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EVOLUTION ψ-HILFER FRACTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. This article concerns evolution equations involving ψ-Hilfer fractional derivative in a Banach space. By using the theory of fractional calculus and ψ-Laplace transform, we firstly derive a definition of mild solutions for these equations. Then we establish theorems for the existence and uniqueness of solutions and the approximate controllability (not the exact controllability) of the ψ-Hilfer fractional differential system under appropriate conditions. We focus on the approximate controllability rather than the exact controllability because the exact controllability cannot be achieved generally for the system in infinite-dimensional spaces. We present a new multidimensional Gronwall-type inequality with multiple singular kernels involving exponential factors, which extends essentially many existing results. We also use the new Gronwall-type inequality to study the dependence of the solution on the order and the initial condition for the fractional integro-differential equations involving ψ-Hilfer fractional derivative. Finally, an example is given to illustrate our main results.

1. Introduction

In the past decades, applications of fractional calculus have gradually expanded and covered fields such as fluid mechanics, rheology, viscoelasticity, fractional control systems and controllers, electroanalytical chemistry, electrical conductivity in biological systems, fractional models of nerves, fractional regression models, etc. When using fractional order derivatives instead of traditional integer order derivatives to describe problems involving hereditary or memory properties, it often not only simplifies the differential equations but also yields results that are closer to reality. We refer the readers to [1, 2, 5, 6, 7, 8, 9, 11, 10, 12, 16, 19, 20, 22, 23, 24, 25, 26, 32, 33, 35, 36, 39] and the reference therein for theory and applications on fractional calculus.

The widely studied fractional derivatives include Riemann-Liouville fractional derivatives and Caputo fractional derivatives, and the solutions of equations containing these two types of derivative operators have significantly different properties. In [16], Hilfer combined the Riemann-Liouville fractional derivative and the Caputo fractional derivative to obtain a derivative, which is later referred to as the Hilfer fractional derivative in many literature. This derivative is an interpolation of the Riemann-Liouville fractional derivative and the Caputo fractional derivative, and he studied differential equations involving this derivative([17]). In [13], the authors studied a class of evolution equations containing Hilfer fractional derivatives in Banach spaces. They introduced the definition of mild solutions for these equations through Laplace transform and the density function, and obtained an existence theorem for mild solutions using non-compactness measures and the Ascoli-Arzela theorem. In [41], the authors introduced ψ-Hilfer fractional derivatives and studied equations involving such derivatives. Compared to traditional derivatives, derivatives containing an arbitrary function $\psi(t)$ are more widely used.

The concept of exact controllability was first proposed by Kalman ([18]) in 1963 and gradually became an active research field due to its enormous applications in the field of physics. The application of certain control to a natural or artificial system to influence its behavior to meet predetermined goals is called a control system. The controllability of control systems is one of

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the fundamental concepts in control theory, which is the basis for studying optimal control and estimation. At present, the controllability problem of differential systems described by differential equations has been studied by many scholars. The controllability problem of differential systems in Banach space is often transformed into a fixed point problem of operators, and the condition of compactness is often indispensable when applying the fixed point theorem. It is well known that the exact controllability cannot be achieved generally for control systems in infinite-dimensional spaces. Therefore, in addition to exact controllability, there is also a more widely used concept of approximate controllability in the controllability of differential systems. Exact controllability refers to the ability of a system to achieve the expected precise value under the influence of control functions, while approximate controllability does not require precise achievement, but only requires the system to reach a region near the expected value. For more research on the controllability of differential systems, please refer to [3, 4, 6, 14, 21, 30, 31] and the references therein.

Integral inequalities serve as fundamental tools for quantitatively analyzing solutions to differential and integral equations. Among these, the Gronwall-Bellman inequality has demonstrated wide applications in deriving estimates across ordinary differential equations, partial differential equations, stochastic differential equations, and integral-differential equations, as it provides explicit bounds of unknown functions $x(t)$. Given its broad applicability and significance, numerous extensions of this classical inequality have been developed. For works on generalized Gronwall-Bellman inequalities, Gronwall-type inequalities, and their applications, we refer readers to [4, 5, 6, 7, 15, 26, 27, 28, 29, 34, 38, 40, 42, 43] and the references therein.

The research on evolution differential equations with ψ-Hilfer fractional derivatives in Banach spaces is blank, and this paper fills this gap. The rest of this paper is organized as follows. In section 2, we introduce some notations, recall some basic known results, and derive a definition of mild solutions for the evolution equations involving ψ-Hilfer fractional derivatives. In section 3, we develop the existence and uniqueness theorem of mild solutions for ψ-Hilfer fractional system, and discuss the approximate controllability of the control problem in the weighted space $C_{1-\alpha-\beta(1-\alpha);\psi}$ under suitable conditions. In section 4, we establish a new multidimensional Gronwall-type inequality involving multiple singular kernels and exponential factors, and use it to study the dependence of solution on the order and the initial condition for the fractional integro-differential ψ-Hilfer fractional equations. Finally, in section 5, an example is given to illustrate the main results.

2. Definition of mild solutions

We begin this part by setting some notation. Suppose that $\psi(t) \in C^1[0,\infty)$ is strictly increasing and satisfies that $\psi(0)=0$ and $\lim_{t\to\infty}\psi(t)=\infty$. Let $\zeta(t)$ be the inverse function of the function $\psi(t)$. Denote by X a Banach space with norm $\|\cdot\|$. We denote by $C(J,X)$ the space of all X-valued continuous functions on J with the natural norm $\|x\|_{C(J,X)} = \sup_{t\in J}\|x(t)\|$. Let

$$C_{1-\gamma;\psi}(J,X) = \left\{x:\ \psi^{1-\gamma}(t)x(t) \in C(J,X)\right\} \quad (\gamma \in (0,1))$$

with the norm

$$\|x\|_{C_{1-\gamma;\psi}} = \sup\left\{\psi^{1-\gamma}(t)\|x(t)\|:\ t \in J\right\}.$$

Obviously, the space $C_{1-\gamma;\psi}(J,X)$ is a Banach space.

Throughout this article, we assume that the semigroup $\{T(t)\}_{t\geq 0}$ is differentiable and uniformly bounded, that is, there is a constant $M>0$ such that $\|T(t)\| \leq M,\ \forall t \geq 0$.

Definition 2.1 ([16])**.** The fractional integral of order $\alpha (0<\alpha<1)$ involving a general function ψ for a function f is defined by

$$I^{\alpha;\psi}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (\psi(t)-\psi(s))^{\alpha-1}f(s)\psi'(s)ds.$$

Remark 2.2. When $\psi(t)=t$, the fractional integral $I^{\alpha;\psi}$ becomes the Riemann-Liouville fractional integral I^{α}([16]).

Definition 2.3. [41] For any $0 < \alpha \le 1$, $0 \le \beta \le 1$, ψ is differentiable, the ψ-Hilfer fractional derivative of order α and type β for a function f is defined by

$$D^{\alpha,\beta;\psi} f(t) = I^{\beta(1-\alpha);\psi} \frac{1}{\psi'(t)} \frac{d}{dt} I^{(1-\beta)(1-\alpha);\psi} f(t).$$

Remark 2.4. When $\psi(t) = t$, the ψ-Hilfer fractional derivative becomes the Hilfer fractional derivative ([16]).

Definition 2.5 ([27])**.** For $f \in L^1_{loc}(\mathbb{R}_+, X)$ and $\lambda \in \mathbb{C}$, the ψ-Laplace transform of f is defined as

$$\mathcal{L}_\psi[f](\lambda) = \tilde{f}(\lambda) := \int_0^\infty e^{-\lambda\psi(t)} f(t)\psi'(t)dt,$$

as long as the integral on the right hand exists as a Bochner integral.

Lemma 2.6. *Let $0 < \alpha < 1$ and $0 \le \beta \le 1$. If $x \in C_{1-\alpha-\beta(1-\alpha);\psi}(J, X)$ and x is a solution of the equations*

$$\begin{gathered} \left(D^{\alpha,\beta;\psi} x\right)(t) = Ax(t) + g(t), \ t \in J', \\ I^{(1-\alpha)(1-\beta);\psi} x(0) = x_0, \end{gathered} \tag{2.1}$$

then x satisfies the equation

$$x(t) = \left(I^{\beta(1-\alpha)} \mathbb{K}_\alpha\right)(\psi(t))x_0 + \int_0^t \mathbb{K}_\alpha(\psi(t) - \psi(s))g(s)\psi'(s)ds,$$

where

$$\begin{gathered} \mathbb{K}_\alpha(t) = \alpha \int_0^\infty t^{\alpha-1}\sigma\xi_\alpha(\sigma)T(\sigma t^\alpha)d\sigma, \\ \xi_q(\sigma) = \frac{1}{q}\varpi_q(\sigma^{-\frac{1}{q}}), \quad \sigma \in (0,\infty), \\ \varpi_q(\tau) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\tau^{-qn-1}\frac{\Gamma(qn+1)}{n!}\sin(n\pi q), \quad \tau \in (0,\infty). \end{gathered}$$

