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LONG-TIME DYNAMICS AND UPPER-SEMICONTINUITY OF ATTRACTORS FOR A POROUS-ELASTIC SYSTEM WITH NONLINEAR LOCALIZED DAMPING

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Abstract. In this article we consider a one-dimensional porous-elastic system with nonlinear localized damping acting in an arbitrarily small region of the interval under consideration. We prove the existence of a smooth global attractor with finite fractal dimension and the existence of exponential attractors via quasi-stability theory recently proposed by Chueshov and Lasiecka. We also prove the continuity of the attractors with respect to two parameters in a residual dense set. Finally, we prove that the family of global attractors is upper-semicontinuous with respect to small perturbations of external forces. These aspects were not previously considered for porous-elastic system with localized damping.

1. Introduction

The study of mathematical models of vibrating flexible structures have been considerably stimulated in recent years by an increasing number of questions of practical concern. Research on stabilization of distributed parameter systems has largely focused on the stabilization of dynamic models of individual structural members such as strings, membranes, and beams. See [15] and references therein.

On the other hand, localized frictional damping has been studied by several authors in one or more space dimension, (see [3, 5, 13, 21, 33, 34]). The main result of the above articles is that localized frictional damping produces exponential decay in time of the solution. A more general result occurs in one-dimensional space where the solution always decays exponentially to zero for any localized frictional damping active over an open subset of the domain.

Motivated by the above, this article is devoted to the study of the porous-elastic system with nonlinear arbitrary localized elastic damping and nonlinear arbitrary

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localized porous dissipation given by

$$
\begin{gathered}
\rho u_{tt} - \mu u_{xx} - b\phi_x + a_1(x)g_1(u_t) + f_1(u,\phi) = \epsilon_1 h_1 \quad \text{in } (0,L)\times(0,\infty),\\
J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + a_2(x)g_2(\phi_t) + f_2(u,\phi) = \epsilon_2 h_2 \quad \text{in } (0,L)\times(0,\infty),\\
u(0,t) = u(L,t) = \phi(0,t) = \phi(L,t) = 0, \quad t>0,\\
(u(x,0),\phi(x,0)) = (u_0(x),\phi_0(x)), \quad \text{in } (0,L),\\
(u_t(x,0),\phi_t(x,0)) = (u_1(x),\phi_1(x)), \quad \text{in } (0,L),
\end{gathered} \tag{1.1}
$$

where the variables u and ϕ represent the displacement of a solid elastic material and the volume fraction, respectively. Here ρ, μ, J, δ, b and ξ are the constitutive coefficients whose physical meaning is well known. The constitutive coefficients, in one-dimensional case, satisfy

$$
\xi>0,\ \delta>0,\ \mu>0,\ \rho>0,\ J>0,\ \mu\xi\ge b^2. \tag{1.2}
$$

The functions $g_1(u_t)$ and $g_2(\phi_t)$ represent the nonlinear damping terms, ϵ_1 and ϵ_2 are positive constants small enough, f_1 and f_2 are nonlinear source terms, a_1 and a_2 are smooth, nonnegative functions responsible by the localized damping effect, h_1 and h_2 represent external forces.

Quintanilla [25] studied the system (1.1) when $\epsilon_1=\epsilon_2=0$, $g_1=0$ and $g_2(s)=\tau s$ with $a_2(x)=1$. He used the Hurtwitz theorem to prove that the system lacks exponential decay when $\frac{\rho}{\mu}\neq\frac{J}{\delta}$. In Magaña and Quintanilla [20] considered the system:

$$
\begin{gathered}
\rho u_{tt} - \mu u_{xx} - b\phi_x - \gamma u_{xxt} = 0 \quad \text{in } (0,L)\times(0,\infty),\\
J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \tau\phi_t = 0 \quad \text{in } (0,L)\times(0,\infty),\\
u(0,t) = u(L,t) = \phi_x(0,t) = \phi_x(L,t) = 0, \quad t>0,\\
(u(x,0),\phi(x,0)) = (u_0(x),\phi_0(x)), \quad \text{in } (0,L)\\
(u_t(x,0),\phi_t(x,0)) = (u_1(x),\phi_1(x)), \quad \text{in } (0,L).
\end{gathered} \tag{1.3}
$$

They proved that the system (1.3) is exponentially stable using the semigroup arguments due to Liu and Zheng [16]. Also, they proved that when $\tau=0$ the system is not exponentially stable. Muñoz Rivera and Quintanilla [22] proved that when $\tau=0$ the energy is controlled by a rate decay of the type $\frac{1}{t}$. Moreover, using a result on [24], they improved the polynomial rate of decay by taking more regular initial data. Santos et al. [28] proved that system (1.3) with $\tau=0$ lacks exponential decay independent of any relation between the coefficients of the wave propagation, and it decays as $\frac{1}{\sqrt{t}}$. In addition they also proved that this rate is optimal. On the other hand, Santos and Almeida Júnior [27] studied the porous-elastic system

$$
\begin{gathered}
\rho u_{tt} - \mu u_{xx} - b\phi_x + \gamma(x)(u_t+\phi_t) = 0 \quad \text{in } \Omega\times(0,\infty),\\
J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \gamma(x)(u_t+\phi_t) = 0 \quad \text{in } \Omega\times(0,\infty),\\
(u(x,0),\phi(x,0)) = (u_0(x),\phi_0(x)), \quad \text{in } \Omega,\\
(u_t(x,0),\phi_t(x,0)) = (u_1(x),\phi_1(x)), \quad \text{in } \Omega
\end{gathered} \tag{1.4}
$$

where the localized damping involves the sum of displacement velocity of a solid elastic material and the volume fraction velocity. Note that, $\Omega=(0,L)$ and $\omega=(L_1,L_2)$ with $0\le L_1<L_2\le L$ and $\gamma\in L^\infty(\Omega)$ is a nonnegative function satisfying

$$
\exists\,\gamma_0>0;\ \gamma(x)\ge\gamma_0,\ \text{a.e. } x\in\omega. \tag{1.5}
$$

The main contribution in [27] has been providing a necessary and sufficient condition for the strong stability and the exponential decay of the porous-elastic system with the rank-one localized damping where the boundary of the damping region must contain at least one of the end points of the spatial domain. Other problems associated with porous elastic systems can be found in references [23, 30, 31, 32].

Feireisl and Zuazua [8] proved the existence of the global attractor with critical semilinear term. The finite fractal dimension and regularity of global attractors for the critical case has been considered by Chueshov, Lasiecka and Toundykov [6]. Finally, very recently, Ma and Huertas [17] proved the continuity of attractors with respect to a parameter forcing in a residual dense set and the existence of generalized exponential attractors.

In [9], the long-time behavior of porous-elastic systems with nonlinear damping and source terms was investigated for the first time. Considering two globally defined nonlinear dampings and arbitrary source terms, the authors show the existence of local and global mild solutions, uniqueness of mild solutions, and continuous dependence of initial data. Under some restrictions on the parameters, they also proved that every mild solution to system blows up in finite time, provided the initial energy is negative and the sources are more dominant than the damping in the system. Additional results are obtained via potential well theory. They proved the existence of a unique global mild solution with initial data coming from the "good" part of the potential well. For such a global solution, we prove that the total energy of the system decays exponentially or algebraically, depending on the behavior of the dissipation in the system near the origin. To our knowledge, the study of global attractors for porous-elastic systems with nonlinear localized damping has not been discussed in the literature. This paper aims to fill this gap.

The purpose of this article is to obtain the existence and upper-semicontinuity of a global attractor for porous-elastic systems subject to a nonlinear localized damping and nonlinear source terms placed in both equations, with a minimal support for the damping. The contributions of the paper are:

(i) The existence of attractors with finite fractal dimension using quasi-stability methods by Chueshov and Lasiecka [7]. Observe that the present result was not previously considered for porous-elastic systems with nonlinear localized damping and nonlinear source terms,

(ii) Stability estimates (see Theorem 4.3) independent of ϵ_1 and ϵ_2. The standard multipliers method leads to terms of the energy level which cannot be directly absorbed (this is not the case when one of the damping functions is supported on the entire domain). In order to handle this, special weight functions are introduced, which eliminate undesirable terms of higher order while contributing lower-order terms,

(iii) The continuity of global attractors, containing residual continuity and upper semicontinuity with respect to the parameters $\epsilon_i \in [0,1]$, $i = 1, 2$.

This article is organized as follows: Section 2 presents assumptions, notations and well-posedness results. In Section 3, we summarize the main results. Section 4 is devoted to prove the existence of attractors and their properties. In the first subsection we prove that the system is gradient by using a unique continuation property proposed by Ma et al. [18]. The second subsection is devoted to prove the stabilizability inequality and quasi-stability of the system. In the third subsection, we prove the Theorem 3.1. More precisely, the existence of finite fractal global attractors with smoothness properties and the existence of a generalized fractal exponential attractor. In the last subsection the upper-semicontinuity of attractor with respect to $\alpha := (\epsilon_1, \epsilon_2)$ is proved (see Theorem 5.4).

