Electronic Journal of Differential Equations, Vol. 2025 (2025), No. ??, pp. 1–35.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, DOI: 10.58997/ejde.2025.??

EXISTENCE OF SOLUTIONS FOR A n-DIMENSIONAL SYSTEMS OF NONLOCAL BOUNDARY VALUE PROBLEMS

GEORGE L. KARAKOSTAS

Abstract. We show the existence of nontrivial solutions of a system of nonlocal boundary value problems for a second order n-dimensional ordinary differential equation with a Caratheodory type response. To do this, we apply the Krasnosel'skii (contraction+compact) fixed point theorem. Examples in two and three dimensional cases illustrate the results, where the best conditions are suggested.

1. Introduction

We study the existence of solutions of a system of n-dimensional second-order ordinary differential equations subject to some nonlocal boundary value conditions. More specifically, our subject is a differential equation of the form

$$(\Theta(t)x'(t))' = (Nx)(t), \quad t \in [0,1] \tag{1.1}$$

where, for each $t \in [0,1]$, the symbol $\Theta(t)$ stands for a $n \times n$-square nonsingular matrix and the function $t \to \Theta(t)$ is assumed to be Lebesgue measurable. N is an operator acting on the space $C(I, \mathbb{R}^n)$ into itself and it satisfies some conditions to be specified below.

We associate equation (1.1) with the two nonlocal boundary conditions

$$A_0x(0) - B_0x'(0) = \psi_0[x] + \zeta_0, \tag{1.2}$$

$$A_1x(1) + B_1x'(1) = \psi_1[x] + \zeta_1. \tag{1.3}$$

Here the coefficients A_0, A_1, B_0, B_1 are $n \times n$ square matrices with real entries and the items ψ_0, ψ_1 are linear functions expressed in the usual Riemann-Stieltjes integral form

$$\psi_i[x] := \int_0^1 d\Psi_i(s)x(s), \quad i = 0, 1,$$

where Ψ_0, Ψ_1 are $n \times n$ matrix valued functions defined on the interval $[0,1]$ and they are of bounded variation. Our purpose is to provide sufficient conditions which guarantee the existence of solutions of the boundary value problem (1.1), (1.2), (1.3). This will be done by applying the well known Krasnosel'skii's (contr.+comp.) fixed point theorem.

In the previous four decades, boundary value problems with nonlocal boundary value conditions have appeared in a great number of scientific works. This is because of the many theoretical problems which study physical phenomena in life sciences. Such related works appear in the literature (see, e.g., [7, 15, 25, 34, 35, 43, 45, 48] and the references therein). It is well known that the study of such problems when (1.1) is linear, was initiated in [4] and then a variety of articles followed them, (see, e.g. [3, 4, 8, 9, 14, 16, 19, 20, 22, 38, 39, 43]) where m-point boundary value problems for nonlinear ordinary differential equations with $\Theta = 1$, were investigated. Apart of the use of the Green's function for the linear case and the upper-lower solutions method for the nonlinear case (developed mainly by Lakshmikantham and Leela in their fruitful work on boundary value problems, published elsewhere; see, also, [7]), many techniques have recently been developed

2020 *Mathematics Subject Classification.* 34B15, 34B18, 34K40.
Key words and phrases. Nonlocal boundary value problems; Krasnoselskii's fixed point theorem; Caratheodory functions; Riemann-Stieltjes integral; contractions; compact operators; positive solutions.

Submitted July 29, 2025. Published November 25, 2025.

to give existence results. These techniques lie in the use of fixed points theorems applied to an appropriate (usually completely continuous) operator. More often the Leray-Schauder's fixed point theorem (see, e.g. [11]) via index theory is applied. Other methods are exhibited elsewhere, see, e.g. the S-type method operators in [10]. A good survey of recent existence results for solutions of first and second order nonlinear differential systems with nonlocal boundary conditions by using methods based on convexity, topological degree and maximum-principle like techniques, is presented in [37].

The motivation of this paper is the work done in [1], where a system of scalar differential equations of the form

$$x''(t) + f(t, x(t), y(t)) = 0, \quad y''(t) - g(t, x(t), y(t)) = 0, \quad t \in I \tag{1.4}$$

is investigated, under the pair of the nonlocal boundary conditions

$$\begin{aligned} &\tilde{a}_0 x(0) - \tilde{b}_0 x'(0) = \tilde{\phi}_0[x], \quad a_0 y(0) - b_0 y'(0) = \phi_0[y] + c_0, \\ &\tilde{a}_1 x(1) + \tilde{b}_1 x'(1) = \tilde{\phi}_1[x], \quad a_1 y(1) + b_1 y'(1) = \phi_1[y] + c_1, \end{aligned} \tag{1.5}$$

where $a_0, \tilde{a}_0, a_1, \tilde{a}_1, b_0, \tilde{b}_0, b_1, \tilde{b}_1$ are positive reals and the functionals $\phi_0[y], etc$, are defined by the type

$$\phi_0[y] := \int_0^1 y(s) d\Phi_0(s), \quad etc.$$

The author proves the existence of positive solutions of the system and, in order to succeed it, suitable conditions are given so that a hybrid-type Krasnosel'skii-Schauder fixed point theorem suggested by Infante, Mascali and Rodriguez-Lopez [26] to be applicable. According to this fixed point theorem, the Krasnoselskii's fixed point theorem on cones in a coordinate and the Schauder's fixed point in the second is applied. Boundary value problems of the same form concerning two-dimensional systems are also studied elsewhere, as e.g. in [23, 47, 52] and the references therein. In the one-dimensional case such problems are studied in many papers, some of which are [2, 5, 6, 9, 12, 13, 17, 18, 21, 27, 28, 31, 32, 33, 36, 38, 39, 40, 41, 42, 44, 45, 46, 49, 50, 51].

BVPs for such equations, in an abstract setting, have been investigated in the literature, see, e.g. [8, 30]. We especially mention to [30], which is concerned with the existence of solutions of boundary value problems for nonlinear second order ordinary differential equations of the type $x'' = H(t, x, x)$, $0 < t < 1$ with the conditions $ax(0) - bx'(0) = x_0$ and $cx(1) + dx'(1) = x_1$ in a general Banach space.

We want to elaborate a little on the form of the boundary value problem (1.1), (1.2), (1.3). The operator N includes the Nemytskii type operators, as well as Volterra integral operators. Also, the case of delay and/or advanced values of the solution is included, see section 10. The coefficient Θ in the left side comes from the Sturm Liouville form. Obviously, the problem (1.1), (1.2), (1.3) is essentially more general than the problem (1.4), (1.5). So, boundary value conditions such as (1.2), (1.3) include the so-called nonlocal and multi-point boundary value problems.

Many of the results obtained in the literature for scalar type equations refer to positive solutions and these results are succeeded by application of the so called Krasnoselskii's fixed point theorem on cones, or some variants of that. The process is quite simple. Formulate a positive cone on a space of functions and seek for a solution in this cone.

In this work we shall use the following so called Krasnoselskii's *contr+comp* fixed point theorem which states as follows:

Theorem 1.1 (Krasnoselskii [29]). *Suppose A is a closed bounded convex subset of a Banach space X. If $T : A \to X$ is a contraction, $C : A \to X$ is compact and $T(A) + C(A) := \{z = T(x) + C(y), \quad x, y \in A\} \subseteq A$, then $T + C$ has a fixed point in A.*

Recall that an operator $C : A \to X$ is said to be compact if it is continuous and maps bounded sets into precompact sets. Concerning the terminology by Krasnoselskii used in his survey [29] (p. 370), an operator $C : A \to X$ is compact in case it is continuous and the set $C(A)$ belongs to a compact set, where A need not be bounded.

In this work we shall give several conditions for the existence of solutions of the problem (1.1), (1.2), (1.3) depending on the regularity of the four matrices A_0, B_0, A_1, B_1. The standard way we shall follow in any case is to transform the boundary value problem in an operator equation written as the sum of two operators $\mathcal{T}$ and $\mathcal{C}$, where the first is a contraction and the second one is compact.

This work is organized as follows: Section 2 deals with some preliminaries needed for the reformulation of the problem. In Sections 3, 4, 5, 6 we discuss the cases when respectively, the matrices A_0, B_0, A_1, B_1 are nonsingular. In the final section 7, we shall present the case where A_0 is nonsingular and the two matrices A_0, A_1 commute. In all cases we use the same symbols with a subindex 1, 2, 3, 4, 5, just to distinguish the conditions given in each section. In section 8 we discuss the problem (1.4)-(1.5), some applications to 2 and 3-dimensional systems are given in sections 9 and 10, while in the last section 11, we present a result concerning the positivity of the solutions and give an example which illustrates the results.

2. Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional real space endowed with its usual euclidean topology and with the euclidean norm. We shall use the same norm $|\cdot|$ as for the real numbers, without confusion. Also, on the space of all $n\times n$ matrices we consider the Euclidean norm, namely, the norm of a matrix $A := (a_{ij})$ is given by $\|A\|_E = \big(\sum_{i=1}^n \sum_{j=0}^n a_{ij}^2\big)^{1/2}$. Thus the norm of the identity matrix $I_{n\times n}$ is equal to $\sqrt{n}$. This norm agrees with the euclidean norm of a vector in the sense that it holds $|Ax| \le \|A\|_E|x|$. Any two $n\times n$ matrices A, B satisfy $\|AB\|_E \le \|A\|_E\|B\|_E$.

We shall work on the space $C(I, \mathbb{R}^n)$ of all n-dimensional continuous functions defined on the interval I. We furnish the space with the usual sup-norm $\|\cdot\|_\infty$. Notice that the total variations of the matrix valued functions appeared in (1.2), (1.3) are defined by

$$V(\Psi_i) := \int_0^1 \|d\Psi_i(s)\|_E, \quad i = 0, 1 \tag{2.1}$$

and, obviously, in case Ψ_i is differentiable, then $V(\Psi_i) := \int_0^1 \|\Psi_i'(s)\|_E ds$. The constants ζ_0, ζ_1 are such that $|\zeta_0| + |\zeta_1| \ne 0$, since, otherwise, zero is a solution of the problem.

Assume that x is a solution of the problem (1.1)-(1.2)-(1.3). From (1.1) we obtain

$$\Theta(t)x'(t) = \Theta(0)x'(0) + \int_0^t (Nx)(u)du$$

and by the nonsingularity of $\Theta(t)$ we can multiply both sides by its inverse to obtain

$$x'(t) = \Theta(t)^{-1}\Theta(0)x'(0) + \Theta(t)^{-1}\int_0^t (Nx)(u)du. \tag{2.2}$$

Thus we have

$$x'(1) = \Theta(1)^{-1}\Theta(0)x'(0) + \Theta(1)^{-1}\int_0^1 (Nx)(u)du. \tag{2.3}$$

One more integration of (2.2) gives

$$x(t) = x(0) + \int_0^t \Theta(s)^{-1}ds\Theta(0)x'(0) + \int_0^t \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds, \tag{2.4}$$

which will be used to express the solution as a fixed point of an operator equation. In the sequel we shall assume that the operator N is defined on the set $C(I, \mathbb{R}^n)$ and it is such that for all $x \in C(I, \mathbb{R}^n)$ the item $(Nx)(\cdot)$ is a measurable n-dimensional function defined on I. Moreover assume that there is function $m : (0, +\infty) \to (0, +\infty)$, which for all $r > 0$ satisfies the condition

$$\|x\|_\infty < r \implies |(Nx)(t)| \le m(r), \quad t \in I. \tag{2.5}$$

(H0) Given the function m and three positive constants ρ, M, K, which will be specified below, we shall assume that there exists a positive real number r such that

$$K + \rho r + Mm(r) < r. \tag{2.6}$$

Notice that such a condition can be implied if, for instance, it holds:

$$\liminf_{r\to+\infty}\frac{m(r)}{r}<\frac{1-\rho}{M}. \tag{2.7}$$

Indeed, if such a relation is true, then we have

$$\begin{aligned}\liminf_{r\to+\infty}\Big[\frac{K}{r}+\rho+M\frac{m(r)}{r}\Big]&=\lim_{r\to+\infty}\frac{K}{r}+\rho+M\liminf_{r\to+\infty}\frac{m(r)}{r}\\&=\rho+M\liminf_{r\to+\infty}\frac{m(r)}{r}<\rho+M\frac{1-\rho}{M}=1.\end{aligned}$$

The latter fact implies that there exists $r>0$ satisfying (2.6). As we shall see, in some cases the first part of (2.7) is zero, so this is satisfied for any value of the factor M, provided that $0<\rho<1$.

Now we proceed to the investigation of the problem by taking into account when the matrices A_0, A_1, B_0, B_1 are nonsingular. To do that we have to solve the system of equations (1.2), (1.3) and (2.4). So, we distinguish the following cases:

3. Case(1): $\det A_0\neq 0$

In this section we shall prove the following theorem.