Proof. Set

$$\rho(\lambda) = \int_0^\infty e^{-\lambda\psi(t)} g(t)\psi'(t)dt.$$

Applying the ψ-Laplace transform to both sides of the first equation of (2.1), we obtain

$$\lambda^\alpha \mathcal{L}_\psi[x](\lambda) - \lambda^{\beta(\alpha-1)} I^{(1-\alpha)(1-\beta);\psi} x(0) = A\mathcal{L}_\psi[x](\lambda) + \rho(\lambda).$$

Then $(\lambda^\alpha I - A)\tilde{x}(\lambda) = \lambda^{\beta(\alpha-1)}x_0 + \rho(\lambda)$; thus

$$\begin{aligned} \tilde{u}(\lambda) &= \lambda^{\beta(\alpha-1)}(\lambda^\alpha I - A)^{-1}x_0 + (\lambda^\alpha I - A)^{-1}\rho(\lambda) \\ &= \lambda^{\beta(\alpha-1)} \int_0^\infty e^{-\lambda^\alpha s}T(s)x_0 ds + \int_0^\infty e^{-\lambda^\alpha s}T(s)\rho(\lambda)ds. \end{aligned}$$

We consider the one-sided stable probability density function in $\mathbb{R}_+$ as

$$\varpi_\alpha(\sigma) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\sigma^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \quad \sigma \in (0,\infty),$$

whose Laplace transform is

$$\int_0^\infty e^{-\lambda\sigma}\varpi_\alpha(\sigma)d\sigma = e^{-\lambda^\alpha}, \quad \alpha \in (0,1). \tag{2.2}$$

Then, using (2.2) we obtain

$$\begin{aligned} \int_0^\infty e^{-\lambda^\alpha s}T(s)x_0 ds &= \int_0^\infty \int_0^\infty e^{-\lambda t\sigma}\alpha t^{\alpha-1}\varpi_\alpha(\sigma)T(t^\alpha)x_0 d\sigma dt \\ &= \int_0^\infty e^{-\lambda t}\Big[\alpha \int_0^\infty \frac{t^{\alpha-1}}{\sigma^\alpha}\varpi_\alpha(\sigma)T\Big(\frac{t^\alpha}{\sigma^\alpha}\Big)x_0 d\sigma\Big]dt \end{aligned}$$

$$= \int_0^\infty e^{-\lambda\psi(t)} \Big[\alpha \int_0^\infty \frac{\psi^{\alpha-1}(t)}{\sigma^\alpha} \varpi_\alpha(\sigma) T\Big(\frac{\psi^\alpha(t)}{\sigma^\alpha}\Big) x_0 d\sigma\Big] \psi'(t) dt,$$

and

$$\begin{aligned}
&\int_0^\infty e^{-\lambda^\alpha s} T(s)\rho(\lambda) ds \\
&= \int_0^\infty \int_0^\infty e^{-\lambda\tau\sigma} \alpha\tau^{\alpha-1} \varpi_\alpha(\sigma) T(\tau^\alpha) \Big(\int_0^\infty e^{-\lambda t} g(\zeta(t)) dt\Big) d\sigma d\tau \\
&= \alpha \int_0^\infty \int_0^\infty e^{-\lambda\vartheta} \frac{\vartheta^{\alpha-1}}{\sigma^\alpha} \varpi_\alpha(\sigma) T\Big(\frac{\vartheta^\alpha}{\sigma^\alpha}\Big) \Big(\int_0^\infty e^{-\lambda t} g(\zeta(t)) dt\Big) d\vartheta d\sigma \\
&= \alpha \int_0^\infty \Big(\int_0^\infty \int_0^\tau e^{-\lambda\tau} \frac{(\tau-t)^{\alpha-1}}{\sigma^\alpha} \varpi_\alpha(\sigma) T\Big(\frac{(\tau-t)^\alpha}{\sigma^\alpha}\Big) g(\zeta(t)) dt d\tau\Big) d\sigma \\
&= \int_0^\infty e^{-\lambda\psi(t)} \Big[\alpha \int_0^{\psi(t)} \int_0^\infty \frac{(\psi(t)-s)^{\alpha-1}}{\sigma^\alpha} \varpi_\alpha(\sigma) T\Big(\frac{(\psi(t)-s)^\alpha}{\sigma^\alpha}\Big) g(\zeta(s)) d\sigma ds\Big] \\
&\quad \times \psi'(t) dt \\
&= \int_0^\infty e^{-\lambda\psi(t)} \Big[\alpha \int_0^t \int_0^\infty (\psi(t)-\psi(s))^{\alpha-1} \sigma\xi_\alpha(\sigma) T(\sigma(\psi(t)-\psi(s))^\alpha) g(s)\psi'(s) d\sigma ds\Big] \psi'(t) dt.
\end{aligned}$$

Set

$$\mathbb{K}_\alpha(t) = \alpha \int_0^\infty t^{\alpha-1} \sigma\xi_\alpha(\sigma) T(\sigma t^\alpha) d\sigma.$$

Since the Laplace transform of

$$\mathcal{F}(t) := \frac{t^{\beta(1-\alpha)-1}}{\Gamma(\beta(1-\alpha))} \quad (0 < \beta \le 1)$$

is $\mathcal{L}[\mathcal{F}(t)](\lambda) = \lambda^{\beta(\alpha-1)}$, we have that for $0 < \beta \le 1$,

$$\begin{aligned}
\lambda^{\beta(\alpha-1)} \int_0^\infty e^{-\lambda^\alpha s} T(s) x_0 ds &= \mathcal{L}[\mathcal{F}(t)](\lambda) \mathcal{L}[\mathbb{K}_\alpha(t)](\lambda) x_0 \\
&= \mathcal{L}\left[(\mathcal{F} * \mathbb{K}_\alpha)(t)\right](\lambda) x_0 \\
&= \mathcal{L}_\psi\Big[\Big(I^{\beta(1-\alpha)} \mathbb{K}_\alpha\Big)(\psi(t))\Big](\lambda) x_0.
\end{aligned}$$

Observe that

$$\int_0^\infty e^{-\lambda^\alpha s} T(s)\rho(\lambda) ds = \int_0^\infty e^{-\lambda\psi(t)} \Big[\int_0^t \mathbb{K}_\alpha(\psi(t)-\psi(s)) g(s)\psi'(s) ds\Big] \psi'(t) dt. \tag{2.3}$$

When $\beta = 0$, (2.3) also holds if we keep in mind that $I^0 \mathbb{K}_\alpha = \mathbb{K}_\alpha$. Therefore, for $0 < \alpha < 1$ and $0 \le \beta \le 1$, we have

$$x(t) = \big(I^{\beta(1-\alpha)} \mathbb{K}_\alpha\big)(\psi(t)) x_0 + \int_0^t \mathbb{K}_\alpha(\psi(t)-\psi(s)) g(s)\psi'(s) ds,$$

which completes the proof. □

Remark 2.7. (i) From [11] we have

$$\|\mathbb{K}_\alpha(t)\| \le \frac{Mt^{\alpha-1}}{\Gamma(\alpha)}, \ t > 0.$$

(ii)

$$\|\big(I^{\beta(1-\alpha)} \mathbb{K}_\alpha\big)(t)\| \le \frac{Mt^{\alpha+\beta(1-\alpha)-1}}{\Gamma(\alpha+\beta(1-\alpha))}, \quad t > 0.$$

Proof.

$$\|\big(I^{\beta(1-\alpha)} \mathbb{K}_\alpha\big)(t)\| = \Big\|\frac{1}{\Gamma(\beta(1-\alpha))} \int_0^t (t-s)^{\beta(1-\alpha)-1} \mathbb{K}_\alpha(s) ds\Big\|$$

$$\le \frac{M}{\Gamma(\alpha)\Gamma(\beta(1-\alpha))}\int_0^t (t-s)^{\beta(1-\alpha)-1}s^{\alpha-1}ds$$
$$= \frac{Mt^{\alpha+\beta(1-\alpha)-1}}{\Gamma(\alpha+\beta(1-\alpha))}. \qquad \square$$

3. Approximate controllability

In this section, we consider the approximate controllability of the following ψ-Hilfer fractional control system in a Banach space X:

$$\begin{gathered}\left(D^{\alpha,\beta;\psi}x\right)(t) = Ax(t) + Bu(t) + f(t,x(t)), \quad t \in J',\\ I^{(1-\alpha)(1-\beta);\psi}x(0) = x_0,\end{gathered} \tag{3.1}$$

where $D^{\alpha,\beta;\psi}$, $\alpha \in (0,1)$, $\beta \in [0,1]$, is the ψ-Hilfer fractional derivative of order α and type β with the lower limit 0; $b > 0$ is a constant, $J = [0,b]$, $J' = (0,b]$; Closed unbounded operator A ($D(A) \subseteq X$) generates a C_0 semigroup $T(t)$ on $[0,\infty)$; The semilinear function $f : J \times X \to X$ is a given function to be specified later; $x_0 \in X$; The control function u takes its value in $V = L^r(J,U)\left(r > \frac{1}{\alpha}\right)$, and U is a Banach space; $B : V \to L^r(J,X)$ is a linear operator. According to Lemma 2.6, we give the following definition.

