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WELL-POSEDNESS OF GENERALIZED MAGNETOHYDRODYNAMIC EQUATIONS IN VARIABLE LEBESGUE SPACES

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Abstract. This article concerns the well-posedness of the generalized magnetohydrodynamic equations in variable Lebesgue spaces. By using some basic properties of variable Lebesgue spaces and decay estimates of the fractional heat kernel, we prove the existence of local and global solutions to the generalized magnetohydrodynamic equations in two different types of variable Lebesgue spaces.

1. Introduction

In this article, we consider the Cauchy problem of the three-dimensional incompressible generalized magnetohydrodynamic equations,

$$\begin{gathered}
\partial_t u + (-\Delta)^\alpha u + (u\cdot\nabla)u - (B\cdot\nabla)B + \nabla P = 0 \quad \text{in } \mathbb{R}^3\times\mathbb{R}_+,\\
\partial_t B + (-\Delta)^\beta B + (u\cdot\nabla)B - (B\cdot\nabla)u = 0 \quad \text{in } \mathbb{R}^3\times\mathbb{R}_+,\\
\operatorname{div} u = 0, \quad \operatorname{div} B = 0 \quad \text{in } \mathbb{R}^3\times\mathbb{R}_+,\\
u|_{t=0} = u_0, \quad B|_{t=0} = B_0 \quad \text{in } \mathbb{R}^3,
\end{gathered} \tag{1.1}$$

where $u = (u_1, u_2, u_3)$ denotes the velocity field of the fluid, $B = (B_1, B_2, B_3)$ denotes the magnetic field, and P represents scalar pressure. The positive parameters α and β represent the fractional dissipations corresponding to the velocity field and magnetic field, respectively. The fractional Laplacian operator $(-\Delta)^\gamma$ is defined through the Fourier transform $\mathcal{F}[(-\Delta)^\gamma f](\xi) = |\xi|^{2\gamma}\mathcal{F}[f](\xi)$.

Problem (1.1) describes the motion of electrically conducting incompressible fluids in a magnetic field such as plasmas and liquid metals. Duvaut and Lions [15] established local well-posedness results of problem (1.1) with $\alpha = \beta = 1$ in $H^s(\mathbb{R}^3)$ with $s \ge 3$ and global well-posedness for the small initial data. Zhai et al. [45] established global well-posedness of problem (1.1) with $\alpha = \beta = 1$ for initial data in critical Besov spaces and relaxed the smallness condition in the third components of the initial velocity field and initial magnetic field. For more relevant studies on the existence of solutions of the classical magnetohydrodynamic equations, we refer to [10, 21, 26, 44]. In [40], Wu established the global existence of weak solutions to problem (1.1) corresponding to arbitrary L^2 initial data. Furthermore if $\alpha, \beta \ge \frac{5}{4}$ and initial data are sufficiently smooth, the weak solutions are actually global classical solutions. Jin et al. [23] proved the unique existence of global solutions to problem (1.1) for small initial data in $H^s(\mathbb{R}^3)$ with $0 < \alpha = \beta < 1$ and $s \ge \frac{5}{2} - \alpha$. In addition, if initial data $(u_0, B_0) \in L^p$ for $1 \le p < 2$, they established optimal decay estimates for the solutions. See also [42, 43] for the global well-posedness of problem (1.1) with $\frac{1}{2} \le \alpha, \beta \le 1$ for small initial data in Lei-Lin spaces and Fourier-Herz spaces. However, it remains open whether there exists a global smooth solution for problem (1.1) with $\alpha + \beta < \frac{5}{2}$ or not.

It is worth mentioning that variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^N)$ were introduced by Orlicz [31], then further studied by Musielak[28] and Nakano[29, 30]. Indeed, the modern development of variable Lebesgue spaces began with Kováčik and Rákosník[24], Cruz-Uribe[13], Diening[16], Fan

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and Zhao[20], et al. So far, there have been extensive literature and monographs on the research of function spaces with variable exponents such as variable Lebesgue spaces [14], variable Sobolev spaces[17], variable Triebel-Lizorkin spaces [18], variable Besov spaces [6, 41], variable Morrey spaces [5] and among others. In addition to theoretical considerations, the study of these function spaces can be motivated by applications to fluid dynamics[3, 34], image processing[12, 22], PDEs and variational calculus [4, 19, 32, 46].

To the best of our knowledge, Ru and Abidin [33] first studied the Cauchy problem of the fractional Navier-Stokes equations in function spaces with variable exponents. More precisely, by introducing the definition of variable Fourier-Besov spaces and exploiting the good properties of Stokes semigroup on the frequency space, they obtained the existence and uniqueness of global solutions to incompressible fractional Navier-Stokes equations for small initial data in variable Fourier-Besov spaces. Subsequently, Wang[39] obtained the global well-posedness and analyticity of the generalized magnetohydrodynamics equations (1.1) in variable Fourier-Besov spaces. Abidin and Chen[1] proved that the fractional Navier-Stokes equations are globally well-posed if the initial data are small in variable Fourier-Besov-Morrey spaces. Abidin et al. [2] studied the global well-posedness for the three-dimensional micropolar fluid equations with small initial data in variable Fourier-Besov spaces.

Very recently, Chamorro and Vergara-Hermosilla [8, 35] established the well-posedness of incompressible (fractional) Navier-Stokes equations in variable Lebesgue spaces. In [9], they also studied the Liouville-type theorems for the stationary Navier-Stokes equations in variable Lebesgue spaces. Vergara-Hermosilla [36] showed the well-posedness results of the generalized nonlinear Heat equations in variable Lebesgue spaces. Chen, Vergara-Hermosilla and Zhao [11] proved the local existence and regularity of solutions to the two-dimensional dissipative surface quasi-geostrophic equation in variable Lebesgue spaces. For the well-posedness results related to the fractional Keller-Segel system, we refer to [37, 38].

Inspired by the above works, it is natural to consider the existence of solutions to the generalized magnetohydrodynamic equations in the framework of the variable Lebesgue spaces. Now we state our main results. The first result is the local existence of solutions for arbitrary initial data in $L^p(\mathbb{R}^3) \times L^q(\mathbb{R}^3)$.

Theorem 1.1. *Let $\alpha, \beta \in (\frac{1}{2}, \frac{5}{4}]$, $\gamma = \min\{\alpha, \beta\}$, $p > \frac{3}{2\gamma-1}$, $q \in [p, 2p]$ and $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}[0, +\infty)$ satisfy $p(\cdot) \le q(\cdot)$, $p^- > 2$ and $\frac{\gamma}{p(\cdot)} + \frac{3}{2p} < \gamma - \frac{1}{2}$. Then for any $(u_0, B_0) \in L^p(\mathbb{R}^3) \times L^q(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$ and $\operatorname{div} B_0 = 0$, there exists a time $T > 0$ such that problem (1.1) possesses a unique local mild solution $(u, B) \in L^{p(\cdot)}([0, T]; L^p(\mathbb{R}^3)) \times L^{q(\cdot)}([0, T]; L^q(\mathbb{R}^3))$.*

Next, we obtain the existence of global solutions with small initial data in $L^{p(\cdot)}(\mathbb{R}^3)$.

Theorem 1.2. *Let $\alpha = \beta \in (\frac{1}{2}, \frac{5}{4})$ and $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^3)$. Then there exists a positive constant ε such that if the initial value $(u_0, B_0) \in L^{p(\cdot)}(\mathbb{R}^3) \cap L^{\frac{3}{2\alpha-1}}(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$ and $\operatorname{div} B_0 = 0$ satisfying*

$$\max\left\{\|(u_0, B_0)\|_{L^{p(\cdot)}}, \|(u_0, B_0)\|_{L^{\frac{3}{2\alpha-1}}}\right\} \le \varepsilon,$$

problem (1.1) possesses a unique global mild solution

$$(u, B) \in L^{p(\cdot)}(\mathbb{R}^3; L^\infty[0, +\infty)) \cap L^{\frac{3}{2\alpha-1}}(\mathbb{R}^3; L^\infty[0, +\infty)).$$

This article is organized as follows. In Section 2, we recall the definitions and properties of the variable Lebesgue spaces. Moreover, we introduce some basic facts on the variable Lebesgue spaces. In Section 3, by using the decay estimates of the fractional heat kernel, we derive some linear and bilinear estimates of solutions. In Section 4, we complete the proofs of Theorems 1.1 and 1.2 by the contraction mapping principle. Throughout this paper the constant C in the estimates may vary from line to line.

