarrow{n\to\infty} f \quad \text{in } Y_\sigma(\mathbb{R}^N), \tag{4.5}$$
in other words, $f \in \omega_\sigma^{\mu,\beta}(u_0)$.

Proof. For every $f \in Y_\sigma^+(\mathbb{R}^N) \cap L^1(\rho_\sigma)$, without loss of generality, we assume that
$$\|f\|_{L^\infty(\rho_\sigma)} \ge 1 \quad \text{and} \quad \|f\|_{L^1(\rho_\sigma)} \ge 1. \tag{4.6}$$
We define
$$\ell = \max\{\|f\|_{L^\infty(\rho_\sigma)}, \|f\|^\eta_{L^\infty(\rho_\sigma)},\ \|f\|^\eta_{L^1(\rho_\sigma)}\}, \tag{4.7}$$
$$\eta = \max\{\eta, \frac{1}{\gamma}, \frac{1}{\gamma-\sigma[m(p-1)-1]}, \frac{1}{2}\} \ge \frac{1}{2}. \tag{4.8}$$
From (4.3) and (4.4) it follows that $\beta > \eta \ge \frac{1}{2}$ and $0 < 2N - \mu\gamma - 2p\beta\sigma < 2p\beta N - p\mu - 2p\beta\sigma$. Next we define a sequence $\lambda_j \to \infty$ as $j \to \infty$ through $\lambda_1 = 2$, and for any $j > 1$,
$$\lambda_j = \max\{2^{\frac{j}{\mu}}\lambda_{j-1}, (2^{j+1}C\ell + 2^{2j}\lambda_{j-1}^{2\beta})^{\frac{1}{2\beta-2\eta}}, j^{\frac{\gamma+2}{2N-\mu\gamma-2p\beta\sigma}}\lambda_{j-1}^{\frac{2p\beta N-p\mu-2p\beta\sigma}{2N-\mu\gamma-2p\beta\sigma}}, 2^{\frac{j}{\beta}}\lambda_{j-1}\}. \tag{4.9}$$
Below we define the initial value $u_0(x)$ as
$$u_0(x) = \sum_{j=1}^\infty \lambda_j^{-\mu}\chi_j\Big(\frac{x}{\lambda_j^{2\beta}}\Big)f(x/\lambda_j^{2\beta}) = \sum_{j=1}^\infty D^{\mu,\beta}_{\lambda_j^{-1}}[\chi_j f(x)]. \tag{4.10}$$
In this context, $\chi_j(x)$ represents the cut-off function, defined within set $\Phi_j = \{2^{-j} < |x| < 2^j\}$ with respect to set $\Psi_j = \{2^{-(j-1)} < |x| < 2^{j-1}\}$, and $\chi_j(x) \in C_0^\infty(\mathbb{R}^N)$, $0 \le \chi_j(x) \le 1$, such that
$$\chi_j(x) = \begin{cases} 1 & \text{for } x \in \Phi_j, \\ 0 & \text{for } x \notin \Psi_j. \end{cases} \tag{4.11}$$
So that
$$u_0(x) \ge 0, \quad u_0(0) = 0. \tag{4.12}$$
Next we consider $0 < \lambda \le 1$, consequently
$$\begin{aligned} \|f\|_{Y_\sigma(\mathbb{R}^N)} &= \sup_{x\in\mathbb{R}^N} |(1+|x|^2)^{-\sigma/2} f(\lambda x)| \\ &\le \sup_{x\in\mathbb{R}^N} \Big| f(\lambda x)(1+|x|^2)^{-\sigma/2}\Big(\frac{1+|\lambda x|^2}{1+|x|^2}\Big)^{\sigma/2}\Big| \\ &\le \sup_{x\in\mathbb{R}^N} \Big| f(\lambda x)\Big(1+|\lambda x|^2\Big)^{-\sigma/2}\Big| \\ &= \|f\|_{Y_\sigma(\mathbb{R}^N)}. \end{aligned} \tag{4.13}$$
Therefore, using (4.10) we have
$$\|u_0\|_{Y_\sigma(\mathbb{R}^N)} \le \sum_{j=1}^\infty 2^{-j}\|f(x)\|_{Y_\sigma(\mathbb{R}^N)} \le \ell. \tag{4.14}$$

Consequently, the series (4.10) is convergent within $Y_\sigma(\mathbb{R}^N)$. Additionally, equation (4.12) allows for the conclusion that

$$u_0 \in Y_\sigma^+(\mathbb{R}^N). \tag{4.15}$$

Now se conduct a detailed analysis of the initial value u_0. This will be broken down into three parts.