2. Assumptions and Preliminary Results

We use throughout this paper the standard Lebesgue spaces $L^p(0,L)$, $p \geq 1$, with the norm denoted by $\|\cdot\|_p$. We denote by $\langle\cdot,\cdot\rangle$ the inner product in $L^2(0,L)$. Let us consider the Hilbert spaces

$$\begin{gathered}\mathcal{H} := H_0^1(0,L)\times H_0^1(0,L)\times L^2(0,L)\times L^2(0,L),\\ \mathcal{V} := H^2(0,L)\cap H_0^1(0,L)\times H^2(0,L)\cap H_0^1(0,L)\times H_0^1(0,L)\times H_0^1(0,L)\end{gathered} \tag{2.1}$$

with inner product in $\mathcal{H}$ given by

$$\begin{aligned}\langle U,V\rangle_{\mathcal{H}} := \rho\langle\varphi,\Phi\rangle + J\langle\psi,\Psi\rangle + \mu\langle u_x,v_x\rangle + \delta\langle\phi_x,w_x\rangle + \xi\langle\phi,w\rangle\\ + b\langle u_x,w\rangle + b\langle\phi,v_x\rangle.\end{aligned} \tag{2.2}$$

for $U = (u,\phi,\varphi,\psi)$, $V = (v,w,\Phi,\Psi)\in\mathcal{H}$.

Remark 2.1. Since, by hypothesis $\mu\xi \geq b^2$, using the same ideas in Raposo et al. [26] we see that (2.2) defines an inner product on $\mathcal{H}$ and that the associated norm $\|\cdot\|_{\mathcal{H}}$ is equivalent to the usual one. In particular, there exists $\gamma_0 > 0$ such that

$$\|u_x\|_2^2 + \|\phi_x\|_2^2 \leq \gamma_0\big(\mu\|u_x\|_2^2 + \delta\|\phi_x\|_2^2 + \xi\|\phi\|_2^2 + 2b\langle u_x,\phi\rangle\big). \tag{2.3}$$

Using the Poincaré's inequality and (2.3), there exists a constant $\gamma_1 > 0$ such that

$$\|u\|_2^2 + \|\phi\|_2^2 \leq \gamma_1\big(\mu\|u_x\|_2^2 + \delta\|\phi_x\|_2^2 + \xi\|\phi\|_2^2 + 2b\langle u_x,\phi\rangle\big). \tag{2.4}$$

If we denote $z = (u,\phi,u_t,\phi_t)$ and $z_0 = (u_0,\phi_0,u_1,\phi_1)$ then system (1.1) can be rewritten as

$$\begin{gathered}\frac{dz}{dt} = (\mathcal{A}+\mathcal{B})z + \mathcal{F}(z), \quad \text{for } t>0,\\ z(0) = z_0 \in \mathcal{H},\end{gathered} \tag{2.5}$$

where

$$\begin{gathered}\mathcal{A}(u,\phi,\varphi,\psi) = \Big(\varphi,\psi,\frac{\mu}{\rho}u_{xx} + \frac{b}{\rho}\phi_x, \frac{\delta}{J}\phi_{xx} - \frac{b}{J}u_x - \frac{\xi}{J}\phi\Big),\\ \text{for } (u,\phi,\varphi,\psi)\in\mathcal{D}(\mathcal{A}) = \mathcal{V};\\ \mathcal{B}(u,\phi,\varphi,\psi) = \Big(0,0,-\frac{1}{\rho}a_1(x)g_1(\varphi), \frac{1}{J}a_2(x)g_2(\psi)\Big), \quad \text{for } (u,\phi,\varphi,\psi)\in\mathcal{H},\\ \mathcal{F}(u,\phi,\varphi,\psi) = \Big(0,0,\frac{1}{\rho}\big(\epsilon_1 h_1 - f_1(u,\phi)\big), \frac{1}{J}\big(\epsilon_2 h_2 - f_2(u,\phi)\big)\Big),\\ \text{for } (u,\phi,\varphi,\psi)\in\mathcal{H}.\end{gathered}$$

We are ready to state the result about the existence of solutions. To this end we introduce the following assumptions:

(i) There exists a function $F\in C^2(\mathbb{R}^2)$ such that

$$\nabla F = (f_1,f_2), \tag{2.6}$$

and for $i=1,2$:

$$|\nabla f_i(u,v)| \leq \beta_0\big(1+|u|^{\theta-1}+|v|^{\theta-1}\big), \quad \forall u,v\in\mathbb{R}, \tag{2.7}$$

with $f_i(0,0)=0$, $\beta_0>0$ and $\theta\geq 1$. Moreover, we assume that there exist constants $\beta_1\geq 0$ and $m_F>0$ such that

$$F(u,v)\geq -\beta_1(|u|^2+|v|^2)-m_F,\quad \forall u,v\in\mathbb{R}, \tag{2.8}$$

$$\nabla F(u,v)\cdot(u,v)-F(u,v)\geq -\beta_1(|u|^2+|v|^2)-m_F,\quad \forall u,v\in\mathbb{R}, \tag{2.9}$$

where $0\leq\beta_1<\frac{1}{2\gamma_1}$.

(ii) The functions $g_i\in C^1(\mathbb{R})$, $i=1,2$, are monotonically increasing with $g_i(0)=0$ and there exist constant $m_i,M_i>0$ such that

$$m_i\leq g_i'(s)\leq M_i,\quad \forall s\in\mathbb{R}. \tag{2.10}$$

(iii) The functions $a_i\in C^\infty(0,L)$, $i=1,2$, are nonnegative and satisfy

$$a_i(x)\geq a_i>0 \text{ in } I_i,\ i=1,2,\quad\text{and}\quad (\alpha_1,\alpha_2)=I_1\cap I_2\neq\emptyset. \tag{2.11}$$

where I_1,I_2 are open intervals contained in $[0,L]$.

(iv) The external forces h_1,h_2 belong to $L^2(0,L)$.

Observe that (2.10) implies the monotonicity property, i.e.

$$(g_i(u)-g_i(v))(u-v)\geq m_i|u-v|^2,\quad \forall u,v\in\mathbb{R}. \tag{2.12}$$

Remark 2.2. The localizing functions allows us to consider damping mechanisms acting in an arbitrarily small region of the string.

Theorem 2.3. *If (i)–(iii) hold, then:*
(a) If initial data $z_0\in\mathcal{H}$, then (2.5) *has a unique mild solution $z(t)\in C([0,\infty),\mathcal{H})$, with $z(0)=z_0$, given by*

$$z(t)=e^{(\mathcal{A}+\mathcal{B})t}z_0+\int_0^t e^{(t-\tau)(\mathcal{A}+\mathcal{B})}\mathcal{F}(z(\tau))d\tau.$$

(b) If $z^1(t)$ and $z^2(t)$ are two mild solutions of (2.5) *then there exists a positive constant $C_0=C(z^1(0),z^2(0))$, such that*

$$\|z^1(t)-z^2(t)\|_{\mathcal{H}}\leq e^{C_0T}\|z^1(0)-z^2(0)\|_{\mathcal{H}},\quad \forall t\in[0,T]. \tag{2.13}$$

Proof. It is easy to see that the operator $\mathcal{A}+\mathcal{B}$ is a maximal monotone operator. In addition, by (2.7), $\mathcal{F}$ is a locally Lipschitz continuous on $\mathcal{H}$. Therefore, applying the theory of maximal nonlinear monotone operators (see [2, 4]) items(a)-(b) follow. The continuous dependence (b) is also obtained by using standard computations in the difference of solutions. □

Next result gives us a relation between mild and strong solutions for (2.5). It says that every mild solution can be obtained as limit of strong solutions.

Lemma 2.4. *Let $z_0=(u_0,\phi_0,u_1,\phi_1)\in\mathcal{H}$ be given and $z=(u,\phi,u_t,\phi_t)\in C(\mathbb{R}^+;\mathcal{H})$ the respective mild solution of* (2.5)*. Then, there exist a sequence of strong solutions $\{z_n\}$ of* (2.5)*, such that*

$$\lim_{n\to\infty}z_n=z\quad \textit{in } C(\mathbb{R}^+;\mathcal{H}).$$

Hence the mild solution is a strong solution.

Proof. Given $z_0\in\mathcal{H}$, we take a sequence of initial data $z_n^0\in D(A)$ such that $z_n^0\to z_0$ in $\mathcal{H}$. The difference $w_n(t)=z_n(t)-z(t)$ can be estimated as

$$\|w_n(t)\|\leq\|e^{t(\mathcal{A}+\mathcal{B})}(z_n^0-z_0)\|+L\int_0^t\|w_n(\tau)\|\,d\tau.$$

By Gronwall's lemma, $w_n(t) \to 0$ uniformly in $t \in \mathbb{R}_+$, hence $z_n(t) \to z(t)$ in $C(\mathbb{R}^+; \mathcal{H})$. The proof is complete. □

The following lemma shows the dissipative property of system (1.1).