2. Preliminaries

Firstly, we recall some standard notation in variable Lebesgue spaces. Let $\mathcal{P}(\mathbb{R}^n)$ be the set of all measurable functions $p(\cdot):\mathbb{R}^n\to[1,+\infty]$ such that

$$p^-:=\operatorname{ess\,inf}_{x\in\mathbb{R}^n}p(x)>1,\quad p^+:=\operatorname{ess\,sup}_{x\in\mathbb{R}^n}p(x)<+\infty.$$

Let $p'(\cdot)$ denote the conjugate exponent of the $p(\cdot)$ with $\frac{1}{p(\cdot)}+\frac{1}{p'(\cdot)}=1$.

Definition 2.1. Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. The Lebesgue space with variable exponent $p(\cdot)$ is defined by

$$L^{p(\cdot)}(\mathbb{R}^n):=\big\{f \text{ is measurable} : \int_{\mathbb{R}^n}|\frac{f(x)}{\lambda}|^{p(x)}dx<\infty,\ \text{for some } \lambda>0\big\}.$$

Then $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}:=\inf\big\{\lambda>0:\int_{\mathbb{R}^n}|\frac{f(x)}{\lambda}|^{p(x)}dx\le 1\big\}.$$

On the basis of classical Lebesgue spaces $L^p(\mathbb{R}^n)$ and variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$, let us state the definition of the following mixed variable Lebesgue spaces introduced in [7].

Definition 2.2. Let $p(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^n)$ and $1<q<+\infty$. Then the mixed Lebesgue space $L^{p(\cdot)}_q(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}_q(\mathbb{R}^n)=L^{p(\cdot)}(\mathbb{R}^n)\cap L^q(\mathbb{R}^n),$$

endowed with the norm

$$\|\cdot\|_{L^{p(\cdot)}_q}=\max\{\|\cdot\|_{L^{p(\cdot)}},\|\cdot\|_{L^q}\}.$$

Next, we recall some properties of the variable Lebesgue spaces such as the Hölder inequality, norm conjugate formula and embedding theorem(see [14, Corollaryies 2.28 and 2.48], [17, Lemma 3.2.20 and Corollary 3.2.14]).

Lemma 2.3. *Let $p(\cdot),p_1(\cdot),p_2(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ satisfy $\frac{1}{p(x)}=\frac{1}{p_1(x)}+\frac{1}{p_2(x)}$. Then there exists a constant C such that for all $f\in L^{p_1(\cdot)}(\mathbb{R}^n)$ and $g\in L^{p_2(\cdot)}(\mathbb{R}^n)$, we have $fg\in L^{p(\cdot)}(\mathbb{R}^n)$ and*

$$\|fg\|_{L^{p(\cdot)}}\le C\|f\|_{L^{p_1(\cdot)}}\|g\|_{L^{p_2(\cdot)}}. \tag{2.1}$$

Lemma 2.4. *Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. Then*

$$\|f\|_{L^{p(\cdot)}}\le\sup_{\|g\|_{L^{p'(\cdot)}}\le1}\int_{\mathbb{R}^n}|f(x)g(x)|dx,\quad \forall f\in L^{p(\cdot)},\quad g\in L^{p'(\cdot)}. \tag{2.2}$$

Lemma 2.5. Given $\Omega\subset\mathbb{R}^n$ and $p_1(\cdot),p_2(\cdot)\in\mathcal{P}(\Omega)$ with $1<p_1^+,p_2^+<+\infty$. Then $L^{p_2(\cdot)}(\Omega)\subset L^{p_1(\cdot)}(\Omega)$ if and only if $p_1(x)\le p_2(x)$ almost everywhere. Furthermore, in this case we have that

$$\|f\|_{L^{p_1(\cdot)}}\le(1+|\Omega|)\|f\|_{L^{p_2(\cdot)}},\quad \forall f\in L^{p_2(\cdot)}(\Omega),$$

where $|\Omega|$ denotes Lebesgue measure of measurable set Ω.

Remark 2.6. Since $L^{p(\cdot)}(\mathbb{R}^n)$ is not invariant to translation, not all properties of the Lebesgue spaces $L^p(\mathbb{R}^n)$ can be generalized to the variable Lebesgue spaces. For example, the Young inequality and the Plancherel formula are not valid anymore, see [17].

Note that the boundedness property of Hardy-Littlewood maximal operator on variable Lebesgue spaces provides a rigorous foundation for numerous conclusions in classical harmonic analysis and function spaces theory. Thus, to guarantee the boundedness of the singular integral operator in the variable Lebesgue spaces, we need to impose the so-called log-Hölder continuity on the exponent functions $p(\cdot)$.

Definition 2.7. Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. Then $p(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^n)$ if the following conditions hold:

- (locally log-Hölder continuous) For all $x,y\in\mathbb{R}^n$, $x\ne y$, $\big|\frac{1}{p(x)}-\frac{1}{p(y)}\big|\le\frac{C}{\log(e+\frac{1}{|x-y|})}$;
- (log-Hölder decay condition) For all $x\in\mathbb{R}^n$, $\big|\frac{1}{p(x)}-\frac{1}{p_\infty}\big|\le\frac{C}{\log(e+|x|)}$, where $\frac{1}{p_\infty}=\lim\limits_{|x|\to\infty}\frac{1}{p(x)}$.

Next, we introduce the following results on the Hardy-Littlewood maximal operator and Riesz transforms (see [14, Theorems 3.16 and 5.42], [17, Theorem 4.3.8 and Corollary 6.3.10], [25, Lemma 7.4]).

Lemma 2.8. *Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$. Then for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$, there exists a positive constant C such that*

$$\|\mathcal{M}(f)\|_{L^{p(\cdot)}} \leq C\|f\|_{L^{p(\cdot)}},$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal operator defined as

$$\mathcal{M}(f)(x) := \sup_{x \in \mathcal{B}} \frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} |f(y)| dy,$$

and $\mathcal{B} \subset \mathbb{R}^n$ is an open ball with center x. Furthermore, we have

$$\|\mathcal{R}_j(f)\|_{L^{p(\cdot)}} \leq C\|f\|_{L^{p(\cdot)}}, \quad \forall 1 \leq j \leq n,$$

where $\mathcal{R}_j := -\partial_{x_j}(-\Delta)^{-\frac{1}{2}}$ is the Riesz transform.

Lemma 2.9. *Let φ is a radially decreasing function on $\mathbb{R}^3$ and f is a locally integrable function, then*

$$|(\varphi * f)(x)| \leq \|\varphi\|_{L^1} \mathcal{M}(f)(x).$$

Next we review the definition of Riesz potential operator and its boundedness property on the variable Lebesgue spaces (see [14, Theorem 5.46], [7, Theorem 4]).

Definition 2.10. Let $0 < \delta < n$, for any measurable function f, the Riesz potential operator $\mathcal{I}_\delta$ is defined as

$$\mathcal{I}_\delta(f)(x) := \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\delta}} dy, \quad x \in \mathbb{R}^n.$$

Lemma 2.11. *Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $0 < \delta < n/p^+$. Then for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$, there exists a positive constant C such that*

$$\|\mathcal{I}_\delta(f)\|_{L^{q(\cdot)}} \leq C\|f\|_{L^{p(\cdot)}} \quad \textit{with } \frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\delta}{n}.$$

Lemma 2.12. *Let $1 < p < +\infty$, $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $0 < \delta < \min\{\frac{n}{p^+}, \frac{n}{p}\}$. Given $f \in L_p^{p(\cdot)}(\mathbb{R}^n)$ and a function $\rho(\cdot)$ satisfying the condition*

$$\rho(\cdot) = \frac{np(\cdot)}{n - \delta p},$$

then there exists a positive constant C such that

$$\|\mathcal{I}_\delta(f)\|_{L^{\rho(\cdot)}} \leq C\|f\|_{L_p^{p(\cdot)}}.$$

Remark 2.13. Note that the mixed spaces $L_p^{p(\cdot)}$ inherit the properties of the spaces $L^{p(\cdot)}$ and L^p. In particular we have the Hölder inequality $\|f\|_{L_p^{p(\cdot)}} \leq \|f\|_{L_q^{q(\cdot)}} \|f\|_{L_r^{r(\cdot)}}$ with $\frac{1}{p(\cdot)} = \frac{1}{q(\cdot)} + \frac{1}{r(\cdot)}$ and $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ and of course the Riesz transforms are also bounded in these spaces.

