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SOLUTIONS TO MAGNETIC SCHRÖDINGER EQUATIONS WITH ARBITRARY GROWTH AT INFINITY

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Abstract. This work addresses the existence of at least one radial solution to the nonlinear magnetic Schrödinger equation

$$\Big(\frac{\varepsilon}{i}\nabla - A(x)\Big)^2 u + V(x)u = f(|u|^2)u \quad \text{in } \mathbb{R}^N,$$

where both the magnetic potential A and the electric potential V are continuous, radial functions. Our main tool is the penalization method developed by del Pino and Felmer [17], which we adapt to the complex-valued setting under magnetic effects. By using the small parameter ε and radial symmetry, we handle nonlinearities with arbitrary growth.

1. Introduction

The purpose of this article is to investigate the magnetic Schrödinger equation

$$\Big(\frac{\varepsilon}{i}\nabla - A(x)\Big)^2 u + V(x)u = f(|u|^2)u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

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where $u : \mathbb{R}^N \to \mathbb{C}$ is an unknown function, $N \geq 3$, i represents the imaginary unit, $A = (A_1, \dots, A_N) : \mathbb{R}^N \to \mathbb{R}^N$ denotes the magnetic (or vector) potential, $V : \mathbb{R}^N \to \mathbb{R}^+$ a potential continuous and the nonlinear term $f:\mathbb{R}^+ \to \mathbb{R}$ is a regular function that satisfies appropriate conditions with arbitrary growth.

A *standing wave* solution for (1.1) is a solution of the form

$$\psi(x,t) = e^{i\omega t}u(x), \tag{1.2}$$

where $\omega \in \mathbb{R}$ is a real frequency, and $u : \mathbb{R}^N \to \mathbb{C}$ is a spatial function to be determined.

Rewriting (1.1) in terms of $\psi(x,t)$, we have

$$\Big(\frac{\varepsilon}{i}\nabla - A(x)\Big)^2 \psi + V(x)\psi = f(|\psi|^2)\psi - i\frac{\partial \psi}{\partial t} \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

Thus, the function ψ satisfies the time-dependent Schrödinger equation with a frequency ω.

The parameter ε in (1.1) represents the Planck constant, typically denoted by $\hbar$ in quantum mechanics. This constant plays a fundamental role in distinguishing quantum mechanics from classical mechanics. In the limit as $\varepsilon \to 0$, the quantum effects vanish, and the equation transitions to a classical regime, where the motion of particles follows Newtonian mechanics. Conversely, for nonzero values of ε, wave-particle duality emerges, and solutions of the equation exhibit interference and localization effects characteristic of quantum systems. The presence of the magnetic potential $A(x)$ further highlights the influence of the electromagnetic field on the quantum behavior of particles, modifying their trajectories according to the principles of gauge theory. For more details on the physical motivations, see [21, 22, 24].

Recently, the study of magnetic Schrödinger equations has been approached from various perspectives, however only a limited number of works have addressed this topic. For instance, in [15], the authors investigate the existence of standing waves for a class of nonlinear Schrödinger equations in $\mathbb{R}^N$, incorporating both electric and magnetic fields. Under suitable non-degeneracy

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assumptions on the critical points of an auxiliary function associated with the electric field, they establish the existence and multiplicity of complex-valued solutions in the semiclassical limit. Moreover, they demonstrate that in this limit, while the presence of a magnetic field induces a phase shift in the complex wave, it does not affect the spatial localization of the wave's modulus peaks.

In [16], a magnetic Schrödinger equation was analyzed in the context of competition between electric potentials. In [7], the authors investigate the existence of infinitely many geometrically distinct solutions to problem (1.1), considering nonlinearities f with either subcritical or critical growth. Using a finite-dimensional reduction approach, [8] and [10] establish a multiplicity result for (1.1). Another multiplicity results can be seen in [4, 5, 6, 18, 23]. For further significant contributions to the study of the magnetic Schrödinger equation, we refer the reader to [3, 9, 11, 12, 14, 25, 28] and the references therein.

Throughout this article, we use the following assumptions:

- (A1) A is a radial continuous function, that is $A \in C(\mathbb{R}^N, \mathbb{R}^N)$ and if $|x| = |y|$ then $A(x) = A(y)$;
- (A2) The potential V is a radial continuous function and there are positive constants $R_1 < r_1 < r_2 < R_2$ such that:
  - (i) $A(x) = (0, \dots, 0) \in \mathbb{R}^N$ in the set $\Omega = \{x \in \mathbb{R}^N : r_1 < |x| < r_2\}$,
  - (ii) $V(x) = 0$ in the set $\Omega = \{x \in \mathbb{R}^N : r_1 < |x| < r_2\}$,
  - (iii) there exists $V_0 > 0$ such that $V(x) \geq V_0$ in the set $\Lambda^c = B_{R_1} \bigcup B^c_{R_2}$;
- (A3) $f \in C(\mathbb{R}^+, \mathbb{R})$ and $\lim_{s \to 0^+} f(s) = 0$ and $f(s) = 0$ for all $s \leq 0$;
- (A4) There exists $\theta > 2$ such that $f(s)s - \theta F(s) > 0$ for every $s \in \mathbb{R}$ with $s > 0$ where $F(s) = \int_0^s f(t)\, dt$;
- (A5) The function $s \to f(s)$ is non-decreasing for $s > 0$.