Theorem 3.1. *Consider the nonlocal boundary value problem* (1.1), (1.2), (1.3), *where N satisfies* (2.5). *Moreover we make the following assumptions:*

- (H1) $\det A_0\neq 0$.
- (H2) *The operator P_1 defined by*

$$P_1:=A_1A_0^{-1}B_0+A_1\int_0^1\Theta(s)^{-1}ds\Theta(0)+B_1\Theta(1)^{-1}\Theta(0),$$

  is nonsingular.
- (H3) *The quantity*

$$\rho_1:=V(A_0^{-1}\Psi_0)+\|(A_0^{-1}B_0+\int_0^1\Theta(s)^{-1}ds\Theta(0))P_1^{-1}\|_E\big(V(A_0\Psi_1)+V(A_1\Psi_0)\big)$$

  satisfies the condition $\rho_1<1$ and moreover condition (H0) *is satisfied with ρ_1 and the constants*

$$M_1:=\|(A_0^{-1}B_0+\int_0^t\Theta(s)^{-1}ds\Theta(0))P^{-1}(\int_u^1\Theta(s)^{-1}ds+B_1\Theta(1)^{-1})\|_E+\int_0^1\|\Theta(s)^{-1}\|_E ds,$$

$$K_1:=|A_0^{-1}\zeta_0|+\|A_0^{-1}B_0+\int_0^1\Theta(s)^{-1}ds\Theta(0)P_1^{-1}\|_E|A_0\zeta_1-A_1\zeta_0|.$$

Then there is a solution of the problem (1.1), (1.2), (1.3).

Proof. In (2.4) we set $t=1$ and obtain the equality

$$x(1)=x(0)+\int_0^1\Theta(s)^{-1}ds\Theta(0)x'(0)+\int_0^1\Theta(s)^{-1}\int_0^s(Nx)(u)\,du\,ds \tag{3.1}$$

From (3.1), (2.3) and (1.3) we take

$$\begin{aligned}&A_1x(0)+A_1\int_0^1\Theta(s)^{-1}ds\Theta(0)x'(0)+A_1\int_0^1\Theta(s)^{-1}\int_0^s(Nx)(u)\,du\,ds\\&+B_1\Theta(1)^{-1}\Theta(0)x'(0)+B_1\Theta(1)^{-1}\int_0^1(Nx)(u)du)\\&=\psi_1[x]+\zeta_1,\end{aligned}$$

which can be written as

$$\begin{aligned} &A_1x(0) + \Big(A_1\int_0^1 \Theta(s)^{-1}ds\Theta(0) + B_1\Theta(1)^{-1}\Theta(0)\Big)x'(0)\\ &\times A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds + B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)\,du\\ &= \psi_1[x] + \zeta_1. \end{aligned} \tag{3.2}$$

Next our goal is to eliminate the values $x(0)$ and $x'(0)$ in (2.4) and to express it in terms of the rest items. To do that, from (1.2) we work as follows: From (1.2) we obtain

$$x(0) = A_0^{-1}B_0x'(0) + A_0^{-1}(\psi_0[x] + \zeta_0) \tag{3.3}$$

and substitute it to (3.2). Then we obtain

$$\begin{aligned} &A_1A_0^{-1}B_0x'(0) + A_1A_0^{-1}(\psi_0[x] + \zeta_0) + \Big(A_1\int_0^1 \Theta(s)^{-1}ds\Theta(0) + B_1\Theta(1)^{-1}\Theta(0)\Big)x'(0)\\ &+ A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds + B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du\\ &= \psi_1[x] + \zeta_1. \end{aligned}$$

To solve this equation with respect to $x'(0)$ we write it in the form

$$P_1x'(0) = -A_1A_0^{-1}(\psi_0[x]+\zeta_0) - A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds - B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du + \psi_1[x] + \zeta_1,$$

where P_1 is the nonsingular matrix defined as in assumption (H3). Next, we solve it with respect to $x'(0)$ as

$$x'(0) = P_1^{-1}\Big(\psi_1[x]+\zeta_1 - A_1A_0^{-1}(\psi_0[x]+\zeta_0) - A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds - B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du\Big).$$

Then from (3.3) we obtain

$$\begin{aligned} x(0) = A_0^{-1}B_0P^{-1}\Big(&\psi_1[x] + \zeta_1 - A_1A_0^{-1}(\psi_0[x] + \zeta_0)\\ &- A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds - B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du\Big) + A_0^{-1}(\psi_0[x] + \zeta_0). \end{aligned}$$

Setting these two values in (2.4) we obtain the expression of the solution as

$$\begin{aligned} x(t) = A_0^{-1}B_0P_1^{-1}\Big(&\psi_1[x] + \zeta_1 - A_1A_0^{-1}(\psi_0[x] + \zeta_0)\\ &- A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds - B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du\Big)\\ &+ A_0^{-1}(\psi_0[x] + \zeta_0) + \int_0^t \Theta(s)^{-1}ds\Theta(0)P_1^{-1}\Big(\psi_1[x] + \zeta_1\\ &- A_1A_0^{-1}(\psi_0[x] + \zeta_0) - A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds\\ &- B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du\Big) + \int_0^t \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds \end{aligned} \tag{3.4}$$

namely,

$$\begin{aligned} x(t) = A_0^{-1}&(\psi_0[x] + \zeta_0) + \Big(A_0^{-1}B_0 + \int_0^t \Theta(s)^{-1}ds\Theta(0)\Big)\\ &\times P_1^{-1}\Big(\psi_1[x] + \zeta_1 - A_1A_0^{-1}(\psi_0[x] + \zeta_0) - A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds\\ &- B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du\Big) + \int_0^t \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds. \end{aligned} \tag{3.5}$$

Inversely: If x is a function satisfying relation (3.5), then it is a solution of the boundary value problem (1.1), (1.2), (1.3). Indeed, the fact that x is differentiable and it satisfies equation (1.1) is obvious. Next, we observe that

$$x(0) = A_0^{-1}(\psi_0[x] + \zeta_0) + A_0^{-1}B_0P_1^{-1}\Big(\psi_1[x] + \zeta_1 - A_1A_0^{-1}(\psi_0[x] + \zeta_0) - A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds - B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du\Big)$$

and

$$x'(0) = P_1^{-1}(\psi_1[x] + \zeta_1 - A_1A_0^{-1}(\psi_0[x] + \zeta_0) - A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds - B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du).$$

Hence, we have

$$\begin{aligned} A_0x(0) - B_0x'(0) = \psi_0[x] + \zeta_0 + B_0P^{-1}\Big(&\psi_1[x] + \zeta_1 - A_1A_0^{-1}(\psi_0[x] + \zeta_0) \\ &- A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds - B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du\Big) \\ &- B_0P^{-1}\Big(\psi_1[x] + \zeta_1 - A_1A_0^{-1}(\psi_0[x] + \zeta_0) \\ &- A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds - B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du\Big)\& = \psi_0[x] + \zeta_0, \end{aligned}$$

and therefore condition (1.2) is satisfied. Also, we have

$$\begin{aligned} x(1) = A_0^{-1}(\psi_0[x] + \zeta_0) + \Big(A_0^{-1}B_0 + \int_0^1 \Theta(s)^{-1}ds\Theta(0)\Big)P_1^{-1}\Big(\psi_1[x] + \zeta_1 \\ - A_1A_0^{-1}(\psi_0[x] + \zeta_0) - A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds \\ - B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du\Big) + \int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds, \end{aligned}$$

$$\begin{aligned} x'(1) = \Theta(1)^{-1}\Theta(0)P_1^{-1}\Big(\psi_1[x] + \zeta_1 - A_1A_0^{-1}(\psi_0[x] + \zeta_0) \\ - A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds - B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du\Big) + \Theta(1)^{-1}\int_0^1 (Nx)(u)du. \end{aligned}$$

So the first part of relation (1.3) becomes

$$\begin{aligned} &A_1x(1) + B_1x'(1) \\ &= A_1A_0^{-1}(\psi_0[x] + \zeta_0) + \Big(A_1A_0^{-1}B_0 + A_1\int_0^1 \Theta(s)^{-1}dsC(0)\Big) \\ &\times P_1^{-1}\Big(\psi_1[x] + \zeta_1 - A_1A_0^{-1}(\psi_0[x] + \zeta_0) - A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds \\ &- B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du\Big) + A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds + B_1\Theta(1)^{-1} \\ &\times \Theta(0)P^{-1}\Big(\psi_1[x] + \zeta_1 - A_1A_0^{-1}(\psi_0[x] + \zeta_0) - A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds \\ &- B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du\Big) + B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du \end{aligned}$$

namely

$$A_1x(1) + B_1x'(1) = A_1A_0^{-1}(\psi_0[x] + \zeta_0) + \Big(A_1A_0^{-1}B_0 + A_1\int_0^1 \Theta(s)^{-1}ds\Theta(0)$$

$$
\begin{aligned}
&+ B_1\Theta(1)^{-1}\Theta(0)\Big)P_1^{-1}\Big(\psi_1[x] + \zeta_1 - A_1A_0^{-1}(\psi_0[x] + \zeta_0)\\
&- A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds - B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du\Big)\\
&+ A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds + B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du\\
={}& A_1A_0^{-1}(\psi_0[x] + \zeta_0) + \psi_1[x] + \zeta_1 - A_1A_0^{-1}(\psi_0[x] + \zeta_0)\\
&- A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds + B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du\\
&+ A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds + B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du\\
={}& \psi_1[x] + \zeta_1.
\end{aligned}
$$

Therefore the function x satisfies both boundary conditions (1.2) and (1.3). To proceed we apply Fubini's theorem and obtain

$$
\int_0^t \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds = \int_0^t\int_u^t \Theta(s)^{-1}ds(Nx)(u)du. \tag{3.6}
$$

Next we write relation (3.5) in the form $x(t) = (\mathcal{T}_1x)(t) + (\mathcal{C}_1x)(t)$, where

$$
\begin{aligned}
(\mathcal{T}_1x)(t) &:= A_0^{-1}(\psi_0[x] + \zeta_0)\\
&\quad + \Big(A_0^{-1}B_0 - \int_0^t \Theta(s)^{-1}ds\Theta(0)\Big)P_1^{-1}\Big(\psi_1[x] + \zeta_1 - A_1A_0^{-1}(\psi_0[x] + \zeta_0)\Big)
\end{aligned} \tag{3.7}
$$

and

$$
\begin{aligned}
(\mathcal{C}_1x)(t) &:= -\Big(A_0^{-1}B_0 + \int_0^t \Theta(s)^{-1}ds\Theta(0)\Big)P_1^{-1}\Big(\int_0^1\int_u^1 \Theta(s)^{-1}ds\\
&\quad + B_1\Theta(1)^{-1}\Big)(Nx)(u)du + \int_0^t\int_u^t \Theta(s)^{-1}ds(Nx)(u)du.
\end{aligned}
$$

The latter can be written in the integral form

$$
(\mathcal{C}_1x)(t) = \int_0^1 \mathcal{G}_1(t,u)(Nx)(u)du,
$$

where the Green's function $\mathcal{G}_1$ is the matrix

$$
\begin{aligned}
&\mathcal{G}_1(t,u)\\
&:= -\Big(A_0^{-1}B_0 + \int_0^t \Theta(s)^{-1}ds\Theta(0)\Big)P_1^{-1}\Big(\int_u^1 \Theta(s)^{-1}ds + B_1\Theta(1)^{-1}\Big) + \chi_{[0,t]}(u)\int_u^t \Theta(s)^{-1}ds.
\end{aligned}
$$

So far we have shown that x is a solution of the original problem if and only it is a fixed point of the operator equation $\mathcal{T}_1x+\mathcal{C}_1x = x$. Hence the existence of a solution of the problem is equivalent to the existence of a fixed point of this operator equation.

In the sequel we shall work on the ball $B_r(0)$ of the space $C([0,1],\mathbb{R}^n)$, where r is a positive real number defined later. To proceed, first we obtain an upper bound of the quantity $\mathcal{G}_1(t,u)$ as follows

$$
\begin{aligned}
\|\mathcal{G}_1(t,u)\|_E &\le \|(A_0^{-1}B_0 + \int_0^t \Theta(s)^{-1}ds\Theta(0))P^{-1}(\int_u^1 \Theta(s)^{-1}ds + B_1\Theta(1)^{-1})\|_E\\
&\quad + \int_u^1 \|\Theta(s)^{-1}\|_E ds =: M_1.
\end{aligned}
$$

and therefore it holds

$$
\|\mathcal{C}_1y\|_\infty \le M_1m(r), \quad y \in B_r(0). \tag{3.8}
$$

Let $x, y \in B_r(0)$ be given. Then we have

$$
|(\mathcal{T}_1x)(t) - (\mathcal{T}_1y)(t)|
$$

$$\le |A_0^{-1}(\psi_0(x-y))| + \|(A_0^{-1}B_0 + \int_0^t \Theta(s)^{-1}ds\Theta(0))P^{-1}\|_E|(A_0\big(\psi_1(x-y)) - A_1(\psi_0(x-y)))|$$

$$\le \big[V(A_0^{-1}\Psi_0) + \|(A_0^{-1}B_0 + \int_0^1 \Theta(s)^{-1}ds\Theta(0))P^{-1}\|_E\big(V(A_0\Psi_1) + V(A_1\Psi_0)\big)\big]\|x-y\|_\infty,$$

namely

$$\|\mathcal{T}_1 x - \mathcal{T}_1 y\|_\infty \le \rho_1\|x-y\|_\infty, \tag{3.9}$$

where ρ_1 is the quantity defined in $(\mathcal{H}3)$ and satisfies $\rho_1 < 1$.