Lemma 2.5. *The energy functional associated with the strong solution of system* (1.1) *satisfies*

$$\frac{d}{dt}\mathcal{E}(t) = -\int_0^L (a_1(x)g_1(u_t)u_t + a_2(x)g_2(\phi_t)\phi_t)\,dx \le 0, \quad \forall t > 0, \tag{2.14}$$

where

$$\mathcal{E}(t) = E(t) + \int_0^L F(u,\phi)\,dx - \int_0^L (\epsilon_1 h_1 u + \epsilon_2 h_2 \phi)\,dx,$$
$$E(t) = \frac{1}{2}\|(u,\phi,u_t,\phi_t)\|_{\mathcal{H}}^2 .$$

Moreover, there exist positive constants C_0, C_1 independent of ϵ_1 and ϵ_2 such that

$$C_0\|(u,\phi,u_t,\phi_t)\|_{\mathcal{H}}^2 - C_1 \le \mathcal{E}(t) \le C_2\big(1 + \|(u,\phi,u_t,\phi_t)\|_{\mathcal{H}}^{\theta+1}\big), \quad \forall t \ge 0. \tag{2.15}$$

Proof. A straightforward computation yields (2.14) by multiplying the first and second equations in (1.1) by u_t and ϕ_t, respectively. It follows from (2.8) and (2.4) that

$$\int_0^L F(u,\phi)\,dx \ge -\beta_1(\|u\|_2^2 + \|\phi\|_2^2) - Lm_F \ge -\beta_1\gamma_1\|z\|_{\mathcal{H}}^2 - Lm_F,$$

and therefore,

$$\mathcal{E}(t) \ge \big(\frac{1}{2} - \beta_1\gamma_1\big)\|(u,\phi,u_t,\phi_t)\|_{\mathcal{H}}^2 - Lm_F - \int_0^L (\epsilon_1 h_1 u + \epsilon_2 h_2\phi)\,dx.$$

Now letting

$$C_0 = \frac{1}{4}\big(1 - 2\beta_1\gamma_1\big) > 0, \tag{2.16}$$

and using the estimate

$$\int_0^L (\epsilon_1 h_1 u + \epsilon_2 h_2 \phi)\,dx \le \frac{C_0}{\gamma_1}\left(\|u\|_2^2 + \|\phi\|_2^2\right) + \frac{\gamma_1}{4C_0}\left(\|h_1\|_2^2 + \|h_2\|_2^2\right), \tag{2.17}$$

the first (or left) inequality in (2.15) is obtained with

$$C_1 = Lm_F + \frac{\gamma_1}{4C_0}\left(\|h_1\|_2^2 + \|h_2\|_2^2\right).$$

Now, using the embedding $H_0^1(0,L) \hookrightarrow L^\infty(0,L)$ and (2.7), we deduce that

$$\int_0^L F(u,\phi)\,dx \le C_2(1 + \|u_x\|_2^{\theta+1} + \|\phi_x\|_2^{\theta+1}).$$

So, using this estimative, we have

$$\mathcal{E}(t) \le C_2\|(u,\phi,u_t,\phi_t)\|_{\mathcal{H}}^{\theta+1} + C_2(1 + \|(u,\phi,u_t,\phi_t)\|_{\mathcal{H}}^{\theta+1})$$

This implies the second inequality in (2.15) holds. The proof is complete. □

3. Main results

First, we observe that the system (1.1) defines a dynamical system $(\mathcal{H}, S_\alpha(t))$, where $\mathcal{H}$ is given in (2.1), and $S_\alpha(t) : \mathcal{H} \to \mathcal{H}$ is the strongly continuous semigroup given by

$$S_\alpha(t)z_0 = (u(t), \phi(t), u_t(t), \phi_t(t)) \quad t \geq 0. \tag{3.1}$$

where $(u(t), \phi(t), u_t(t), \phi_t(t))$ is the unique mild solution of the system (1.1) with the initial data $z_0 = (u_0, \phi_0, u_1, \phi_1) \in \mathcal{H}$ and $\alpha = (\epsilon_1, \epsilon_2) \in \Lambda = [0,1] \times [0,1]$.

The main result for long-time dynamics is given by the following theorem whose proof will be provided in the next section.

Theorem 3.1. *Suppose that assumptions of Theorem 2.3 hold and $\alpha = (\epsilon_1, \epsilon_2) \in \Lambda$. Then*

(i) The dynamical system $(\mathcal{H}, S_\alpha(t))$ is quasi-stable (uniformly in α) on any bounded positively invariant set $B \subset \mathcal{H}$.

(ii) The dynamical system $(\mathcal{H}, S_\alpha(t))$ possesses a unique compact global attractor $\mathcal{A}_\alpha \subset \mathcal{H}$, which is characterized by the unstable manifold $\mathcal{A}_\alpha = \mathbb{M}_+(\mathcal{N}_\alpha)$ of the set of stationary solutions

$$\begin{aligned}\mathcal{N}_\alpha = \big\{(u, \phi, 0, 0) \in \mathcal{H} : -\mu u_{xx} - b\phi_x + f_1(u, \phi) = \epsilon_1 h_1 \\ -\delta\phi_{xx} + bu_x + \xi\phi + f_2(u, \phi) = \epsilon_2 h_2\big\}\end{aligned}$$

(iii) The dynamical system $(\mathcal{H}, S_\alpha(t))$ has a bounded absorbing set $\mathcal{B}$ independent of α. In particular,

$$\mathcal{A}_\alpha \subset \mathcal{B}, \quad \forall \alpha \in \Lambda.$$

(iv) The attractor $\mathcal{A}_\alpha$ has finite fractal and Hausdorff dimension $dim^f_{\mathcal{H}} \mathcal{A}_\alpha$.

(v) The global attractor $\mathcal{A}_\alpha$ is bounded in

$$\mathcal{V} = (H^2(0, L) \cap H^1_0(0, L))^2 \times (H^1_0(0, L))^2.$$

Moreover, every trajectory $z = (u, \phi, u_t, \phi_t)$ in $\mathcal{A}_\alpha$ satisfies

$$\|(u, \phi)\|^2_{(H^2 \cap H_0)^2} + \|(u_t, \phi_t)\|^2_{(H^1_0)^2} + \|(u_{tt}, \phi_{tt})\|^2_{(L^2)^2} \leq R_1^2, \tag{3.2}$$

for some constant $R_1 > 0$ independent of α.

(vi) The dynamical system $(\mathcal{H}, S_\alpha(t))$ possesses a generalized fractal exponential attractor. More precisely, for any $\delta \in (0, 1]$, there exists a generalized exponential attractor $\mathcal{A}^{\exp}_{\alpha,\delta} \subset \mathcal{H}$, with finite fractal dimension in the extended space $\widetilde{\mathcal{H}}_{-\delta}$, defined as interpolation of

$$\widetilde{\mathcal{H}}_0 := \mathcal{H}, quad and \quad \widetilde{\mathcal{H}}_{-1} := (L^2(0, L))^2 \times (H^{-1}(0, L))^2.$$

4. Proofs of main results

4.1. **Gradient system and stationary solutions.** We recall that a dynamical system $(H, S(t))$ is gradient if it possesses a strict Lyapunov functional. That is, a functional $\Phi : H \to \mathbb{R}$ is a strict Lyapunov function for a system $(H, S(t))$ if,

(i) the map $t \to \Phi(S(t)z)$ is non-increasing for each $z \in H$,
(ii) if $\Phi(S(t)z) = \Phi(z)$ for some $z \in H$ and for all t, then z is a stationary point of $S(t)$, that is, $S(t)z = z$.

Lemma 4.1. *Suppose that assumptions (i)–(iii) hold. Then the dynamical system* $(\mathcal{H}, S_\alpha(t))$ *is gradient, that is, there exists a strict Lyapunov function* Φ *defined in* $\mathcal{H}$*. In addition,*

$$\Phi(z) \to \infty \quad \text{if and only if} \quad \|z\|_{\mathcal{H}} \to \infty. \tag{4.1}$$

Proof. Let us define the function $\Phi : \mathcal{H} \to \mathbb{R}$ by

$$\begin{aligned}\Phi(S_\alpha(t)z) = \frac{1}{2}\|(u(t), \phi(t), u_t(t), \phi_t(t))\|^2_{\mathcal{H}} + \int_0^L F(u(t), \phi(t))\,dx \\ - \int_0^L (\epsilon_1 h_1 u + \epsilon_2 h_2 \phi)\,dx.\end{aligned} \tag{4.2}$$

From (2.14) we have

$$\frac{d}{dt}\Phi(S_\alpha(t)z) = -\int_0^L (a_1(x)g_1(u_t)u_t + a_2(x)g_2(\phi_t)\phi_t)\,dx \le 0, \quad \forall t \ge 0, \tag{4.3}$$

which shows that $t \mapsto \Phi(S_\alpha(t)z)$ is a non-increasing function.