Finally, by Duhamel's principle, the mild solutions (u, B) for the equations (1.1) can be represented as

$$\begin{aligned} u(t) &= G_t^\alpha * u_0(x) - \int_0^t G_{t-\tau}^\alpha * \mathbb{P}\nabla \cdot [u \otimes u - B \otimes B](x,\tau) d\tau, \\ B(t) &= G_t^\beta * B_0(x) - \int_0^t G_{t-\tau}^\beta * \mathbb{P}\nabla \cdot [u \otimes B - B \otimes u](x,\tau) d\tau, \end{aligned} \tag{2.3}$$

where $\mathbb{P} := Id - \nabla(-\Delta)^{-1} \operatorname{div}$ is the Leray-Hopf projection. In our proof we need the following decay estimates for the fractional heat kernel $G_t^\alpha(x)$, see [27].

Lemma 2.14. *The fractional heat kernel $G_t^\alpha(x)$ satisfies the point-wise estimate*
$$|\nabla G_t^\alpha(x)| \le C\frac{1}{(t^{\frac{1}{2\alpha}}+|x|)^{n+1}},$$
where
$$G_t^\alpha(x)=\mathcal{F}^{-1}(e^{-t|\xi|^{2\alpha}})=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}e^{-t|\xi|^{2\alpha}}d\xi,\quad x\in\mathbb{R}^n.$$

Lemma 2.15. *For all $\alpha>0$, $\nu>0$, $1\le p\le q\le\infty$. Then for any $f\in L^p(\mathbb{R}^n)$, we have the L^p-L^q estimates*
$$\|G_t^\alpha * f\|_{L^q}\le Ct^{-\frac{n}{2\alpha}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p},$$
$$\|(-\Delta)^{\frac{\nu}{2}}G_t^\alpha * f\|_{L^q}\le Ct^{-\frac{\nu}{2\alpha}-\frac{n}{2\alpha}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p}.$$

3. Key estimates

The aim of this section is to derive estimates used in the proof of Theorems 1.1 and 1.2. Firstly, we establish the linear and bilinear estimates of the integral equations (2.3) for the local well-posedness issue. For $T>0$, we introduce the functional space $L^{p(\cdot)}\big([0,T);L^p(\mathbb{R}^3)\big)$ with the norm
$$\|f\|_{L^{p(\cdot)}([0,T);L^p)}=\inf\big\{\lambda>0:\int_0^T\big|\frac{\|f(\cdot,t)\|_{L^p}}{\lambda}\big|^{p(t)}dt\le 1\big\}.$$

Lemma 3.1. *Let $\alpha\in(\frac12,\frac54]$, $p(\cdot)\in\mathcal{P}^{\log}[0,+\infty)$ and $p\in[1,+\infty]$. Then for any $f\in L^p(\mathbb{R}^3)$, there exists a positive constant C such that*
$$\|G_t^\alpha * f\|_{L^{p(\cdot)}([0,T);L^p)}\le C\max\{T^{\frac{1}{p^-}},T^{\frac{1}{p^+}}\}\|f\|_{L^p}.$$

Proof. By applying the Young inequality, we obtain
$$\|G_t^\alpha * f\|_{L^p_x}\le\|G_t^\alpha(x)\|_{L^1_x}\|f\|_{L^p_x}=\|f\|_{L^p_x}.\tag{3.1}$$
Then taking the $L^{p(\cdot)}$-norm to both sides of (3.1) with respect to the time variable t yields
$$\|G_t^\alpha * f\|_{L_t^{p(\cdot)}(L^p_x)}\le C\|1\|_{L_t^{p(\cdot)}}\|f\|_{L^p_x}.$$
Using the classical estimate
$$\|1\|_{L^{p(\cdot)}[0,T)}\le C\max\{T^{\frac{1}{p^-}},T^{\frac{1}{p^+}}\},$$
we deduce that
$$\|G_t^\alpha * f\|_{L_t^{p(\cdot)}(L^p_x)}\le C\max\{T^{\frac{1}{p^-}},T^{\frac{1}{p^+}}\}\|f\|_{L^p_x}.$$
This completes the proof. □

Lemma 3.2. *Let $\alpha\in(\frac12,\frac54]$, $p>\frac{3}{2\alpha-1}$ and $p(\cdot)\in\mathcal{P}^{\log}[0,+\infty)$ satisfy $p^->2$ and $\frac{\alpha}{p(\cdot)}+\frac{3}{2p}<\alpha-\frac12$. Then for any $T>0$, there exists a positive constant C such that*
$$\|\int_0^t G_{t-\tau}^\alpha * \mathbb{P}\nabla\cdot(u\otimes u)(\tau)d\tau\|_{L^{p(\cdot)}([0,T);L^p)}\le C(1+T)\|u\|^2_{L^{p(\cdot)}([0,T);L^p)}.\tag{3.2}$$

Proof. For any $0<t<T$, by using Lemma 2.15 and the Hölder inequality, together with the boundedness property of the Leray projector $\mathbb{P}$, we obtain
$$\begin{aligned}\|\int_0^t G_{t-\tau}^\alpha * \mathbb{P}\nabla\cdot(u\otimes u)(\tau)d\tau\|_{L^p_x}&\le\int_0^t\|\nabla G_{t-\tau}^\alpha *(u\otimes u)(\tau)\|_{L^p_x}d\tau\\&\le\int_0^t\frac{1}{(t-\tau)^{\frac{1}{2\alpha}+\frac{3}{2\alpha p}}}\|(u\otimes u)(\tau)\|_{L_x^{\frac{p}{2}}}d\tau\\&\le\int_0^t\frac{1}{(t-\tau)^{\frac{1}{2\alpha}+\frac{3}{2\alpha p}}}\|u(\tau)\|^2_{L^p_x}d\tau.\end{aligned}$$

Then taking the $L^{p(\cdot)}$-norm with respect to the time variable t and using the norm conjugate formula (2.2), we see that

$$
\begin{aligned}
&\| \int_0^t G^\alpha_{t-\tau} * \mathbb{P}\nabla \cdot (u \otimes u)(\tau) d\tau \|_{L_t^{p(\cdot)}(L_x^p)} \\
&\le C \| \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2\alpha}+\frac{3}{2\alpha p}}} \|u(\tau)\|^2_{L^p_x} d\tau \|_{L_t^{p(\cdot)}} \\
&\le C \sup_{\|\psi\|_{L_t^{p'(\cdot)}} \le 1} \int_0^T \int_0^t \frac{|\psi(t)|}{|t-\tau|^{\frac{1}{2\alpha}+\frac{3}{2\alpha p}}} \|u(\tau)\|^2_{L^p_x} d\tau dt \\
&= C \sup_{\|\psi\|_{L_t^{p'(\cdot)}} \le 1} \int_0^T \int_0^T \frac{1_{\{0<\tau<t\}} |\psi(t)|}{|t-\tau|^{\frac{1}{2\alpha}+\frac{3}{2\alpha p}}} dt \|u(\tau)\|^2_{L^p_x} d\tau.
\end{aligned}
\tag{3.3}
$$

For using lemma 2.11, we extend the function $\psi(t)$ by zero on $\mathbb{R}\backslash[0,T)$, and see that the right-hand side of (3.3) can be represented as

$$
\begin{aligned}
&\sup_{\|\psi\|_{L_t^{p'(\cdot)}} \le 1} \int_0^T \int_0^T \frac{1_{\{0<\tau<t\}} |\psi(t)|}{|t-\tau|^{\frac{1}{2\alpha}+\frac{3}{2\alpha p}}} dt \|u(\tau)\|^2_{L^p_x} d\tau \\
&= \sup_{\|\psi\|_{L_t^{p'(\cdot)}} \le 1} \int_0^T \Big(\int_{-\infty}^{+\infty} \frac{|\psi(t)|}{|t-\tau|^{\frac{1}{2\alpha}+\frac{3}{2\alpha p}}} dt \Big) \|u(\tau)\|^2_{L^p_x} d\tau \\
&= \sup_{\|\psi\|_{L_t^{p'(\cdot)}} \le 1} \int_0^T \mathcal{I}_{\delta_1}(|\psi|) \|u(\tau)\|^2_{L^p_x} d\tau,
\end{aligned}
\tag{3.4}
$$

where $\delta_1 = 1 - \frac{1}{2\alpha} - \frac{3}{2\alpha p}$. Furthermore, for the right-hand side of (3.4), by using the Hölder inequality (2.1) with $\frac{2}{p(\cdot)} + \frac{1}{\widetilde{p_1}(\cdot)} = 1$ yields

$$
\begin{aligned}
\sup_{\|\psi\|_{L_t^{p'(\cdot)}} \le 1} \int_0^T \mathcal{I}_{\delta_1}(|\psi|) \|u(\tau)\|^2_{L^p_x} d\tau &\le C \sup_{\|\psi\|_{L_t^{p'(\cdot)}} \le 1} \|\mathcal{I}_{\delta_1}(|\psi|)\|_{L_t^{\widetilde{p_1}(\cdot)}} \|u\|^2_{L_t^{p(\cdot)}(L^p_x)} \\
&\le C \sup_{\|\psi\|_{L_t^{p'(\cdot)}} \le 1} \|\psi\|_{L_t^{r_1(\cdot)}} \|u\|^2_{L_t^{p(\cdot)}(L^p_x)},
\end{aligned}
\tag{3.5}
$$

where $\frac{1}{\widetilde{p_1}(\cdot)} = \frac{1}{r_1(\cdot)} - (1 - \frac{1}{2\alpha} - \frac{3}{2\alpha p})$.