Now we give typical examples of functions $A(x)$ and $V(x)$ satisfying conditions (A1) and (A2). Let

$$A_i(x) = \begin{cases} r_1 - |x|, & |x| < r_1, \\ 0, & r_1 \leq |x| \leq r_2, \\ |x| - r_2, & |x| > r_2, \end{cases}$$

for $i = 1, \dots N$. In this case A is radial and

$$A(x) = (0, \dots, 0) \quad \text{if } r_1 < |x| < r_2,$$

satisfying condition (A2)(i). Let

$$V(x) - \begin{cases} \exp(\frac{1}{|x|^2 - r_1}), & x \in B_{r_1}(0), \\ 0, & x \in B_{r_2} \cap B^c_{r_1}, \\ |x|^2 - 2r_2 x + r_2^2, & x \in B^c_{r_2}. \end{cases}$$

This function is continuous and satisfies the conditions (ii) and (iii) in (A2). Both functions A and V are continuous and radial, ensuring symmetry with respect to the origin. Now we state the main result in this article.

mainresult **Theorem 1.1.** *Suppose that $N \geq 3$ and* (A1)–(A5) *hold. Then, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ problem* (1.1) *admits a nontrivial complex solution u_ε with $|u_\varepsilon(x)| \to 0$ as $|x| \to +\infty$ and $|u_\varepsilon| \in L^\infty(\mathbb{R}^N) \cap C^{1,\lambda}_{\rm loc}(\mathbb{R}^N)$, for $\lambda \in (0, 1)$.*

Elliptic problems with arbitrary growth can be found as partial differential equations where the growth term can be nonlinear and increase uncontrollably wit respect to the unknown variable or its derivatives. For example, the authors in [2], analyzed the problem

$$-\varepsilon^2 \Delta u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N,$$

which exhibits superlinear growth at infinity, without imposing any constraints on the growth of the function f. Utilizing the force of the parameter ε, the space of radial functions, and the penalization method introduced by Del Pino and Felmer [17], they establish the existence of solutions. The particular case $f(t) = t^p$ with $p > 1$ was studied in [1]. Additionally, an extension of [2] to a Hamiltonian system was investigated in [13].

Other approaches have been used. For instance, in [30], the existence of infinitely many solutions was demonstrated for various elliptic problems under Dirichlet and Neumann boundary conditions. this was also done for a Hamiltonian system, considering nonlinearities exhibiting sublinear behavior at the origin. The methodology involved modifying the nonlinearity and obtaining solutions with small L^∞ norms, ensuring that each solution of the modified problem corresponded to a solution of the original problem. The case studied in [30], where the nonlinearity could change sign, was explored in [20]. In [19], a Kirchhoff problem was analyzed, considering nonlinearities that exhibit both sublinear and linear behavior at the origin, employing the strategy introduced in [30].

The main contributions of this article are as follows:

(1) When the growth of the nonlinearity is not bounded by a polynomial function of controlled order, classical methods such as Sobolev embeddings may not directly apply. To overcome this difficulty we are adapting to our case the arguments that can be found in [1, 2, 13]. However, with respect to [1, 2, 13], due to the presence of the magnetic potential A and the fact that the solutions assume complex values, a more careful analysis will be needed and some refined estimates will be given in Lemmas 3.3, 4.1, and 4.2.

(2) Proving the existence of weak solutions can be challenging, especially if the growth of the nonlinearity does not satisfy suitable structural conditions. For this reason, it was necessary to put forward appropriate hypotheses on the magnetic potential A that were compatible with the existing hypotheses about the electrical potential V. This is another point that differs our article from the articles [1], [2] and [13] that consider solutions that assume real values and $A = 0$.

(3) Even if existence is guaranteed, proving sufficient regularity for the solutions can be complicated, as arbitrary growth may generate instabilities or singularities. Once again, due to the presence of the magnetic potential A and the fact that the solutions assume complex values, it was necessary to use Moser's iteration method in Lemma 4.2.

This article is organized as follows. In Section 2 we introduce the Banach space where we will look for the solution to the problem and we recall an important inequality called the diamagnetic inequality. In Section 3 we use the Del Pino Felmer Penalization [17]. With this penalty and the fact that we are in the space of radial functions, we get around the difficulty of having an arbitrary growth. In fact, it is possible to obtain a functional associated to the penalized problem. In this section we show that this functional has the Geometry of the Mountain Pass Theorem and that Palais-Smale sequences have strongly convergent subsequences. In Section 4, using the strength of the parameter ε and the fact that we obtain a radial solution to the penalized problem, we show that the solution to the penalized problem is a solution to the original problem.

tools

2. Notation and variational tools

To introduce the variational structure of the problem, we define the Hilbert space $H^1_{A,\varepsilon}(\mathbb{R}^N,\mathbb{C})$ obtained by the closure of $C_0^\infty(\mathbb{R}^N,\mathbb{C})$ under the scalar product

$$(u,v)_{A,\varepsilon} = \operatorname{Re}\int_{\mathbb{R}^N} \big(\nabla_{A,\varepsilon}u\cdot\overline{\nabla_{A,\varepsilon}v} + V(x)u\overline{v}\big)\,dx \quad \text{for all } u,v\in H^1_{A,\varepsilon}(\mathbb{R}^N,\mathbb{C})$$

where $\nabla_{A,\varepsilon}u = (D_1^\varepsilon u,\dots,D_N^\varepsilon u)$, $D_j^\varepsilon = \frac{\varepsilon}{i}\partial_j - A_j(x)$, and Re and the bar denote the real part of a complex number and the complex conjugation respectively. The norm induced by this inner product is

$$\|u\|_{A,\varepsilon} = \Big(\int_{\mathbb{R}^N}(|\nabla_{A,\varepsilon}u|^2 + V(x)|u|^2)\,dx\Big)^{1/2} \quad \text{for } u\in H^1_{A,\varepsilon}(\mathbb{R}^N,\mathbb{C}).$$