Now, let $x \in B_r(0)$. Then from (3.7) we have

$$\begin{aligned}
&|(\mathcal{T}_1 x)(t)|\\
&\le |A_0^{-1}(\psi_0[x]+\zeta_0)| + \|(A_0^{-1}B_0 + \int_0^1 \Theta(s)^{-1}ds\Theta(0))P_1^{-1}\|_E|A_0\big(\psi_1[x]+\zeta_1) - A_1(\psi_0[x]+\zeta_0)\big)|\\
&\le V(A_0^{-1}\Psi_0)\|x\|_\infty + |A_0^{-1}\zeta_0| + \|(A_0^{-1}B_0 + \int_0^1 \Theta(s)^{-1}ds\Theta(0))P_1^{-1}\|_E\big(V(A_0\Psi_1 - A_1\Psi_0)\|x\|_\infty\\
&\quad + |A_0\zeta_1 - A_1\zeta_0|\big),
\end{aligned}$$

and therefore

$$|(\mathcal{T}_1 x)(t)| \le K_1 + \rho_1 r,$$

where K_1 and ρ_1 are defined in $(\mathcal{H}3)$. From this and (3.8) we see that

$$\|x\|_\infty, \|y\|_\infty \le r \implies |(\mathcal{T}_1 x)(t) + (\mathcal{C}_1 y)(t)| \le K_1 + \rho_1 r + M_1 m(r).$$

Because of condition (H0) there exists $r_1 > 0$ such that $K_1 + \rho_1 r_1 + M_1 m(r_1) \le r_1$, which means that $\mathcal{T}_1 x + \mathcal{C}_1 y \in B_{r_1}(0)$, for all $x, y \in B_{r_1}(0)$.

We shall show that the operator $\mathcal{C}_1$ is compact. First of all it is easy to prove that $\mathcal{C}_1$ is a continuous operator. Let W be a bounded subset of $B_{r_1}(0)$. We shall show that $\mathcal{C}_1 W$ is compact. Boundedness of this set is obvious, by (2.5) and (3.8). To show equicontinuity, we consider an $x \in W$ and two points $0 \le t_1 \le t_2 \le 1$ and observe that

$$\begin{aligned}
&|(\mathcal{C}_1 x)(t_1) - (\mathcal{C}_1 x)(t_2)|\\
&\le \Big|\int_0^1 \mathcal{G}_1(t_1,u)(Nx)(u)du - \int_0^1 \mathcal{G}_1(t_2,u)(Nx)(u)du|\\
&= | - \Big(A_0^{-1}B_0 + \int_0^{t_1}\Theta(s)^{-1}ds\Theta(0)\Big)P_1^{-1}\Big(\int_0^1\int_u^1 \Theta(s)^{-1}ds + B_1\Theta(1)^{-1}\Big)(Nx)(u)du\\
&\quad + \int_0^{t_1}\int_u^{t_1}\Theta(s)^{-1}ds(Nx)(u)du + \Big(A_0^{-1}B_0 + \int_0^{t_2}\Theta(s)^{-1}ds\Theta(0)\Big)\\
&\quad \times P_1^{-1}\Big(\int_0^1\int_u^1 \Theta(s)^{-1}ds + B_1\Theta(1)^{-1}\Big)(Nx)(u)du - \int_0^{t_2}\int_u^{t_2}\Theta(s)^{-1}ds(Nx)(u)du\Big|\\
&\le \Big|\int_{t_1}^{t_2}\Theta(s)^{-1}ds\Theta(0)P_1^{-1}\Big(\int_0^1\int_u^1\Theta(s)^{-1}ds + B_1\Theta(1)^{-1}\Big)(Nx)(u)du\Big|\\
&\quad + \Big|\int_0^{t_2}\int_0^s\Theta(s)^{-1}(Nx)(u)\,du\,ds - \int_0^{t_1}\int_0^s\Theta(s)^{-1}(Nx)(u)\,du\,ds\Big|\\
&\le \int_{t_1}^{t_2}\|\Theta(s)^{-1}ds\Theta(0)P_1^{-1}\|_E\Big(\int_0^1\int_0^1\|\Theta(s)^{-1}\|_E ds + \|B_1\Theta(1)^{-1}\|_E\Big)|(Nx)(u)du|\\
&\quad + |\int_{t_1}^{t_2}\int_0^s\Theta(s)^{-1}(Nx)(u)\,du\,ds|\\
&\le m(r_1)\int_{t_1}^{t_2}\Big(\|\Theta(s)^{-1}ds\Theta(0)P_1^{-1}\|_E\Big(\int_0^1\int_u^1\|\Theta(s)^{-1}\|_E ds + B_1\Theta(1)^{-1}\|_E\Big)\\
&\quad + m(r_1)\int_{t_1}^{t_2}\|\Theta(s)^{-1}\|_E ds.
\end{aligned}$$

This relation proves the (uniform) equicontinuity of the family $\{\mathcal{C}_1 x : x \in B_{r_1}(0)\}$ and so $\mathcal{C}_1$ is a compact mapping. Also, from relation (3.9) and condition ($\mathcal{H}3a$) it follows that the operator T is a contraction. By the Krasnosel'skii fixed point theorem (1.1), the mapping $\mathcal{T}_1 + \mathcal{C}_1$ admits a fixed point $\bar{x}$ in $B_{r_1}(0)$, which is a solution of the original problem and the proof is complete. □

4. Case(2): $\det B_0 \neq 0$

The result in this section is given in the following theorem:

Theorem 4.1. *Consider the nonlocal boundary value problem* (1.1), (1.2), (1.3), *where* N *satisfies* (2.5). *Moreover we make the following assumptions:*

- (H4) $\det B_0 \neq 0$.
- (H5) *The operator* P_2 *defined by*

$$P_2 := A_1 + A_1 \int_0^1 \Theta(s)^{-1} ds \Theta(0) B_0^{-1} A_0 + B_1 \Theta(1)^{-1} \Theta(0) B_0^{-1} A_0$$

is nonsingular.

- (H5) *The quantity*

$$\rho_2 := \Big\{ \sqrt{n} + \int_0^1 \|\Theta(s)^{-1} ds \Theta(0) B_0^{-1} A_0\|_E \Big\} \|P_2^{-1}\|_E \\ \times \big[\|A_1 \int_0^1 \Theta(s)^{-1} ds + B_1 \Theta(1)^{-1}\|_E \|\Theta(0) B_0^{-1}\|_E V(\Psi_0) + V(\Psi_1) \big]$$

satisfies the condition $\rho_2 < 1$ *and moreover condition (H0) is satisfied with* ρ_2,

$$M_2 := (\sqrt{n} + \int_0^1 \|\Theta(s)^{-1}\|_E ds \|\Theta(0) B_0^{-1} A_0) P_2^{-1}\|_E \Big[\|A_1\|_E \int_0^1 \|\Theta(s)^{-1}\|_E ds \\ + \|B_1 \Theta(1)^{-1}\|_E \Big] + \int_0^1 \|\Theta(s)^{-1}\|_E ds$$

and

$$K_2 := \| \Big[I_{n\times n} + \int_0^t \Theta(s)^{-1} ds \Theta(0) B_0^{-1} A_0 \Big] \|_E \|P_2^{-1}\|_E \Big[\| \Big(A_1 \int_0^1 \Theta(s)^{-1} ds \\ + B_1 \Theta(1)^{-1} \Big) \|_E \|\Theta(0) B_0^{-1}\|_E |\zeta_0| + |\zeta_1| \Big].$$

Then there is a solution of the problem (1.1), (1.2), (1.3).

Proof. Let x be a solution of the problem. From (1.2) we have

$$x'(0) = B_0^{-1} A_0 x(0) - B_0^{-1}(\Psi_0[x] + \zeta_0) \tag{4.1}$$

and so (2.4) becomes

$$x(t) = x(0) + \int_0^t \Theta(s)^{-1} ds \Theta(0) \Big[B_0^{-1} A_0 x(0) - B_0^{-1}(\Psi_0[x] + \zeta_0) \Big] + \int_0^t \Theta(s)^{-1} \int_0^s (Nx)(u)\, du\, ds.$$

Putting $t = 1$ we obtain

$$x(1) = x(0) + \int_0^1 \Theta(s)^{-1} ds \Theta(0) \Big[B_0^{-1} A_0 x(0) - B_0^{-1}(\Psi_0[x] + \zeta_0) \Big] + \int_0^1 \Theta(s)^{-1} \int_0^s (Nx)(u)\, du\, ds,$$

while from (2.3) and (4.1) we obtain

$$x'(1) = \Theta(1)^{-1} \Theta(0) \Big[B_0^{-1} A_0 x(0) - B_0^{-1}(\Psi_0[x] + \zeta_0) \Big] + \Theta(1)^{-1} \int_0^1 (Nx)(u)\, du\, ds.$$

Then from (1.3) we obtain that

$$\psi_1[x] + \zeta_1 = A_1 x(1) + B_1 x'(1)$$

$$
\begin{aligned}
&= A_1x(0) + A_1\int_0^1 \Theta(s)^{-1}ds\Theta(0)\Big[B_0^{-1}A_0x(0) - B_0^{-1}(\psi_0[x]+\zeta_0)\Big]\\
&\quad + A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds + B_1\Theta(1)^{-1}\Theta(0)\Big[B_0^{-1}A_0x(0)\\
&\quad - B_0^{-1}(\psi_0[x]+\zeta_0)\Big] + B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)\,du\,ds,
\end{aligned}
$$

which can be written as

$$
\begin{aligned}
P_2x(0) &= (A_1\int_0^1 \Theta(s)^{-1}ds + B_1\Theta(1)^{-1})\Theta(0)B_0^{-1}(\psi_0[x]+\zeta_0)\\
&\quad - A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds - B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du + \psi_1[x]+\zeta_1,
\end{aligned}
$$

where P_2 is defined in (H5). Thus

$$
\begin{aligned}
x(0) &= P_2^{-1}\Big[\Big(A_1\int_0^1 \Theta(s)^{-1}ds + B_1\Theta(1)^{-1}\Big)\Theta(0)B_0^{-1}(\psi_0[x]+\zeta_0)\\
&\quad - A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds - B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du + \psi_1[x]+\zeta_1\Big].
\end{aligned}
\tag{4.2}
$$

Substituting this value to (4.1) we obtain

$$
\begin{aligned}
x'(0) &= B_0^{-1}A_0P_2^{-1}\Big[\Big(A_1\int_0^1 \Theta(s)^{-1}ds + B_1\Theta(1)^{-1}\Big)\Theta(0)B_0^{-1}(\psi_0[x]+\zeta_0)\\
&\quad - A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds - B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du + \psi_1[x]+\zeta_1\Big]\\
&\quad - B_0^{-1}(\Psi_0[x]+\zeta_0)
\end{aligned}
\tag{4.3}
$$

and so from (2.4) we obtain the expression of the solution as

$$
\begin{aligned}
x(t) &= \Big\{I_{n\times n} + \int_0^t \Theta(s)^{-1}ds\Theta(0)B_0^{-1}A_0\Big\}P_2^{-1}\Big[\Big(A_1\int_0^1 \Theta(s)^{-1}ds + B_1\Theta(1)^{-1}\Big)\\
&\quad \times \Theta(0)B_0^{-1}(\psi_0[x]+\zeta_0) - A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds\\
&\quad - B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du + \psi_1[x]+\zeta_1\Big] - \int_0^t \Theta(s)^{-1}ds\Theta(0)B_0^{-1}(\psi_0[x]\\
&\quad + \zeta_0) + \int_0^t \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds.
\end{aligned}
\tag{4.4}
$$

Until now we have proved that if x is a solution of the original problem, then it satisfies the operator equation (4.4). We shall show the inverse, namely, any function satisfying equation (4.4) is a solution of the boundary value problem (1.1), (1.2), (1.3). Indeed, let x be such a function. Then from (4.3) and (4.4) we obtain

$$
\begin{aligned}
&A_0x(0) - B_0x'(0)\\
&= A_0P_2^{-1}\Big[\Big(A_1\int_0^1 \Theta(s)^{-1}ds + B_1\Theta(1)^{-1}\Big)\Theta(0)B_0^{-1}(\psi_0[x]+\zeta_0)\\
&\quad - A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)du - B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du\Big] + \psi_1[x]+\zeta_1\Big]\\
&\quad - A_0P_2^{-1}\Big[\Big(A_1\int_0^1 \Theta(s)^{-1}ds + B_1\Theta(1)^{-1}\Big)\Theta(0)B_0^{-1}(\psi_0[x]+\zeta_0)\\
&\quad + B_0A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds + B_0B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du + \psi_1[x]+\zeta_1\Big]\\
&\quad + \psi_0[x]+\zeta_0 = \psi_0[x]+\zeta_0,
\end{aligned}
\tag{4.5}
$$

namely condition (1.2) is satisfied.