Definition 3.1. A function $x \in C_{1-\alpha-\beta(1-\alpha);\psi}(J,X)$ is called a mild solution of problem (3.1) if it satisfies the integral equation

$$x(t) = \left(I^{\beta(1-\alpha)}\mathbb{K}_\alpha\right)(\psi(t))x_0 + \int_0^t \mathbb{K}_\alpha(\psi(t)-\psi(s))[Bu(s) + f(s,x(s))]\psi'(s)ds.$$

Definition 3.2. Let $x(\cdot;u)$ be a mild solution of problem (3.1) corresponding to the control $u(\cdot) \in V$ and the initial value $x_0 \in X$. The set

$$K_b(f) := \{x(b;u) : u(\cdot) \in V\}$$

is called the reachable set of problem (3.1) at terminal time b. If $\overline{K_b(f)} = X$, problem (3.1) is said to be approximately controllable on J.

Before we give the existence and uniqueness lemma of mild solutions of problem (3.1), we pose the following assumptions:

(H1) There exist a function $\mu(\cdot) \in L^r(J,\mathbb{R}_+)$ and a positive constant ℓ_1 such that

$$\|f(t,x)\| \le \mu(t) + \ell_1\psi^{1-\alpha-\beta(1-\alpha)}(t)\|x\|,$$

for a.e. $t \in J$ and each $x \in X$.

(H2) There exists a positive constant ℓ_2 such that

$$\|f(t,x_1) - f(t,x_2)\| \le \ell_2\|x_1 - x_2\|, \quad \forall x_i \in X (i = 1,2).$$

Lemma 3.3. *If* (H1), (H2) *are satisfied, then for any control function $u(\cdot) \in V$, there exists a unique mild solution for the control problem* (3.1) *on $C_{1-\alpha-\beta(1-\alpha);\psi}(J,X)$.*

Proof. Define the operator $\mathbb{T}$ as follows:

$$(\mathbb{T}x)(t) = \left(I^{\beta(1-\alpha)}\mathbb{K}_\alpha\right)(\psi(t))x_0 + \int_0^t \mathbb{K}_\alpha(\psi(t)-\psi(s))[Bu(s) + f(s,x(s))]\psi'(s)ds. \tag{3.2}$$

From our hypotheses, it follows that $\mathbb{T}$ maps $C_{1-\alpha-\beta(1-\alpha);\psi}(J,X)$ into itself. Then, we show that $\mathbb{T}^n$ is a contraction mapping on $C_{1-\alpha-\beta(1-\alpha);\psi}(J,X)$.

As a matter of fact, for each $x_1, x_2 \in C_{1-\alpha-\beta(1-\alpha);\psi}(J,X)$ and $t \in J$, we can obtain

$$\begin{aligned}&\psi^{1-\alpha-\beta(1-\alpha)}(t)\|(\mathbb{T}x_1)(t) - (\mathbb{T}x_2)(t)\|\\ &\le \psi^{1-\alpha-\beta(1-\alpha)}(t)\int_0^t \|\mathbb{K}_\alpha(\psi(t)-\psi(s))[f(s,x_1(s)) - f(s,x_2(s))]\|\psi'(s)ds\\ &\le \frac{M\ell_2}{\Gamma(\alpha)}\psi^{1-\alpha-\beta(1-\alpha)}(t)\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\psi^{\alpha+\beta(1-\alpha)-1}(s)\end{aligned}$$

$$\begin{aligned}&\times \psi^{1-\alpha-\beta(1-\alpha)}(s)\|x_1(s)-x_2(s)\|\psi'(s)ds\\&\le \frac{\Gamma(\alpha+\beta(1-\alpha))M\ell_2\psi^\alpha(t)}{\Gamma(2\alpha+\beta(1-\alpha))}\|x_1-x_2\|_{C_{1-\alpha-\beta(1-\alpha);\psi}}.\end{aligned}\tag{3.3}$$

Using (3.2), (3.3), and arguing by induction on n, we easily see that

$$\begin{aligned}&\psi^{1-\alpha-\beta(1-\alpha)}(t)\|(\mathbb{T}^n x_1)(t)-(\mathbb{T}^n x_2)(t)\|\\&\le \frac{\Gamma(\alpha+\beta(1-\alpha))\left(M\ell_2\psi^\alpha(t)\right)^n}{\Gamma((n+1)\alpha+\beta(1-\alpha))}\|x_1-x_2\|_{C_{1-\alpha-\beta(1-\alpha);\psi}}.\end{aligned}$$

Hence, we can obtain

$$\|(\mathbb{T}^n x_1)(t)-(\mathbb{T}^n x_2)(t)\|_{C_{1-\alpha-\beta(1-\alpha);\psi}} \le \frac{\Gamma(\alpha+\beta(1-\alpha))\left(M\ell_2\psi^\alpha(t)\right)^n}{\Gamma((n+1)\alpha+\beta(1-\alpha))}\|x_1-x_2\|_{C_{1-\alpha-\beta(1-\alpha);\psi}}.$$

Because

$$\lim_{n\to\infty}\frac{\left(M\ell_2\psi^\alpha(t)\right)^n}{\Gamma((n+1)\alpha+\beta(1-\alpha))}=0,$$

there exists a positive integer N such that

$$\frac{\Gamma(\alpha+\beta(1-\alpha))\left(M\ell_2\psi^\alpha(t)\right)^N}{\Gamma((N+1)\alpha+\beta(1-\alpha))}<1.$$

So $\mathbb{T}^N$ is a contraction mapping on $C_{1-\alpha-\beta(1-\alpha);\psi}(J,X)$. Then by a well-known extension of the Banach contraction mapping theorem, $\mathbb{T}$ has a unique fixed point $x(t)$ on $C_{1-\alpha-\beta(1-\alpha);\psi}(J,X)$, which is the unique mild solution of problem (3.1). $\square$

Next, we study the approximate controllability of the evolution ψ-Hilfer fractional differential equations (3.1) in the Banach space X.

We denote the Nemytskil operator associated with the semilinear function f by

$$\mathcal{N}_f:\ C_{1-\alpha-\beta(1-\alpha);\psi}(J,X)\to L^r(J,X),\quad \mathcal{N}_f(x)(t)=f(t,x(t)).$$

The linear bounded operator $\mathbb{H}:\ L^r(J,X)\to X$ is defined as

$$\mathbb{H}g=\int_0^b \mathbb{K}_\alpha(\psi(b)-\psi(s))\psi'(s)g(s)ds,\quad g(\cdot)\in L^r(J,X).$$

By Definition 3.2, we easily know that if for any $x_0\in X$ and $u(\cdot)\in V$, $\overline{K_b(f)}=X$, then problem (3.1) is approximately controllable on J. Therefore, if for any target state $\xi\in X$ and each $\varepsilon>0$, there exists a control function $u_\varepsilon(\cdot)\in V$, such that the mild solution of problem (3.1) satisfies

$$\left\|\xi-\left(I^{\beta(1-\alpha)}\mathbb{K}_\alpha\right)(\psi(b))x_0-\mathbb{H}\mathcal{N}_f(x_\varepsilon)-\mathbb{H}Bu_\varepsilon\right\|<\varepsilon,\tag{3.4}$$

where $x_\varepsilon(t)=x(t;u_\varepsilon),\ t\in(0,b]$, then problem (3.1) is approximately controllable on J.

To analyze the approximate controllability of problem (3.1), we introduce the assumption

(H3) There exists a positive constant ℓ_3 such that

$$\|f(t,x_1)-f(t,x_2)\|\le \ell_3\psi^{1-\alpha-\beta(1-\alpha)}(t)\|x_1-x_2\|,\quad \forall x_i\in X(i=1,2),\ t\in J.$$

The following lemma will be used to establish the approximate controllability of problem (3.1).

Lemma 3.4. *If* (H1) *and* (H3) *are satisfied, then*

$$\begin{gathered}\|x\|_{C_{1-\alpha-\beta(1-\alpha);\psi}}\le \sigma E_\alpha\left(M\ell_1\psi^{1-\beta(1-\alpha)}(b)\right),\\ \|x-y\|_{C_{1-\alpha-\beta(1-\alpha);\psi}}\le \phi E_\alpha\left(M\ell_3\psi^{1-\beta(1-\alpha)}(b)\right)\|Bu-Bv\|_{L^r(J,X)},\end{gathered}$$

where x and y are the unique mild solutions of problem (3.1) *with respect to u and v $(u,v\in V)$, respectively, E_α is the Mittag-Leffler function defined by*

$$E_\alpha(z)=\sum_{i=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)},$$

$$\phi = \frac{M\psi^{1-1/r}(b)}{\Gamma(\alpha)}(\frac{r-1}{r\alpha-1})^{1-1/r},$$
$$\sigma = \frac{M}{\Gamma(\alpha+\beta(1-\alpha))}\|x_0\| + \frac{M\psi^{1-1/r}(b)}{\Gamma(\alpha)}\Big(\frac{r-1}{r\alpha-1}\Big)^{1-1/r}\left(\|Bu\|_{L^r(J,X)} + \|\mu\|_{L^r(J,X)}\right).$$

Proof. Since x is the unique mild solution of (3.1) with respect to $u \in V$ in $C_{1-\alpha-\beta(1-\alpha);\psi}(J,X)$, we have

$$x(t) = \big(I^{\beta(1-\alpha)}\mathbb{K}_\alpha\big)(\psi(t))x_0 + \int_0^t \mathbb{K}_\alpha(\psi(t)-\psi(s))[Bu(s)+f(s,x(s))]\psi'(s)ds.$$