Since $\frac{1}{\widetilde{p_1}(\cdot)} = 1 - \frac{2}{p(\cdot)}$ and $\frac{1}{p'(\cdot)} = 1 - \frac{1}{p(\cdot)}$, we can deduce that $r_1(\cdot) < p'(\cdot)$ under the condition $\frac{\alpha}{p(\cdot)} + \frac{3}{2p} < \alpha - \frac{1}{2}$. By using Lemma 2.5 with $r_1(\cdot) < p'(\cdot)$ and $\Omega = [0,T)$, we obtain

$$
\begin{aligned}
\sup_{\|\psi\|_{L_t^{p'(\cdot)}} \le 1} \|\psi\|_{L_t^{r_1(\cdot)}} \|u\|^2_{L_t^{p(\cdot)}(L^p_x)} &\le (1+T) \sup_{\|\psi\|_{L_t^{p'(\cdot)}} \le 1} \|\psi\|_{L_t^{p'(\cdot)}} \|u\|^2_{L_t^{p(\cdot)}(L^p_x)} \\
&\le (1+T) \|u\|^2_{L_t^{p(\cdot)}(L^p_x)}.
\end{aligned}
\tag{3.6}
$$

Finally, taking estimates (3.4)–(3.6) into (3.3), we obtain (3.2). This complete the proof. $\square$

Lemma 3.3. *Let $\alpha \in (\frac{1}{2}, \frac{5}{4}]$, $p > \frac{3}{2\alpha-1}$, $q \in [p, 2p]$, $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}[0,+\infty)$ satisfy $p(\cdot) \le q(\cdot)$, $q^- > 2$ and $\frac{\alpha}{p(\cdot)} + \frac{3}{2p} < \alpha - \frac{1}{2}$. Then for any $T > 0$, there exists a positive constant C such that*

$$
\| \int_0^t G^\alpha_{t-\tau} * \mathbb{P}\nabla \cdot (B \otimes B)(\tau) d\tau \|_{L^{p(\cdot)}([0,T);L^p)} \le C(1+T) \|B\|^2_{L^{q(\cdot)}([0,T);L^q)}. \tag{3.7}
$$

Proof. For any $0 < t < T$, by using Lemma 2.15 and the Hölder inequality, along with the boundedness property of the Leray projector $\mathbb{P}$, it follows that

$$
\| \int_0^t G^\alpha_{t-\tau} * \mathbb{P}\nabla \cdot (B \otimes B)(\tau) d\tau \|_{L^p_x} d\tau \le C \int_0^t \|\nabla G^\alpha_{t-\tau} * (B \otimes B)(\tau)\|_{L^p_x} d\tau
$$

$$
\begin{aligned}
&\le C\int_0^t \frac{1}{(t-\tau)^{\frac{1}{2\alpha}+\frac{3}{2\alpha}(\frac{2}{q}-\frac{1}{p})}}\|(B\otimes B)(\tau)\|_{L_x^{\frac{q}{2}}}d\tau\\
&\le C\int_0^t \frac{1}{(t-\tau)^{\frac{1}{2\alpha}+\frac{3}{2\alpha}(\frac{2}{q}-\frac{1}{p})}}\|B(\tau)\|_{L_x^q}^2 d\tau.
\end{aligned}
$$

Then taking $L^{p(\cdot)}$-norm with respect to the time variable t and using the norm conjugate formula (2.2), we have

$$
\begin{aligned}
&\|\int_0^t G_{t-\tau}^\alpha * \mathbb{P}\nabla\cdot(B\otimes B)(\tau)d\tau\|_{L_t^{p(\cdot)}(L_x^p)}\\
&\le C\|\int_0^t \frac{1}{(t-\tau)^{\frac{1}{2\alpha}+\frac{3}{2\alpha}(\frac{2}{q}-\frac{1}{p})}}\|B(\tau)\|_{L_x^q}^2 d\tau\|_{L_t^{p(\cdot)}}\\
&\le C\sup_{\|\eta\|_{L_t^{p'(\cdot)}}\le 1}\int_0^T\int_0^t \frac{|\eta(t)|}{|t-\tau|^{\frac{1}{2\alpha}+\frac{3}{2\alpha}(\frac{2}{q}-\frac{1}{p})}}\|B(\tau)\|_{L_x^q}^2 d\tau dt\\
&= C\sup_{\|\eta\|_{L_t^{p'(\cdot)}}\le 1}\int_0^T\int_0^T \frac{1_{\{0<\tau<t\}}|\eta(t)|}{|t-\tau|^{\frac{1}{2\alpha}+\frac{3}{2\alpha}(\frac{2}{q}-\frac{1}{p})}}dt\|B(\tau)\|_{L_x^q}^2 d\tau.
\end{aligned}
\tag{3.8}
$$

For using lemma 2.11, we extend the function $\eta(t)$ by zero on $\mathbb{R}\setminus[0,T)$, and see that the right-hand side of (3.8) can be represented as

$$
\begin{aligned}
&\sup_{\|\eta\|_{L_t^{p'(\cdot)}}\le 1}\int_0^T\int_0^T \frac{1_{\{0<\tau<t\}}|\eta(t)|}{|t-\tau|^{\frac{1}{2\alpha}+\frac{3}{2\alpha}(\frac{2}{q}-\frac{1}{p})}}dt\|B(\tau)\|_{L_x^q}^2 d\tau\\
&= \sup_{\|\eta\|_{L_t^{p'(\cdot)}}\le 1}\int_0^T\Big(\int_{-\infty}^{+\infty} \frac{|\eta(t)|}{|t-\tau|^{\frac{1}{2\alpha}+\frac{3}{2\alpha}(\frac{2}{q}-\frac{1}{p})}}dt\Big)\|B(\tau)\|_{L_x^q}^2 d\tau\\
&= \sup_{\|\eta\|_{L_t^{p'(\cdot)}}\le 1}\int_0^T \mathcal{I}_{\delta_2}(|\eta|)\|B(\tau)\|_{L_x^q}^2 d\tau,
\end{aligned}
\tag{3.9}
$$

where $\delta_2 = 1-\frac{1}{2\alpha}-\frac{3}{2\alpha}(\frac{2}{q}-\frac{1}{p})$. Furthermore, for the right-hand side of (3.9), by using the Hölder inequality (2.1) with $\frac{2}{q(\cdot)}+\frac{1}{\widetilde{p_2}(\cdot)}=1$ yields

$$
\begin{aligned}
\sup_{\|\eta\|_{L_t^{p'(\cdot)}}\le 1}\int_0^T \mathcal{I}_{\delta_2}(|\eta|)\|B(\tau)\|_{L_x^q}^2 d\tau &\le C\sup_{\|\eta\|_{L_t^{p'(\cdot)}}\le 1}\|\mathcal{I}_{\delta_2}(|\eta|)\|_{L_t^{\widetilde{p_2}(\cdot)}}\|B\|_{L_t^{q(\cdot)}(L_x^q)}^2\\
&\le C\sup_{\|\eta\|_{L_t^{p'(\cdot)}}\le 1}\|\eta\|_{L_t^{r_2(\cdot)}}\|B\|_{L_t^{q(\cdot)}(L_x^q)}^2,
\end{aligned}
\tag{3.10}
$$

where $\frac{1}{\widetilde{p_2}(\cdot)}=\frac{1}{r_2(\cdot)}-\big(1-\frac{1}{2\alpha}-\frac{3}{2\alpha}(\frac{2}{q}-\frac{1}{p})\big)$.