Now suppose that $\Phi(S_\alpha(t)z) = \Phi(z)$ for all $t \ge 0$. Then (4.3) implies that

$$\int_0^L (a_1(x)g_1(u_t)u_t + a_2(x)g_2(\phi_t)\phi_t)\,dx = 0, \quad t \ge 0.$$

Then using (2.11) and (2.10), we can deduce for all $T > 0$ that

$$\begin{gathered} u_t = \phi_t = 0 \quad \text{a.e. in} \quad (\alpha_1, \alpha_2) \times (0, T), \\ a_1(x)g_1(u_t) = a_2(x)g_2(\phi_t) = 0 \quad \text{a.e. in } (0, L) \times (0, T). \end{gathered}$$

This means that $z(t) = (u(t), \phi(t), u_t(t), \phi_t(t))$ is a solution of

$$\begin{gathered} \rho u_{tt} - \mu u_{xx} - b\phi_x + f_1(u, \phi) = \epsilon_1 h_1 \quad \text{in } (0, L) \times (0, T), \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + f_2(u, \phi) = \epsilon_2 h_2 \quad \text{in } (0, L) \times (0, T), \\ u_t = \phi_t = 0 \quad \text{in } (\alpha_1, \alpha_2) \times (0, T). \end{gathered} \tag{4.4}$$

Taking the derivative of (4.4) with respect to the variable t in distributional sense and defining $v = u_t$ and $w = \phi_t$ yields

$$\begin{gathered} \rho v_{tt} - \mu v_{xx} - bw_x + p_1(x, t)v + q_1(x, t)w = 0 \quad \text{in } (0, L) \times (0, T), \\ Jw_{tt} - \delta w_{xx} + bv_x + \xi w + p_2(x, t)v + q_2(x, t)w = 0 \quad \text{in } (0, L) \times (0, T), \\ v = w = 0 \quad \text{in } (\alpha_1, \alpha_2) \times (0, T). \end{gathered} \tag{4.5}$$

where $p_i = \partial_u f_i(u, \phi)$, $q_i = \partial_v f_i(u, \phi)$ for $i = 1, 2$. From assumption (2.7) we can deduce that $p_i, q_i \in L^2(0, T; L^2(0, L))$. Using the unique continuation property in [18, Theorem 3.2.]), we conclude that $v = w = 0$ in $(0, L) \times (0, T)$. Therefore,

$$u_t = \phi_t = 0 \quad \text{in } (0, L) \times (0, T).$$

Therefore $z = (u_0, \phi_0, 0, 0)$ is a stationary solution of $S_\alpha(t)$. This proves that Φ is a strict Lyapunov function.

Now, by the second inequality in (2.15), we have

$$\Phi(z) \le C_2(1 + \|z\|^{\theta+1}_{\mathcal{H}}).$$

Considering the last estimate and taking $\Phi(z) \to +\infty$ we have $\|z\|_{\mathcal{H}} \to +\infty$. On the other hand, by the first inequality in (2.15) we obtain

$$\|z\|^2_{\mathcal{H}} \le \frac{1}{C_0}\left(\Phi(z) + C_1\right),$$

from here we conclude that $\|z\|_{\mathcal{H}} \to +\infty$ implies $\Phi(z) \to +\infty$, proving (4.1). The proof is complete. □

Lemma 4.2. *Suppose that assumptions (i)–(iii) hold. Then the set $\mathcal{N}_\alpha$ of the stationary points of $(\mathcal{H}, S_\alpha(t))$ is bounded in $\mathcal{H}$ uniformly in $\alpha \in \Lambda$.*

Proof. Let $z \in \mathcal{N}_\alpha$ be arbitrary. We know that $z = (u, \phi, 0, 0)$ and z satisfies the system

$$\begin{gathered} -\mu u_{xx} - b\phi_x + f_1(u,\phi) = \epsilon_1 h_1, \\ -\delta\phi_{xx} + bu_x + \xi\phi + f_2(u,\phi) = \epsilon_2 h_2. \end{gathered} \tag{4.6}$$

Multiplying the first equation in (4.6) by u and the second by ϕ, respectively, taking the sum and integrating over $(0, L)$, we obtain

$$\begin{aligned} &\mu\|u_x\|_2^2 + \delta\|\phi_x\|_2^2 + \xi\|\phi\|_2^2 + 2b\langle u_x, \phi\rangle \\ &= -\int_0^L \nabla F(u,\phi)\cdot(u,\phi)\,dx + \int_0^L (\epsilon_1 h_1 u + \epsilon_2 h_2 \phi)\,dx. \end{aligned} \tag{4.7}$$

Hence, using (2.4), (2.8), and (2.9), we obtain

$$\begin{aligned} &-\int_0^L \nabla F(u,\phi)\cdot(u,\phi)\,dx \\ &\le 2\beta_1\gamma_1\big(\mu\|u_x\|_2^2 + \delta\|\phi_x\|_2^2 + \xi\|\phi\|_2^2 + 2b\langle u_x, \phi\rangle\big) + 2Lm_F. \end{aligned} \tag{4.8}$$

Combining (4.7) and (4.8) on account of (2.16) yields

$$\begin{aligned} &4C_0\big(\mu\|u_x\|_2^2 + \delta\|\phi_x\|_2^2 + \xi\|\phi\|_2^2 + 2b\langle u_x, \phi\rangle\big) \\ &\le 2Lm_F + \int_0^L (\epsilon_1 h_1 u + \epsilon_2 h_2 \phi)\,dx. \end{aligned} \tag{4.9}$$

Hence, using the estimate (2.17), we deduce

$$3C_0\|z\|_{\mathcal{H}}^2 \le 2m_F L + \frac{\gamma_1}{4C_0}\left(\|h_1\|_2^2 + \|h_2\|_2^2\right), \tag{4.10}$$

which shows that the set $\mathcal{N}_\alpha$ is bounded in $\mathcal{H}$ uniformly in $\alpha \in \Lambda$. The proof is complete. □

4.2. **Uniform stabilizability inequality.** The following theorem plays an important role to prove the existence of a global attractor and its properties. We usually call it the stabilizability estimate. An important fact is that this estimate is independent of the parameter $\alpha = (\epsilon_1, \epsilon_2) \in \Lambda = [0,1] \times [0,1]$.

Theorem 4.3. *Suppose that assumptions (i)–(iii) hold. Let $B \subset \mathcal{H}$ be a bounded positively invariant set and let $S_\alpha(t)z^i = (u^i(t), \phi^i(t), u_t^i(t), \phi_t^i(t))$, $i = 1, 2$, be mild solutions of (1.1) with initial conditions $z^i \in B$. Then, there exist constants $\vartheta_B, \eta_B, C_B > 0$, depending on B yet independent of α, such that*

$$E(t) \le \vartheta_B E(0) e^{-\eta_B t} + C_B \sup_{s\in[0,t]} \left(\|u(s)\|_{2\theta}^2 + \|\phi(s)\|_{2\theta}^2\right), \tag{4.11}$$

for all $t \ge 0$, where $u = u^1 - u^2$ and $\phi = \phi^1 - \phi^2$.

Proof. For $u = u^1 - u^2$ and $\phi = \phi^1 - \phi^2$, the following notation is adopted

$$\begin{gathered} F_i(u,\phi) = f_i(u^1,\phi^1) - f_i(u^2,\phi^2), \quad G_1(u_t) = g_1(u_t^1) - g_1(u_t^2), \\ G_2(\phi_t) = g_2(\phi_t^1) - g_2(\phi_t^2). \end{gathered}$$

Then, (u, ϕ, u_t, ϕ_t) solves the system

$$
\begin{gathered}
\rho u_{tt} - \mu u_{xx} - b\phi_x + a_1(x)G_1(u_t) + F_1(u,\phi) = 0 \quad \text{in } (0,L)\times(0,\infty), \\
J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + a_2(x)G_2(\phi_t) + F_2(u,\phi) = 0 \quad \text{in } (0,L)\times(0,\infty), \\
u(0,t) = u(L,t) = \phi(0,t) = \phi(L,t) = 0, \quad t > 0, \\
(u(x,0),\phi(x,0)) = (u_0(x),\phi_0(x)), \quad \text{in } (0,L), \\
(u_t(x,0),\phi_t(x,0)) = (u_1(x),\phi_1(x)), \quad \text{in } (0,L).
\end{gathered}
\tag{4.12}
$$

Take $|\Sigma| = \alpha_2 - \alpha_1$. Let us consider ϵ_0, small enough, such that $0 < \epsilon_0 < \frac{|\Sigma|}{2}$ and we define the auxiliary function, as in [13],

$$
h_\lambda(x) = \begin{cases} (\lambda-1)x, & x \in [0,\alpha_1+\epsilon_0), \\ \lambda(x-\alpha_1-\epsilon_0) + \frac{\alpha_1-\alpha_2+2\epsilon_0}{L}(\alpha_1+\epsilon_0), & x \in [\alpha_1+\epsilon_0, \alpha_2-\epsilon_0], \\ (\lambda-1)(x-L), & x \in (\alpha_2-\epsilon_0, L], \end{cases}
\tag{4.13}
$$

with $\lambda := \frac{L-(\alpha_2-\alpha_1-2\epsilon_0)}{L} \in (0,1)$ and $0 \le \alpha_1 < \alpha_2 \le L$.