Since there is no relationship between $H^1_{A,\varepsilon}(\mathbb{R}^N,\mathbb{C})$ and $H^1(\mathbb{R}^N,\mathbb{R})$; that is, $H^1_{A,\varepsilon}(\mathbb{R}^N,\mathbb{C}) \not\subset H^1(\mathbb{R}^N,\mathbb{R})$ and $H^1(\mathbb{R}^N,\mathbb{C}) \not\subset H^1_{A,\varepsilon}(\mathbb{R}^N,\mathbb{C})$, we will frequently use in this paper the following diamagnetic inequality (see [25, Theorem 7.21])

$$\varepsilon|\nabla|u|(x)| \le |\nabla_{A,\varepsilon}u(x)| \quad \text{for almost every } x\in\mathbb{R}^N\,. \tag{2.1}$$ diam

This implies that, if $u \in H^1_{A,\varepsilon}(\mathbb{R}^N, \mathbb{C})$ then $|u| \in H^1(\mathbb{R}^N, \mathbb{R})$. Therefore, $u \in L^p(\mathbb{R}^N, \mathbb{C})$ for any $p \in [2, 2^*]$.

We denote by $H^1_{A,\varepsilon,\mathrm{rad}}(\mathbb{R}^N, \mathbb{C})$ the subspace of $H^1_{A,\varepsilon}(\mathbb{R}^N, \mathbb{C})$ formed by the radial functions, that is

$$H^1_{A,\varepsilon,\mathrm{rad}}(\mathbb{R}^N, \mathbb{C}) = \{u \in H^1_{A,\varepsilon}(\mathbb{R}^N, \mathbb{C}) : u(x) = u(|x|) \text{ for } x \in \mathbb{R}^N\}.$$

Auxiliary

3. Auxiliary problems

Note that, since the functional has arbitrary growth, it is not possible to obtain a functional directly associated to this problem. To overcome this difficulty, we will use the Del Pino and Felmer prenalization method [17] and make strong use of the radiality of the problem to obtain a functional that satisfies the Mountain Pass Geometry and that has compactness.

We choose $k > \theta/(\theta - 2)$, where θ is given by (A4), and using (A5), take $a > 0$ to be the unique number such that $f(a) = V_0/k$, with V_0 given by (A2). We set

$$\widehat{f}(s) := \begin{cases} f(s), & \text{if } s \leq a, \\ V_0/k, & \text{if } s > a. \end{cases}$$

Let χ_Λ denotes the characteristic function of the set Λ. We introduce the penalized nonlinearity $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ by setting

$$g(x, s) := \chi_\Lambda(x) f(s) + (1 - \chi_\Lambda(x))\widehat{f}(s). \tag{3.1}$$ def_g

Now, we shall consider the modified problem

$$\begin{gathered} (\frac{\varepsilon}{i}\nabla - A(x))^2 u + V(x)u = g(x, |u|^2)u, \quad x \in \mathbb{R}^N, \\ u \in H^1_{A,\varepsilon}(\mathbb{R}^N, \mathbb{C}). \end{gathered} \tag{3.2}$$ tDea

Notice that if u_ε is a solution of the above problem such that $|u_\varepsilon(x)| < a^{1/2}$ in Λ^c ,then, in view of the definition of g, there holds $g(x, |u|^2)u = f(|u|^2)u$ for each $x \in \mathbb{R}^N$. Thus, the function u is also a solution of the original problem (1.1).

In view of the above comment, in the sequel we study the modified problem (3.2). In the next result we prove that some properties on the continuous function $g(x, s)$.

lem3.1 **Lemma 3.1.** *The continuous function $g(x, s)$ satisfies the following properties uniformly in $x \in \mathbb{R}^N$:*

1. *$g(x, s) = 0$ for each $s \leq 0$;*
2. *$\lim_{s\to 0^+} g(x, s) = 0$, g and $G(s) = \int_0^s g(t)dt$ are radials in x;*
3. (i) *$0 \leq \frac{\theta}{2} G(x, s) < g(x, s)s$, for each $x \in \Lambda$, $s > 0$,*
   (ii) *$0 \leq G(x, s) \leq \frac{V(x)}{k} s$ and $0 \leq g(x, s) \leq \frac{V(x)}{k}$, for each $x \in \Lambda^c$, $s > 0$;*
4. *the function $s \mapsto g(x, s)$ is non-decreasing for $s > 0$;*
5. *for all $u \in H^1_{A,\varepsilon,rad}(\mathbb{R}^N, \mathbb{C})$, we have $\int_{\mathbb{R}^N} G(x, |u|^2)dx < \infty$ and $\int_{\mathbb{R}^N} g(x, |u|^2)|u|^2 dx < \infty$.*

Proof. Note that from (1)–(4) follow by the definition of g. Now, for $u \in H^1_{A,\varepsilon,\mathrm{rad}}(\mathbb{R}^N, \mathbb{C})$, from [27], there exists $C_N > 0$, which depends just the dimension N such that

$$|u(x)| \leq \frac{C_N \|u\|_{A,\varepsilon}}{|x|^{(N-1)/2}}. \tag{3.3}$$ Decaimento

Then

$$\begin{aligned} \int_{\mathbb{R}^N} G(x, |u|^2)dx &\leq \int_\Lambda G\Big(x, \frac{C_N^2 \|u\|^2_{A,\varepsilon}}{R_1}\Big)dx + \int_{\Lambda^c} G(x, |u|^2)dx \\ &\leq \max_{\overline{\Lambda}} G\Big(\cdot, \frac{C_N^2 \|u\|^2_{A,\varepsilon}}{R_1}\Big)|\Lambda| + \frac{V_0}{k}\int_{\Lambda^c} |u|^2 dx < \infty. \end{aligned} \tag{3.4}$$ calhou

Using the same argument we can conclude that $\int_{\mathbb{R}^N} g(x, |u|^2)|u|^2 dx < \infty$. $\square$