Since $\frac{1}{\widetilde{p_2}(\cdot)}=1-\frac{2}{q(\cdot)}$ and $\frac{1}{p'(\cdot)}=1-\frac{1}{p(\cdot)}$, we can deduce that $r_2(\cdot)<p'(\cdot)$ under the conditions $p(\cdot)\le q(\cdot)$ and $p\le q$. By using Lemma 2.5 with $r_2(\cdot)<p'(\cdot)$ and $\Omega=[0,T)$, we obtain

$$
\begin{aligned}
\sup_{\|\eta\|_{L_t^{p'(\cdot)}}\le 1}\|\eta\|_{L_t^{r_2(\cdot)}}\|B\|_{L_t^{q(\cdot)}(L_x^q)}^2 &\le C(1+T)\sup_{\|\eta\|_{L_t^{p'(\cdot)}}\le 1}\|\eta\|_{L_t^{p'(\cdot)}}\|B\|_{L_t^{q(\cdot)}(L_x^q)}^2\\
&\le C(1+T)\|B\|_{L_t^{q(\cdot)}(L_x^q)}^2.
\end{aligned}
\tag{3.11}
$$

Finally, taking estimates (3.9)–(3.11) into (3.8), we obtain (3.7). This completes the proof. □

Lemma 3.4. *Let $\beta\in(\frac{1}{2},\frac{5}{4}]$, $q\in[1,+\infty]$ and $q(\cdot)\in\mathcal{P}^{\log}[0,+\infty)$. Then for each $f\in L^q(\mathbb{R}^3)$, there exists a positive constant C such that*

$$
\|G_t^\beta * f\|_{L^{q(\cdot)}([0,T);L^q)}\le C\max\{T^{\frac{1}{q^-}},T^{\frac{1}{q^+}}\}\|f\|_{L^q}.
$$

The proof of the above lemma is similar to Lemma 3.1, we omit it.

Lemma 3.5. *Let $\beta \in (\frac{1}{2}, \frac{5}{4}]$, $p > \frac{3}{2\beta-1}$, $q \geq \frac{p}{p-1}$, $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}[0,+\infty)$ satisfy $\frac{\beta}{p(\cdot)} + \frac{3}{2p} < \beta - \frac{1}{2}$. Then for any $T > 0$, there exists a positive constant C such that*

$$\| \int_0^t G^\beta_{t-\tau} * \mathbb{P}\nabla \cdot (u \otimes B)(\tau) d\tau \|_{L^{q(\cdot)}([0,T);L^q)} \leq C(1+T) \|u\|_{L^{p(\cdot)}([0,T);L^p)} \|B\|_{L^{q(\cdot)}([0,T);L^q)}. \tag{3.12}$$

Proof. For any $0 < t < T$, by using Lemma 2.15 and the Hölder inequality, with the boundedness property of the Leray projector $\mathbb{P}$, we obtain

$$\begin{aligned}
\| \int_0^t G^\beta_{t-\tau} * \mathbb{P}\nabla \cdot (u \otimes B)(\tau) d\tau \|_{L^q_x} d\tau &\leq C \int_0^t \|\nabla G^\beta_{t-\tau} * (u \otimes B)(\tau)\|_{L^q_x} d\tau \\
&\leq C \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2\beta} + \frac{3}{2\beta p}}} \|(u \otimes B)(\tau)\|_{L_x^{\frac{pq}{p+q}}} d\tau \\
&\leq C \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2\beta} + \frac{3}{2\beta p}}} \|u(\tau)\|_{L^p_x} \|B(\tau)\|_{L^q_x} d\tau.
\end{aligned}$$

Then taking $L^{q(\cdot)}$-norm with respect to the time variable t and using the norm conjugate formula (2.2), we see that

$$\begin{aligned}
&\| \int_0^t G^\beta_{t-\tau} * \mathbb{P}\nabla \cdot (u \otimes B)(\tau) d\tau \|_{L_t^{q(\cdot)}(L^q_x)} \\
&\leq C \| \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2\beta} + \frac{3}{2\beta p}}} \|u(\tau)\|_{L^p_x} \|B(\tau)\|_{L^q_x} d\tau \|_{L_t^{q(\cdot)}} \\
&\leq C \sup_{\|\phi\|_{L_t^{q'(\cdot)}} \leq 1} \int_0^T \int_0^t \frac{|\phi(t)|}{|t-\tau|^{\frac{1}{2\beta} + \frac{3}{2\beta p}}} \|u(\tau)\|_{L^p_x} \|B(\tau)\|_{L^q_x} d\tau dt \\
&= C \sup_{\|\phi\|_{L_t^{q'(\cdot)}} \leq 1} \int_0^T \int_0^T \frac{1_{\{0<\tau<t\}} |\phi(t)|}{|t-\tau|^{\frac{1}{2\beta} + \frac{3}{2\beta p}}} dt \|u(\tau)\|_{L^p_x} \|B(\tau)\|_{L^q_x} d\tau.
\end{aligned} \tag{3.13}$$

For using lemma 2.11, we extend the function $\phi(t)$ by zero on $\mathbb{R} \backslash [0,T)$, and see that the right-hand side of (3.13) can be represented as

$$\begin{aligned}
&\sup_{\|\phi\|_{L_t^{q'(\cdot)}} \leq 1} \int_0^T \int_0^T \frac{1_{\{0<\tau<t\}} |\phi(t)|}{|t-\tau|^{\frac{1}{2\beta} + \frac{3}{2\beta p}}} dt \|u(\tau)\|_{L^p_x} \|B(\tau)\|_{L^q_x} d\tau \\
&= \sup_{\|\phi\|_{L_t^{q'(\cdot)}} \leq 1} \int_0^T \Big(\int_{-\infty}^{+\infty} \frac{|\phi(t)|}{|t-\tau|^{\frac{1}{2\beta} + \frac{3}{2\beta p}}} dt \Big) \|u(\tau)\|_{L^p_x} \|B(\tau)\|_{L^q_x} d\tau \\
&= \sup_{\|\phi\|_{L_t^{q'(\cdot)}} \leq 1} \int_0^T \mathcal{I}_{\delta_3}(|\phi|) \|u(\tau)\|_{L^p_x} \|B(\tau)\|_{L^q_x} d\tau,
\end{aligned} \tag{3.14}$$

where $\delta_3 = 1 - \frac{1}{2\beta} - \frac{3}{2\beta p}$. Furthermore, for the right-hand side of (3.14), using the Hölder inequality (2.1) with $\frac{1}{q(\cdot)} + \frac{1}{p(\cdot)} + \frac{1}{\widetilde{p_3}(\cdot)} = 1$ yields

$$\begin{aligned}
&\sup_{\|\phi\|_{L_t^{q'(\cdot)}} \leq 1} \int_0^T \mathcal{I}_{\delta_3}(|\phi|) \|u(\tau)\|_{L^p_x} \|B(\tau)\|_{L^q_x} d\tau \\
&\leq C \sup_{\|\phi\|_{L_t^{q'(\cdot)}} \leq 1} \|\mathcal{I}_{\delta_3}(|\phi|)\|_{L_t^{\widetilde{p_3}(\cdot)}} \|u\|_{L_t^{p(\cdot)}(L^p_x)} \|B\|_{L_t^{q(\cdot)}(L^q_x)} \\
&\leq C \sup_{\|\phi\|_{L_t^{q'(\cdot)}} \leq 1} \|\phi\|_{L_t^{r_3(\cdot)}} \|u\|_{L_t^{p(\cdot)}(L^p_x)} \|B\|_{L_t^{q(\cdot)}(L^q_x)},
\end{aligned} \tag{3.15}$$

where $\frac{1}{\widetilde{p_3}(\cdot)} = \frac{1}{r_3(\cdot)} - \big(1 - \frac{1}{2\beta} - \frac{3}{2\beta p}\big)$.