$$u_n(x) = \sum_{j=1}^{n-1} \lambda_j^{-\mu} \chi_j\Big(\frac{x}{\lambda_j^{2\beta}}\Big) f(x/\lambda_j^{2\beta}) = \sum_{j=1}^{n-1} D^{\mu,\beta}_{\lambda_j^{-1}}[\chi_j(x) f(x)], \tag{4.16}$$

$$v_n(x) = D^{\mu,\beta}_{\lambda_n^{-1}}[\chi_n(x) f(x)], \tag{4.17}$$

$$w_n(x) = \sum_{j=n+1}^{\infty} \lambda_j^{-\mu} \chi_j\Big(\frac{x}{\lambda_j^{2\beta}}\Big) f(x/\lambda_j^{2\beta}) = \sum_{j=1}^{n+1} D^{\mu,\beta}_{\lambda_j^{-1}}[\chi_j(x) f(x)]. \tag{4.18}$$

From Theorems 3.3 and 3.4, we have

$$\operatorname{supp}[S(\lambda_n^2) u_n(x)] \subset \{x : |x| \le 2^{n-1}\lambda_{n-1}^{2\beta} + C\lambda_n^{2\eta}\ell\}, \tag{4.19}$$

$$\operatorname{supp}[S(\lambda_n^2) v_n(x)] \subset \{x : 2^{-n}\lambda_n^{2\beta} - C\lambda_n^{2\eta}\ell \le |x| \le 2^{n}\lambda_n^{2\beta} + C\lambda_n^{2\eta}\ell\}, \tag{4.20}$$

$$\operatorname{supp}[S(\lambda_n^2) w_n(x)] \subset \{x : |x| \ge 2^{-n-1}\lambda_{n+1}^{2\beta} - C\lambda_n^{2\eta}\ell\}. \tag{4.21}$$

From (4.9), it is apparent that $(2^{j+1}C\ell + 2^{2j}\lambda_{j-1}^{2\beta})^{\frac{1}{2\beta-2\eta}} \le \lambda_j$, so that for $j > 1$,

$$2^{j-1}\lambda_{j-1}^{2\beta} + 2C\lambda_j^{2\eta}\ell < 2^{j-1}\lambda_{j-1}^{2\beta}\lambda_j^{2\eta} + 2C\lambda_j^{2\eta}\ell \le 2^{-j}\lambda_j^{2\beta},$$

hence

$$2^{n-1}\lambda_{n-1}^{2\beta} + C\lambda_n^{2\eta}\ell < 2^{-n}\lambda_n^{2\beta} - C\lambda_n^{2\eta}\ell < 2^{n}\lambda_n^{2\beta} + C\lambda_n^{2\eta}\ell < 2^{-n-1}\lambda_{n+1}^{2\beta} - C\lambda_n^{2\eta}\ell.$$

It can be demonstrated from equations (4.19), (4.20) and (4.21) that

$$\operatorname{supp}[S(\lambda_n^2) u_n(x)] \cap \operatorname{supp}[S(\lambda_n^2) v_n(x)] = \emptyset,$$
$$\operatorname{supp}[S(\lambda_n^2) v_n(x)] \cap \operatorname{supp}[S(\lambda_n^2) w_n(x)] = \emptyset.$$

Thus

$$S(\lambda_n^2) u_0 = S(\lambda_n^2) u_n(x) + S(\lambda_n^2) v_n(x) + S(\lambda_n^2) w_n(x). \tag{4.22}$$

Applying the exchange relationship in Definition2.6 to (4.22), we have

$$\begin{aligned} D^{\mu,\beta}_{\lambda_n} S(\lambda_n^2) u_0 &= D^{\mu,\beta}_{\lambda_n}\big[S(\lambda_n^2) u_n(x)\big] + D^{\mu,\beta}_{\lambda_n}\big[S(\lambda_n^2) v_n(x)\big] + D^{\mu,\beta}_{\lambda_n}\big[S(\lambda_n^2) w_n(x)\big] \\ &= S\big(\lambda_n^{2-2p\beta-\mu[m(p-1)-1]}\big)[D^{\mu,\beta}_{\lambda_n} u_n(x)] + S\big(\lambda_n^{2-2p\beta-\mu[m(p-1)-1]}\big) \\ &\quad \times [S(\lambda_n^2) v_n(x)] + S\big(\lambda_n^{2-2p\beta-\mu[m(p-1)-1]}\big)\big[S(\lambda_n^2) w_n(x)\big]. \end{aligned} \tag{4.23}$$