For $t \in J$, we have

$$\begin{aligned}
&\psi^{1-\alpha-\beta(1-\alpha)}(t)\|x(t)\| \\
&\le \psi^{1-\alpha-\beta(1-\alpha)}(t)\|\big(I^{\beta(1-\alpha)}\mathbb{K}_\alpha\big)(\psi(t))x_0\| + \psi^{1-\alpha-\beta(1-\alpha)}(t) \\
&\quad \times \int_0^t \|\mathbb{K}_\alpha(\psi(t)-\psi(s))\psi'(s)Bu(s)\|\,ds \\
&\quad + \psi^{1-\alpha-\beta(1-\alpha)}(t)\int_0^t \|\mathbb{K}_\alpha(\psi(t)-\psi(s))\psi'(s)f(s,x(s))\|\,ds \\
&\le \frac{M}{\Gamma(\alpha+\beta(1-\alpha))}\|x_0\| + \frac{M}{\Gamma(\alpha)}\psi^{1-\alpha-\beta(1-\alpha)}(t)\int_0^t(\psi(t)-\psi(s))^{1-\alpha}\psi'(s) \\
&\quad \times \|Bu(s)\|ds + \frac{M}{\Gamma(\alpha)}\psi^{1-\alpha-\beta(1-\alpha)}(t)\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s) \\
&\quad \times \Big[\mu(s) + \ell_1\psi^{1-\alpha-\beta(1-\alpha)}(s)\|x(s)\|\Big]\,ds \\
&\le \frac{M}{\Gamma(\alpha+\beta(1-\alpha))}\|x_0\| + \frac{M\psi^{1-1/r}(b)}{\Gamma(\alpha)} \\
&\quad \times \Big(\frac{r-1}{r\alpha-1}\Big)^{1-1/r}\left(\|Bu\|_{L^r(J,X)} + \|\mu\|_{L^r(J,X)}\right) \\
&\quad + \frac{M\ell_1\psi^{1-\alpha-\beta(1-\alpha)}(b)}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)\psi^{1-\alpha-\beta(1-\alpha)}(s)\|x(s)\|ds.
\end{aligned}$$

Setting $\mathbb{M}(t) = \psi^{1-\alpha-\beta(1-\alpha)}(t)\|x(t)\|$, in the above inequality we obtain

$$\mathbb{M}(t) \le \sigma + \frac{M\ell_1\psi^{1-\alpha-\beta(1-\alpha)}(b)}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)\mathbb{M}(s)ds.$$

By [27, Theorem 3], we can obtain

$$\mathbb{M}(t) \le \sigma E_\alpha\Big(M\ell_1\psi^{1-\alpha-\beta(1-\alpha)}(b)\psi^\alpha(t)\Big) \le \sigma E_\alpha\left(M\ell_1\psi^{1-\beta(1-\alpha)}(b)\right).$$

So

$$\|x\|_{C_{1-\alpha-\beta(1-\alpha);\psi}} = \sup_{t\in J}\psi^{1-\alpha-\beta(1-\alpha)}(t)\|x(t)\| \le \sigma E_\alpha\left(M\ell_1\psi^{1-\beta(1-\alpha)}(b)\right).$$

A parallel argument yields

$$\|x-y\|_{C_{1-\alpha-\beta(1-\alpha);\psi}} \le \phi E_\alpha\left(M\ell_3\psi^{1-\beta(1-\alpha)}(b)\right)\|Bu-Bv\|_{L^r(J,X)}.$$

This completes the proof. □

Moreover, to discuss the approximate controllability of problem (3.1), we assume that

(H4) For each $\varepsilon > 0$ and $\delta \in L^r(J,X)$, there exists a control function $u \in L^r(J,U)$ such that

$$\|\mathbb{H}\delta - \mathbb{H}Bu\| < \varepsilon, \tag{3.5}$$
$$\|Bu\|_{L^r(J,X)} < C\|\delta\|_{L^r(J,X)}, \tag{3.6}$$

where C is a positive constant independent of $\delta \in L^r(J,X)$, and

$$C\ell_3 E_\alpha\Big(M\ell_3\psi^{1-\beta(1-\alpha)}(b)\Big)\frac{M\psi^{1-1/r}(b)}{\Gamma(\alpha)}\Big(\frac{r-1}{r\alpha-1}\Big)^{1-1/r} < 1. \tag{3.7}$$

Theorem 3.5. *Suppose that the hypotheses of Lemma 3.4 and* (H4) *hold. Then problem* (3.1) *is approximately controllable on* J.

Proof. As the domain $D(A)$ of the operator A is dense in X, we need only prove that $D(A) \subset K_b(f)$, i.e., for any $\xi \in D(A)$ and each $\varepsilon > 0$, there exits a $u_\varepsilon \in V$, such that

$$\|\xi - \big(I^{\beta(1-\alpha)}\mathbb{K}_\alpha\big)(\psi(b))x_0 - \mathbb{H}\mathcal{N}_f(x_\varepsilon) - \mathbb{H}Bu_\varepsilon\| < \varepsilon, \tag{3.8}$$

where $x_\varepsilon(t) = x(t;u_\varepsilon)$, $t \in (0,b]$.

First, for each $x_0 \in X$, due to the differentiability of the C_0-semigroup $T(t)(t>0)$, we know that $(I^{\beta(1-\alpha)}\mathbb{K}_\alpha)(\psi(b))x_0 \in D(A)$. So, for any given $\xi \in D(A)$, there exists a function $\omega \in L^r(J,X)$ such that

$$\mathbb{H}\omega = \xi - \big(I^{\beta(1-\alpha)}\mathbb{K}_\alpha\big)(\psi(b))x_0,$$

for example, we can take

$$\begin{aligned}\omega(t) = {}& \frac{[\Gamma(\alpha)]^2(\psi(b)-\psi(t))^{1-\alpha}}{\psi(b)}\big[W_\alpha(\psi(b)-\psi(t)) + 2\psi(t)\frac{dW_\alpha(\psi(b)-\psi(t))}{dt}\big]\\ &\times\big[\xi - (I^{\beta(1-\alpha)}\mathbb{K}_\alpha)(\psi(b))x_0\big], \quad t \in (0,b),\end{aligned}$$

where $W_\alpha(t) = t^{1-\alpha}\mathbb{K}_\alpha(t)$.

Now, we show that there exists a control function $u_\varepsilon \in V$ such that the inequality (3.8) holds. Indeed, for any given $\varepsilon > 0$ and $u_1 \in V$, by (H4), there exists a $u_2 \in V$, such that

$$\|\xi - \big(I^{\beta(1-\alpha)}\mathbb{K}_\alpha\big)(\psi(b))x_0 - \mathbb{H}\mathcal{N}_f(x_1) - \mathbb{H}Bu_2\| < \frac{\varepsilon}{2^2},$$

where $x_1(t) = x(t;u_1)$, $t \in (0,b]$. Denote $x_2(t) = x(t;u_2)$, $t \in (0,b]$. Using hypothesis (H4) and Lemma 3.4 again, we have that there exists $v_2 \in V$ such that

$$\|\mathbb{H}Bv_2 - [\mathbb{H}\mathcal{N}_f(x_2) - \mathbb{H}\mathcal{N}_f(x_1)]\| < \frac{\varepsilon}{2^3}.$$

and

$$\begin{aligned}\|Bv_2\|_{L^r(J,X)} &\le C\,\|\mathcal{N}_f(x_2)(\cdot) - \mathcal{N}_f(x_1)(\cdot)\|\\ &\le C\ell_3\|x_2 - x_1\|_{C_{1-\alpha-\beta(1-\alpha);\psi}}\\ &\le C\ell_3E_\alpha\big(M\ell_3\psi^{1-\beta(1-\alpha)}(b)\big)\frac{M\psi^{1-1/r}(b)}{\Gamma(\alpha)}\big(\frac{r-1}{r\alpha-1}\big)^{1-1/r}\|Bu_2 - Bu_1\|_{L^r(J,X)}.\end{aligned}$$

Let $u_3(t) = u_2(t) - v_2(t),\ u_3 \in V$. Then

$$\begin{aligned}&\|\xi - \big(I^{\beta(1-\alpha)}\mathbb{K}_\alpha\big)(\psi(b))x_0 - \mathbb{H}\mathcal{N}_f(x_2) - \mathbb{H}Bu_3\|\\ &\le \|\xi - \big(I^{\beta(1-\alpha)}\mathbb{K}_\alpha\big)(\psi(b))x_0 - \mathbb{H}\mathcal{N}_f(x_1) - \mathbb{H}Bu_2\| + \|\mathbb{H}Bv_2 - [\mathbb{H}\mathcal{N}_f(x_2) - \mathbb{H}\mathcal{N}_f(x_1)]\|\\ &\le \Big(\frac{1}{2^2} + \frac{1}{2^3}\Big)\varepsilon.\end{aligned}$$