Also we have

$$\begin{aligned} x(1) = \Big(I_{n\times n} + \int_0^t \Theta(s)^{-1} ds \Theta(0) B_0^{-1} A_0\Big) P_2^{-1} \Big[\Big(A_1 \int_0^1 \Theta(s)^{-1} ds + B_1 \Theta(1)^{-1}\Big) \\ \times \Theta(0) B_0^{-1}(\psi_0[x] + \zeta_0) - A_1 \int_0^1 \Theta(s)^{-1} \int_0^s (Nx)(u)\, du\, ds \\ - B_1 \Theta(1)^{-1} \int_0^1 (Nx)(u) du + \psi_1[x] + \zeta_1\Big] \\ - \int_0^1 \Theta(s)^{-1} ds \Theta(0) B_0^{-1}(\psi_0[x] + \zeta_0) + \int_0^1 \Theta(s)^{-1} \int_0^s (Nx)(u)\, du\, ds. \end{aligned}$$

and

$$\begin{aligned} x'(1) = \Theta(1)^{-1}\Theta(0) B_0^{-1} A_0 P_2^{-1} \Big[\Big(A_1 \int_0^1 \Theta(s)^{-1} ds + B_1 \Theta(1)^{-1}\Big) \\ \times \Theta(0) B_0^{-1}(\psi_0[x] + \zeta_0) - A_1 \int_0^1 \Theta(s)^{-1} \int_0^s (Nx)(u) du \\ - B_1 \Theta(1)^{-1} \int_0^1 (Nx)(u) du + \psi_1[x] + \zeta_1\Big] - \Theta(1)^{-1}\Theta(0) B_0^{-1}(\psi_0[x] \\ + \zeta_0) + \Theta(1)^{-1} \int_0^1 (Nx)(u)\, du\, ds. \end{aligned}$$

Then we see that relation (1.3) is satisfied, because it holds that

$$\begin{aligned} A_1 x(1) + B_1 x'(1) &= A_1 \Big\{\Big(I_{n\times n} + \int_0^t \Theta(s)^{-1} ds \Theta(0) B_0^{-1} A_0\Big) P_2^{-1} \\ &\quad \times \Big[\Big(A_1 \int_0^1 \Theta(s)^{-1} ds + B_1 \Theta(1)^{-1}\Big) \Theta(0) B_0^{-1}(\psi_0[x] + \zeta_0) \\ &\quad - A_1 \int_0^1 \Theta(s)^{-1} \int_0^s (Nx)(u)\, du\, ds - B_1 \Theta(1)^{-1} \int_0^1 (Nx)(u)) du + \psi_1[x] + \zeta_1\Big] \\ &\quad - \int_0^1 \Theta(s)^{-1} ds \Theta(0) B_0^{-1}(\psi_0[x] + \zeta_0) + \int_0^1 \Theta(s)^{-1} \int_0^s (Nx)(u)\, du\, ds\Big\} \\ &\quad + B_1 \Big\{\Theta(1)^{-1}\Theta(0) B_0^{-1} A_0 P_2^{-1} \Big[\Big(A_1 \int_0^1 \Theta(s)^{-1} ds \\ &\quad + B_1 \Theta(1)^{-1}\Big) \Theta(0) B_0^{-1}(\psi_0[x] + \zeta_0) - A_1 \int_0^1 \Theta(s)^{-1} \int_0^s (Nx)(u)\, du\, ds \\ &\quad - B_1 \Theta(1)^{-1} \int_0^1 (Nx)(u) du + \psi_1[x] + \zeta_1\Big] - \Theta(1)^{-1}\Theta(0) B_0^{-1}(\psi_0[x] + \zeta_0) \\ &\quad + \Theta(1)^{-1} \int_0^1 (Nx)(u)\, du\, ds\Big\} \\ &= \Big[A_1 + (A_1 \int_0^1 \Theta(s)^{-1} ds + B_1 \Theta(1)^{-1}) \Theta(0) B_0^{-1} A_0\Big] \\ &\quad \times P_2^{-1} \Big[A_1 \int_0^1 \Theta(s)^{-1} ds \Theta(0) B_0^{-1}(\psi_0[x] + \zeta_0) \\ &\quad + B_1 \Theta(1)^{-1}\Theta(0) B_0^{-1}(\psi_0[x] + \zeta_0) - A_1 \int_0^1 \Theta(s)^{-1} \int_0^s (Nx)(u)\, du\, ds \\ &\quad - B_1 \Theta(1)^{-1} \int_0^1 (Nx)(u) du + \psi_1[x] + \zeta_1\Big] - A_1 \int_0^1 \Theta(s)^{-1} ds \end{aligned}$$

$$
\begin{aligned}
&\times \Theta(0)B_0^{-1}(\psi_0[x]+\zeta_0) + A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds \\
&- B_1\Theta(1)^{-1}\Theta(0)B_0^{-1}(\psi_0[x]+\zeta_0) + B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du \\
= &A_1\int_0^1 \Theta(s)^{-1}ds\Theta(0)B_0^{-1}(\psi_0[x]+\zeta_0) + B_1\Theta(1)^{-1}\Theta(0)B_0^{-1}(\psi_0[x]+\zeta_0) \\
&- A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds - B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du \\
&+ \psi_1[x] + \zeta_1 - A_1\int_0^1 \Theta(s)^{-1}ds\Theta(0)B_0^{-1}(\psi_0[x]+\zeta_0) \\
&+ A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds - B_1\Theta(1)^{-1}\Theta(0)B_0^{-1}(\psi_0[x]+\zeta_0) \\
&+ B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du \\
= &\psi_1[x] + \zeta_1.
\end{aligned}
$$

That a function x satisfying (4.4) satisfies (1.1), also, is obvious.

Now we write equation (4.4) in the form $x(t) = (\mathcal{T}_2x)(t) + (\mathcal{C}_2x)(t)$, where

$$
\begin{aligned}
(\mathcal{T}_2x)(t) = &\Big\{I_{n\times n} + \int_0^t \Theta(s)^{-1}ds\Theta(0)B_0^{-1}A_0\Big\}P_2^{-1} \\
&\times \Big[\Big(A_1\int_0^1 \Theta(s)^{-1}ds + B_1\Theta(1)^{-1}\Big)\Theta(0)B_0^{-1}(\psi_0[x]+\zeta_0) + \psi_1[x] + \zeta_1\Big].
\end{aligned} \tag{4.6}
$$

and

$$
\begin{aligned}
(\mathcal{C}_2x)(t) = &\Big\{I_{n\times n} + \int_0^t \Theta(s)^{-1}ds\Theta(0)B_0^{-1}A_0\Big\}P_2^{-1}\Big[-A_1\int_0^1 \Theta(s)^{-1} \\
&\times \int_0^s (Nx)(u)\,du\,ds - B_1\Theta(1)^{-1}\int_0^1 (Nx)(u))du\Big] + \int_0^t \Theta(s)^{-1}\int_0^s (Nx)(u)du))\,du\,ds.
\end{aligned}
$$

By (3.6), the latter can be written in the form $(\mathcal{C}_2x)(t) = \int_0^1 \mathcal{G}_2(t,u)(Nx)(u)du$, where

$$
\begin{aligned}
\mathcal{G}_2(t,u) := &-(I_{n\times n} + \int_0^t \Theta(s)^{-1}ds\Theta(0)B_0^{-1}A_0)P_2^{-1}\Big[A_1\int_u^1 \Theta(s)^{-1}ds + B_1\Theta(1)^{-1}\Big] \\
&+ \chi_{0,t]}(u)\int_u^t \Theta(s)^{-1}ds.
\end{aligned}
$$

Hence the existence of a solution of the problem is equivalent to the existence of a fixed point of the operator equation $x = \mathcal{T}_2x + \mathcal{C}_2x$. To proceed we observe that

$$
\begin{aligned}
\|\mathcal{G}_2(t,u)\|_E \le &(\sqrt{n} + \int_0^1 \|\Theta(s)^{-1}\|_E ds\|\Theta(0)B_0^{-1}A_0\|_E)\|P_2^{-1}\|_E \\
&\times \Big[\|A_1\|_E\int_0^1 \|\Theta(s)^{-1}\|_E ds + \|B_1\Theta(1)^{-1}\|_E\Big] + \int_0^1 \|\Theta(s)^{-1}\|_E ds =: M_2
\end{aligned}
$$

and therefore, for any $x \in B_r(0)$, it holds $\|\mathcal{C}_2x\|_\infty \le M_2m(r)$. Let $x, y \in B_r(0)$. Then we have

$$
\begin{aligned}
|(\mathcal{T}_2x)(t) - (\mathcal{T}_2y)(t)| \le &\Big\{\sqrt{n} + \int_0^1 \|\Theta(s)^{-1}ds\Theta(0)B_0^{-1}A_0\|_E\Big\}\|P_2^{-1}\|_E\Big[\|A_1\int_0^1 \Theta(s)^{-1}ds \\
&\times + B_1\Theta(1)^{-1}\|_E\|\Theta(0)B_0^{-1}\|_E|\psi_0(x-y)| + |\psi_1(x-y)|\Big].
\end{aligned}
$$

and so

$$
\|T_2x - T_2y\|_\infty \le \rho_2\|x-y\|_\infty, \tag{4.7}
$$

where ρ_2 is defined in (H6) which, as we have assumed is smaller than 1. Also for any $x \in B_r(0)$, we have

$$|(\mathcal{T}_2 x)(t)| \le \|(I_{n\times n} + \int_0^t \Theta(s)^{-1} ds \Theta(0) B_0^{-1} A_0)\|_E \|P_2^{-1}\|_E \Big[\|\Big(A_1 \int_0^1 \Theta(s)^{-1} ds + B_1 \Theta(1)^{-1}\|_E\Big)\|\Theta(0)B_0^{-1}\|_E (V(\Psi_0)|x| + |\zeta_0|) + V(\Psi_1)|x| + |\zeta_1|\Big],$$

namely $|(\mathcal{T}_2 x)(t) \le K_2 + \rho_2 r$, where K_2 and ρ_2 are defined in (H6).

Finally, we see that for all $x, y \in B_r(0)$ it holds $\|\mathcal{T}_2 x + \mathcal{C}_2 y\|_\infty \le K_2 + L_2 r + M_2 m(r)$. Now, from condition (H0) we can conclude that there exists $r_2 > 0$ such that $K_2 + \rho_2 r_2 + M_2 m(r_2) \le r_2$. Then, from the previous arguments we conclude that $\mathcal{T}_2 x + \mathcal{C}_2 y \in B_{r_2}(0)$, for all $x, y \in B_{r_2}(0)$. The relations we have found so far prove the (uniform) equicontinuity of the family $\{\mathcal{C}_2 x : x \in B_{r_2}(0)\}$ and so $\mathcal{C}_2$ is a compact mapping. Also, from relation (4.7) and condition (H6) it follows that the operator $\mathcal{T}_2$ is a contraction. By the Krasnosel'skii fixed point theorem the mapping $\mathcal{T}_2 + \mathcal{C}_2$ admits a fixed point in $B_{r_2}(0)$, which is a solution of the original problem. □

5. Case(3): $\det A_1 \neq 0$

Here we prove the following theorem.

Theorem 5.1. *Consider the nonlocal boundary value problem* (1.1), (1.2), (1.3), *where N satisfies* (2.5). *Moreover we make the following Assumptions:*

- (H7) $\det A_1 \neq 0$.
- (H8) *The operator P_3 defined by*

$$P_3 := A_0 \int_0^1 \Theta(s)^{-1} ds \Theta(0) + A_0 A_1^{-1} B_1 \Theta(1)^{-1} \Theta(0) + B_0$$

is nonsingular.

- (H9) *The quantity*

$$\rho_3 := \Big(\int_0^1 \|\Theta(s)^{-1} ds \Theta(0)\|_E + \|A_1^{-1} B_1 \Theta(1)^{-1} \Theta(0)\|_E + \int_0^1 \|\Theta(s)^{-1}\|_E ds \|\Theta(0)\|_E\Big)\|P_3^{-1}\|_E (V(A_0 A_1^{-1}\Psi_1) + V(\Psi_0)) + V(A_1^{-1}\Psi_1)$$

satisfies the inequality $\rho_3 < 1$ and moreover condition (H0), where

$$M_3 := \Big(\int_0^1 \|\Theta(s)^{-1}\|_E ds \|\Theta(0)\|_E + \|A_1^{-1} B_1 \Theta(1)^{-1}\Theta(0)\|_E + \int_0^1 \|\Theta(s)^{-1}\|_E ds \|\Theta(0)\|_E\Big)\|P_3^{-1}\Big[A_0 A_1^{-1} B_1 \Theta(1)^{-1} + A_0 \int_0^1 \Theta(s)^{-1} ds\Big]\|_E + \|A_1^{-1} B_1 \Theta(1)^{-1}\|_E + \int_0^1 \|\Theta(s)^{-1}\|_E ds + \|\int_0^1 E^{-1}(s)\|_E ds$$

and

$$K_3 := \Big(\int_0^1 \|\Theta(s)^{-1}\|_E ds \|\Theta(0)\|_E + \|A_1^{-1} B_1 \Theta(1)^{-1}\Theta(0)\|_E + \int_0^t \|\Theta(s)^{-1}\|_E ds \|\Theta(0)\|_E\Big)\|\|P_3^{-1}\|_E (|A_0 A_1^{-1}\zeta_1| + |\zeta_0|) + |A_1^{-1}\zeta_1|.$$

Then there is a solution of the problem (1.1), (1.2), (1.3).