Now we consider the case of $\lambda \ge 1$.

$$\begin{aligned} \|f(\lambda\cdot)\|_{L^1(\rho_\sigma)} &= \int_{\mathbb{R}^N} \big|(1+|x|^2)^{-\sigma/2} f(\lambda x)\big| dx \\ &\le \int_{\mathbb{R}^N} \Big| f(\lambda x)(1+|x|^2)^{-\sigma/2} \Big(\frac{1+|\lambda x|^2}{1+|x|^2}\Big)^{\sigma/2}\Big| dx \\ &\le \lambda^{\sigma}\Big(\int_{\mathbb{R}^N} |f(\lambda x)(1+|\lambda x|^2)^{-\sigma/2}| dx\Big) \\ &= \lambda^{\sigma-N}\|f\|_{L^1(\rho_\sigma)}. \end{aligned} \tag{4.24}$$

It can be deduced from (4.9) and (4.16) that

$$\|D^{\mu,\beta}_{\lambda_n} u_n(x)\|_{L^1(\rho_\sigma)} \le n\big(\frac{\lambda_n}{\lambda_{n-1}}\big)^{\mu-2\beta N+2\beta\sigma}\|f\|_{L^1(\rho_\sigma)} < \infty, \tag{4.25}$$

because $D^{\mu,\beta}_{\lambda_n} u_n(x) \in L^1(\rho_\sigma)$; furthermore, the results in section (4.3) show that $2 - \mu[m(p-1) - 1] - 2p\beta < 0$. It follows again from (4.9) that there exists an integer N, such that if $n > N$, then

$$0 < \lambda_n^{2-\mu[m(p-1)-1]-2p\beta} < C\|D^{\mu,\beta}_{\lambda_n} u_n(x)\|^{-[m(p-1)-1]}_{L^1(\rho_\sigma)}. \tag{4.26}$$

An application of the weighted L^1-L^∞ estimates, as proposed in Theorem3.5, to Equation (4.25), yields

$$\|S(\lambda_n^{2-2p\beta-\mu[m(p-1)-1]})D_{\lambda_n}^{\mu,\beta}u_n(x)\|_{Y_\sigma(\mathbb{R}^N)} \le Cn^{\frac{p}{\gamma}}\lambda_n^{\mu+\frac{2p\beta\sigma-2N}{\gamma}}\lambda_{n-1}^{\frac{2p\beta N-\mu p-2p\beta\sigma}{\gamma}}. \tag{4.27}$$

From (4.9) we conclude that

$$\|S\big(\lambda_n^{2-2p\beta-\mu[m(p-1)-1]}\big)D_{\lambda_n}^{\mu,\beta}u_n(x)\|_{Y_\sigma(\mathbb{R}^N)} \le Cn^{-1}\ell \to 0 \quad \text{as } n\to\infty. \tag{4.28}$$

To arrive at the solution for (4.18), we combine equations (4.9) and (4.13), for $n>1$,

$$\begin{aligned}\|D_{\lambda_n}^{\mu,\beta}w_n(x)\|_{Y_\sigma(\mathbb{R}^N)} &\le \|\sum_{j=n+1}^{\infty}(\frac{\lambda_n}{\lambda_j})^{\mu}f((\frac{\lambda_n}{\lambda_j})^{2\beta}\cdot)\|_{Y_\sigma(\mathbb{R}^N)}\\ &\le \sum_{j=n+1}^{\infty}(\frac{\lambda_n}{\lambda_j})^{\mu}\|f\|_{Y_\sigma(\mathbb{R}^N)}\\ &\le \|f\|_{L^1(\rho_\sigma)}<\infty,\end{aligned} \tag{4.29}$$

thereby $D_{\lambda_n}^{\mu,\beta}w_n(x)\in Y_\sigma(\mathbb{R}^N)$. Applying Theorem3.4, we obtain

$$S\big(\lambda_n^{2-2p\beta-\mu[m(p-1)-1]}\big)[D_{\lambda_n}^{\mu,\beta}w_n(x)]\in Y_\sigma(\mathbb{R}^N). \tag{4.30}$$