By Lemma 3.1(5), the functional associated with (3.2), namely

$$J_\varepsilon(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla_{A,\varepsilon} u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^N} G(x,|u|^2)dx, \quad u \in H^1_{A,\varepsilon,\mathrm{rad}}(\mathbb{R}^N,\mathbb{C})$$

belongs to $C^1(H^1_{A,\varepsilon,\mathrm{rad}}(\mathbb{R}^N,\mathbb{C}),\mathbb{R})$ with Gâteaux differential given by

$$J'_\varepsilon(u)v := \mathrm{Re}\Big(\int_{\mathbb{R}^N} \nabla_{A,\varepsilon} u\overline{\nabla_{A,\varepsilon} v}dx + \int_{\mathbb{R}^N} V(x)u\bar{v}dx - \int_{\mathbb{R}^N} g(x,|u|^2)u\bar{v}dx\Big), \quad u,v \in H^1_{A,\varepsilon,\mathrm{rad}}(\mathbb{R}^N,\mathbb{C}).$$

Moreover, its critical points are the weak solutions of the modified problem (3.2).

Now we prove that the associated functional J_ε satisfies the Mountain Pass Geometry.

mountpassIA **Lemma 3.2.** *Suppose (1)–(5) of Lemma 3.1 hold. Then, the functional J_ε has a Mountain Pass Geometry, that is*

(i) $J_\varepsilon(0) = 0$;
(ii) *there exist $\rho, \delta > 0$ such that $J_\varepsilon(u) \geq \delta$ for all $u \in H^1_{A,\varepsilon,}(\mathbb{R}^N,\mathbb{C})$ with $\|u\|_{A,\varepsilon} = \rho$;*
(iii) *there exists $u_0 \in H^1_{A,\varepsilon,\mathrm{rad}}(\mathbb{R}^N,\mathbb{C})$ such that $\|u\|_{A,\varepsilon} > \rho_0$ and $J_\varepsilon(u_0) \leq 0$.*

Proof. Note that by (2) of Lemma 3.1, given $\Upsilon > 0$, there exists $\delta > 0$ such that for all $0 \leq s < \delta$. Then we have

$$G(s) = F(s) \leq \frac{\Upsilon}{2}|s|^2.$$

Considering (3.3) and $\|u\|_{A,\varepsilon} = \rho$ with ρ enough small such that $|u(x)| \leq \frac{C_N\|u\|_{A,\varepsilon}}{R_1^{1/2}} \leq \delta$, there exists $C > 0$ such that

$$\begin{aligned} J_\varepsilon(u) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla_{A,\varepsilon} u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u|^2 dx - \frac{1}{2}\int_{\Lambda} G(x,|u|^2)dx - \frac{1}{2}\int_{\Lambda^c} G(x,|u|^2)dx \\ &\geq \frac{1}{2}\|u\|^2_{A,\varepsilon} - \frac{\Upsilon}{4}\int_{\mathbb{R}^N} |u|^2 dx - \frac{1}{2k}\int_{\mathbb{R}^N} V(x)|u|^2 dx \geq C\|u\|^2_{A,\varepsilon}, \end{aligned}$$

and the proof of (ii) is complete, decreasing ρ if necessary.

(iii) Note that, for all $x \in \Lambda$, from (G_3), there exist $C > 0$ and $s_0 > 0$ such that $G(x,s) \leq Cs^\theta$ for all $s \geq s_0$. Now consider $v_0 \in C_0^\infty(\Lambda)$ and note that

$$J_\varepsilon(sv_0) \leq \frac{s^2}{2}\|v_0\|^2_{A,\varepsilon} - \frac{s^\theta}{2}\int_\Lambda |v_0|^\theta\, dx.$$

Since $\theta > 2$, we obtain $J_\varepsilon(sv_0) \to -\infty$ as $s \to +\infty$ thus, taken $u_0 = sv_0$ for s sufficiently large (iii) is proved. □

By [29, Theorem 1.15] we assert the existence of a Palais-Smale sequence (u_n) in $H^1_{A,\varepsilon,\mathrm{rad}}(\mathbb{R}^N,\mathbb{C})$ at level d_ε, that is, a sequence with the property

$$J_\varepsilon(u_n) \to d_\varepsilon \quad \text{and} \quad J'_\varepsilon(u_n) \to 0,$$

where d_ε is the minimax level of the mountain pass theorem related to J_ε, namely

$$d_\varepsilon = \inf_{\varsigma\in\mathfrak{J}} \max_{t\in[0,1]} J_\varepsilon(\varsigma(t)), \tag{3.5}$$ beta0

where

$$\mathfrak{J} = \left\{\varsigma \in C([0,1], H^1_{A,\varepsilon,\mathrm{rad}}(\mathbb{R}^N,\mathbb{C}) : \varsigma(0) = 0 \quad \text{and} \quad J_\varepsilon(\varsigma(1)) < 0\right\}.$$

By a reasoning similar to the one in [29, Theorem 4.2], we have

$$d_\varepsilon = \inf_{u\in H^1_{A,\varepsilon,\mathrm{rad}}(\mathbb{R}^N,\mathbb{C}), u\neq 0} \sup_{t\geq 0} \Phi_\varepsilon(tu). \tag{3.6}$$ beta-e

The main feature of the modified functional is that it satisfies the Palais-Smale condition, as we can see from the next result.

vit5 **Lemma 3.3.** *The functional J_ε satisfies the $(PS)_{d_\varepsilon}$ condition for any level $d_\varepsilon \in \mathbb{R}$.*