Since $\frac{1}{p_3(\cdot)} = 1 - \frac{1}{p(\cdot)} - \frac{1}{q(\cdot)}$ and $\frac{1}{q'(\cdot)} = 1 - \frac{1}{q(\cdot)}$, we can deduce that $r_3(\cdot) < q'(\cdot)$ under the condition $\frac{\beta}{p(\cdot)} + \frac{3}{2p} < \beta - \frac{1}{2}$. By using Lemma 2.5 with $r_3(\cdot) < q'(\cdot)$ and $\Omega = [0, T)$, we obtain

$$\begin{aligned} &\sup_{\|\phi\|_{L_t^{q'(\cdot)}} \leq 1} \|\phi\|_{L_t^{r_3(\cdot)}} \|u\|_{L_t^{p(\cdot)}(L_x^p)} \|B\|_{L_t^{q(\cdot)}(L_x^q)} \\ &\leq C(1+T) \sup_{\|\phi\|_{L_t^{q'(\cdot)}} \leq 1} \|\phi\|_{L_t^{q'(\cdot)}} \|u\|_{L_t^{p(\cdot)}(L_x^p)} \|B\|_{L_t^{q(\cdot)}(L_x^q)} \\ &\leq C(1+T) \|u\|_{L_t^{p(\cdot)}(L_x^p)} \|B\|_{L_t^{q(\cdot)}(L_x^q)}. \end{aligned} \tag{3.16}$$

Finally, taking estimates (3.14)–(3.16) into (3.13), we obtain (3.12). We complete the proof of Lemma 3.5. □

Secondly, we shall establish the linear and bilinear estimates for global well-posedness issue. For $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^3)$, the function space $L^{p(\cdot)}(\mathbb{R}^3; L^\infty[0, +\infty))$ is defined to be the set of all measurable functions $f : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ such that the following norm is finite

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^3; L^\infty)} = \inf \Big\{\lambda > 0 : \int_{\mathbb{R}^3} \Big| \frac{\|f(\cdot, x)\|_{L^\infty}}{\lambda} \Big|^{p(x)} \mathrm{d}x \leq 1 \Big\}.$$

Based on definition 2.2, for $p \in (1, +\infty)$, we introduce the mixed variable Lebesgue space

$$L_p^{p(\cdot)}(\mathbb{R}^3; L^\infty[0, +\infty)) := L^{p(\cdot)}(\mathbb{R}^3; L^\infty[0, +\infty)) \cap L^p(\mathbb{R}^3; L^\infty[0, +\infty)),$$

with the norm

$$\| \cdot \|_{L_p^{p(\cdot)}(\mathbb{R}^3; L^\infty)} = \max \big\{ \| \cdot \|_{L_x^{p(\cdot)}(L_t^\infty)}, \| \cdot \|_{L_x^p(L_t^\infty)} \big\}.$$

Lemma 3.6. *Let $\alpha = \beta \in (\frac{1}{2}, \frac{5}{4})$, $p(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^3)$. Then for any $f \in L^{p(\cdot)}_{\frac{3}{2\alpha-1}}(\mathbb{R}^3)$, there exists a positive constant C such that*

$$\|G_t^\alpha * f\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}(\mathbb{R}^3; L^\infty)} \leq C \|f\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}}.$$

Proof. The proof is similar to that in [35, Proposition 3.4]. Since $f \in L^{\frac{3}{2\alpha-1}} \subset L^1_{\rm loc}$ and the fractional heat kernel $G_t^\alpha(x)$ is a radially decreasing function, by Lemma 2.9 we see that

$$|(G_t^\alpha * f)(t, x)| \leq \|G_t^\alpha\|_{L_x^1} \mathcal{M}(f)(x) = \mathcal{M}(f)(x),$$

which shows that

$$\|G_t^\alpha * f\|_{L_t^\infty} \leq C \mathcal{M}(f)(x). \tag{3.17}$$

Then, taking the $L^{p(\cdot)}_{\frac{3}{2\alpha-1}}$-norm to the both sides of (3.17) with respect to the space variable x and using Lemma 2.8, together with the boundedness property of the maximal operator $\mathcal{M}$ in $L^{\frac{3}{2\alpha-1}}(\mathbb{R}^3)$, we obtain

$$\begin{aligned} \|G_t^\alpha * f\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}(\mathbb{R}^3; L^\infty)} &\leq C \|\mathcal{M}(f)\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}} \\ &\leq C \max \Big\{ \|\mathcal{M}(f)\|_{L^{p(\cdot)}}, \|\mathcal{M}(f)\|_{L^{\frac{3}{2\alpha-1}}} \Big\} \\ &\leq C \max \Big\{ \|f\|_{L^{p(\cdot)}}, \|f\|_{L^{\frac{3}{2\alpha-1}}} \Big\} \\ &\leq C \|f\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}}. \end{aligned}$$

This completes the proof. □

Lemma 3.7. *Let $\alpha = \beta \in (\frac{1}{2}, \frac{5}{4})$ and $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^3)$. Then there exists a positive constant C such that*

$$\| \int_0^t G_{t-\tau}^\alpha * \mathbb{P} \nabla \cdot (v \otimes w)(\tau) d\tau \|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}(\mathbb{R}^3; L^\infty)} \leq C \|v\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}(\mathbb{R}^3; L^\infty)} \|w\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}(\mathbb{R}^3; L^\infty)}. \tag{3.18}$$

Proof. Our proof is similar to that in [35, Proposition 3.6]. From the properties of the Leray projector $\mathbb{P}$, we have

$$\int_0^t G^\alpha_{t-\tau} * \mathbb{P}\nabla\cdot(v\otimes w)(\tau)d\tau = \int_0^t \mathbb{P}\left(G^\alpha_{t-\tau} * \nabla\cdot(v\otimes w)(\tau)\right)d\tau = \mathbb{P}\Big(\int_0^t G^\alpha_{t-\tau} * \nabla\cdot(v\otimes w)(\tau)d\tau\Big).$$

Then, using the Fubini theorem and Lemma 2.14, we obtain

$$\begin{aligned}
&|\int_0^t G^\alpha_{t-\tau} * \nabla\cdot(v\otimes w)(\tau)d\tau| \\
&\le C\int_0^t\int_{\mathbb{R}^3} |\nabla G^\alpha_{t-\tau}(x-y)||v(\tau,y)||w(\tau,y)|dyd\tau \\
&\le C\int_{\mathbb{R}^3}\int_0^t \frac{1}{(|t-\tau|^{\frac{1}{2\alpha}}+|x-y|)^4}|v(\tau,y)||w(\tau,y)|d\tau dy \\
&\le C\int_{\mathbb{R}^3}\int_0^t \frac{1}{(|t-\tau|^{\frac{1}{2\alpha}}+|x-y|)^4}d\tau\|v(\cdot,y)\|_{L^\infty_t}\|w(\cdot,y)\|_{L^\infty_t}dy.
\end{aligned} \tag{3.19}$$

Note that there exists a positive constant C such that

$$\begin{aligned}
\int_0^t \frac{1}{(|t-\tau|^{\frac{1}{2\alpha}}+|x-y|)^4}d\tau &\le \int_0^{+\infty}\frac{1}{(\tau^{\frac{1}{2\alpha}}+|x-y|)^4}d\tau \\
&= \int_0^{+\infty}\frac{|x-y|^{2\alpha}}{\big((|x-y|^{2\alpha}s)^{\frac{1}{2\alpha}}+|x-y|\big)^4}ds \\
&= \frac{1}{|x-y|^{4-2\alpha}}\int_0^{+\infty}\frac{1}{(1+s^{\frac{1}{2\alpha}})^4}ds \\
&\le \frac{C}{|x-y|^{4-2\alpha}}.
\end{aligned} \tag{3.20}$$

Substituting (3.20) into (3.19) and by Definition 2.10 of the Riesz potential operator with $n=3$, we have

$$\begin{aligned}
\mathbb{P}\Big(\int_0^t G^\alpha_{t-\tau} * \nabla\cdot(v\otimes w)(\tau)d\tau\Big) &\le C\mathbb{P}\Big(\int_{\mathbb{R}^3}\frac{1}{|x-y|^{4-2\alpha}}\|v(\cdot,y)\|_{L^\infty_t}\|w(\cdot,y)\|_{L^\infty_t}dy\Big) \\
&= C\mathbb{P}\left(\mathcal{I}_{2\alpha-1}\big(\|v\|_{L^\infty_t}\|w\|_{L^\infty_t}\big)(x)\right).
\end{aligned} \tag{3.21}$$