Multiplying the first and second equations of the system (4.12) by $u_x h_\lambda$ and $\phi_x h_\lambda$, respectively, and integrating by parts, we have

$$
\begin{aligned}
&\int_0^T \int_0^L \Big(\frac{\rho}{2}u_t^2 + \frac{\mu}{2}u_x^2 + \frac{J}{2}\phi_t^2 + \frac{\delta}{2}\phi_x^2 + \frac{\xi}{2}\phi^2 + bu_x\phi\Big) h_\lambda'\,dx\,dt \\
&= -\Big[\rho\int_0^L u_t u_x h_\lambda\,dx\Big]_0^T - \Big[J\int_0^L \phi_t\phi_x h_\lambda\,dx\Big]_0^T \\
&\quad + \xi\int_0^T\int_0^L \phi^2 h_\lambda'\,dx\,dt + b\int_0^T\int_0^L u_x\phi h_\lambda'\,dx\,dt \\
&\quad + \int_0^T\int_0^L (a_1(x)G_1(u_t)u_x + a_2(x)G_2(\phi_t)\phi_x)h_\lambda\,dxdt \\
&\quad + \int_0^T\int_0^L (F_1(u,\phi)u_x + F_2(u,\phi)\phi_x)h_\lambda\,dx\,dt.
\end{aligned}
\tag{4.14}
$$

Observing that

$$
h_\lambda'(x) = \begin{cases} \lambda, & x \in (\alpha_1+\epsilon_0, \alpha_2-\epsilon_0), \\ (\lambda-1), & x \in [0,\alpha_1+\epsilon_0) \cup (\alpha_2-\epsilon_0, L], \end{cases}
\tag{4.15}
$$

from the above equality we have

$$
\begin{aligned}
&(1-\lambda)\int_0^T E(t)\,dt \\
&= \Big[\int_0^L (\rho u_t u_x + J\phi_t\phi_x)h_\lambda\,dx\Big]_0^T \\
&\quad - \xi\int_0^T\int_0^L \phi^2 h_\lambda'\,dx\,dt - b\int_0^T\int_0^L u_x\phi h_\lambda'\,dx\,dt \\
&\quad + \frac{1}{2}\int_0^T\int_{\alpha_1+\epsilon_0}^{\alpha_2-\epsilon_0} (\rho u_t^2 + J\phi_t^2)\,dx\,dt + \frac{1}{2}\int_0^T\int_{\alpha_1+\epsilon_0}^{\alpha_2-\epsilon_0} (\mu u_x^2 + \delta\phi_x^2)\,dx\,dt \\
&\quad + \frac{\xi}{2}\int_0^T\int_{\alpha_1+\epsilon_0}^{\alpha_2-\epsilon_0} \phi^2\,dx\,dt + b\int_0^T\int_{\alpha_1+\epsilon_0}^{\alpha_2-\epsilon_0} u_x\phi\,dx\,dt
\end{aligned}
\tag{4.16}
$$

$$
\begin{aligned}
&- \int_0^T \int_0^L (a_1(x)G_1(u_t)u_x - a_2(x)G_2(\phi_t)\phi_x)h_\lambda \,dx\,dt \\
&- \int_0^T \int_0^L (F_1(u,\phi)u_x + F_2(u,\phi)\phi_x)h_\lambda \,dx\,dt.
\end{aligned} \tag{4.17}
$$

Let us estimate the right-hand side of (4.17). Using the equivalence between the norm of the energy and the usual norm in $\mathcal{H}$, we obtain

$$
\Big[\int_0^L (\rho u_t u_x + \phi_t\phi_x)h_\lambda \,dx\Big]_0^T \le C(E(0)+E(T)). \tag{4.18}
$$

On the other hand, since $L^{2\theta}(0,L) \hookrightarrow L^2(0,L)$ we obtain

$$
-\xi \int_0^T \int_0^L \phi^2 h_\lambda' \,dx \le C\int_0^T \|\phi\|_{2\theta}^2 \,dt, \tag{4.19}
$$

and for $\epsilon > 0$,

$$
-b\int_0^T \int_0^L u_x\phi h_\lambda' \,dx\,dt \le \epsilon \int_0^T E(t)\,dt + C_\epsilon \int_0^T \|\phi\|_{2\theta}^2 \,dt. \tag{4.20}
$$

Using (2.10) and applying Young's inequality, we obtain

$$
\begin{aligned}
- \int_0^T \int_0^L a_1(x)G_1(u_t)u_x h_\lambda \,dx\,dt &\le M_1 \int_0^T \int_0^L a_1(x)|u_t||u_x h_\lambda| \,dx\,dt \\
&\le C_\epsilon \int_0^T \int_0^L a_1(x)|u_t|^2 \,dx\,dt + \epsilon \int_0^T E(t)\,dt \\
&\le C_\epsilon \int_0^T \int_0^L a_1(x)G_1(u_t)u_t \,dx\,dt + \epsilon \int_0^T E(t)\,dt.
\end{aligned}
$$

Analogously,

$$
- \int_0^T \int_0^L a_2(x)G_2(\phi_t)\phi_x h_\lambda \,dx\,dt \le C_\epsilon \int_0^T \int_0^L a_2(x)G_2(\phi_t)\phi_t \,dx\,dt + \epsilon \int_0^T E(t)\,dt.
$$

Then the two inequalities above imply

$$
\begin{aligned}
&- \int_0^T \int_0^L (a_1(x)G_1(u_t)u_x + a_2(x)G_2(\phi_t)\phi_x)h_\lambda \,dxdt \\
&\le C_\epsilon \int_0^T \int_0^L (a_1(x)G_1(u_t)u_t + a_2(x)G_2(\phi_t)\phi_t)\,dx\,dt + \epsilon \int_0^T E(t)\,dt.
\end{aligned} \tag{4.21}
$$

Using (2.7), we have

$$
\begin{aligned}
&\int_0^T \int_0^L (F_1(u,\phi)u_x + F_2(u,\phi)\phi_x)h_\lambda \,dx\,dt \\
&\le C_B \int_0^T (\|u\|_{2\theta} + \|\phi\|_{2\theta})(\|u_x\|_2 + \|\phi_x\|_2)\,dt \\
&\le C_{B,\epsilon} \int_0^T (\|u\|_{2\theta}^2 + \|\phi\|_{2\theta}^2)\,dt + \epsilon \int_0^T E(t)\,dt.
\end{aligned} \tag{4.22}
$$

Inserting the estimates (4.18)-(4.22) into (4.17) with $\epsilon > 0$ small enough, we have

$$\begin{aligned}\int_0^T E(t)\,dt &\le C(E(T)+E(0)) + \frac{1}{2}\int_0^T\int_{\alpha_1+\epsilon_0}^{\alpha_2-\epsilon_0}(\rho u_t^2 + J\phi_t^2)\,dx\,dt \\ &\quad + \frac{1}{2}\int_0^T\int_{\alpha_1+\epsilon_0}^{\alpha_2-\epsilon_0}(\mu u_x^2 + \delta\phi_x^2)\,dx\,dt + \frac{\xi}{2}\int_0^T\int_{\alpha_1+\epsilon_0}^{\alpha_2-\epsilon_0}\phi^2\,dx\,dt \\ &\quad + C\int_0^T\int_0^L (a_1(x)G_1(u_t)u_t + a_2(x)G_2(\phi_t)\phi_t)\,dx\,dt \\ &\quad + C_B\int_0^T (\|u\|_{2\theta}^2 + \|\phi\|_{2\theta}^2)\,dt.\end{aligned} \tag{4.23}$$

Now, let us consider a cut-off function $\eta \in C_0^\infty(0,L)$ such that

$$\eta(x) = \begin{cases} 1, & x \in [\alpha_1+\epsilon_0, \alpha_2-\epsilon_0], \\ 0, & x \in [0,\alpha_1)\cup(\alpha_2, L] \\ 0 \le \eta(x) \le 1, & x \in [0,L]. \end{cases} \tag{4.24}$$