Proof. Suppose that $(u_n) \subset H^1_{A,\varepsilon,\mathrm{rad}}(\mathbb{R}^N, \mathbb{C})$ is a $(\mathrm{PS})_{d_\varepsilon}$ sequence for J_ε, that is, $J_\varepsilon(u_n) \to d_\varepsilon$ and $J'_\varepsilon(u_n) \to 0$. We first prove that (u_n) is bounded in $H^1_{A,\varepsilon}(\mathbb{R}^N, \mathbb{C})$. Indeed, by using (3) of lemma 3.1 we obtain

$$\begin{aligned} d_\varepsilon + o_n(1)\|u_n\|_{A,\varepsilon} &\geq J_\varepsilon(u_n) - \frac{1}{\theta} J'_\varepsilon(u_n)u_n \\ &\geq \big(\frac{1}{2} - \frac{1}{\theta}\big)\|u_n\|^2_{A,\varepsilon} + \frac{1}{\theta}\int_{\Lambda^c_\varepsilon} g(x, |u_n|^2)|u_n|^2 dx - \frac{1}{2}\int_{\Lambda^c_\varepsilon} G(x, |u_n|^2) dx \\ &\geq \frac{1}{2}\big(\frac{\theta - 2}{\theta} - \frac{1}{k}\big)\|u_n\|^2_{A,\varepsilon}, \end{aligned}$$

where $o_n(1)$ denotes a quantity approaching zero as $n \to \infty$. Since $k > \theta/(\theta - 2)$ we conclude from the above inequality that (u_n) is bounded in $H^1_{A,\varepsilon}(\mathbb{R}^N, \mathbb{C})$.

Claim. For any given $\zeta > 0$, there exists $R = R(\zeta) > 0$ such that $R > 4R_2$ and

$$\limsup_{n\to\infty} \int_{B_R(0)^c} (|\nabla_{A,\varepsilon} u_n|^2 + V(x)|u_n|^2) dx \leq \zeta. \tag{3.7}$$

Crucial

To prove the claim we consider $\eta_R \in C^\infty(\mathbb{R}^N, \mathbb{R})$ such that $0 \leq \eta_R \leq 1$, $\eta_R \equiv 0$ in $B_{R/2}(0)$, $\eta_R \equiv 1$ in $B_R(0)^c$ and $|\nabla \eta_R| \leq C/R$, where $C > 0$ is a constant independent of R. Since the sequence $(\eta_R u_n)$ is bounded in $H^1_{A,\varepsilon}(\mathbb{R}^N, \mathbb{C})$, we have that $J'_\varepsilon(u_n)(\eta_R u_n) = o_n(1)$, that is,

$$\mathrm{Re}\Big(\int_{\mathbb{R}^N} \nabla_{A,\varepsilon} u_n \overline{\nabla_{A,\varepsilon}(u_n \eta_R)} dx\Big) + \int_{\mathbb{R}^N} V(x)|u_n|^2 \eta_R dx = \int_{\mathbb{R}^N} g(x, |u_n|^2)|u_n|^2 \eta_R dx + o_n(1).$$

Since η_R take values in $\mathbb{R}$, a direct calculation shows that

$$\overline{\nabla_{A,\varepsilon}(u_n \eta_R)} = i\overline{u_n}\nabla \eta_R + \eta_R \overline{\nabla_{A,\varepsilon} u_n}.$$

The two above equalities and Lemma 3.1(3)(ii) imply that

$$\begin{aligned} &\int_{\mathbb{R}^N} \Big(|\nabla_{A,\varepsilon} u_n|^2 + V(x)|u_n|^2\Big)\eta_R dx \\ &\leq \frac{1}{k}\int_{\mathbb{R}^N} V(x)|u_n|^2 \eta_R dx + \mathrm{Re}\Big(\int_{\mathbb{R}^N} -i\overline{u_n}\nabla_{A,\varepsilon} u_n \nabla \eta_R\Big) dx + o_n(1). \end{aligned}$$

By using the definition of η_R, Hölder's inequality and the boundedness of (u_n) we obtain

$$\begin{aligned} \Big(1 - \frac{1}{k}\Big)\int_{B_R(0)^c} \left(|\nabla_{A,\varepsilon} u_n|^2 + V(x)|u_n|^2\right) dx &\leq \frac{C}{R}\|\overline{u_n}\|_{L^2}\|\nabla_{A,\varepsilon} u_n\|_{L^2} + o_n(1) \\ &\leq \frac{C_1}{R} + o_n(1). \end{aligned}$$

So, for any fixed $\zeta > 0$, we can choose $R > 0$ large enough, such that

$$\limsup_{n\to\infty} \int_{B_R(0)^c} (|\nabla_{A,\varepsilon} u_n|^2 + V(x)|u_n|^2) \leq \zeta.$$

This completes the proof of the claim.

Now note that

$$\begin{aligned} \int_{\mathbb{R}^N} \left(g\left(x, |u_n|^2\right)|u_n|^2 - g(x, |u|^2)|u|^2\right) dx &= \int_{B_{R_1}} \left(g\left(x, |u_n|^2\right)|u_n|^2 - g(x, |u|^2)|u|^2\right) dx \\ &\quad + \int_{B_R \setminus B_{R_1}} \left(g\left(x, |u_n|^2\right)|u_n|^2 - g(x, |u|^2)|u|^2\right) dx \\ &\quad + \int_{B^c_R} \left(g\left(x, |u_n|^2\right)|u_n|^2 - g(x, |u|^2)|u|^2\right) dx. \end{aligned}$$

We shall prove that each of these terms approaches zero as $n \to \infty$. From the boundedness of $B_{R_1} \subset \Lambda^c$, we have $u_n \to u$ in $L^2(B_{R_1})$. By Lemma 3.1(3) it follows that

$$\int_{B_{R_1}} \left(g\left(x, |u_n|^2\right)|u_n|^2 - g(x, |u|^2)|u|^2\right) dx = o_n(1).$$