By induction we can obtain a sequence $\{u_n(\cdot)\} \subset V$ satisfying

$$\|\xi - \big(I^{\beta(1-\alpha)}\mathbb{K}_\alpha\big)(\psi(b))x_0 - \mathbb{H}\mathcal{N}_f(x_n) - \mathbb{H}Bu_{n+1}\| < \Big(\frac{1}{2^2} + \dots + \frac{1}{2^{n+1}}\Big)\varepsilon,$$

where $x_n(t) = x(t;u_n)$, $t \in (0,b]$, and

$$\begin{aligned}&\|Bu_{n+1} - Bu_n\|_{L^r(J,X)}\\ &\le C\ell_3E_\alpha\big(M\ell_3\psi^{1-\beta(1-\alpha)}(b)\big)\frac{M\psi^{1-1/r}(b)}{\Gamma(\alpha)}\Big(\frac{r-1}{r\alpha-1}\Big)^{1-1/r}\|Bu_n - Bu_{n-1}\|_{L^r(J,X)}.\end{aligned}$$

From (3.7), we see that the sequence $\{Bu_n : n \in \mathbb{N}_+\}$ is a Cauchy sequence in the Banach space $L^r(J,X)$. Hence, there exists a function $\tau(\cdot) \in L^r(J,X)$, such that $Bu_n(\cdot) = \tau(\cdot)$ in $L^r(J,X)$, which implies that for each $\varepsilon > 0$, there exists an integer $N > 0$, such that

$$\|\mathbb{H}Bu_{N+1} - \mathbb{H}Bu_N\| \le \frac{\varepsilon}{2}.$$

Therefore,

$$\begin{aligned}
&\|\xi - \big(I^{\beta(1-\alpha)}\mathbb{K}_\alpha\big)(\psi(b))x_0 - \mathbb{H}\mathcal{N}_f(x_N) - \mathbb{H}Bu_N\| \\
&\le \|\xi - \big(I^{\beta(1-\alpha)}\mathbb{K}_\alpha\big)(\psi(b))x_0 - \mathbb{H}\mathcal{N}_f(x_N) - \mathbb{H}Bu_{N+1}\| + \|\mathbb{H}Bu_{N+1} - \mathbb{H}Bu_N\| \\
&\le \Big(\frac{1}{2^2} + \dots + \frac{1}{2^{N+1}}\Big)\varepsilon + \frac{\varepsilon}{2} < \varepsilon,
\end{aligned}$$

which yields the approximate controllability of problem (3.1). □

4. A new Gronwall-type inequality and the dependence of solution on the order and the initial condition

First, we present the following multivariate Gronwall-type inequality with multiple different singular kernels involving exponential factors, which generalizes many existing results.

Theorem 4.1. *Suppose that $m, p \in \mathbb{N}$, $\beta_{ki} > 0$, $a_i \ge 0$, $\psi_i(t)$ is an increasing and positive monotone differentiable function on $(a_i, T_i]$, $\zeta_i(t)$ is a locally integrable function on $(a_i, T_i]$ $(1 \le i \le m, 1 \le k \le p)$, $a(t_1, t_2, \dots, t_m)$ is a nonnegative locally integrable function on $[a_1, T_1) \times [a_2, T_2) \times \dots \times [a_m, T_m)$ (some $T_i \le +\infty$) and $g_k(t_1, t_2, \dots, t_m)(1 \le k \le p)$ is a nonnegative, nondecreasing continuous function defined on $[a_1, T_1) \times [a_2, T_2) \times \dots \times [a_m, T_m)$, $g_k(t_1, t_2, \dots, t_m) \le \mathrm{C}$ (constant), and suppose $u(t_1, t_2, \dots, t_m)$ is nonnegative and locally integrable on $[a_1, T_1) \times [a_2, T_2) \times \dots \times [a_m, T_m)$ with*

$$\begin{aligned}
u(t_1, t_2, \dots, t_m) &\le a(t_1, t_2, \dots, t_m) \\
&\quad + \sum_{k=1}^{p} g_k(t_1, t_2, \dots, t_m) \int_{a_m}^{t_m} \dots \int_{a_2}^{t_2} \int_{a_1}^{t_1} \prod_{i=1}^{m} \mathrm{e}^{\zeta(s_i)-\zeta_i(t_i)} \psi_i'(s_i) \\
&\quad \times (\psi_i(t_i) - \psi_i(s_i))^{\beta_{ki}-1} u(s_1, s_2, \dots, s_m) ds_1 \dots ds_m,
\end{aligned}$$

on $[a_1, T_1) \times [a_2, T_2) \times \dots \times [a_m, T_m)$. Then

$$\begin{aligned}
u(t_1, t_2, \dots, t_m) &\le a(t_1, t_2, \dots, t_m) + \sum_{n=1}^{\infty} \sum_{\substack{0 \le l_1, \dots, l_p \le n \\ l_1 + \dots + l_p = n}} \binom{n}{l_1, \dots, l_p} \prod_{k=1}^{p} (g_k(t_1, t_2, \dots, t_m))^{l_k} \\
&\quad \times \int_{a_m}^{t_m} \dots \int_{a_2}^{t_2} \int_{a_1}^{t_1} \prod_{i=1}^{m} \mathrm{e}^{\zeta_i(s_i)-\zeta_i(t_i)} \psi_i'(s_i) \\
&\quad \times \Big[\frac{\prod_{k=1}^{p} (\Gamma(\beta_{ki}))^{l_k}}{\Gamma(\sum_{k=1}^{p} l_k \beta_{ki})} (\psi_i(t_i) - \psi_i(s_i))^{\sum_{k=1}^{p} l_k \beta_{ki} - 1}\Big] \\
&\quad \times a(s_1, s_2, \dots, s_m) ds_1 \dots ds_m, \quad a_i \le t_i < T_i \ (1 \le i \le m),
\end{aligned} \tag{4.1}$$

where

$$\binom{n}{l_1, \dots, l_p} = \frac{n!}{l_1! \dots l_p!}, \quad l_1 + \dots + l_p = n.$$

Proof. Define an operator P by

$$\begin{aligned}
(Pu)(t_1, t_2, \dots, t_m) &= \sum_{k=1}^{p} g_k(t_1, t_2, \dots, t_m) \int_{a_m}^{t_m} \dots \int_{a_2}^{t_2} \int_{a_1}^{t_1} \prod_{i=1}^{m} \mathrm{e}^{\zeta(s_i)-\zeta_i(t_i)} \psi_i'(s_i) \\
&\quad \times (\psi_i(t_i) - \psi_i(s_i))^{\beta_{ki}-1} u(s_1, s_2, \dots, s_m) ds_1 \dots ds_m,
\end{aligned}$$

for each locally integrable function $u(t_1, t_2, \dots, t_m)$. Then the fact

$$u(t_1, t_2, \dots, t_m) \le f(t_1, t_2, \dots, t_m) + (Pu)(t_1, t_2, \dots, t_m)$$

implies

$$u(t_1, t_2, \dots, t_m) \le \sum_{i=0}^{n} (P^i f)(t_1, t_2, \dots, t_m) + (P^{n+1} u)(t_1, t_2, \dots, t_m), \tag{4.2}$$

for all $n \in \mathbb{N}$.

Next, we want to prove the following (4.3) by induction:

$$\begin{aligned}(P^n u)(t_1, t_2, \dots, t_m) \le &\sum_{\substack{0 \le l_1, \dots, l_p \le n \\ l_1 + \dots + l_p = n}} \binom{n}{l_1, \dots, l_p} \prod_{k=1}^{p} (g_k(t_1, t_2, \dots, t_m))^{l_k} \\ &\times \int_{a_m}^{t_m} \cdots \int_{a_2}^{t_2} \int_{a_1}^{t_1} \prod_{i=1}^{m} \mathrm{e}^{\zeta_i(s_i) - \zeta_i(t_i)} \psi_i'(s_i) \Big[\frac{\prod_{k=1}^{p} (\Gamma(\beta_{ki}))^{l_k}}{\Gamma(\sum_{k=1}^{p} l_k \beta_{ki})} \Big] \\ &\times (\psi_i(t_i) - \psi_i(s_i))^{\sum_{k=1}^{p} l_k \beta_{ki} - 1} u(s_1, s_2, \dots, s_m) ds_1 \dots ds_m,\end{aligned} \tag{4.3}$$

for all $n \in \mathbb{N}$. Clearly, (4.3) is true for $n = 1$. Suppose that (4.3) holds for $n = l$. We want to prove that (4.3) also holds for $n = l + 1$. In fact, we have

$$\begin{aligned}(P^{l+1} u)(t) = &\sum_{j=1}^{p} g_j(t_1, t_2, \dots, t_m) \int_{a_m}^{t_m} \cdots \int_{a_2}^{t_2} \int_{a_1}^{t_1} \prod_{i=1}^{m} \mathrm{e}^{\zeta_i(s_i) - \zeta_i(t_i)} \psi_i'(s_i) \\ &\times (\psi_i(t_i) - \psi_i(s_i))^{\beta_{ji} - 1} (P^l u)(s_1, s_2, \dots, s_m) ds_1 \dots ds_m \\ \le &\sum_{j=1}^{p} \sum_{\substack{0 \le l_1, \dots, l_p \le l \\ l_1 + \dots + l_p = l}} \binom{l}{l_1, \dots, l_p} g_j(t_1, t_2, \dots, t_m) \int_{a_m}^{t_m} \cdots \int_{a_2}^{t_2} \int_{a_1}^{t_1} \prod_{i=1}^{m} \mathrm{e}^{\zeta_i(s_i) - \zeta_i(t_i)} \\ &\times \psi_i'(s_i) (\psi_i(t_i) - \psi_i(s_i))^{\beta_{ji} - 1} \prod_{k=1}^{p} (g_k(s_1, s_2, \dots, s_m))^{l_k} \int_{a_m}^{s_m} \cdots \int_{a_2}^{s_2} \int_{a_1}^{s_1} \\ &\times \prod_{i=1}^{m} \mathrm{e}^{\zeta_i(\tau_i) - \zeta_i(s_i)} \psi_i'(\tau_i) \Big[\frac{\prod_{k=1}^{p} (\Gamma(\beta_{ki}))^{l_k}}{\Gamma(\sum_{k=1}^{p} l_k \beta_{ki})} (\psi_i(s_i) - \psi_i(\tau_i))^{\sum_{k=1}^{p} l_k \beta_{ki} - 1} \Big] \\ &\times u(\tau_1, \tau_2, \dots, \tau_m) d\tau_1 \dots d\tau_m ds_1 \dots ds_m.\end{aligned}$$