Proof. Let x be a solution of the problem. From (1.1) we have

$$x'(t) = \Theta(t)^{-1}\big(\Theta(0)x'(0) + \int_0^t (Nx)(u) du\big).$$

This and (1.3) give

$$\begin{aligned} x(1) &= -A_1^{-1}B_1x'(1) + A_1^{-1}(\psi_1[x]+\zeta_1)\\ &= -A_1^{-1}B_1\Theta(1)^{-1}\big(\Theta(0)x'(0) + \int_0^1 (Nx)(u)du\big) + A_1^{-1}(\psi_1[x]+\zeta_1). \end{aligned} \tag{5.1}$$

Then relations (5.1) and (3.1) give

$$\begin{aligned} &x(0) + \Big(\int_0^1 \Theta(s)^{-1}ds\Theta(0) + A_1^{-1}B_1\Theta(1)^{-1}\Theta(0)\Big)x'(0)\\ &= -A_1^{-1}B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du + A_1^{-1}(\psi_1[x]+\zeta_1) - \int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds. \end{aligned} \tag{5.2}$$

The unique solution of the system of equations (1.2)-(5.2) is given by

$$\begin{aligned} x'(0) = -P_3^{-1}\Big[&A_0A_1^{-1}B_1\Theta(1)^{-1}\int_0^1 (Nx)(u))du\\ &+ A_0\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds - A_0A_1^{-1}(\psi_1[x]+\zeta_1) + \psi_0[x]+\zeta_0\Big] \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} x(0) = &\Big(\int_0^1 \Theta(s)^{-1}ds\Theta(0) + A_1^{-1}B_1\Theta(1)^{-1}\Theta(0)\Big)P_3^{-1}\\ &\times\Big[A_0A_1^{-1}B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du + A_0\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds\\ &- A_0A_1^{-1}(\psi_1[x]+\zeta_1) + \psi_0[x]+\zeta_0\Big] - A_1^{-1}B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du\\ &+ A_1^{-1}(\psi_1[x]+\zeta_1) - \int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds. \end{aligned} \tag{5.4}$$

Therefore, by (2.4) it follows that the solution x can be expressed in the form

$$\begin{aligned} x(t) = &\Big(\int_0^1 \Theta(s)^{-1}ds\Theta(0) + A_1^{-1}B_1\Theta(1)^{-1}\Theta(0)\Big)\\ &\times P_3^{-1}\Big[A_0A_1^{-1}B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du + A_0\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds\\ &- A_0A_1^{-1}(\psi_1[x]+\zeta_1) + \psi_0[x]+\zeta_0\Big] - A_1^{-1}B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du\\ &+ A_1^{-1}(\psi_1[x]+\zeta_1) - \int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds\\ &- \int_0^t \Theta(s)^{-1}ds\Theta(0)P_3^{-1}\Big[A_0A_1^{-1}B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du\\ &+ A_0\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds - A_0A_1^{-1}(\Psi_1[x]+\zeta_1) + \psi_0[x]+\zeta_0\Big]\\ &+ \int_0^t \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds, \end{aligned} \tag{5.5}$$

namely

$$
\begin{aligned}
x(t) = &\Big(\int_0^1 \Theta(s)^{-1} ds\Theta(0) + A_1^{-1}B_1\Theta(1)^{-1}\Theta(0) - \int_0^t \Theta(s)^{-1} ds\Theta(0)\Big) \\
&\times P_3^{-1}\Big[A_0A_1^{-1}B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du + A_0\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds \\
&- A_0A_1^{-1}(\psi_1[x]+\zeta_1) + \psi_0[x] + \zeta_0\Big] - A_1^{-1}B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du \\
&+ A_1^{-1}(\psi_1[x]+\zeta_1) - \int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds \\
&+ \int_0^t \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds.
\end{aligned} \tag{5.6}
$$

So, we proved that if x is a solution of the original problem, then it satisfies the operator equation (5.6). We shall show that the inverse, is also true, namely, we shall show that if a function satisfies equation (5.6) then it is a solution of the boundary value problem (1.1), (1.2), (1.3).

To do that we observe that if x is a function satisfying (5.6), then the values $x(0)$ and $x'(0)$ given in (5.3) and (5.4) satisfy the boundary condition (1.2). This is true since these values are obtained as the solutions of the system (1.2)-(5.2). To see that x satisfies (1.3), too, we obtain the values $x(1)$ and $x'(1)$ as follows

$$
\begin{aligned}
x(1) = &A_1^{-1}B_1\Theta(1)^{-1}\Theta(0)P_3^{-1}\Big[A_0A_1^{-1}B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du \\
&+ A_0\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds - A_0A_1^{-1}(\psi_1[x]+\zeta_1) + \psi_0[x] + \zeta_0\Big] \\
&- A_1^{-1}B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du + A_1^{-1}(\psi_1[x]+\zeta_1).
\end{aligned}
$$

and

$$
\begin{aligned}
x'(1) = &-\Theta(1)^{-1}\Theta(0)P_3^{-1}\Big[A_0A_1^{-1}B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du + A_0\int_0^1 \Theta(s)^{-1} \\
&\times \int_0^s (Nx)(u)\,du\,ds - A_0A_1^{-1}(\psi_1[x]+\zeta_1) + \psi_0[x] + \zeta_0\Big] + \Theta(1)^{-1}\int_0^1 (Nx)(u)\,du\,ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
A_1x(1) + B_1x'(1) = &B_1\Theta(1)^{-1}\Theta(0)P_3^{-1}\Big(A_0A_1^{-1}B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du + \\
&+ A_0\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds - A_0A_1^{-1}(\psi_1[x]+\zeta_1) + \psi_0[x] + \zeta_0\Big) \\
&- B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du + (\psi_1[x]+\zeta_1) + B_1\Theta(1)^{-1}\Theta(0) \\
&\times P_3^{-1}\Big[A_0A_1^{-1}B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du + A_0\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds \\
&- A_0A_1^{-1}(\psi_1[x]+\zeta_1) + \psi_0[x] + \zeta_0\Big] + B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)\,du\,ds \\
= &\psi_1[x] + \zeta_1).
\end{aligned}
$$

Next we write equation (5.6) in the form $x(t) = \mathcal{T}_3 x + \mathcal{C}_3 x$, where

$$\begin{aligned} \mathcal{T}_3 x(t) := \Big(\int_0^1 \Theta(s)^{-1} ds \Theta(0) + A_1^{-1} B_1 \Theta(1)^{-1}\Theta(0) - \int_0^t \Theta(s)^{-1} ds \Theta(0)\Big) \\ \times P_3^{-1}\Big[-A_0 A_1^{-1}(\psi_1[x] + \zeta_1) + \psi_0[x] + \zeta_0\Big] + A_1^{-1}(\psi_1[x] + \zeta_1) \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} \mathcal{C}_3 x(t) := &\Big(\int_0^1 \Theta(s)^{-1} ds \Theta(0) + A_1^{-1} B_1 \Theta(1)^{-1}\Theta(0) - \int_0^t \Theta(s)^{-1} ds \Theta(0)\Big) \\ &\times P_3^{-1}\Big[A_0 A_1^{-1} B_1 \Theta(1)^{-1} \int_0^1 (Nx)(u) du + A_0 \int_0^1 \Theta(s)^{-1} \int_0^s (Nx)(u)\, du\, ds\Big] \\ &- A_1^{-1} B_1 \Theta(1)^{-1} \int_0^1 (Nx)(u) du - \int_0^1 \Theta(s)^{-1} \int_0^s (Nx)(u)\, du\, ds \\ &+ \int_0^t \Theta(s)^{-1} \int_0^s (Nx)(u)\, du\, ds. \end{aligned}$$

By relation (3.6), the operator $\mathcal{C}_3$ can be written as

$$\begin{aligned} \mathcal{C}_3 x(t) = &\Big(\int_0^1 \Theta(s)^{-1} ds \Theta(0) + A_1^{-1} B_1 \Theta(1)^{-1}\Theta(0) - \int_0^t \Theta(s)^{-1} ds \Theta(0)\Big) \\ &\times P_3^{-1}\Big[A_0 A_1^{-1} B_1 \Theta(1)^{-1} \int_0^1 (Nx)(u) du + A_0 \int_0^1 \int_u^1 \Theta(s)^{-1} ds (Nx)(u) du\Big] \\ &- A_1^{-1} B_1 \Theta(1)^{-1} \int_0^1 (Nx)(u) du - \int_0^1 \int_u^1 \Theta(s)^{-1} ds (Nx)(u) du \\ &+ \int_0^t \int_u^t E^{-1}(s) ds (Nx)(u) du \\ = &\int_0^1 \mathcal{G}_3(t,u)(Nx)(u) du, \end{aligned} \tag{5.8}$$

where the Greens' function $\mathcal{G}_3$ is defined by

$$\begin{aligned} \mathcal{G}_3(t,u) = &\Big(\int_0^1 \Theta(s)^{-1} ds \Theta(0) + A_1^{-1} B_1 \Theta(1)^{-1}\Theta(0) - \int_0^t \Theta(s)^{-1} ds \Theta(0)\Big) \\ &\times P_3^{-1}\Big[A_0 A_1^{-1} B_1 \Theta(1)^{-1} + A_0 \int_u^1 \Theta(s)^{-1} ds\Big] \\ &- A_1^{-1} B_1 \Theta(1)^{-1} - \int_u^1 \Theta(s)^{-1} ds + \chi_{[0,t]}(u) \int_u^t E^{-1}(s) ds. \end{aligned}$$

To proceed we observe that

$$\begin{aligned} \|\mathcal{G}_3(t,u)\|_E \le &\Big(\int_0^1 \|\Theta(s)^{-1}\|_E ds \|\Theta(0)\|_E + \|A_1^{-1} B_1 \Theta(1)^{-1}\Theta(0)\|_E \\ &+ \int_0^1 \|\Theta(s)^{-1}\|_E ds \|\Theta(0)\|_E\Big) \|P_3^{-1}\Big[A_0 A_1^{-1} B_1 \Theta(1)^{-1} + A_0 \int_u^1 \Theta(s)^{-1} ds\Big]\|_E \\ &+ \|A_1^{-1} B_1 \Theta(1)^{-1}\|_E + \int_0^1 \|\Theta(s)^{-1}\|_E ds + \|\int_0^1 E^{-1}(s) ds\|_E = M_3 \end{aligned}$$

and therefore for any $x \in B_r(0)$ it holds that $\|C_3 x\|_\infty \le M_3 m(r)$.

r Also, we can see that, for all fixed $x, y \in B_r(0)$ we have

$$\begin{aligned} &|\mathcal{T}_3 x(t) - \mathcal{T}_3 y(t)| \\ &= \Big\{ \Big(\int_0^1 \|\Theta(s)^{-1} ds \Theta(0)\|_E + \|A_1^{-1} B_1 \Theta(1)^{-1}\Theta(0)\|_E \\ &\quad + \int_0^1 \|\Theta(s)^{-1}\|_E ds \|\Theta(0)\|_E \Big) \|P_3^{-1}\|_E \Big(V(A_0 A_1^{-1}\Psi_1) + V(\Psi_0) \Big) + V(A_1^{-1}\Psi_1) \Big\} \|x-y\|_\infty \\ &= \rho_3 \|x-y\|_\infty, \end{aligned} \tag{5.9}$$

where ρ_3 is defined in (H9) and satisfies $\rho_3 < 1$. On the other hand, given any $x \in B_r(0)$, we have

$$\begin{aligned} |\mathcal{T}_3 x(t)| \le &\Big(\int_0^1 \|\Theta(s)^{-1}\|_E ds \|\Theta(0)\|_E + \|A_1^{-1} B_1 \Theta(1)^{-1}\Theta(0)\|_E \\ &+ \int_0^1 \|\Theta(s)^{-1}\|_E ds \|\Theta(0)\|_E \Big) \|P_3^{-1}\| \Big[V(A_0 A_1^{-1}\Psi_1) r + |A_0 A_1^{-1}\zeta_1| \Big] \\ &+ V(\Psi_0) r + |\zeta_0|) + V(A_1^{-1}\Psi_1) r + |A_1^{-1}\zeta_1| \end{aligned}$$

namely $|\mathcal{T}_3 x(t)| \le K_3 + \rho_3 r$, where K_3 and ρ_3 is defined in (H9). Finally, we see that for all $x, y \in B_r(0)$ it holds $|(\mathcal{T}_3 x)(t) + (\mathcal{C}_3 y)(t)| \le K_3 + \rho_3 r + M_3 m(r)$.

Now, from condition (H8) there exists $r_3 > 0$ such that $K_3 + \rho_3 r_3 + M_3 m(r_3) \le r_3$. Then, from the previous arguments we conclude that $\mathcal{T}_3 x + \mathcal{C}_3 y \in B_{r_3}(0)$ for all $x, y \in B_{r_3}(0)$. The relations we have found so far prove the (uniform) equicontinuity of the family $\{\mathcal{C}_3 x : x \in B_{r_3}(0)\}$ and so $\mathcal{C}_3$ is a compact mapping. Also, from relation (5.9) and condition ($\mathcal{H}9a$) it follows that the operator $\mathcal{T}_3$ is a contraction. By the Krasnosel'skii fixed point theorem the mapping $\mathcal{T}_3 + \mathcal{C}_3$ admits a fixed point in $B_{r_3}(0)$, which is a solution of the original problem. □

6. Case(4): $\det B_1 \neq 0$

In this section we prove the following theorem.