Using the proof of lemma 3.1(5) and Lebesgue's Dominated Convergence Theorem, we conclude that

$$\int_{B_R\setminus B_{R_1}} \left(g\left(x,|u_n|^2\right)|u_n|^2 - g(x,|u|^2)|u|^2\right)dx = o_n(1).$$

Now note that

$$\begin{aligned}
&\Big|\int_{B_R^c} g\left(x,|u_n|^2\right)|u_n|^2 - g(x,|u|^2)|u|^2dx\Big| \\
&\le \int_{B_R^c} g\left(x,|u_n|^2\right)|u_n|^2dx + \int_{B_R^c} g(x,|u|^2)|u|^2dx \\
&\le \int_{B_R^c} (|\nabla_{A,\varepsilon}u_n|^2 + V(x)|u_n|^2) + \int_{B_R^c} g\left(x,|u_n|^2\right)|u_n|^2dx.
\end{aligned}$$

Since $\int_{B_R^c} g\left(x,|u_n|^2\right)|u_n|^2dx$ is integrable and using (3.7), we obtain

$$\int_{B_R^c} \left(g\left(x,|u_n|^2\right)|u_n|^2 - g(x,|u|^2)|u|^2\right)dx = o_n(1).$$

Observe that $J_\varepsilon'(u_n)u = o_n(1)$, which implies that

$$\|u\|_{A,\varepsilon}^2 = \int_{\mathbb{R}^N} g(x,|u|^2)|u|^2dx.$$

Then

$$\lim_{n\to\infty}\|u_n\|_{A,\varepsilon}^2 = \lim_{n\to\infty}\int_{\mathbb{R}^N} g(x,|u_n|^2)|u_n|^2dx = \int_{\mathbb{R}^N} g(x,|u|^2)|u|^2dx = \|u\|_{A,\varepsilon}^2. \qquad \square$$

4. Proof of Theorem 1.1

From Lemmas 3.2 and 3.3, for each $\varepsilon > 0$, there exists $u_\varepsilon \in H^1_{A,\varepsilon,\mathrm{rad}}(\mathbb{R}^N,\mathbb{C})$ weak solution of problem (3.2). That is,

$$J_\varepsilon(u_\varepsilon) = d_\varepsilon \quad \text{and} \quad J_\varepsilon'(u_\epsilon)v = 0, \forall v \in H^1_{A,\varepsilon,\mathrm{rad}}(\mathbb{R}^N,\mathbb{C}).$$

Note that, by the Principle of Symmetric Criticality [29, Theorem 1.28], we have that u_ε is in fact a critical point of J_ε in the space $H^1_{A,\varepsilon}(\mathbb{R}^N,\mathbb{C})$. The next result is crucial for this section.

vit55 **Lemma 4.1.** *$\|u_\varepsilon\|^2_{A,\varepsilon} \to 0$ as $\varepsilon \to 0$.*

Proof. Taking $\psi \in C^\infty_{0,\,\mathrm{rad}}(\Omega,\mathbb{R})$, a nonnegative function with $\operatorname{supp}\psi \subset \Omega$, there is a unique $t_\epsilon \in \mathbb{R}^+$ such that

$$J_\epsilon(t_\epsilon\psi) = \max_{t\ge 0} J_\epsilon(t\psi).$$

Then, from (A2),

$$\epsilon^2\int_\Omega |\nabla\psi|^2dx = \int_\Omega f\left(|t_\epsilon\psi|^2\right)|\psi|^2dx,$$

and choosing $\Omega_1 \subset \Omega$ such that $\psi(x) \ge \psi_0 > 0$ for all $x \in \Omega_1$, it follows that

$$\epsilon^2\int_\Omega |\nabla\psi|^2dx \ge \int_{\Omega_1} f\left(|t_\epsilon\psi|^2\right)|\psi|^2 \ge \psi_0^2\int_{\Omega_1} f\left(|t_\epsilon\psi|^2\right)dx.$$

Thus, from (A5), we conclude that $t_\varepsilon \to 0$ as $\varepsilon \to 0$. Furthermore,

$$0 < d_\varepsilon \le J_\epsilon(t_\varepsilon\psi) \le \frac{t_\varepsilon^2}{2}\int_\Omega \epsilon^2|\nabla\psi|^2dx,$$

which implies that $d_\varepsilon \to 0$ as $\varepsilon \to 0$. On the other hand, there exists $C > 0$ such that

$$\begin{aligned}
d_\varepsilon &= J_\epsilon(u_\epsilon) = J_\epsilon(u_\epsilon) - \frac{1}{\theta}J_\epsilon'(u_\epsilon)u_\epsilon \\
&\ge \left(\frac{1}{2}-\frac{1}{\theta}\right)\Big(\int_{\mathbb{R}^N}\Big(|\nabla_{A,\varepsilon}u_\varepsilon|^2 + \left(1-\frac{1}{k}\right)V(x)|u_\epsilon|^2\Big)dx\Big) \\
&\ge C\|u_\varepsilon\|^2_{A,\varepsilon}.
\end{aligned}$$

The proof is complete. $\square$

Now we prove a regularity result. Its proof follows the argument found in [5, Lemma 4.1.]. We will give the proof here for completeness of this work.

leminha1 **Lemma 4.2.** *Let (ε_n) be a sequence of positive numbers with $\varepsilon_n \to 0^+$ and $u_n \in H^1_{A,\varepsilon_n,\mathrm{rad}}(\mathbb{R}^N,\mathbb{C})$ be a solution of (3.2). Then $|u_n| \in L^\infty(\mathbb{R}^N) \cap C^{1,\lambda}_{\mathrm{loc}}(\mathbb{R}^N)$, for $\lambda \in (0,1)$. Moreover, there exists a constant $C > 0$, independent on n, such that $|u_n| \le C\|u_n\|_{A,\varepsilon_n}$.*