By exchanging the integration order and noting that the functions g_i are nondecreasing, we obtain

$$\begin{aligned}&\int_{a_1}^{t_1} \mathrm{e}^{\zeta_1(s_1) - \zeta_1(t_1)} \psi_1'(s_1) (\psi_1(t_1) - \psi_1(s_1))^{\beta_{j1} - 1} \prod_{k=1}^{p} (g_k(s_1, s_2, \dots, s_m))^{l_k} \\ &\times \int_{a_1}^{s_1} \mathrm{e}^{\zeta_1(\tau_1) - \zeta_1(s_1)} \psi_1'(\tau_1) \Big[\frac{\prod_{k=1}^{p} (\Gamma(\beta_{k1}))^{l_k}}{\Gamma(\sum_{k=1}^{p} l_k \beta_{k1})} (\psi_1(s_1) - \psi_1(\tau_1))^{\sum_{k=1}^{p} l_k \beta_{k1} - 1} \Big] \\ &\times u(\tau_1, \tau_2, \dots, \tau_m) d\tau_1 ds_1 \\ &\le \prod_{k=1}^{p} (g_k(t_1, s_2, \dots, s_m))^{l_k} \frac{[\prod_{k \ne j} (\Gamma(\beta_{k1}))^{l_k}] (\Gamma(\beta_{j1}))^{l_j + 1}}{\Gamma(\sum_{k \ne j} l_k \beta_{k1} + (l_j + 1) \beta_{j1})} \\ &\times \int_{a_1}^{t_1} \mathrm{e}^{\zeta_1(s_1) - \zeta_1(t_1)} \psi_1'(s_1) (\psi_1(t_1) - \psi_1(s_1))^{\beta_{j1} + \sum_{k=1}^{p} l_k \beta_{k1} - 1} u(s_1, \tau_2, \dots, \tau_m) ds_1,\end{aligned}$$

where we use that

$$\begin{aligned}&\int_{\tau_1}^{t_1} (\psi_1(t_1) - \psi_1(s_1))^{\beta_{j1} - 1} (\psi_1(s_1) - \psi_1(\tau_1))^{\sum_{k=1}^{p} l_k \beta_{k1} - 1} \psi_1'(s) ds \\ &= (\psi_1(t_1) - \psi_1(\tau_1))^{\beta_{j1} + \sum_{k=1}^{p} l_k \beta_{k1} - 1} B(\beta_{j1}, \sum_{k=1}^{p} l_k \beta_{k1})\end{aligned}$$

and

$$B(\beta_{j1},\sum_{k=1}^{p} l_k\beta_{k1}) = \frac{\Gamma(\beta_{j1})\Gamma(\sum_{k=1}^{p} l_k\beta_{k1})}{\Gamma(\sum_{k\neq j} l_k\beta_{k1} + (l_j+1)\beta_{j1})}.$$

Repeating this process for m times, we obtain

$$\begin{aligned}
(P^{l+1}u)(t) \le \sum_{j=1}^{p} &\sum_{\substack{0\le l_1,\dots,l_p\le l\\ l_1+\dots+l_p=l}} \binom{l}{l_1,\dots,l_p}\Big[\prod_{k\neq j}(g_k(t_1,s_2,\dots,s_m))^{l_k}\Big](g_j(t_1,s_2,\dots,s_m))^{l_j+1}\\
&\times \int_{a_m}^{t_m}\dots\int_{a_2}^{t_2}\int_{a_1}^{t_1}\prod_{i=1}^{m}\frac{[\prod_{k\neq j}(\Gamma(\beta_{ki}))^{l_k}](\Gamma(\beta_{ji}))^{l_j+1}}{\Gamma(\sum_{k\neq j} l_k\beta_{ki}+(l_j+1)\beta_{ji})}\mathrm{e}^{\zeta_i(s_i)-\zeta_i(t_i)}\psi_i'(s_i)\\
&\times(\psi_i(t_i)-\psi_i(s_i))^{\sum_{k\neq j} l_k\beta_{ki}+(l_j+1)\beta_{ji}-1}u(s_1,s_2,\dots,s_m)ds_1\dots ds_m.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(P^{l+1}u)(t) &= \sum_{\substack{0\le l_1,\dots,l_p\le l+1\\ l_1+\dots+l_p=l+1}}\sum_{j=1}^{p}\binom{l}{l_1,\dots,l_{j-1},l_j-1,l_{j+1},\dots,l_p}\prod_{k=1}^{p}(g_k(t_1,s_2,\dots,s_m))^{l_k}\\
&\quad\times\int_{a_m}^{t_m}\dots\int_{a_2}^{t_2}\int_{a_1}^{t_1}\prod_{i=1}^{m}\frac{\prod_{k=1}^{p}(\Gamma(\beta_{ki}))^{l_k}}{\Gamma(\sum_{k=1}^{p} l_k\beta_{ki})}\mathrm{e}^{\zeta_i(s_i)-\zeta_i(t_i)}\psi_i'(s_i)\\
&\quad\times(\psi_i(t_i)-\psi_i(s_i))^{\sum_{k=1}^{p} l_k\beta_{ki}-1}u(s_1,s_2,\dots,s_m)ds_1\dots ds_m\\
&= \sum_{\substack{0\le l_1,\dots,l_p\le l+1\\ l_1+\dots+l_p=l+1}}\binom{l+1}{l_1,\dots,l_p}\prod_{k=1}^{p}(g_k(t_1,s_2,\dots,s_m))^{l_k}\\
&\quad\times\int_{a_m}^{t_m}\dots\int_{a_2}^{t_2}\int_{a_1}^{t_1}\prod_{i=1}^{m}\frac{\prod_{k=1}^{p}(\Gamma(\beta_{ki}))^{l_k}}{\Gamma(\sum_{k=1}^{p} l_k\beta_{ki})}\mathrm{e}^{\zeta_i(s_i)-\zeta_i(t_i)}\psi_i'(s_i)\\
&\quad\times(\psi_i(t_i)-\psi_i(s_i))^{\sum_{k=1}^{p} l_k\beta_{ki}-1}u(s_1,s_2,\dots,s_m)ds_1\dots ds_m,
\end{aligned}$$

where we have used

$$\sum_{j=1}^{m}\binom{n}{l_1,\dots,l_{j-1},l_j-1,l_{j+1},\dots,l_m} = \binom{n+1}{l_1,\dots,l_m},$$

in which $\sum_{j=1}^{m} l_j = n+1$, and we take

$$\binom{n}{l_1,\dots,l_m} = 0$$

when $l_i=-1$ for some i. Thus we show that (4.3) also holds for $n=l+1$. Therefore, (4.3) is true.

By (4.3) we can see that $(P^n u)(t)\to 0$ as $n\to\infty$. From (4.2) and (4.3), we see that (4.1) holds. The proof is complete □

Remark 4.2. Theorem 4.1 extends essentially many existing results. For example, taking $\zeta_i\equiv 0$ $(1\le i\le m)$ in Theorem 4.1, we obtain the following inequality in [27], while taking $\zeta_i\equiv 0$ $(1\le i\le m)$ and $p=1$ in Theorem 4.1, we obtain the inequality in [26]; taking $m=p=1$, $a=0$, $\zeta\equiv 0$, and $\psi(t)=t$ in Theorem 4.1, we obtain the inequality in [43]; and taking $m=p=1$, $g(t)\equiv b$, $a=0$, $\zeta\equiv 0$, and $\psi(t)=t$ in Theorem 4.1, we obtain the inequality in [15].