Theorem 6.1. *Consider the nonlocal boundary value problem* (1.1), (1.2), (1.3), *where* N *satisfies* (2.5). *Moreover we make the following assumptions:*

- (H10) $\det B_1 \neq 0$.
- (H11) *The operator* P_4 *defined by*

$$P_4 := A_0 + A_0 \int_0^1 \Theta(s)^{-1} ds \Theta(1) B_1^{-1} A_1 + B_0 \Theta(0)^{-1}\Theta(1) B_1^{-1} A_1.$$

  is nonsingular.
- (H12) *The quantity*

$$\begin{aligned} \rho_4 := &\Big(\sqrt{n} + \int_0^1 \|\Theta(s)^{-1}\|_E ds \|\Theta(1) B_1^{-1} A_1\|_E \Big) \|P_4^{-1}\|_E \Big[\|A_0 \int_0^1 \Theta(s)^{-1} ds \Theta(1) \\ &\times B_1^{-1}\|_E V(\Psi_1) + \|B_0 \Theta(0)^{-1}\Theta(1) B_1^{-1}\|_E (V(\Psi_1) + V(\Psi_0)) \Big] \\ &+ \int_0^1 \|\Theta(s)^{-1}\|_E ds \|\Theta(1) B_1^{-1}\|_E V(\Psi_1). \end{aligned}$$

  satisfies the the inequality $\rho_4 < 1$ *and condition (H0), where*

$$\begin{aligned} M_4 := &\Big(\int_0^1 \|\Theta(s)^{-1}\|_E ds \|\Theta(1) B_1^{-1} A_1\|_E + \sqrt{n} \Big) \|P_4^{-1}\|_E \Big[\int_0^1 \|A_0 \Theta(s)^{-1}\|_E ds \\ &+ \|B_0 \Theta(0)^{-1}\|_E \Big] + \int_0^1 \|\Theta(s)^{-1}\|_E ds, \end{aligned}$$

  and

$$K_4 := \Big(\sqrt{n} + \int_0^1 \|\Theta(s)^{-1}\|_E ds \|\Theta(1) B_1^{-1} A_1\|_E \Big) \|P_4^{-1}\|_E$$

$$
\begin{aligned}
&\times \Big[\|A_0\int_0^1 \Theta(s)^{-1}ds\Theta(1)B_1^{-1}\|_E|\zeta_1| + \|B_0\Theta(0)^{-1}\Theta(1)B_1^{-1}\|_E|\zeta_1| + |\zeta_0|\Big]\\
&+ \int_0^1 \|\Theta(s)^{-1}\|_E ds\|\Theta(1)B_1^{-1}\|_E|\zeta_1|.
\end{aligned}
$$

Then there is a solution of the problem (1.1), (1.2), (1.3).

Proof. In this section we assume that the matrix B_1 is nonsingular. From equation (1.1) we obtain

$$
x'(t) = \Theta(t)^{-1}\Theta(1)x'(1) - \Theta(t)^{-1}\int_t^1 (Nx)(u)du
$$

and

$$
x(t) = x(1) - \int_t^1 \Theta(s)^{-1}ds\Theta(1)x'(1) + \int_t^1 \Theta(s)^{-1}\int_s^1 (Nx)(u)\,du\,ds. \tag{6.1}
$$

From the first of these relations we take

$$
x'(0) = \Theta(0)^{-1}\Theta(1)x'(1) - \Theta(0)^{-1}\int_0^1 (Nx)(u)du \tag{6.2}
$$

and from the second one,

$$
x(0) = x(1) - \int_0^1 \Theta(s)^{-1}ds\Theta(1)x'(1) + \int_0^1 \Theta(s)^{-1}\int_s^1 (Nx)(u)\,du\,ds.
$$

By using (1.3) and (6.2) we have

$$
x'(0) = \Theta(0)^{-1}\Theta(1)[B_1^{-1}(\psi_1[x] + \zeta_1) - B_1^{-1}A_1x(1)] - \Theta(0)^{-1}\int_0^1 (Nx)(u)du
$$

and

$$
\begin{aligned}
x(0) &= x(1) - \int_0^1 \Theta(s)^{-1}ds\Theta(1)[B_1^{-1}(\psi_1[x] + \zeta_1) - B_1^{-1}A_1x(1)] + \int_0^1 \Theta(s)^{-1}\int_s^1 (Nx)(u)\,du\,ds\\
&= -\int_0^1 \Theta(s)^{-1}ds\Theta(1)B_1^{-1}(\psi_1[x] + \zeta_1)\\
&\quad + [I_{n\times n} + \int_0^1 \Theta(s)^{-1}ds\Theta(1)B_1^{-1}A_1]x(1) + \int_0^1 \Theta(s)^{-1}\int_s^1 (Nx)(u)\,du\,ds.
\end{aligned}
$$

We put these values to the boundary condition (1.2) and obtain

$$
\begin{aligned}
&A_0\Big\{-\int_0^1 \Theta(s)^{-1}ds\Theta(1)B_1^{-1}(\psi_1[x] + \zeta_1) + [I_{n\times n}\\
&+ \int_0^1 \Theta(s)^{-1}ds\Theta(1)B_1^{-1}A_1]x(1) + \int_0^1 \Theta(s)^{-1}\int_s^1 (Nx)(u)\,du\,ds\Big\}\\
&- B_0\Big(\Theta(0)^{-1}\Theta(1)[B_1^{-1}(\psi_1[x] + \zeta_1) - B_1^{-1}A_1x(1)] - \Theta(0)^{-1}\int_0^1 (Nx)(u)du\Big\}\\
&= \psi_0[x] + \zeta_0.
\end{aligned}
$$

The latter relation can be written in the form

$$
\begin{aligned}
P_4x(1) &= A_0\int_0^1 \Theta(s)^{-1}ds\Theta(1)B_1^{-1}(\psi_1[x] + \zeta_1) - A_0\int_0^1 \Theta(s)^{-1}\int_s^1 (Nx)(u)du\\
&\quad + B_0\Theta(0)^{-1}\Theta(1)B_1^{-1}(\psi_1[x] + \zeta_1) - B_0\Theta(0)^{-1}\int_0^1 (Nx)(u)du + \psi_0[x] + \zeta_0,
\end{aligned}
$$

where P_4 is defined in (H11). So

$$
\begin{aligned}
x(1) = P_4^{-1}\Big[&A_0\int_0^1 \Theta(s)^{-1}ds\Theta(1)B_1^{-1}(\psi_1[x] + \zeta_1) - A_0\int_0^1 \Theta(s)^{-1}\int_s^1 (Nx)(u)du\\
&+ B_0\Theta(0)^{-1}\Theta(1)B_1^{-1}(\psi_1[x] + \zeta_1) - B_0\Theta(0)^{-1}\int_0^1 (Nx)(u)du + \psi_0[x] + \zeta_0\Big].
\end{aligned} \tag{6.3}
$$

Then from (1.3) we obtain

$$
\begin{aligned}
x'(1) = B_1^{-1}(\psi_1[x]+\zeta_1) - B_1^{-1}A_1P_4^{-1}\Big[A_0\int_0^1 \Theta(s)^{-1}ds\Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1) \\
- A_0\int_0^1 \Theta(s)^{-1}\int_s^1 (Nx)(u)du + B_0\Theta(0)^{-1}\Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1) \\
- B_0\Theta(0)^{-1}\int_0^1 (Nx)(u)du + \psi_0[x]+\zeta_0\Big].
\end{aligned} \tag{6.4}
$$

Finally, from equation (6.1) we see that the solution x can be expressed in the form

$$
\begin{aligned}
x(t) = P_4^{-1}\Big[A_0\int_0^1 \Theta(s)^{-1}ds\Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1) - A_0\int_0^1 \Theta(s)^{-1}\int_s^1 (Nx)(u)du \\
+ B_0\Theta(0)^{-1}\Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1) - B_0\Theta(0)^{-1}\int_0^1 (Nx)(u)du + \psi_0[x]+\zeta_0\Big] \\
- \int_t^1 \Theta(s)^{-1}ds\Theta(1)\Big\{B_1^{-1}(\psi_1[x]+\zeta_1) - B_1^{-1}A_1P_4^{-1}\Big[A_0\int_0^1 \Theta(s)^{-1}ds\Theta(1) \\
\times B_1^{-1}(\psi_1[x]+\zeta_1) - A_0\int_0^1 \Theta(s)^{-1}\int_s^1 (Nx)(u)du + B_0\Theta(0)^{-1}\Theta(1)B_1^{-1}(\psi_1[x] \\
+ \zeta_1) - B_0\Theta(0)^{-1}\int_0^1 (Nx)(u)du + \psi_0[x]+\zeta_0\Big]\Big\} + \int_t^1 \Theta(s)^{-1}\int_s^1 (Nx)(u)\,du\,ds.
\end{aligned} \tag{6.5}
$$

We proved that if x is a solution of the original problem, then it satisfies relation (6.5) Now, we shall show that the inverse, is true, namely, we shall show that if a function satisfies equation (6.5), then it is a solution of the boundary value problem (1.1), (1.2), (1.3).

To do that we can easily see that if x is such a function then the values $x(1)$ and $x'(1)$ given in (6.3) and (6.4) satisfy the boundary condition (1.3). To check relation (1.2), from (6.5) we obtain

$$
\begin{aligned}
x(0) &= P_4^{-1}\Big[A_0\int_0^1 \Theta(s)^{-1}ds\Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1) - A_0\int_0^1 \Theta(s)^{-1}\int_s^1 (Nx)(u)du \\
&\quad + B_0\Theta(0)^{-1}\Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1) - B_0\Theta(0)^{-1}\int_0^1 (Nx)(u)du + \psi_0[x]+\zeta_0\Big] \\
&\quad - \int_0^1 \Theta(s)^{-1}ds\Theta(1)\Big\{B_1^{-1}(\psi_1[x]+\zeta_1) - B_1^{-1}A_1P_4^{-1}\Big[A_0\int_0^1 \Theta(s)^{-1}ds \\
&\quad \times \Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1) - A_0\int_0^1 \Theta(s)^{-1}\int_s^1 (Nx)(u)du \\
&\quad + B_0\Theta(0)^{-1}\Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1) \\
&\quad - B_0\Theta(0)^{-1}\int_0^1 (Nx)(u)du + \psi_0[x]+\zeta_0\Big\} + \int_0^1 \Theta(s)^{-1}\int_s^1 (Nx)(u)\,du\,ds \\
&= \Big[I_{n\times n} + \int_0^1 \Theta(s)^{-1}ds\Theta(1)B_1^{-1}A_1\Big]P_4^{-1}\Big[A_0\int_0^1 \Theta(s)^{-1}ds\Theta(1) \\
&\quad \times B_1^{-1}(\psi_1[x]+\zeta_1) - A_0\int_0^1 \Theta(s)^{-1}\int_s^1 (Nx)(u)du + B_0\Theta(0)^{-1}\Theta(1) \\
&\quad \times B_1^{-1}(\psi_1[x]+\zeta_1) - B_0\Theta(0)^{-1}\int_0^1 (Nx)(u)du + \psi_0[x]+\zeta_0\Big] \\
&\quad - \int_0^1 \Theta(s)^{-1}ds\Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1) + \int_0^1 \Theta(s)^{-1}\int_s^1 (Nx)(u)\,du\,ds
\end{aligned} \tag{6.6}
$$

and

$$
\begin{aligned}
x'(0) &= \Theta(0)^{-1}\Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1) - \Theta(0)^{-1}\Theta(1)B_1^{-1}A_1P_4^{-1}\Big[A_0\int_0^1\Theta(s)^{-1}ds \\
&\quad \times \Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1) - A_0\int_0^1\Theta(s)^{-1}\int_s^1(Nx)(u)du + B_0\Theta(0)^{-1}\Theta(1)B_1^{-1}(\psi_1[x] \\
&\quad +\zeta_1) - B_0\Theta(0)^{-1}\int_0^1(Nx)(u)du + \psi_0[x]+\zeta_0\Big] - \Theta(0)^{-1}\int_0^1(Nx)(u)\,du\,ds.
\end{aligned} \tag{6.7}
$$