As a consequence, for all $n\ge 1$ and any $\varepsilon>0$, using Comparison Principle, there exists a constant $M>0$ such that $|x|>M$. Then

$$|(1+|x|^2)^{-\sigma/2}S\big(\lambda_n^{2-2p\beta-\mu[m(p-1)-1]}\big)[D_{\lambda_n}^{\mu,\beta}w_n(x)](x)|<\frac{\varepsilon}{2}. \tag{4.31}$$

Since

$$D_{\lambda_n}^{\mu,\beta}w_n(x)\subset\{x:|x|\ge 2^{-n-1}(\frac{\lambda_{n+1}}{\lambda_n})^{2\beta}\}, \tag{4.32}$$

from Theorem3.3, it follows that

$$\begin{aligned}&\operatorname{supp}\big\{S\big(\lambda_n^{2-2p\beta-\mu[m(p-1)-1]}\big)[D_{\lambda_n}^{\mu,\beta}w_n(x)]\big\}\\ &\subset\big\{x:|x|\ge 2^{-n-1}\big(\frac{\lambda_{n+1}}{\lambda_n}\big)^{2\beta}-C\lambda_n^{\eta[2-2p\beta-\mu(m(p-1)-1)]}\ell\big\}.\end{aligned} \tag{4.33}$$

Note that $\lambda_n^{\eta[2-2p\beta-\mu(m(p-1)-1)]}<1$, Since (4.9) implies $2^{2n}\le(\frac{\lambda_{n+1}}{\lambda_n})^{2\beta}$, the existence of an integer N, such that if $n>N$, then

$$2^{-n-1}(\frac{\lambda_{n+1}}{\lambda_n})^{2\beta}-C\lambda_n^{\eta 2-2p\beta-\mu[m(p-1)-1]}\ell>M. \tag{4.34}$$

The preceding equation can be expressed as follows: if $n>N$, then

$$\|S\big(\lambda_n^{2-2p\beta-\mu[m(p-1)-1]}\big)[D_{\lambda_n}^{\mu,\beta}w_n(x)]\|_{Y_\sigma(\mathbb{R}^N)}<\varepsilon. \tag{4.35}$$

Subsequently, we can show that

$$\|S\big(\lambda_n^{2-2p\beta-\mu[m(p-1)-1]}\big)D_{\lambda_n}^{\mu,\beta}v_n(x)-f\|_{Y_\sigma(\mathbb{R}^N)}\xrightarrow{n\to\infty}0, \tag{4.36}$$

because $D_{\lambda_n}^{\mu,\beta}v_n(x)$ implies

$$D_{\lambda_n}^{\mu,\beta}v_n(x)=f(x)\quad\text{for } x\in\Phi_j, \tag{4.37}$$

and

$$D_{\lambda_n}^{\mu,\beta}v_n(x)\le f(x)\quad\text{for } x\in\mathbb{R}^N. \tag{4.38}$$

We conclude from Theorem3.5 that

$$S\big(\lambda_n^{2-2p\beta-\mu[m(p-1)-1]}\big)[D_{\lambda_n}^{\mu,\beta}v_n(x)]\in Y_\sigma(\mathbb{R}^N). \tag{4.39}$$

Note that $f\in Y_\sigma(\mathbb{R}^N)$. As a result, for all $n\ge 1,\varepsilon>0$, using comparison principle, there exists a constant $M>0$ such that

$$\sup_{|x|>M}(1+|x|^2)^{-\sigma/2}\big|S\big(\lambda_n^{2-2p\beta-\mu[m(p-1)-1]}\big)[D_{\lambda_n}^{\mu,\beta}v_n](x)|<\frac{\varepsilon}{3}, \tag{4.40}$$

and

$$\sup |x| > M(1+|x|^2)^{-\sigma/2}|f(x)| < \frac{\varepsilon}{3}. \tag{4.41}$$

Then $f \in C(\mathbb{R}^N)$ implies that $S(t)f \in C([0,\infty)\times\mathbb{R}^N)$. Note that $\lambda_n^{2-2p\beta-\mu(m(p-1)-1)} \to 0$ as $n \to \infty$. Thereupon, for any $\varepsilon > 0$, there exists a integer N_1, such that if $n > N_1$, then

$$\sup_{|x|>M}(1+|x|^2)^{-\sigma/2}\big|\big[S\big(\lambda_n^{2-2p\beta-\mu[m(p-1)-1]}\big)f\big](x) - f(x)\big| < \frac{\varepsilon}{18}. \tag{4.42}$$