Corollary 4.3. *Suppose that $m,p\in\mathbb{N}$, $\beta_{ki}>0$, $a_i\ge 0$, $\psi_i(t)$ is an increasing and positive monotone differentiable function on $(a_i,T_i]$ $(1\le i\le m, 1\le k\le p)$, $a(t_1,t_2,\dots,t_m)$ is a nonnegative locally integrable function on $[a_1,T_1)\times[a_2,T_2)\times\dots\times[a_m,T_m)$ (some $T_i\le+\infty$) and $g_k(t_1,t_2,\dots,t_m)(1\le k\le p)$ is a nonnegative, nondecreasing continuous function defined on*

$[a_1,T_1)\times[a_2,T_2)\times\cdots\times[a_m,T_m)$, $g_k(t_1,t_2,\dots,t_m)\le \mathrm{C}$ (constant), and suppose $u(t_1,t_2,\dots,t_m)$ is nonnegative and locally integrable on $[a_1,T_1)\times[a_2,T_2)\times\cdots\times[a_m,T_m)$ with

$$\begin{aligned}
&u(t_1,t_2,\dots,t_m)\\
&\le a(t_1,t_2,\dots,t_m)+\sum_{k=1}^{p}g_k(t_1,t_2,\dots,t_m)\int_{a_m}^{t_m}\cdots\int_{a_2}^{t_2}\int_{a_1}^{t_1}\prod_{i=1}^{m}\psi_i'(s_i)(\psi_i(t_i)-\psi_i(s_i))^{\beta_{ki}-1}\\
&\quad\times u(s_1,s_2,\dots,s_m)ds_1\dots ds_m,
\end{aligned}$$

on $[a_1,T_1)\times[a_2,T_2)\times\cdots\times[a_m,T_m)$. Then

$$\begin{aligned}
u(t_1,t_2,\dots,t_m)&\le a(t_1,t_2,\dots,t_m)+\sum_{n=1}^{\infty}\sum_{\substack{0\le l_1,\dots,l_p\le n\\ l_1+\dots+l_p=n}}\binom{n}{l_1,\dots,l_p}\prod_{k=1}^{p}(g_k(t_1,t_2,\dots,t_m))^{l_k}\\
&\quad\times\int_{a_m}^{t_m}\cdots\int_{a_2}^{t_2}\int_{a_1}^{t_1}\prod_{i=1}^{m}\psi_i'(s_i)\Big[\frac{\prod_{k=1}^{p}(\Gamma(\beta_{ki}))^{l_k}}{\Gamma(\sum_{k=1}^{p}l_k\beta_{ki})}(\psi_i(t_i)-\psi_i(s_i))^{\sum_{k=1}^{p}l_k\beta_{ki}-1}\Big]\\
&\quad\times a(s_1,s_2,\dots,s_m)ds_1\dots ds_m,\ a_i\le t_i<T_i\ (1\le i\le m),
\end{aligned}$$

where

$$\binom{n}{l_1,\dots,l_p}=\frac{n!}{l_1!\dots l_p!},\quad l_1+\dots+l_p=n.$$

Using Theorem 4.1, we obtain the following result of the dependence of the solution on the order and the initial condition for the fractional Cauchy problem.

Theorem 4.4. *Let $f:[0,b]\times X\times X\to X$ be a bounded function and satisfy the Lipschitz-type condition with respect to the second and third variable, i.e., for any $t\in[0,b]$, $x_i,\ y_i\in X$ $(i=1,2)$, there exist a $L>0$ and an integrable function $\zeta(t)\le 0$, a.e. $t\in[0,b]$ such that*

$$\|f(t,x_1,x_2)-f(t,y_1,y_2)|\le L\mathrm{e}^{\zeta(t)}\left(\|x_1-y_1\|+\|x_2-y_2\|\right).$$

Assume that $x(t)$ and $y(t)$ are the solutions of the following initial-value integro-differential equations (4.4) and (4.5), respectively:

$$\begin{gathered}
\left(D^{\alpha,\beta;\psi}x\right)(t)=Ax(t)+f\Big(t,x(t),\int_0^t\rho(t,s)x(s)ds\Big),\quad 0<\alpha<1,0\le\beta\le1,\ t\in(0,b],\\
I^{1-\gamma;\psi}x(0)=x_0,\quad \alpha\le\gamma=\alpha+\beta-\alpha\beta<1,
\end{gathered}\tag{4.4}$$

and

$$\begin{gathered}
\Big(D^{\alpha',\beta';\psi}y\Big)(t)=Ay(t)+f\Big(t,y(t),\int_0^t\rho(t,s)y(s)ds\Big),\quad 0<\alpha'<1,\ 0\le\beta'\le1,\ t\in(0,b],\\
I^{1-\gamma';\psi}y(0)=y_0,\quad \alpha'\le\gamma'=\alpha'+\beta'-\alpha'\beta'<1,
\end{gathered}\tag{4.5}$$

where $b>0$, and

$$\rho(t,s)=\mathrm{e}^{\zeta(s)-\zeta(t)}(\psi(t)-\psi(s))^{r-1}\psi'(s)\ (r\in(0,1]).$$

Then, for $t\in(0,b]$,

$$\begin{aligned}
\|x(t)-y(t)\|\le A(t)&+\sum_{n=1}^{\infty}(LM)^n\sum_{k=0}^{\infty}C_n^k\frac{(\Gamma(r))^{n-k}}{(\Gamma(\alpha'))^k(\Gamma(\alpha'+r))^{n-k}}\\
&\times\int_0^t\mathrm{e}^{\zeta(s)-\zeta(t)}(\psi(t)-\psi(s))^{n\alpha'+(n-k)r-1}A(s)\psi'(s)ds,
\end{aligned}$$

where

$$\begin{aligned}
A(t)=\|&\big(I^{\beta(1-\alpha)}\mathbb{K}_\alpha\big)(\psi(t))x_0-\big(I^{\beta(1-\alpha)}\mathbb{K}_\alpha\big)(\psi(t))y_0\|\\
&+\int_0^t\|\mathbb{K}_\alpha(\psi(t)-\psi(s))-\mathbb{K}_{\alpha'}(\psi(t)-\psi(s))\|\psi'(s)ds\cdot\|f\|,
\end{aligned}$$

and $\|f\|=\sup_{(t,x_1,x_2)\in(0,b]\times X\times X}\|f(t,x_1,x_2)\|$.

Proof. By Lemma 2.6, the solution of (4.4) satisfies

$$x(t)=\big(I^{\beta(1-\alpha)}\mathbb{K}_\alpha\big)(\psi(t))x_0+\int_0^t\mathbb{K}_\alpha(\psi(t)-\psi(s))f\Big(s,x(s),\int_0^s\rho(s,\tau)x(\tau)d\tau\Big)\psi'(s)ds.$$

Also byLemma 2.6, the solution of (4.5) satisfies

$$y(t)=\Big(I^{\beta'(1-\alpha')}\mathbb{K}_{\alpha'}\Big)(\psi(t))y_0+\int_0^t\mathbb{K}_{\alpha'}(\psi(t)-\psi(s))f\Big(s,y(s),\int_0^s\rho(s,\tau)y(\tau)d\tau\Big)\psi'(s)ds.$$

Then we have

$$\begin{aligned}
&\|x(t)-y(t)\|\\
&\le\|\big(I^{\beta(1-\alpha)}\mathbb{K}_\alpha\big)(\psi(t))x_0-\big(I^{\beta'(1-\alpha')}\mathbb{K}_{\alpha'}\big)(\psi(t))y_0\|\\
&\quad+\int_0^t\|\mathbb{K}_\alpha(\psi(t)-\psi(s))-\mathbb{K}_{\alpha'}(\psi(t)-\psi(s))\|\Big\|f\Big(s,x(s),\int_0^s\rho(s,\tau)x(\tau)d\tau\Big)\Big\|\psi'(s)ds\\
&\quad+\int_0^t\|\mathbb{K}_{\alpha'}(\psi(t)-\psi(s))\|\Big\|f\Big(s,x(s),\int_0^s\rho(s,\tau)x(\tau)d\tau\Big)\\
&\quad-f\Big(s,y(s),\int_0^s\rho(s,\tau)y(\tau)d\tau\Big)\psi'(s)ds\Big\|\\
&\le A(t)+\frac{LM}{\Gamma(\alpha')}\int_0^t e^{\zeta(s)-\zeta(t)}(\psi(t)-\psi(s))^{\alpha'-1}\|x(s)-y(s)\|\psi'(s)ds\\
&\quad+\frac{LM}{\Gamma(\alpha')}\int_0^t e^{\zeta(s)-\zeta(t)}(\psi(t)-\psi(s))^{\alpha'-1}\\
&\quad\times\Big[\int_0^s e^{\zeta(\tau)-\zeta(s)}(\psi(s)-\psi(\tau))^{r-1}\|x(\tau)-y(\tau)\|\psi'(\tau)d\tau\Big]\psi'(s)ds\\
&\le A(t)+\frac{LM}{\Gamma(\alpha')}\int_0^t e^{\zeta(s)-\zeta(t)}(\psi(t)-\psi(s))^{\alpha'-1}\|x(s)-y(s)\|\psi'(s)ds\\
&\quad+\frac{LM\Gamma(r)}{\Gamma(\alpha'+r)}\int_0^t e^{\zeta(s)-\zeta(t)}(\psi(t)-\psi(s))^{\alpha'+r-1}\|x(s)-y(s)\|\psi'(s)ds,
\end{aligned}$$

where

$$\begin{aligned}
A(t)&=\|\big(I^{\beta(1-\alpha)}\mathbb{K}_\alpha\big)(\psi(t))x_0-\big(I^{\beta'(1-\alpha')}\mathbb{K}_{\alpha'}\big)(\psi(t))y_0\|\\
&\quad+\int_0^t\|\mathbb{K}_\alpha(\psi(t)-\psi(s))-\mathbb{K}_{\alpha'}(\psi(t)-\psi(s))\|\psi'(s)ds\cdot\|f\|.
\end{aligned}$$

An application of Theorem 4.1 (with $m=1$ and $p=2$) yields

$$\begin{aligned}
\|x(t)-y(t)\|\le A(t)&+\sum_{n=1}^{\infty}(LM)^n\sum_{k=0}^{\infty}C_n^k\frac{(\Gamma(r))^{n-k}}{(\Gamma(\alpha'))^k(\Gamma(\alpha'+r))^{n-k}}\\
&\times\int_0^t e^{\zeta(s)-\zeta(t)}(\psi(t)-\psi(s))^{n\alpha'+(n-k)r-1}A(s)\psi'(s)ds,
\end{aligned}$$

where $C_n^m=\frac{n!}{m!(n-m)!}$ is a binomial coefficient. Thus, the proof is complete. □

Remark 4.5. In Theorem 4.4, we do not require the function $\zeta(t)$ to be nondecreasing on $[0,b]$, so using Corollary 4.3 without exponential factors cannot lead to the conclusion of Theorem 4.4.