Hence it holds that

$$
\begin{aligned}
&A_0x(0) - B_0x'(0) \\
&= A_0\Big[I_{n\times n} + \int_0^1\Theta(s)^{-1}ds\Theta(1)B_1^{-1}A_1\Big]P_4^{-1}\Big[A_0\int_0^1\Theta(s)^{-1}ds\Theta(1) \\
&\quad \times B_1^{-1}(\psi_1[x]+\zeta_1) - A_0\int_0^1\Theta(s)^{-1}\int_s^1(Nx)(u)du + B_0\Theta(0)^{-1}\Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1) \\
&\quad - B_0\Theta(0)^{-1}\int_0^1(Nx)(u)du + \psi_0[x]+\zeta_0\Big] - A_0\int_0^1\Theta(s)^{-1}ds\Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1) \\
&\quad + A_0\int_0^1\Theta(s)^{-1}\int_s^1(Nx)(u)\,du\,ds - B_0\Theta(0)^{-1}\Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1) \\
&\quad + B_0\Theta(0)^{-1}\Theta(1)B_1^{-1}A_1P_4^{-1}\Big[A_0\int_0^1\Theta(s)^{-1}ds\Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1) \\
&\quad - A_0\int_0^1\Theta(s)^{-1}\int_s^1(Nx)(u)du + B_0\Theta(0)^{-1}\Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1) \\
&\quad - B_0\Theta(0)^{-1}\int_0^1(Nx)(u)du + \psi_0[x]+\zeta_0\Big] + B_0\Theta(0)^{-1}\int_0^1(Nx)(u)\,du\,ds \\
&= \Big[A_0 + A_0\int_0^1\Theta(s)^{-1}ds\Theta(1)B_1^{-1}A_1 + B_0\Theta(0)^{-1}\Theta(1)B_1^{-1}A_1\Big] \\
&\quad \times P_4^{-1}\Big[A_0\int_0^1\Theta(s)^{-1}ds\Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1) - A_0\int_0^1\Theta(s)^{-1}\int_s^1(Nx)(u)du \\
&\quad + B_0\Theta(0)^{-1}\Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1) - B_0\Theta(0)^{-1}\int_0^1(Nx)(u)du + \psi_0[x]+\zeta_0\Big] \\
&\quad - A_0\int_0^1\Theta(s)^{-1}ds\Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1) + A_0\int_0^1\Theta(s)^{-1}\int_s^1(Nx)(u)\,du\,ds \\
&\quad - B_0\Theta(0)^{-1}\Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1) + B_0\Theta(0)^{-1}\int_0^1(Nx)(u)\,du\,ds \\
&= A_0\int_0^1\Theta(s)^{-1}ds\Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1) - A_0\int_0^1\Theta(s)^{-1}\int_s^1(Nx)(u)du \\
&\quad + B_0\Theta(0)^{-1}\Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1) - B_0\Theta(0)^{-1}\int_0^1(Nx)(u)du + \psi_0[x]+\zeta_0 \\
&\quad - A_0\int_0^1\Theta(s)^{-1}ds\Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1) + A_0\int_0^1\Theta(s)^{-1}\int_s^1(Nx)(u)\,du\,ds \\
&\quad - B_0\Theta(0)^{-1}\Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1) + B_0\Theta(0)^{-1}\int_0^1(Nx)(u)\,du\,ds \\
&= \psi_0[x]+\zeta_0.
\end{aligned}
$$

Therefore the condition (1.2) is satisfied.

Next write equation (6.5) in the form $x(t) = \mathcal{T}_4 x(t) + \mathcal{C}_4 x(t)$, where

$$
\begin{aligned}
\mathcal{T}_4 x(t) &:= P_4^{-1}\Big[A_0\int_0^1 \Theta(s)^{-1}ds\Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1) + B_0\Theta(0)^{-1}\Theta(1)B_1^{-1}(\psi_1[x] \\
&\quad + \zeta_1) + \psi_0[x]+\zeta_0\Big] - \int_t^1 \Theta(s)^{-1}ds\Theta(1)\Big\{B_1^{-1}(\psi_1[x]+\zeta_1) \\
&\quad - B_1^{-1}A_1P_4^{-1}\Big[A_0\int_0^1 \Theta(s)^{-1}ds\Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1) \\
&\quad + B_0\Theta(0)^{-1}\Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1)+\psi_0[x]+\zeta_0\Big]\Big\} \\
&= \Big(I_{n\times n} + \int_t^1 \Theta(s)^{-1}ds\Theta(1)B_1^{-1}A_1\Big)P_4^{-1}\Big[A_0\int_0^1 \Theta(s)^{-1}ds\Theta(1)B_1^{-1} \\
&\quad \times(\psi_1[x]+\zeta_1) + B_0\Theta(0)^{-1}\Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1)+\psi_0[x]+\zeta_0\Big] \\
&\quad - \int_t^1 \Theta(s)^{-1}ds\Theta(1)B_1^{-1}(\psi_1[x]+\zeta_1).
\end{aligned}
\tag{6.8}
$$

and

$$
\begin{aligned}
\mathcal{C}_4 x(t) &:= -P_4^{-1}\Big[A_0\int_0^1 \Theta(s)^{-1}\int_s^1 (Nx)(u)du + B_0\Theta(0)^{-1}\int_0^1 (Nx)(u)du\Big] \\
&\quad + \int_t^1 \Theta(s)^{-1}ds\Theta(1)\Big\{B_1^{-1}A_1P_4^{-1}\Big[A_0\int_0^1 \Theta(s)^{-1}\int_s^1 (Nx)(u)du + B_0\Theta(0)^{-1} \\
&\quad \times\int_0^1 F(u,x(u))du\Big]\Big\} + \int_t^1 \Theta(s)^{-1}\int_s^1 (Nx)(u)\,du\,ds \\
&= \int_0^1 \mathcal{G}_4(t,u)(Nx)(u)du.
\end{aligned}
$$

Here the kernel $\mathcal{G}_4$ is defined by

$$
\begin{aligned}
\mathcal{G}_4(t,u) &:= -P_4^{-1}\Big[A_0\int_0^u \Theta(s)^{-1}ds + B_0\Theta(0)^{-1}\Big] + \int_t^1 \Theta(s)^{-1}ds\Theta(1)B_1^{-1}A_1 \\
&\quad P_4^{-1}\Big[A_0\int_0^u \Theta(s)^{-1}ds + B_0\Theta(0)^{-1}\Big] + \chi_{[t,1]}(u)\int_0^u \Theta(s)^{-1}ds \\
&= \Big(\int_t^1 \Theta(s)^{-1}ds\Theta(1)B_1^{-1}A_1 - I_{n\times n}\Big)P_4^{-1}\Big[A_0\int_0^u \Theta(s)^{-1}ds + B_0\Theta(0)^{-1}\Big] \\
&\quad + \chi_{[t,1]}(u)\int_0^u \Theta(s)^{-1}ds,
\end{aligned}
\tag{6.9}
$$

where we have applied the Fubini's Theorem twice.

For the kernel we observe that

$$
\begin{aligned}
\|\mathcal{G}_4(t,u)\|_E &\le \Big(\int_0^1 \|\Theta(s)^{-1}\|_E ds\|\Theta(1)B_1^{-1}A_1\|_E + \sqrt{n}\Big)\|P_4^{-1}\|_E \\
&\quad \times\Big[\int_0^1 \|A_0\Theta(s)^{-1}\|_E ds + \|B_0\Theta(0)^{-1}\|_E\Big] + \int_0^1 \|\Theta(s)^{-1}\|_E ds = M_4
\end{aligned}
$$

and therefore, for any $x \in B_r(0)$, $\|\mathcal{C}_4 x\|_\infty \le M_4 m(r)$. Also, for all $x, y \in B_r(0)$, we have

$$
\begin{aligned}
|\mathcal{T}_4 x(t) - \mathcal{T}_4 y(t)| &= \Big|\Big(I_{n\times n} + \int_t^1 \Theta(s)^{-1}ds\Theta(1)B_1^{-1}A_1\Big)P_4^{-1}\Big\{A_0\int_0^1 \Theta(s)^{-1}ds \\
&\quad \times\Theta(1)B_1^{-1}\Psi_1[x-y] + B_0\Theta(0)^{-1}\Theta(1)B_1^{-1}(\psi_1(x-y)+\psi_0(x-y))\Big\} \\
&\quad - \int_t^1 \Theta(s)^{-1}ds\Theta(1)B_1^{-1}\Psi_1[x-y]\Big|
\end{aligned}
$$

$$
\begin{aligned}
&\le \Big(\sqrt{n}+\int_0^1 \|\Theta(s)^{-1}\|_E ds\|\Theta(1)B_1^{-1}A_1\|_E\Big)\\
&\quad\times \|P_4^{-1}\|_E\Big[\|A_0\int_0^1 \Theta(s)^{-1}ds\Theta(1)B_1^{-1}\|_E V(\Psi_1)\|x-y\|_\infty\\
&\quad+\|B_0\Theta(0)^{-1}\Theta(1)B_1^{-1}\|_E(V(\Psi_1)\|x-y\|_\infty+V(\Psi_0)\|x-y\|_\infty)\Big]\\
&\quad+\|\int_0^1 \Theta(s)^{-1}ds\Theta(1)B_1^{-1}\|_E V(\Psi_1)\|x-y\|_\infty,
\end{aligned}
$$

which implies that

$$
\|\mathcal{T}_4x-\mathcal{T}_4y\|_\infty \le \rho_4\|x-y\|_\infty, \tag{6.10}
$$

where ρ_4 is defined in (H12) and it satisfies the inequality $\rho_4<1$.

On the other hand, for any $x\in B_r(0)$ and $t\in I$, it holds that

$$
\begin{aligned}
|\mathcal{T}_4x(t)| &\le \Big(\sqrt{n}+\int_0^1 \|\Theta(s)^{-1}\|_E ds\|\Theta(1)B_1^{-1}A_1\|_E\Big)\|P_4^{-1}\|_E\\
&\quad\times\Big[\|A_0\int_0^1 \Theta(s)^{-1}ds\Theta(1)B_1^{-1}\|_E|\zeta_1|+\|B_0\Theta(0)^{-1}\Theta(1)B_1^{-1}\|_E|\zeta_1|+|\zeta_0|\Big]\\
&\quad+\int_0^1 \|\Theta(s)^{-1}\|_E ds\|\Theta(1)B_1^{-1}\|_E|\zeta_1|\Big(\sqrt{n}+\int_0^1 \|\Theta(s)^{-1}\|_E ds\|\Theta(1)B_1^{-1}A_1\|_E\Big)\\
&\quad\times\|P_4^{-1}\|_E\Big[\|A_0\int_0^1 \Theta(s)^{-1}ds\Theta(1)B_1^{-1}\|_E V(\Psi_1)r+\|B_0\Theta(0)^{-1}\Theta(1)B_1^{-1}\|_E V(\Psi_1)r\\
&\quad+V(\Psi_0)r\Big]+\int_0^1 \|\Theta(s)^{-1}\|_E ds\|\Theta(1)B_1^{-1}\|_E V(\psi_1)r,
\end{aligned}
$$

and so

$$
\|\mathcal{T}_4x\|_\infty \le K_4+\rho_4 r,
$$

where K_4 and ρ_4 are defined in (H12). Finally, we see that for all $x,y\in B_r(0)$ it holds $\|\mathcal{T}_4x+\mathcal{C}_4y\|_\infty \le K_4+\rho_4r+M_4m(r)$.

Now, from condition (H0) there exists $r_4>0$ such that

$$
K_4+\rho_4r_4+M_4m(r_4)\le r_4.
$$

Then, from the previous arguments, we conclude that $\mathcal{T}_4x+\mathcal{C}_4y\in B_{r_4}(0)$ for all $x,y\in B_{r_4}(0)$. The relations we have found so far prove the (uniform) equicontinuity of the family $\{\mathcal{C}_4x : x\in B_{r_4}(0)\}$ and so $\mathcal{C}_4$ is a compact mapping. Also, from relation (6.10) and condition (H12) it follows that the operator $\mathcal{T}_4$ is a contraction. By the Krasnosel'skii fixed point theorem the mapping $\mathcal{T}_4+\mathcal{C}_4$ admits a fixed point $\bar{x}$ in $B_{r_4}(0)$, which is a solution of the original problem. □

7. Case(5): $\det A_0\neq 0$ and $A_0A_1=A_1A_0$

In this section we prove the following theorem.

Theorem 7.1. *Consider the nonlocal boundary value problem* (1.1), (1.2), (1.3), *where N satisfies* (2.5). *Moreover we make the following conditions:*

- (H13) $\det A_0\neq 0$, $\det A_1\neq 0$ *and* $A_0A_1=A_1A_0$.
- (H14) *The operator P_5 defined by*

$$
P_5:=A_0B_1\Theta(1)^{-1}\Theta(0)+A_0A_1\int_0^1 \Theta(s)^{-1}ds\Theta(0)+A_1B_0
$$

  is nonsingular.
- (H15) *The quantity*

$$
\rho_5:=(\|A_0^{-1}B_0\|_E+\int_0^1 \|\Theta(s)^{-1}\|_E ds\|\Theta(0)\|_E)\|P_5^{-1}\|_E\Big[V(A_0\Psi_1)+V(A_1\Psi_0)\Big]+V(A_0^{-1}\Psi_0)
$$

satisfies the inequality $\rho_5 < 1$ and condition (H0) where

$$M_5 := (\|A_0^{-1}B_0\|_E + \int_0^1 \|\Theta(s)^{-1}\|_E ds \|\Theta(0)\|_E)\|P_5^{-1}\|_E \times \big(\|A_0A_1\|_E \int_0^1 \|\Theta(s)^{-1}\|_E ds + \|A_0B_1\Theta(1)^{-1}\|_E\big) + \int_0^1 \|\Theta(s)^{-1}\| ds$$

and

$$K_5 := (\|A_0^{-1}B_0\|_E + \int_0^1 \|\Theta(s)^{-1}\|_E ds \|\Theta(0)\|_E)\|P_5^{-1}\|_E \Big[|A_0\zeta_1| + |A_1\zeta_0|)\Big] + |A_0^{-1}\zeta_0|).$$

Then there is a solution of the problem.