Furthermore, applying Lemma 2.12 and the boundedness property of the Leray projector $\mathbb{P}$, we obtain

$$\begin{aligned}
\big\|\mathbb{P}\Big(\int_0^t G^\alpha_{t-\tau} * \nabla\cdot(v\otimes w)(\tau)d\tau\Big)\big\|_{L^{p(\cdot)}_x(L^\infty_t)} &\le C\|\mathcal{I}_{2\alpha-1}(\|v\|_{L^\infty_t}\|w\|_{L^\infty_t})\|_{L^{p(\cdot)}_x} \\
&\le C\|\|v\|_{L^\infty_t}\|w\|_{L^\infty_t}\|_{L^{\frac{p(\cdot)}{2}}_{\frac{3}{2(2\alpha-1)},x}} \\
&\le C\|v\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1},x}(L^\infty_t)}\|w\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1},x}(L^\infty_t)},
\end{aligned} \tag{3.22}$$

since $\frac{1}{2}<\alpha<\frac{5}{4}$ and $\frac{2(2\alpha-1)}{3}=\frac{2\alpha-1}{3}+\frac{2\alpha-1}{3}$, the Hardy-Littlewood-Sobolev inequality and the Hölder inequality imply that

$$\begin{aligned}
\big\|\mathbb{P}\Big(\int_0^t G^\alpha_{t-\tau} * \nabla\cdot(v\otimes w)(\tau)d\tau\Big)\big\|_{L_x^{\frac{3}{2\alpha-1}}(L^\infty_t)} &\le C\|\mathcal{I}_{2\alpha-1}(\|v\|_{L^\infty_t}\|w\|_{L^\infty_t})\|_{L_x^{\frac{3}{2\alpha-1}}} \\
&\le C\|\|v\|_{L^\infty_t}\|w\|_{L^\infty_t}\|_{L_x^{\frac{3}{2(2\alpha-1)}}} \\
&\le C\|v\|_{L_x^{\frac{3}{2\alpha-1}}(L^\infty_t)}\|w\|_{L_x^{\frac{3}{2\alpha-1}}(L^\infty_t)}.
\end{aligned} \tag{3.23}$$

Combining the above estimates (3.22) and (3.23), we obtain (3.18). This completes the proof. □

4. Proof of main results

Proof of Theorem 1.1. For $(u_0, B_0) \in L^p(\mathbb{R}^3) \times L^q(\mathbb{R}^3)$, it follows from Lemmas 3.1 and 3.4 that there exist positive constants C_0 and C_1 such that

$$\|G_t^\alpha * u_0\|_{L^{p(\cdot)}([0,T);L^p)} \le C_0 \max\{T^{\frac{1}{p^-}}, T^{\frac{1}{p^+}}\}\|u_0\|_{L^p}, \tag{4.1}$$

$$\|G_t^\beta * B_0\|_{L^{q(\cdot)}([0,T);L^q)} \le C_1 \max\{T^{\frac{1}{q^-}}, T^{\frac{1}{q^+}}\}\|B_0\|_{L^q}. \tag{4.2}$$

We define the mapping $\mathscr{B}$ and the solutions space Z by

$$\mathscr{B}(u,B)(t) := \big(\mathscr{B}_1(u,B)(t), \mathscr{B}_2(u,B)(t)\big),$$

$$\begin{aligned} Z := \Big\{&(u,B) \in X \times Y := L^{p(\cdot)}\big([0,T); L^p(\mathbb{R}^3)\big) \times L^{q(\cdot)}\big([0,T); L^q(\mathbb{R}^3)\big) : \\ &\|u\|_X \le 2C_0 \max\{T^{\frac{1}{p^-}}, T^{\frac{1}{p^+}}\}\|u_0\|_{L^p},\ \|B\|_Y \le 2C_1 \max\{T^{\frac{1}{q^-}}, T^{\frac{1}{q^+}}\}\|B_0\|_{L^q}\Big\}, \end{aligned}$$

endowed with the norm $\|(u,B)\|_Z := \|u\|_X + \|B\|_Y$, where T will be determined later,

$$\begin{aligned} \mathscr{B}_1(u,B)(t) &:= G_t^\alpha * u_0(x) - \int_0^t G_{t-\tau}^\alpha * \mathbb{P}\nabla \cdot [u \otimes u - B \otimes B](x,\tau)d\tau, \\ \mathscr{B}_2(u,B)(t) &:= G_t^\beta * B_0(x) - \int_0^t G_{t-\tau}^\beta * \mathbb{P}\nabla \cdot [u \otimes B - B \otimes u](x,\tau)d\tau. \end{aligned}$$

Taking $\gamma = \min\{\alpha, \beta\}$ and let $p > \frac{3}{2\gamma-1}$ satisfying $\frac{\gamma}{p(\cdot)} + \frac{3}{2p} < \gamma - \frac{1}{2}$, it holds that $p \ge \frac{p}{p-1}$, applying lemmas 3.2, 3.3, and 3.5 yields that, for any $0 < T < +\infty$, there exist positive constants C_2, C_3 and C_4 such that

$$\begin{aligned} \|\mathscr{B}_1(u,B)\|_X &\le C_0 \max\{T^{\frac{1}{p^-}}, T^{\frac{1}{p^+}}\}\|u_0\|_{L^p} + C_2(1+T)\|u\|_X^2 + C_3(1+T)\|B\|_Y^2 \\ &\le C_0 \max\{T^{\frac{1}{p^-}}, T^{\frac{1}{p^+}}\}\|u_0\|_{L^p}\Big\{1 + 4C_0C_2(1+T)\max\{T^{\frac{1}{p^-}}, T^{\frac{1}{p^+}}\}\|u_0\|_{L^p} \\ &\quad + 4C_0^{-1}C_1^2C_3(1+T)\frac{\max\{T^{\frac{2}{q^-}}, T^{\frac{2}{q^+}}\}}{\max\{T^{\frac{1}{p^-}}, T^{\frac{1}{p^+}}\}}\|u_0\|_{L^p}^{-1}\|B_0\|_{L^q}^2\Big\} \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} \|\mathscr{B}_2(u,B)\|_Y &\le C_1 \max\{T^{\frac{1}{q^-}}, T^{\frac{1}{q^+}}\}\|B_0\|_{L^q} + C_4(1+T)\|u\|_X\|B\|_Y \\ &\le C_1 \max\{T^{\frac{1}{q^-}}, T^{\frac{1}{q^+}}\}\|B_0\|_{L^q}\big\{1 + 4C_0C_4(1+T)\max\{T^{\frac{1}{p^-}}, T^{\frac{1}{p^+}}\}\|u_0\|_{L^p}\big\} \end{aligned} \tag{4.4}$$

for all $(u,B) \in Z$.

On the other hand, under the similar argument, for all $(u_1,B_1),(u_2,B_2) \in Z$, it is easy to see that

$$\begin{aligned} &\|\mathscr{B}(u_1,B_1) - \mathscr{B}(u_2,B_2)\|_Z \\ &\le \Big\|\int_0^t G_{t-\tau}^\alpha * \mathbb{P}\nabla \cdot [u_1(\tau) \otimes (u_1(\tau) - u_2(\tau)) + (u_1(\tau) - u_2(\tau)) \otimes u_2(\tau)]d\tau\Big\|_X \\ &\quad + \Big\|\int_0^t G_{t-\tau}^\alpha * \mathbb{P}\nabla \cdot [B_1(\tau) \otimes (B_1(\tau) - B_2(\tau)) + (B_1(\tau) - B_2(\tau)) \otimes B_2(\tau)]d\tau\Big\|_X \\ &\quad + \Big\|\int_0^t G_{t-\tau}^\beta * \mathbb{P}\nabla \cdot [u_1(\tau) \otimes (B_1(\tau) - B_2(\tau)) + (u_1(\tau) - u_2(\tau)) \otimes B_2(\tau)]d\tau\Big\|_Y \\ &\quad + \Big\|\int_0^t G_{t-\tau}^\beta * \mathbb{P}\nabla \cdot [B_1(\tau) \otimes (u_1(\tau) - u_2(\tau)) + (B_1(\tau) - B_2(\tau)) \otimes u_2(\tau)]d\tau\Big\|_Y \\ &\le \Big[C_2(1+T)(\|u_1\|_X + \|u_2\|_X) + C_4(1+T)(\|B_1\|_Y + \|B_2\|_Y)\Big]\|u_1 - u_2\|_X \\ &\quad + \Big[C_4(1+T)(\|u_1\|_X + \|u_2\|_X) + C_3(1+T)(\|B_1\|_Y + \|B_2\|_Y)\Big]\|B_1 - B_2\|_Y \\ &\le \big\{4C_0C_2C_T^p\|u_0\|_{L^p} + 4C_1C_4C_T^q\|B_0\|_{L^q}\big\}\|u_1 - u_2\|_X \end{aligned}$$

$$+\big\{4C_0C_4C_T^p\|u_0\|_{L^p}+4C_1C_3C_T^q\|B_0\|_{L^q}\big\}\|B_1-B_2\|_Y, \tag{4.5}$$

where $C_T^p:=(1+T)\max\{T^{\frac{1}{p^-}},T^{\frac{1}{p^+}}\}$, $C_T^q:=(1+T)\max\{T^{\frac{1}{q^-}},T^{\frac{1}{q^+}}\}$.