Proof. We define $u_{L,n} \in H^1_{A,\varepsilon_n,\mathrm{rad}}(\mathbb{R}^N,\mathbb{C})$ and $z_{L,n} \in H^1_{A,\varepsilon_n,\mathrm{rad}}(\mathbb{R}^N,\mathbb{C})$ by setting

$$u_{L,n}(x) := \min\{|u_n(x)|, L\}, \quad z_{L,n} := u_{L,n}^{2(\beta-1)} u_n,$$

with $\beta > 1$ to be determined later. By using the calculation performed in [14, equation (2.2)] and the diamagnetic inequality we obtain

$$\mathrm{Re}\left(\nabla_{A,\varepsilon_n} u_n \overline{\nabla_{A,\varepsilon_n} z_{L,n}}\right) \ge u_{L,n}^{2(\beta-1)} |\nabla_{A,\varepsilon_n} u_n|^2 \ge u_{L,n}^{2(\beta-1)} \varepsilon_n^2 \big|\nabla|u_n|\big|^2.$$

This inequality, the definition of $z_{L,n}$ and $J'_{\varepsilon_n}(u_n) z_{L,n} = 0$ imply that

$$\int_{\mathbb{R}^N} u_{L,n}^{2(\beta-1)} |\nabla|u_n||^2 dx \le \int_{\mathbb{R}^N} \left(g(x,|u_n|^2) - V(x)\right) |u_n|^2 u_{L,n}^{2(\beta-1)} dx. \tag{4.1}$$ moser1

Arguing as in (3.4), using Lemma 4.1 and Lemma 3.1(1), we obtain $C_1 > 0$ such that

$$g(x,s) \le \frac{V_0}{2} + C_1 |s|^{(2^*-2)/2}, \text{ for all } (x,s) \in \mathbb{R}^N \times \mathbb{R}.$$

This, (4.1), and $V(x) \ge V_0$ provide

$$\begin{aligned} \int_{\mathbb{R}^N} u_{L,n}^{2(\beta-1)} \varepsilon_n^2 \big|\nabla|u_n|\big|^2 dx &\le \int_{\mathbb{R}^N} \Big(\frac{V_0}{2} + C_1 |u_n|^{2^*-2} - V(x)\Big) |u_n|^2 u_{L,n}^{2(\beta-1)} dx \\ &\le C_1 \int_{\mathbb{R}^N} |u_n|^{2^*} u_{L,n}^{2(\beta-1)} dx. \end{aligned} \tag{4.2}$$ 2moser

Let S be the best constant of the embedding $\mathcal{D}^{1,2}(\mathbb{R}^N,\mathbb{R}) \hookrightarrow L^{2^*}(\mathbb{R}^N,\mathbb{R})$ and define $\widehat{u}_{L,n} := |u_n| u_{L,n}^{\beta-1}$. We have that

$$S^{-1} \|\widehat{u}_{L,n}\|^2_{L^{2^*}} \le \int_{\mathbb{R}^N} \big|\nabla\big(|u_n| u_{L,n}^{\beta-1}\big)\big|^2 dx.$$

But

$$\begin{aligned} \int_{\mathbb{R}^N} \big|\nabla\big(|u_n| u_{L,n}^{\beta-1}\big)\big|^2 dx &= \int_{\{|u_n| \le L\}} \big|\nabla\big(|u_n| u_{L,n}^{\beta-1}\big)\big|^2 dx + \int_{\{|u_n| > L\}} \big|\nabla\big(|u_n| u_{L,n}^{\beta-1}\big)\big|^2 dx \\ &= \int_{\{|u_n| \le L\}} \big|\nabla |u_n|^\beta\big|^2 \, dx + \int_{\{|u_n| > L\}} L^{2(\beta-1)} \, |\nabla|u_n||^2 \, dx \\ &\le \beta^2 \int_{\mathbb{R}^N} u_{L,n}^{2(\beta-1)} \varepsilon_n^2 \big|\nabla|u_n|\big|^2 dx, \end{aligned}$$

and therefore

$$\|\widehat{u}_{L,n}\|^2_{L^{2^*}} \le C_3 \beta^2 \int_{\mathbb{R}^N} u_{L,n}^{2(\beta-1)} \varepsilon_n^2 \big|\nabla|u_n|\big|^2 dx.$$

This and (4.2) yield

$$\|\widehat{u}_{L,n}\|^2_{L^{2^*}} \le C_4 \beta^2 \int_{\mathbb{R}^N} |u_n|^{2^*} u_{L,n}^{2(\beta-1)} dx, \tag{4.3}$$ 4moser

for all $\beta > 1$. The above expression and $u_{L,n} \le |u_n|$, imply that

$$\|\widehat{u}_{L,n}\|^2_{L^{2^*}} \le C_4 \beta^2 \int_{\mathbb{R}^N} |u_n|^{2^*-2} |u_n|^{2\beta} dx. \tag{4.4}$$ 5moser

Now, setting

$$t := \frac{2^* 2^*}{2(2^*-2)} > 1, \quad \alpha := \frac{2t}{t-1} < 2^*, \tag{4.5}$$ def_t

we can apply Hölder's inequality with exponents $t/(t-1)$ and t in (4.4), to obtain

$$\|\widehat{u}_{L,n}\|^2_{L^{2^*}} \le C_4\beta^2\|u_n\|^{2\beta}_{L^{\beta\alpha}}\Big(\int_{\mathbb{R}^N}|u_n|^{2^*(2^*/2)}dx\Big)^{1/t}. \tag{4.6}$$