So multiplying the first and second equations of (4.12) by $u\eta$ and $\phi\eta$, respectively, and integrating by parts, we obtain

$$\begin{aligned}&\int_0^T\int_0^L (\rho u_t^2 + J\phi_t^2 + \mu u_x^2 + \delta\phi_x^2 + \xi\phi^2)\eta\,dx\,dt \\ &= -\Big[\int_0^L (\rho u_t u + J\phi_t\phi)\eta\,dx\Big]_0^T + \int_0^T\int_0^L (2\rho u_t^2 + 2J\phi_t^2)\eta\,dx\,dt \\ &\quad + \frac{1}{2}\int_0^T\int_0^L (\mu u^2 + \delta\phi^2)\eta_{xx}\,dx\,dt + b\int_0^T\int_0^L (\phi_x u - u_x\phi)\eta\,dx\,dt \\ &\quad - \int_0^T\int_0^L (G_1(u_t)u + G_2(\phi_t)\phi)\eta\,dx\,dt \\ &\quad - \int_0^T\int_0^L (F_1(u,\phi)u + F_2(u,\phi)\phi)\eta\,dx\,dt.\end{aligned} \tag{4.25}$$

Consequently, by calculations to the ones before, we infer that

$$\begin{aligned}&\int_0^T\int_0^L (\rho u_t^2 + J\phi_t^2 + \mu u_x^2 + \delta\phi_x^2 + \xi\phi^2)\eta\,dx\,dt \\ &\le C(E(T)+E(0)) + C\int_0^T\int_0^L (a_1(x)G_1(u_t)u_t + a_2(x)G_2(\phi_t)\phi_t)\,dxdt \\ &\quad + \epsilon\int_0^T E(t)\,dt + C_B\int_0^T (\|u\|_{2\theta}^2 + \|\phi\|_{2\theta}^2)\,dt.\end{aligned}$$

Substituting the last estimate in (4.23) with $\epsilon > 0$ small enough (4.17) and using the fact that η has support contained in $[\alpha_1, \alpha_2]$, we obtain

$$\begin{aligned}\int_0^T E(t)\,dt &\le C(E(T)+E(0)) \\ &\quad + C\int_0^T\int_0^L (a_1(x)G_1(u_t)u_t + a_2(x)G_2(\phi_t)\phi_t)\,dx\,dt \\ &\quad + C_B\int_0^T (\|u\|_{2\theta}^2 + \|\phi\|_{2\theta}^2)\,dt.\end{aligned} \tag{4.26}$$

Next, multiplying the first and second equations in (4.12) by u_t and ϕ_t, and integrate by parts over $[0, L] \times [s, T]$ so that

$$\begin{aligned} &\int_s^T \int_0^L (a_1(x)G_1(u_t)u_t + a_2(x)G_2(\phi_t)\phi_t)\,dx\,dt \\ &= E(s) - E(T) - \int_0^T \int_0^L (F_1(u,\phi)u_t + F_2(u,\phi)\phi_t)\,dx\,dt. \end{aligned} \tag{4.27}$$

For each $\epsilon > 0$, we have

$$\begin{aligned} &\int_0^T \int_0^L (F_1(u,\phi)u_t + F_2(u,\phi)\phi_t)\,dx\,dt \\ &\le C_B \int_0^T (\|u\|_{2\theta} + \|\phi\|_{2\theta})(\|u_t\|_2 + \|\phi_t\|_2)\,dt \\ &\le C_{B,\epsilon} \int_0^T (\|u\|_{2\theta}^2 + \|\phi\|_{2\theta}^2)\,dt + \epsilon \int_0^T E(t)\,dt. \end{aligned} \tag{4.28}$$

Now we use (4.27) and (4.28) to obtain

$$\begin{aligned} &\int_0^T \int_0^L (a_1(x)G_1(u_t)u_t + a_2(x)G_2(\phi_t)\phi_t)\,dt \\ &\le E(0) + E(T) + \epsilon \int_0^T E(t)\,dt + C_{B,\epsilon} \int_0^T (\|u\|_{2\theta}^2 + \|\phi\|_{2\theta}^2)\,dt. \end{aligned} \tag{4.29}$$

Next, we combine estimates (4.26) and (4.29) for $\epsilon > 0$ small enough to obtain

$$\int_0^T E(t)\,dt \le C(E(T) + E(0)) + C_B \int_0^T (\|u\|_{2\theta}^2 + \|\phi\|_{2\theta}^2)\,dt. \tag{4.30}$$

Now, integrate the energy equality (4.27) with respect to s so that

$$\begin{aligned} TE(T) = \int_0^T E(t)\,dt - \int_0^T \int_s^T (a_1(x)G_1(u_t)u_t + a_2(x)G_2(\phi_t)\phi_t)\,dt\,ds \\ - \int_0^T \int_s^T \int_0^L (F_1(u,\phi)u_t + F_2(u,\phi)\phi_t)\,dx\,dt\,ds. \end{aligned}$$

By (4.28) and that $a_1(x)G_1(u_t)u_t + a_2(x)G_2(v_t)v_t \ge 0$, the following is immediate,

$$TE(T) \le 2 \int_0^T E(t)\,dt + C_{B,T} \int_0^T (\|u\|_{2\theta}^2 + \|\phi\|_{2\theta}^2)\,dt. \tag{4.31}$$

Substituting (4.30) in (4.31) yields

$$TE(T) \le C(E(T) + E(0)) + C_{B,T} \int_0^T (\|u\|_{2\theta}^2 + \|\phi\|_{2\theta}^2)\,dt.$$

We choose $T > 2C$ to deduce that

$$E(T) \le \gamma_T E(0) + C_{B,T} \sup_{s\in[0,T]} (\|u(s)\|_{2\theta}^2 + \|\phi(s)\|_{2\theta}^2), \tag{4.32}$$

where

$$\gamma_T = \frac{C}{T - C} < 1.$$

Then a standard argument (see [19, Lemma 4.6]) shows that there exist $\vartheta_B, \eta_B, C_B > 0$ such that

$$E(t) \le \vartheta_B E(0)e^{-\eta_B t} + C_B \sup_{\sigma\in[0,t]} (\|u(s)\|_{2\theta}^2 + \|\phi(s)\|_{2\theta}^2), \quad \forall t \ge 0.$$

The proof is complete. □

Proof of Theorem 3.1. (i) Consider a bounded positively invariant set $B \subset \mathcal{H}$ with respect to $S_\alpha(t)$, and call it $S_\alpha(t)z^i = (u^i(t), \phi^i(t), u_t^i(t), \phi_t^i(t))$ for $z^i \in B$, $i = 1, 2$. Set also $u = u^1 - u^2, \phi = \phi^1 - \phi^2$, as before. It follows from (2.13) that

$$\|S_\alpha(t)z^1 - S_\alpha(t)z^2\|_{\mathcal{H}} \le a(t)\|z^1 - z^2\|_{\mathcal{H}} \tag{4.33}$$

with $a(t) = e^{C_0 T}$. Now let $X = H_0^1(0, L) \times H_0^1(0, L)$, and define the semi-norm

$$n_X(u, v) := (\|u\|_{2\theta}^2 + \|\phi\|_{2\theta}^2)^{1/2}$$

Since the embedding (in 1D) $H_0^1(0, L) \hookrightarrow L^{2\theta}(0, L)$ is compact, we know that n_X is a compact semi-norm on X.

By (4.11) we deduce that

$$\|S_\alpha(t)z^1 - S_\alpha(t)z^2\|_{\mathcal{H}}^2 \le b(t)\|z_1 - z_2\|_{\mathcal{H}}^2 + c(t) \sup_{s\in[0,t]} [n_X(u(s), \phi(s))]^2, \tag{4.34}$$

where $b(t) = \vartheta_B e^{-\eta_B t}$ and $c(t) = C_B$. Clearly,

$$b(t) \in L^1(\mathbb{R}^+) \quad \text{and} \quad \lim_{t\to\infty} b(t) = 0.$$

Since $B \subset \mathcal{H}$ is bounded, we know that $c(t)$ is locally bounded on $[0, \infty)$. We now have that the dynamical system $(\mathcal{H}, S_\alpha(t))$ is quasi-stable on any bounded positively invariant set $B \subset \mathcal{H}$ by [7, Definition 7.9.2].

(ii) Since $(\mathcal{H}, S_\alpha(t))$ is quasi-stable, applying [7, Proposition 7.9.4], we have that $(\mathcal{H}, S_\alpha(t))$ is asymptotically smooth. Thus, noting Lemmas 4.1 and 4.2 and using [7, Corollary 7.5.7], we know that $(\mathcal{H}, S_\alpha(t))$ has a compact global attractor given by $\mathcal{A}_\alpha = \mathbb{M}_+(\mathcal{N}_\alpha)$.