If we choose $T>0$ small enough such that

$$\|u_0\|_{L^p}\le\frac{1}{C_T^p}\min\big\{\frac{1}{16C_0C_2},\frac{1}{16C_0C_4}\big\}$$

and

$$\|B_0\|_{L^q}\le\frac{1}{C_T^q}\min\big\{\frac{1}{16C_1C_4},\frac{1}{16C_1C_3},\frac{C_0^{1/2}(1+T)^{1/2}\max\{T^{\frac{1}{2p^-}},T^{\frac{1}{2p^+}}\}\|u_0\|_{L^p}^{1/2}}{2\sqrt{2}C_1C_3^{1/2}}\big\},$$

then (4.3)–(4.5) imply

$$\|\mathscr{B}_1(u,B)\|_X\le 2C_0\max\{T^{\frac{1}{p^-}},T^{\frac{1}{p^+}}\}\|u_0\|_{L^p},$$
$$\|\mathscr{B}_2(u,B)\|_Y\le 2C_1\max\{T^{\frac{1}{q^-}},T^{\frac{1}{q^+}}\}\|B_0\|_{L^q},$$
$$\|\mathscr{B}(u_1,B_1)-\mathscr{B}(u_2,B_2)\|_Z<\frac12\|(u_1,B_1)-(u_2,B_2)\|_Z$$

for all (u_1,B_1) and $(u_2,B_2)\in Z$. Therefore, by the contraction mapping principle, there exists a unique solution $(u,B)\in Z$ satisfying (2.3). This completes the proof. □

Proof of Theorem 1.2. For $(u_0,B_0)\in L^{p(\cdot)}(\mathbb{R}^3)\cap L^{\frac{3}{2\alpha-1}}(\mathbb{R}^3)$, it follows from Lemma 3.6 that there exists a positive constant C_0 such that

$$\|G_t^\alpha * u_0\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}(\mathbb{R}^3;L^\infty)}\le C_0\|u_0\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}}, \tag{4.6}$$
$$\|G_t^\alpha * B_0\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}(\mathbb{R}^3;L^\infty)}\le C_0\|B_0\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}}. \tag{4.7}$$

We define the mapping $\mathscr{B}$ and the solutions space $\mathcal{X}$ by

$$\mathscr{B}(u,B)(t):=\Big(\mathscr{B}_3(u,B)(t),\mathscr{B}_4(u,B)(t)\Big),$$
$$\mathcal{X}:=\Big\{(u,B)\in L^{p(\cdot)}_{\frac{3}{2\alpha-1}}(\mathbb{R}^3;L^\infty[0,+\infty)):\|u\|_{\mathcal{X}}\le 2C_0\|u_0\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}},\|B\|_{\mathcal{X}}\le 2C_0\|B_0\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}}\Big\},$$

endowed with the norm

$$\|(u,B)\|_{\mathcal{X}}:=\|u\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}(\mathbb{R}^3;L^\infty)}+\|B\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}(\mathbb{R}^3;L^\infty)},$$

where

$$\mathscr{B}_3(u,B)(t):=G_t^\alpha * u_0(x)-\int_0^t G_{t-\tau}^\alpha * \mathbb{P}\nabla\cdot[u\otimes u-B\otimes B](\tau,x)d\tau,$$
$$\mathscr{B}_4(u,B)(t):=G_t^\alpha * B_0(x)-\int_0^t G_{t-\tau}^\alpha * \mathbb{P}\nabla\cdot[u\otimes B-B\otimes u](\tau,x)d\tau.$$

Now applying lemma 3.7, (4.6) and (4.7) yields that, there exists a positive constant C_1 such that

$$\begin{aligned}\|\mathscr{B}_3(u,B)\|_{\mathcal{X}}&\le C_0\|u_0\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}}+C_1\|u\|_{\mathcal{X}}^2+C_1\|B\|_{\mathcal{X}}^2\\&\le C_0\|u_0\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}}\Big\{1+4C_0C_1\|u_0\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}}+4C_0C_1\|u_0\|^{-1}_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}}\|B_0\|^2_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}}\Big\}\end{aligned} \tag{4.8}$$

and

$$\|\mathscr{B}_4(u,B)\|_{\mathcal{X}}\le C_0\|B_0\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}}+2C_1\|u\|_{\mathcal{X}}\|B\|_{\mathcal{X}}\le C_0\|B_0\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}}\Big\{1+8C_0C_1\|u_0\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}}\Big\} \tag{4.9}$$

for all $(u,B)\in\mathcal{X}$.

On the other hand, under the similar argument, for all $(u_1,B_1),(u_2,B_2)\in\mathcal{X}$, it is easy to see that

$$
\begin{aligned}
&\|\mathscr{B}(u_1,B_1)-\mathscr{B}(u_2,B_2)\|_{\mathcal{X}}\\
&\le \Big\|\int_0^t G^\alpha_{t-\tau}*\mathbb{P}\nabla\cdot[u_1(\tau)\otimes(u_1(\tau)-u_2(\tau))+(u_1(\tau)-u_2(\tau))\otimes u_2(\tau)]d\tau\Big\|_{\mathcal{X}}\\
&\quad+\Big\|\int_0^t G^\alpha_{t-\tau}*\mathbb{P}\nabla\cdot[B_1(\tau)\otimes(B_1(\tau)-B_2(\tau))+(B_1(\tau)-B_2(\tau))\otimes B_2(\tau)]d\tau\Big\|_{\mathcal{X}}\\
&\quad+\Big\|\int_0^t G^\alpha_{t-\tau}*\mathbb{P}\nabla\cdot[u_1(\tau)\otimes(B_1(\tau)-B_2(\tau))+(u_1(\tau)-u_2(\tau))\otimes B_2(\tau)]d\tau\Big\|_{\mathcal{X}}\\
&\quad+\Big\|\int_0^t G^\alpha_{t-\tau}*\mathbb{P}\nabla\cdot[B_1(\tau)\otimes(u_1(\tau)-u_2(\tau))+(B_1(\tau)-B_2(\tau))\otimes u_2(\tau)]d\tau\Big\|_{\mathcal{X}}\\
&\le C_1(\|u_1\|_{\mathcal{X}}+\|u_2\|_{\mathcal{X}}+\|B_1\|_{\mathcal{X}}+\|B_2\|_{\mathcal{X}})(\|u_1-u_2\|_{\mathcal{X}}+\|B_1-B_2\|_{\mathcal{X}})\\
&\le \Big\{4C_0C_1\|u_0\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}}+4C_0C_1\|B_0\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}}\Big\}\|(u_1,B_1)-(u_2,B_2)\|_{\mathcal{X}}.
\end{aligned}\tag{4.10}
$$

If $(u_0,B_0)\in L^{p(\cdot)}_{\frac{3}{2\alpha-1}}(\mathbb{R}^3)$ satisfies

$$
\|u_0\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}}\le\frac{1}{16C_0C_1},
$$

and

$$
\|B_0\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}}\le\min\Big\{\frac{1}{16C_0C_1},\frac{1}{2\sqrt{2}C_0^{1/2}C_1^{1/2}}\|u_0\|^{1/2}_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}}\Big\},
$$

then (4.8)–(4.10) imply that

$$
\begin{gathered}
\|\mathscr{B}_3(u,B)\|_{\mathcal{X}}\le 2C_0\|u_0\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}},\\
\|\mathscr{B}_4(u,B)\|_{\mathcal{X}}\le 2C_0\|B_0\|_{L^{p(\cdot)}_{\frac{3}{2\alpha-1}}},\\
\|\mathscr{B}(u_1,B_1)-\mathscr{B}(u_2,B_2)\|_{\mathcal{X}}<\frac{1}{2}\|(u_1,B_1)-(u_2,B_2)\|_{\mathcal{X}}
\end{gathered}
$$

for all (u_1,B_1) and $(u_2,B_2)\in\mathcal{X}$. Therefore, by the contraction mapping principle, there exists a unique solution $(u,B)\in\mathcal{X}$ satisfying (2.3) with $\alpha=\beta$. This completes the proof. □

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