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Claim. There exist $n_0\in\mathbb{N}$ and $K>0$ such that, for any $n\ge n_0$, it holds

$$\int_{\mathbb{R}^N}|u_n|^{2^*(2^*/2)}dx\le K.$$

Assuming the claim is true, we can use (4.6) to conclude that

$$\|\widehat{u}_{L,n}\|^2_{L^{2^*}}\le C_6\beta^2\|u_n\|^{2\beta}_{L^{\beta\alpha}}.$$

Since

$$\|u_{L,n}\|^{2\beta}_{L^{\beta 2^*}}=\Big(\int_{\mathbb{R}^N}u_{L,n}^{\beta 2^*}dx\Big)^{2/2^*}\le\Big(\int_{\mathbb{R}^N}|u_n|^{2^*}u_{L,n}^{2^*(\beta-1)}dx\Big)^{2/2^*}=\|\widehat{u}_{L,n}\|^2_{L^{2^*}}\le C_6\beta^2\|u_n\|^{2\beta}_{L^{\beta\alpha}},$$

we can apply Fatou's lemma in the variable L to obtain

$$\||u_n|\|_{L^{\beta 2^*}}\le C_7^{1/\beta}\beta^{1/\beta}\||u_n|\|_{L^{\beta\alpha}}.$$

We now set $\beta:=2^*/\alpha>1$ and note that, since $|u_n|\in L^{2^*}(\mathbb{R}^N)$, the above inequality holds for this choice of β. Moreover, since $\beta^2\alpha=\beta 2^*$, it follows that the inequality also holds with β replaced by β^2. Hence,

$$\||u_n|\|_{L^{\beta^2 2^*}}\le C_7^{1/\beta^2}\beta^{2/\beta^2}\||u_n|\|_{L^{\beta^2\alpha}}.$$

By iterating this process and recalling that $\beta\alpha=2^*$ we obtain, for $k\in\mathbb{N}$,

$$\||u_n|\|_{L^{\beta^k 2^*}}\le C_7^{\sum_{i=1}^k\beta^{-i}}\beta^{\sum_{i=1}^m i\beta^{-i}}\||u_n|\|_{L^{2^*}}.$$

Since $\beta>1$ we can take the limit as $k\to\infty$ to obtain

$$\||u_n|\|_{L^\infty}\le C_8\||u_n|\|_{L^{2^*}}.$$

From the Sobolev imbedding, there exists a constant positive C, independent on n such that

$$\||u_n|\|_{L^\infty}\le C\||u_n|\|_{A,\varepsilon_n}. \tag{4.7}$$

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By the elliptic regularity $|u_n|\in L^\infty(\mathbb{R}^N)\cap C^{1,\lambda}_{\rm loc}(\mathbb{R}^N)$, for $\lambda\in(0,1)$.

It remains to prove the claim. In fact, let $\beta=2^*/2$. From (4.3), we have

$$|\widehat{u}_{L,n}|^2_{2^*}\le C\beta^2\int_{\mathbb{R}^N}u_n^{2^*}u_{L,n}^{(2^*-2)}dx,$$

or equivalently

$$|\widehat{u}_{L,n}|^2_{2^*}\le C\beta^2\int_{\mathbb{R}^N}u_n^2u_{L,n}^{(2^*-2)}u_n^{(2^*-2)}dx.$$

Using the Hölder inequality with exponents $\frac{2^*}{2}$ and $\frac{2^*}{2^*-2}$

$$|\widehat{u}_{L,n}|^2_{2^*}\le C\beta^2\Big(\int_{\mathbb{R}^N}\big[u_nu_{L,n}^{\frac{(2^*-2)}{2}}\big]^{2^*}dx\Big)^{2/2^*}\Big(\int_{\mathbb{R}^N}u_n^{2^*}dx\Big)^{\frac{2^*-2}{2^*}}.$$

From definition of $\widehat{u}_{L,n}$, we have

$$\Big(\int_{\mathbb{R}^N}\Big[u_nu_{L,n}^{\frac{(2^*-2)}{2}}\Big]^{2^*}dx\Big)^{2/2^*}\le C\beta^2\Big(\int_{\mathbb{R}^N}\big[u_nu_{L,n}^{\frac{(2^*-2)}{2}}\big]^{2^*}dx\Big)^{2/2^*}\Big(\int_{\mathbb{R}^N}u_n^{2^*}dx\Big)^{\frac{2^*-2}{2^*}}.$$

From Lemma 4.1, we conclude that

$$\Big(\int_{\mathbb{R}^N}\big[u_nu_{L,n}^{\frac{(2^*-2)}{2}}\big]^{2^*}dx\Big)^{2/2^*}\le C\beta^p\int_{\mathbb{R}^N}u_n^2u_{L,n}^{(2^*-2)}dx,$$

or equivalently

$$\Big(\int_{\mathbb{R}^N}\big[u_nu_{L,n}^{\frac{(2^*-2)}{2}}\big]^{2^*}dx\Big)^{2/2^*}\le C\beta^p\int_{\mathbb{R}^N}u_n^{2^*}dx\le K<\infty.$$

Using the Fatou's lemma in the variable L, we have

$$\int_{\mathbb{R}^N} u_n^{2^*(2^*/2)} dx \le K < \infty,$$

and therefore the claim holds. $\square$

4.1. **Proof of Theorem 1.1.** From (4.7) in Lemma 4.2, we have

$$|u_\varepsilon| \le C\|u_\varepsilon\|_{A,\epsilon}.$$

From Lemma 4.1, for $\varepsilon > 0$ sufficiently small, we have that

$$|u_\varepsilon| < a^{1/2}$$

and this completes the proof.

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