(iii) Let Φ be the Lyapunov functional given in (4.2). By (2.15) and [7, Remark 7.5.8], we obtain

$$\begin{aligned}\sup_{z\in\mathcal{A}_\alpha} \|z\|_{\mathcal{H}}^2 &\le \frac{\sup_{z\in\mathcal{A}_\alpha} \Phi(z) + C_1}{C_0} \le \frac{\sup_{z\in\mathcal{N}} \Phi(z) + C_1}{C_0} \\ &\le \frac{C_2(1 + \sup_{z\in\mathcal{N}} \|z\|_{\mathcal{H}}^{\theta+1}) + C_1}{C_0}.\end{aligned}$$

Hence, by (4.10), we conclude that there exists a constant $R > 0$ independent of α such that

$$\sup_{z\in\mathcal{A}_\alpha} \|z\|_{\mathcal{H}}^2 \le R.$$

Therefore, the closed ball $\mathcal{B} = B(0, R_0)$ in $\mathcal{H}$ of center zero and radius $R_0 > R$ is a bounded absorbing independent of $\alpha \in \Lambda$.

(iv) From the above, $(\mathcal{H}, S_\alpha(t))$ is quasi-stable on the attractor $\mathcal{A}_\alpha$. Thus, using in [7, Theorem 7.9.6], we know that the attractor $\mathcal{A}_\alpha$ has finite fractal dimension $\dim_{\mathcal{H}}^f \mathcal{A}_\alpha$.

(v) Since the system $(\mathcal{H}, S_\alpha(t))$ is quasi-stable on the attractor $\mathcal{A}_\alpha$ with $c_\infty = \sup_{t\in\mathbb{R}^+} c(t) = C_{\mathcal{A}_\alpha} < \infty$, it follows from [7, Theorem 7.9.8] that any complete trajectory $z = (u, \phi, u_t, \phi_t)$ in $\mathcal{A}_\alpha$ has the following regularity properties

$$v_t, \phi_t \in L^\infty(\mathbb{R}, H_0^1(0, L)) \cap C(\mathbb{R}, L^2(0, L)),\\ v_{tt}, p_{tt} \in L^\infty(\mathbb{R}, L^2(0, L)).$$

Thus, since $\mathcal{A}_\alpha \subset \mathcal{B}$ for all $\alpha \in \Lambda$ by (iii), there exists $C_{\mathcal{B}} > 0$ such that

$$\|(u_t, \phi_t)\|^2_{H_0^1\times H_0^1} + \|(u_{tt}, \phi_{tt})\|^2_{L^2\times L^2} \le C_{\mathcal{B}}.$$

Hence, using (1.1) and noting that the nonlinear terms are continuous, we conclude there exists a constant $C'_{\mathcal{B}} > 0$ such that

$$\|(u, \phi)\|^2_{H^2\cap H_0^1} \le C'_{\mathcal{B}}.$$

Therefore (3.2) holds. Since the global attractors $\mathcal{A}_\alpha$ are also characterized by

$$\mathcal{A}_\alpha = \{z(0) : z \text{ is a bounded full trajectory of } S_\alpha(t)\},$$

we conclude the $\mathcal{A}_\alpha$ is bounded in $\mathcal{H}_1$.

(vi) Let $\mathcal{B}$ be the bounded absorbing of $(\mathcal{H}, S_\alpha(t))$ given by (iii). Hence the system $(\mathcal{H}, S_\alpha(t))$ is quasi-stable on $\mathcal{B}$. For the solution $z(t)$ with initial data $z_0 = z(0) \in \mathcal{B}$, there exists $C_{\mathcal{B}} > 0$ such that for any $0 \le t \le T$,

$$\|z_t(t)\|_{\widetilde{\mathcal{H}}_{-1}} \le C_{\mathcal{B}}$$

which leads to

$$\|S_\alpha(t_1)z_0 - S_\alpha(t_2)z_0\|_{\widetilde{\mathcal{H}}_{-1}} \le \int_{t_1}^{t_2} \|z_t(\tau)\|_{\widetilde{\mathcal{H}}_{-1}}\, d\tau \le C_{\mathcal{B}}|t_1 - t_2| \tag{4.35}$$

for each $0 \le t_1 < t_2 \le T$. From (4.35), we conclude that for any $z_0 \in \mathcal{B}$, the map $t \mapsto S_\alpha(t)z_0$ is Hölder continuous in the extended space $\widetilde{\mathcal{H}}$ with the exponent $\delta = 1$. Then, the existence of a generalized exponential attractor, whose fractal dimension is finite, is immediate in $\widetilde{\mathcal{H}}_{-1}$.

Following the similar arguments in [19, Theorem 5.1], the existence of exponential attractors is obtained in $\widetilde{\mathcal{H}}_{-\delta}$ with $\delta \in (0, 1)$. The proof of Theorem 3.1 is complete. □

5. Continuity and upper-semicontinuity of attractors

Let X be a complete metric space and $\mathcal{A}_\lambda$ be a family of global attractors for a semigroup $S_\lambda(t)$ on X, where λ belongs to a complete metric space Λ.

Definition 5.1. We say that the global attractor $\mathcal{A}_\lambda$ is

- Upper semicontinuous at $\lambda_0 \in \Lambda$ if
$$\lim_{\lambda\to\lambda_0} \operatorname{dist}_X(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0}) = 0.$$
- Lower semicontinuous at $\lambda_0 \in \Lambda$ if
$$\lim_{\lambda\to\lambda_0} \operatorname{dist}_X(\mathcal{A}_{\lambda_0}, \mathcal{A}_\lambda) = 0.$$

- Continuous at $\lambda_0 \in \Lambda$ if it
$$\lim_{\lambda \to \lambda_0} \mathrm{d}_X(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0}) = 0,$$
where $\mathrm{d}_X(A,B) = \max\{\mathrm{dist}_X(A,B), \mathrm{dist}_X(B,A)\}$ denotes the denotes the Hausdorff metric in X.

Note that upper semicontinuity is typically easier to obtain than lower semicontinuity and the key ingredient of the proof are the a priori estimates on the attractor and no knowledge on the attractor structure is needed. On the other hand, the lower semicontinuity of attractors need a careful description of the structure of the attractor for the limit equation, which is then transferred to the attractors under perturbation (see [12]).

We use the recent results in [14] on the continuity of attractors with respect to a parameter, where the results were obtained as a extension of the previous results in [1]. Let $S_\lambda(t)$ be a family of parametrized semigroups defined on X, where λ belongs to a complete metric space Λ.

The result in [14, Theorem 5.2] provides sufficient conditions for the continuity of global attractors on a residual dense subset.

Theorem 5.2. *Suppose that*

(1) $S_\lambda(t)$ *has a global attractor* $\mathcal{A}_\lambda$ *for every* $\lambda \in \Lambda$,
(2) *There is a bounded subset* D *of* X *such that* $\mathcal{A}_\lambda \subset D$ *for every* $\lambda \in \Lambda$,
(3) *For* $t > 0$, $S_\lambda(t)x$ *is continuous in* λ, *uniformly for* x *in bounded subsets of* X.

Then $\mathcal{A}_\lambda$ *is continuous on* J *where* J *is a "residual" set dense in* Λ.

Theorem 5.3. *Under the assumptions of Theorem 3.1, there exists a set* J *dense in* $\Lambda = [0,1] \times [0,1]$ *such that* $\mathcal{A}_\alpha$, *where* $\alpha = (\epsilon_1, \epsilon_2) \in \Lambda$, *is continuous at* $\alpha_0 = (\epsilon_1^0, \epsilon_2^0) \in J$, *that is,*
$$\lim_{\alpha \to \alpha_0} d_{\mathcal{H}}(\mathcal{A}_\alpha, \mathcal{A}_{\alpha_0}) = 0, \quad \forall \alpha_0 \in J. \tag{5.1}$$

Proof. We shall apply the Theorem 5.2 with $\Lambda = [0,1] \times [0,1]$. Theorem 3.1 indicates that (1) holds. The property (2) follows promptly from Theorem 3.1 (iii).