Corollary 4.6. *Under the hypotheses of Theorem 4.4, if $\alpha=\alpha'$ and $\beta=\beta'$, then*

$$\begin{aligned}
\|x(t)-y(t)\|\le\frac{1}{\Gamma(\alpha+\beta(1-\alpha))}\Big[&M(\psi(t))^{\alpha+\beta(1-\alpha)-1}+\sum_{n=1}^{\infty}L^nM^{n+1}\sum_{k=0}^{\infty}C_n^k\\
&\times\frac{(\Gamma(r))^{n-k}}{(\Gamma(\alpha))^k(\Gamma(\alpha+r))^{n-k}}\int_0^t e^{\zeta(s)-\zeta(t)}(\psi(t)-\psi(s))^{n\alpha+(n-k)r-1}
\end{aligned}$$

$$\times (\psi(s))^{\alpha+\beta(1-\alpha)-1}\psi'(s)ds\Big]\|x_0-y_0\|,$$

for $t\in(0,b]$.

Proof. If $\alpha=\alpha'$ and $\beta=\beta'$, then

$$A(t)=\|\big(I^{\beta(1-\alpha)}\mathbb{K}_\alpha\big)(\psi(t))(x_0-y_0)\|\le\frac{M(\psi(t))^{\alpha+\beta(1-\alpha)-1}}{\Gamma(\alpha+\beta(1-\alpha))}\|x_0-y_0\|.$$

By Theorem 4.4, we have

$$\begin{aligned}\|x(t)-y(t)\|&\le A(t)+\sum_{n=1}^{\infty}(LM)^n\sum_{k=0}^{\infty}C_n^k\frac{(\Gamma(r))^{n-k}}{(\Gamma(\alpha))^k(\Gamma(\alpha+r))^{n-k}}\\&\quad\times\int_0^t e^{\zeta(s)-\zeta(t)}(\psi(t)-\psi(s))^{n\alpha+(n-k)r-1}A(s)\psi'(s)ds\\&=\frac{1}{\Gamma(\alpha+\beta(1-\alpha))}\Big[M(\psi(t))^{\alpha+\beta(1-\alpha)-1}+\sum_{n=1}^{\infty}L^nM^{n+1}\sum_{k=0}^{\infty}C_n^k\\&\quad\times\frac{(\Gamma(r))^{n-k}}{(\Gamma(\alpha))^k(\Gamma(\alpha+r))^{n-k}}\int_0^t e^{\zeta(s)-\zeta(t)}(\psi(t)-\psi(s))^{n\alpha+(n-k)r-1}\\&\quad\times(\psi(s))^{\alpha+\beta(1-\alpha)-1}\psi'(s)ds\Big]\|x_0-y_0\|,\end{aligned}$$

for $t\in(0,b]$. The proof is complete. □

5. An example

In this section, we give an example to show the applicability of the results obtained in previous sections. Let $X=\{u(t):u(t)\in L^2[0,\pi],\ u(t)\text{ is a real function}\}$ and $U=X$. We define the inner product and norm on X respectively, for $u_1,\ u_2\in X$, by

$$\langle u_1,u_2\rangle=\int_0^\pi u_1(t)u_2(t)dt,\quad \|u_1\|_X=\Big(\int_0^\pi u_1^2(t)dt\Big)^{1/2}.$$

We define the operator $A:D(A)\subset X\to X$ by

$$D(A):=\{v\in X:v''\in X,\ v(0)=v(\pi)=0\},\ Au=\frac{\partial^2u}{\partial x^2}.$$

From [34] we know that $-A$ has eigenvalues of the form n^2 $(n\in\mathbb{N}_+)$, and the corresponding normalized eigenfunctions are given by $e_n=\sqrt{\frac{2}{\pi}}\sin(nx)$ $(n\in\mathbb{N}_+)$. Moreover, A generates a compact analytic semigroup $\{T(t)\}_{t\ge0}$ on X, and

$$T(t)u=\sum_{n=1}^{\infty}e^{-n^2t}\langle u,e_n\rangle e_n.$$

We can verify that $\|T(t)\|\le e^{-t}$ for all $t\ge0$, and take $M=1$. Furthermore, by [37] we know that $\{T(t)\}_{t\ge0}$ is continuous in the uniform operator topology for $t>0$.

For each $u(\cdot)\in V=L^2(J,U)$, we have

$$u(t)=\sum_{n=1}^{\infty}u_n(t)e_n,\quad u_n(t)=<u(t),e_n>.$$

The operator B is defined by

$$Bu(t)=\sum_{n=1}^{\infty}v_n(t)e_n,$$

where

$$v_n(t)=\begin{cases}0, & t\in[0,1-\frac{1}{n^3}],\\ u_n(t), & t\in(1-\frac{1}{n^3},1],\end{cases}$$

where $n\in\mathbb{N}_+$. Then we can see that $\|Bu(\cdot)\|\le\|u(\cdot)\|$, which means that $B\in L\big(V,L^2(J,X)\big)$.

We consider the following fractional differential control problem involving ψ-Hilfer fractional derivative:

$$\begin{gathered}(D^{\alpha,\beta;\psi}x)(t) = Ax(t) + f(t,x(t)) + Bu(t), \quad t \in J' = (0,1], \\ I^{(1-\alpha)(1-\beta);\psi}x(0) = x_0,\end{gathered} \tag{5.1}$$

where $\alpha = \frac{4}{5}$, $\beta = \frac{3}{4}$, $x_0 \in X$, $\psi(t) = t^3$, and $f(t,x) = Lt^{3/20}\sin(x)$, $t \in (0,1]$.

Next, we verify that hypothesis (H4) holds. To do this, for each $h(\cdot) \in L^2(J,X)$, let

$$l = \int_0^1 \mathbb{K}_\alpha(\psi(1) - \psi(s))\psi'(s)h(s)ds = \sum_{n=1}^{\infty} l_n e_n,$$

where $l_n = \langle l, e_n\rangle$. We can take

$$\begin{gathered}\overline{u}_n(t) = \frac{3n^3}{1 - e^{-\psi(3)}} l_n e^{-n^3(1-\psi(t))}, \quad 1 - \frac{1}{n^3} \le t \le 1, \\ l_n = \int_{1-\frac{1}{n^3}}^1 \int_0^\infty (1-\psi(t))^{-\frac{1}{5}} \sigma \xi_{\frac{4}{5}} e^{-n^3\sigma(1-\psi(t))^{\frac{4}{5}}} \overline{u}(t)\psi'(t)d\sigma dt.\end{gathered}$$

And define

$$u(t) := \sum_{n=1}^{\infty} \tilde{u}_n(t)e_n,$$

where

$$\tilde{u}_n(t) = \begin{cases} 0, & t \in [0, 1 - \frac{1}{n^3}], \\ \overline{u}_n(t), & t \in (1 - \frac{1}{n^3}, 1], \end{cases}$$

for $n \in \mathbb{N}_+$. So, for each given function $h(\cdot) \in L^2(J,X)$, there exists $u(\cdot) \in V$ such that

$$\int_0^1 \mathbb{K}_\alpha(1-\psi(s))\psi'(s)Bu(s)ds = \int_0^1 \mathbb{K}_\alpha(1-\psi(s))\psi'(s)h(s)ds,$$

which implies that condition (3.5) of (H4) holds. Moreover, we can obtain

$$\begin{aligned}\|Bu(\cdot)\|^2 &= \sum_{n=1}^{\infty} \int_{1-\frac{1}{n^3}}^1 |\overline{u}(t)|^2 dt \\ &= \left(1 - e^{-\psi(3)}\right)^{-1} \sum_{n=1}^{\infty} 3n^3 l_n^2 \\ &= \frac{4}{3}\left(1 - e^{-\psi(3)}\right)^{-1} \sum_{n=1}^{\infty} \left(1 - e^{-\psi(3)n^3}\right) \int_0^1 |h_n(t)|^2 dt \\ &\le \frac{4}{3}\left(1 - e^{-\psi(3)}\right)^{-1} |h(\cdot)|^2.\end{aligned}$$

Hence, condition (3.6) of (H4) is also satisfied, and if

$$\frac{4\sqrt{15}}{9(1 - e^{-\psi(3)})\Gamma(\frac{4}{5})} \ell_3 E_{\frac{4}{5}}(\ell_3) < 1,$$

then problem (5.1) is approximately controllable on J.

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