It follows from (4.9) that there exists a integerN_2, if $n > N_2$, so that

$$2^{n-1} > M, \tag{4.43}$$

based on the assumption $f(0) = 0$ that there exists a integerN_3, if $n > N_3$, such that

$$\sup_{|x|>2^{-N_3+1}}\big(1+|x|^2\big)^{-\sigma/2}|f(x)| < \frac{\varepsilon}{18}. \tag{4.44}$$

By (4.43), if $n > N_1$, then

$$\sup_{|x|>2^{-N_3+1}}\big(1+|x|^2\big)^{-\sigma/2}\big|\big[S\big(\lambda_n^{2-2p\beta-\mu[m(p-1)-1]}\big)f\big](x)\big| < \frac{\varepsilon}{9}. \tag{4.45}$$

By combining equations (4.38) and (4.45) with the comparison principle, if $n > N_1$, we have

$$\sup_{|x|>2^{-N_3+1}}(1+|x|^2)^{-\sigma/2}|[S(\lambda_n^{2-2p\beta-\mu[m(p-1)-1]})D^{\mu,\beta}_{\lambda_n}v_n(x)](x)| < \frac{\varepsilon}{9}. \tag{4.46}$$

Combine equations (4.37), (4.42), (4.43), (4.45) and (4.46) that if $n > \max\{N_1, N_2, N_3\}$, the reason why

$$\begin{aligned}
&\sup_{|x|\le M}\big(1+|x|^2\big)^{-\sigma/2}\big|S\big(\lambda_n^{2-2p\beta-\mu[m(p-1)-1]}\big)\big[D^{\mu,\beta}_{\lambda_n}v_n\big](x) - f(x)\big| \\
&\le \sup_{|x|\le M}\big(1+|x|^2\big)^{-\sigma/2}\big|\big[S\big(\lambda_n^{2-2p\beta-\mu[m(p-1)-1]}\big)f\big](x) - f(x)\big| \\
&\quad + \sup_{2^{-n+1}\le|x|\le M}\big(1+|x|^2\big)^{-\sigma/2}\big|S\big(\lambda_n^{2-2p\beta-\mu[m(p-1)-1]}\big) \\
&\quad \times [D^{\mu,\beta}_{\lambda_n}v_n](x) - S\big(\lambda_n^{2-2p\beta-\mu[m(p-1)-1]}\big)f(x)\big| \\
&\quad + \sup_{|x|\le 2^{-n+1}}(1+|x|^2)^{-\sigma/2}\big|S\big(\lambda_n^{2-2p\beta-\mu[m(p-1)-1]}\big)[D^{\mu,\beta}_{\lambda_n}v_n](x)\big| \\
&\quad + \sup_{|x|\le 2^{-n+1}}(1+|x|^2)^{-\sigma/2}\big|\big[S\big(\lambda_n^{2-2p\beta-\mu[m(p-1)-1]}\big)f\big](x)\big| \\
&< \frac{\varepsilon}{3}.
\end{aligned} \tag{4.47}$$

It follows from (4.40), (4.41) and (4.47) that (4.36) holds. Finally, it follows from (4.23), (4.28), (4.35), and (4.36) that

$$\begin{aligned}
&\|S\big(\lambda_n^{2-2p\beta-\mu[m(p-1)-1]}\big)[D^{\mu,\beta}_{\lambda_n}v_n] - f\|_{Y_\sigma(\mathbb{R}^N)} \\
&\le \sup_{|x|\le M}(1+|x|^2)^{-\sigma/2}\big|S\big(\lambda_n^{2-2p\beta-\mu[m(p-1)-1]}\big)[D^{\mu,\beta}_{\lambda_n}v_n](x) - f(x)\big| \\
&\quad + \sup_{|x|>M}(1+|x|^2)^{-\sigma/2}\big|S\big(\lambda_n^{2-2p\beta-\mu[m(p-1)-1]}\big)[D^{\mu,\beta}_{\lambda_n}v_n](x)\big| \\
&\quad + \sup_{|x|>M}(1+|x|^2)^{-\sigma/2}|f(x)| < \varepsilon.
\end{aligned} \tag{4.48}$$

It can therefore

$$D^{\mu,\beta}_{\lambda_n}S(\lambda_n^2)u_0 \xrightarrow{n\to\infty} f \quad \text{in } Y_\sigma(\mathbb{R}^N), \tag{4.49}$$

setting $t_n = \lambda_n^2$, so that (4.7) follows from (4.49) and the proof is complete. ☐

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