Now, we shall prove the condition (3). Let D be a bounded set of $\mathcal{H}$. Given $\alpha^1 = (\epsilon_1, \epsilon_2), \alpha_2 = (\epsilon_1', \epsilon_2') \in \Lambda$ and $z \in D$, let us denote
$$\begin{gathered} S_{\alpha_i}(t)z = (u^i(t), \phi^i(t), u_t^i(t), \phi_t^i(t)), \quad i = 1,2, \\ u = u^1 - u^2, \quad \phi = \phi^1 - \phi^2. \end{gathered}$$

Then $z(t) = (u(t), \phi(t), u_t(t), \phi_t(t))$ satisfies the system
$$\begin{gathered} \rho u_{tt} - \mu u_{xx} - b\phi_x + a_1(x)G_1(u_t) + F_1(u,\phi) = (\epsilon_1 - \epsilon_1')h_1, \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + a_2(x)G_2(\phi_t) + F_2(u,\phi) = (\epsilon_2 - \epsilon_2')h_2, \end{gathered} \tag{5.2}$$
where
$$\begin{gathered} F_i(u,\phi) = f_i(u^1, \phi^1) - f_i(u^2, \phi^2), \quad i = 1,2; \\ G_1(u_t) = g_1(u_t^1) - g_1(u_t^2), \quad G_2(\phi_t) = g_2(\phi_t^1) - g_2(\phi_t^2). \end{gathered}$$

Multiplying the first equation in (5.2) by u_t, the second by ϕ_t, respectively, and using integration by parts, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|U\|_{\mathcal{H}}^2 &= -\int_0^L (F_1(u,\phi)u_t + F_2(u,\phi)\phi_t)\,dx \\ &\quad -\int_0^L (a_1(x)G_1(u_t)u_t + a_2(x)G_2(v_t)v_t)\,dx \\ &\quad +\int_0^L \big((\epsilon_1-\epsilon_1')h_1u_t + (\epsilon_2-\epsilon_2')h_2\phi_t\big)\,dx.\end{aligned}\tag{5.3}$$

Using (2.7), Hölder's inequality and the embedding $H_0^1(0,L)\hookrightarrow L^\infty(0,L)$, we deduce that

$$\begin{aligned}&\int_0^L F_1(u,\phi)u_t\,dx \\ &\le C(1+\|S_{\sigma_1}(t)z\|_{\mathcal{H}}^{\theta-1} + \|S_{\sigma_2}(t)z\|_{\mathcal{H}}^{\theta-1})(\|u\|_2+\|\phi\|_2)\|u_t\|_2 \\ &\le C(1+\|S_{\sigma_1}(t)z\|_{\mathcal{H}}^{\theta-1} + \|S_{\sigma_2}(t)z\|_{\mathcal{H}}^{\theta-1})(\|u_x\|_2+\|\phi_x\|_2)\|u_t\|_2.\end{aligned}\tag{5.4}$$

Using that $\mathcal{E}(t)$ is a non-increasing function and (2.15), we find that for $i=1,2$,

$$\|S_{\sigma_i}(t)z_0\|_{\mathcal{H}}^{p-1} \le \frac{\mathcal{E}(0)+C_1}{C_0} \le \frac{C_2(1+\|z\|_{\mathcal{H}}^{p+1})+C_1}{C_0} \le C_D, \quad \forall z\in D.$$

Inserting the above estimate into (5.4) and using Young's inequality, we see that

$$\begin{aligned}\int_0^L F_1(u,\phi)u_t\,dx &\le C_D(\|u_x\|_2+\|\phi_x\|_2)\|u_t\|_2 \\ &\le C_D(\|u_x\|_2^2+\|\phi_x\|_2^2) + \rho\|u_t\|_2^2.\end{aligned}$$

Analogously,

$$\int_0^L F_2(u,\phi)\phi_t\,dx \le C_D(\|u_x\|_2^2+\|\phi_x\|_2^2) + J\|\phi_t\|_2^2.$$

Adding the last two estimates and using (2.3), we conclude that

$$\int_0^L (F_1(u,\phi)u_t + F_2(u,\phi)\phi_t)\,dx \le C_D\|z\|_{\mathcal{H}}^2.\tag{5.5}$$

By the monotonicity property (2.12), we obtain

$$-\int_0^L (a_1(x)G_1(u_t)u_t + a_2(x)G_2(v_t)v_t)\,dx \le 0.\tag{5.6}$$

In addition,

$$\begin{aligned}&\int_0^L \big((\epsilon_1-\epsilon_1')h_1u_t + (\epsilon_2-\epsilon_2')h_2\phi_t\big)\,dx \\ &\le \frac{1}{4}(\rho\|u_t\|_2^2 + J\|\phi_t\|_2^2) + \frac{1}{\rho}|\epsilon_1-\epsilon_1'|^2\|h_1\|^2 + \frac{1}{J}|\epsilon_2-\epsilon_2'|^2\|h_2\|_2^2 \\ &\le \frac{1}{4}\|z\|_{\mathcal{H}}^2 + \frac{1}{\rho}|\epsilon_1-\epsilon_1'|^2\|h_1\|^2 + \frac{1}{J}|\epsilon_2-\epsilon_2'|^2\|h_2\|_2^2.\end{aligned}\tag{5.7}$$

Substituting the estimates (5.5)-(5.7) into (5.3), we obtain

$$\frac{d}{dt}\|z\|_{\mathcal{H}}^2 \le C_D\|z\|_{\mathcal{H}}^2 + \frac{1}{\rho}|\epsilon_1-\epsilon_1'|^2\|h_1\|^2 + \frac{1}{J}|\epsilon_2-\epsilon_2'|^2\|h_2\|_2^2.\tag{5.8}$$

Applying Gronwall's inequality to (5.8) and using that $\|z(0)\|_{\mathcal{H}}^2 = 0$, we conclude that

$$\|z(t)\|_{\mathcal{H}}^2 \le C\left(e^{Ct}-1\right)\left(|\epsilon_1-\epsilon_1'|^2\|h_1\|^2+|\epsilon_2-\epsilon_2'|^2\|h_2\|_2^2\right), \quad t>0.$$

This implies

$$\|S_{\alpha_1}(t)z - S_{\alpha_2}(t)z\|_{\mathcal{H}} \le \sqrt{C\left(e^{Ct}-1\right)\left(|\epsilon_1-\epsilon_1'|^2\|h_1\|^2+|\epsilon_2-\epsilon_2'|^2\|h_2\|_2^2\right)}, \quad t>0.$$

Therefore (3) holds. As a conclusion, by applying Theorem 5.2, there exists a dense set $J \subset \Lambda$ such that (5.1) holds. The proof is complete. □

The next result deals with the upper-semicontinuity of the attractor with respect to parameter α.

Theorem 5.4. *Suppose that assumptions (i)–(iii) hold. Then, the attractor $\mathcal{A}_\alpha$ is s upper semicontinuous with respect to the pair $\alpha = (\epsilon_1, \epsilon_2)$ in $\Lambda = [0,1]\times[0,1]$, i.e.*

$$\lim_{\alpha\to\alpha_0} \operatorname{dist}_{\mathcal{H}}(\mathcal{A}_\alpha, \mathcal{A}_{\alpha_0}) = 0, \quad \forall \alpha_0 = (\epsilon_1^0, \epsilon_2^0) \in \Lambda. \tag{5.9}$$

Proof. We proceed by contradiction as in [10, 11]. Suppose that (5.9) does not hold. Then, there exist an $\epsilon > 0$ and a sequence $\alpha_n = (\epsilon_1^n, \epsilon_2^n) \to \alpha_0$ such that

$$\operatorname{dist}_{\mathcal{H}}(\mathcal{A}_{\alpha_n}, \mathcal{A}_{\alpha_0}) \ge \epsilon > 0, \quad \forall n \in \mathbb{N}.$$

Thus, there exists a sequence $\{z_0^n\} \in \mathcal{A}_{\alpha_n}$ by the compactness of $\mathcal{A}_\alpha$ such that

$$\operatorname{dist}_{\mathcal{H}}(z_0^n, \mathcal{A}_{\alpha_0}) \ge \epsilon > 0, \quad \forall n. \tag{5.10}$$

Let $z^n(t) = (u^n(t), \phi^n(t), u_t^n(t), \phi_t^n(t))$ be a full trajectory from the attractor $\mathcal{A}_{\alpha_n}$ such that $z^n(0) = z_0^n$. We know by the Theorem 3.1-(iv) that

$$\{z^n\} \text{ is uniformly bounded in } L^\infty(\mathbb{R}; \mathcal{V}). \tag{5.11}$$

Since $\mathcal{V}$ is compactly embedded into $\mathcal{H}$, using Simon's Compactness Theorem (see [29]), we obtain a subsequence $\{z^{n_k}\}$ and $z \in C([-T,T];\mathcal{H})$ such that

$$\lim_{k\to\infty} \max_{t\in[-T,T]} \|z^{n_k}(t) - z(t)\|_{\mathcal{H}} = 0. \tag{5.12}$$

By (5.11) and (5.12), we conclude that $\sup_{t\in\mathbb{R}} \|z(t)\|_{\mathcal{H}} < \infty$.

Using the same argument as in the proof of property (3) in Theorem 5.2, we can see that

$$z(t) = (u(t), \phi(t), u_t(t), \phi_t(t))$$

solves (in distributional sense) the limiting equations ($\alpha = \alpha_0$)

$$\rho u_{tt} - \mu u_{xx} - b\phi_x + a_1(x)g_1(u_t) + f_1(u,\phi) = \epsilon_1^0 h_1,$$
$$J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + a_2(x)g_2(\phi_t) + f_2(u,\phi) = \epsilon_2^0 h_2.$$

Thus, $z(t)$ is a bounded full trajectory for the limiting semi-flow $S_{\alpha_0}(t)$. Finally, the limit (5.12) implies

$$z_0^{n_k} \to z(0) \in \mathcal{A}_{\alpha_0},$$

which is contradict (5.10). The proof is complete. □

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