Proof. From (1.1) we have the expression of x' as in (2.2) and so (2.3) holds. Thus

$$B_1x'(1) = B_1\Theta(1)^{-1}\Theta(0)x'(0) + B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du.$$

Then from (1.3) it follows that

$$\Psi_1[x] + \zeta_1 - A_1x(1) = B_1\Theta(1)^{-1}\Theta(0)x'(0) + B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du,$$

which implies that

$$A_1x(1) = \psi_1[x] + \zeta_1 - B_1\Theta(1)^{-1}\Theta(0)x'(0) - B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du.$$

From (2.4) we obtain

$$A_1x(1) = A_1x(0) + A_1\int_0^1 \Theta(s)^{-1}ds\Theta(0)x'(0) + A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds$$

and therefore

$$\begin{aligned}&\Psi_1[x] + \zeta_1 - B_1\Theta(1)^{-1}\Theta(0)x'(0) - B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du\\ &= A_1x(0) + A_1\int_0^1 \Theta(s)^{-1}ds\Theta(0)x'(0) + A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds.\end{aligned}$$

Hence

$$\begin{aligned}A_1x(0) &= \psi_1[x] + \zeta_1 - B_1\Theta(1)^{-1}\Theta(0)x'(0) - B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du\\ &\quad - A_1\int_0^1 \Theta(s)^{-1}ds\Theta(0)x'(0) - A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds\\ &= \psi_1[x] + \zeta_1 - \Big(B_1\Theta(1)^{-1} + A_1\int_0^1 \Theta(s)^{-1}ds\Big)\Theta(0)x'(0)\\ &\quad - B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du - A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds.\end{aligned} \tag{7.1}$$

By condition ($\mathcal{H}13$), the matrices A_0, A_1 satisfy the condition $A_0A_1 = A_1A_0$. Notice that in this case we have

$$A_0A_1A_0^{-1} = A_1 \implies A_1A_0^{-1} = A_0^{-1}A_1, \tag{7.2}$$

Then from (1.2) we have

$$A_0A_1x(0) = A_1B_0x'(0) + A_1(\psi_0[x] + \zeta_0),$$

which, due to (7.1), gives

$$\begin{aligned}P_5x'(0) &= A_0(\psi_1[x] + \zeta_1) - A_1(\psi_0[x] + \zeta_0)\\ &\quad - A_0A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds - A_0B_1\Theta(1)^{-1}\int_0^1 (Nx)(u))du,\end{aligned}$$

where P_5 is defined in $\mathcal{H}_{14}$. From ($\mathcal{H}_{14}$) we obtain

$$\begin{aligned}x'(0) = P_5^{-1}\Big[&A_0(\psi_1[x]+\zeta_1) - A_1(\psi_0[x]+\zeta_0)\\ &- A_0A_1\int_0^1\Theta(s)^{-1}\int_0^s(Nx)(u)\,du\,ds - A_0B_1\Theta(1)^{-1}\int_0^1(Nx)(u)du\Big].\end{aligned}$$

From (1.2) and (2.4) it follows that

$$\begin{aligned}x(t) &= x(0) + \int_0^t\Theta(s)^{-1}ds\Theta(0)x'(0) + \int_0^t\Theta(s)^{-1}\int_0^s(Nx)(u)\,du\,ds\\ &= (A_0^{-1}B_0+\\ &+\int_0^t\Theta(s)^{-1}ds\Theta(0))x'(0) + A_0^{-1}(\Psi_0[x]+\zeta_0) + \int_0^t\Theta(s)^{-1}\int_0^s(Nx)(u)\,du\,ds\\ &= (A_0^{-1}B_0 + \int_0^t\Theta(s)^{-1}ds\Theta(0))P_5^{-1}\Big[A_0(\psi_1[x]+\zeta_1) - A_1(\psi_0[x]+\zeta_0)\\ &\quad - A_0A_1\int_0^1\Theta(s)^{-1}\int_0^s(Nx)(u)\,du\,ds - A_0B_1\Theta(1)^{-1}\int_0^1(Nx)(u)du\Big]\\ &\quad + A_0^{-1}(\psi_0[x]+\zeta_0) + \int_0^t\Theta(s)^{-1}\int_0^s(Nx)(u)\,du\,ds.\end{aligned}\tag{7.3}$$

We proved that if x is a solution of the original problem, then it satisfies relation (7.3). Now, we shall show that the inverse is true, namely, we shall show that if a function satisfies equation (7.3), then it is a solution of the boundary value problem (1.1), (1.2), (1.3).

Indeed, from (7.3) we obtain

$$\begin{aligned}x(0) = A_0^{-1}B_0P_5^{-1}\Big[&A_0(\psi_1[x]+\zeta_1) - A_1(\psi_0[x]+\zeta_0)\\ &- A_0A_1\int_0^1\Theta(s)^{-1}\int_0^s(Nx)(u)\,du\,ds - A_0B_1\Theta(1)^{-1}\int_0^1(Nx)(u)du\Big] + A_0^{-1}(\psi_0[x]+\zeta_0)\end{aligned}$$

and

$$\begin{aligned}x'(0) = P_5^{-1}\Big[&A_0(\psi_1[x]+\zeta_1) - A_1(\psi_0[x]+\zeta_0)\\ &- A_0A_1\int_0^1\Theta(s)^{-1}\int_0^s(Nx)(u)\,du\,ds - A_0B_1\Theta(1)^{-1}\int_0^1(Nx)(u)du\Big].\end{aligned}$$

Then we can easily see that condition (1.2) is satisfied.

Also, we have

$$\begin{aligned}x(1) = (A_0^{-1}B_0 + \int_0^1\Theta(s)^{-1}ds\Theta(0))P_5^{-1}\Big[&A_0(\psi_1[x]+\zeta_1) - A_1(\psi_0[x]+\zeta_0)\\ &- A_0A_1\int_0^1\Theta(s)^{-1}\int_0^s(Nx)(u)\,du\,ds - A_0B_1\Theta(1)^{-1}\int_0^1(Nx)(u)du\Big]\\ &+ A_0^{-1}(\psi_0[x]+\zeta_0) + \int_0^1\Theta(s)^{-1}\int_0^s(Nx)(u)\,du\,ds\end{aligned}$$

and

$$\begin{aligned}x'(1) = \Theta(1)^{-1}\Theta(0)P_5^{-1}\Big[&A_0(\psi_1[x]+\zeta_1) - A_1(\psi_0[x]+\zeta_0)\\ &- A_0A_1\int_0^1\Theta(s)^{-1}\int_0^s(Nx)(u)\,du\,ds - A_0B_1\Theta(1)^{-1}\int_0^1(Nx)(u)du\Big]\\ &+ \Theta(1)^{-1}\int_0^1(Nx)(u)du.\end{aligned}$$

Therefore, by (7.2), it holds

$$A_1x(1) + B_1x'(1)$$

$$
\begin{aligned}
&= A_1\Big\{(A_0^{-1}B_0 + \int_0^1 \Theta(s)^{-1}ds\Theta(0))P_5^{-1}\Big[A_0(\psi_1[x]+\zeta_1) \\
&\quad - A_1(\psi_0[x]+\zeta_0) - A_0A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u))\,du\,ds - A_0B_1\Theta(1)^{-1} \\
&\quad \times \int_0^1 (Nx)(u)du\Big] + A_0^{-1}(\psi_0[x]+\zeta_0) + \int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds\Big\} + B_1\Big\{\Theta(1)^{-1} \\
&\quad \times \Theta(0)P_5^{-1}\Big[A_0(\psi_1[x]+\zeta_1) - A_1(\psi_0[x]+\zeta_0) - A_0A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds \\
&\quad - A_0B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du\Big] + \Theta(1)^{-1}\int_0^1 (Nx)(u)du \\
&= \Big[A_1A_0^{-1}B_0 + A_1\int_0^1 \Theta(s)^{-1}ds\Theta(0) + B_1\Theta(1)^{-1}\Theta(0)\Big]P_5^{-1}\Big[A_0(\psi_1[x]+\zeta_1) \\
&\quad - A_1(\psi_0[x]+\zeta_0) - A_0A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds \\
&\quad - A_0B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du\Big] + A_1A_0^{-1}(\psi_0[x]+\zeta_0) + A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)du.
\end{aligned}
$$

The first factor in the big parenthesis can be written as

$$
A_0^{-1}\Big[A_1B_0 + A_0A_1\int_0^1 \Theta(s)^{-1}ds\Theta(0) + A_0B_1\Theta(1)^{-1}\Theta(0)\Big],
$$

which is equal to $A_0^{-1}P_5$. Therefore,

$$
\begin{aligned}
A_1x(1)+B_1x'(1) &= A_0^{-1}\Big[A_0(\psi_1[x]+\zeta_1) - A_1(\psi_0[x]+\zeta_0) \\
&\quad - A_0A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds - A_0B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du\Big] \\
&\quad + A_1A_0^{-1}(\psi_0[x]+\zeta_0) + A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)du \\
&= \psi_1[x]+\zeta_1.
\end{aligned}
$$

Thus conditions (1.2) and (1.3) are satisfied. The fact that (1.1) is, also, satisfied, is obvious.

Now write equation (7.3) in the form

$$
x = \mathcal{T}_5x + \mathcal{C}_5, \tag{7.4}
$$

where the operators $\mathcal{T}_5$ and $\mathcal{C}_5$ are defined as follows:

$$
\mathcal{T}_5x(t) = (A_0^{-1}B_0 + \int_0^t \Theta(s)^{-1}ds\Theta(0))P_5^{-1}\Big[A_0(\psi_1[x]+\zeta_1) - A_1(\psi_0[x]+\zeta_0)\Big] + A_0^{-1}(\psi_0[x]+\zeta_0),
$$

and

$$
\begin{aligned}
\mathcal{C}_5x(t) &= (A_0^{-1}B_0 + \int_0^t \Theta(s)^{-1}ds\Theta(0))P_5^{-1}\Big[-A_0A_1\int_0^1 \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds \\
&\quad - A_0B_1\Theta(1)^{-1}\int_0^1 (Nx)(u)du\Big] + \int_0^t \Theta(s)^{-1}\int_0^s (Nx)(u)\,du\,ds.
\end{aligned}
$$

Hence, to show the existence of solutions of the original problem is sufficient to seek for the existence of fixed points of equation (7.4).

By Fubini's theorem the operator $\mathcal{C}_5$ can be written in the form

$$
\mathcal{C}_5x(t) = \int_0^1 \mathcal{G}_5(t,u)(Nx)(u)du,
$$

where the kernel is defined by

$$\mathcal{G}_5(t,u) = (A_0^{-1}B_0 + \int_0^t \Theta(s)^{-1}ds\Theta(0))P_5^{-1}\Big[-A_0A_1\int_u^1 \Theta(s)^{-1}ds - A_0B_1\Theta(1)^{-1}\Big] + \int_u^t \Theta(s)^{-1}ds.$$

For this function we have

$$\begin{aligned}\|\mathcal{G}_5(t,u)\|_E &\le (\|A_0^{-1}B_0\|_E + \int_0^1 \|\Theta(s)^{-1}\|_E ds\|\Theta(0)\|_E)\|P_5^{-1}\|_E \\ &\quad\times \big(\|A_0A_1\|_E \int_0^1 \|\Theta(s)^{-1}\|_E ds + \|A_0B_1\Theta(1)^{-1}\|_E\big) + \int_0^1 \|\Theta(s)^{-1}\|_E ds = M_5.\end{aligned}$$

Then by (2.5), for all $r > 0$ and $x \in B_r(0)$, we obtain

$$\|\mathcal{C}_5 x\|_\infty \le M_5 m(r). \tag{7.5}$$

Also, for all x, y we have

$$\mathcal{T}_5x(t) - \mathcal{T}_5y(t) = (A_0^{-1}B_0 + \int_0^t \Theta(s)^{-1}ds\Theta(0))P_5^{-1}\Big[A_0\psi_1(x-y) - A_1\psi_0(x-y)\Big] + A_0^{-1}\psi_0(x-y),$$

and therefore

$$\begin{aligned}|\mathcal{T}_5x(t) - \mathcal{T}_5y(t)| &\le \Big\{(\|A_0^{-1}B_0\|_E + \int_0^1 \|\Theta(s)^{-1}\|_E ds\|\Theta(0)\|_E)\|P_5^{-1}\|_E \\ &\quad\times \Big[V(A_0\Psi_1) + V(A_1\Psi_0)\Big] + V(A_0^{-1}\Psi_0)\Big\}\|x-y\|_\infty,\end{aligned}$$

namely

$$\|\mathcal{T}_5x - \mathcal{T}_5y\|_\infty \le \rho_5\|x-y\|_\infty, \tag{7.6}$$

where ρ_5 is defined in (H15).

Also, for any $x \in B_r(0)$ we have

$$\begin{aligned}|\mathcal{T}_5x(t) + \mathcal{C}_5y(t)| &\le (\|A_0^{-1}B_0\|_E + \int_0^t \|\Theta(s)^{-1}\|_E ds\|\Theta(0)\|_E)\|P_5^{-1}\|_E\Big|\big[A_0(\psi_1[x] + \zeta_1) \\ &\quad - A_1(\psi_0[x] + \zeta_0)\big]\Big| + |A_0^{-1}(\psi_0[x] + \zeta_0)| + |\mathcal{C}_5y(t)| \\ &\le K_5 + \rho_5 r + M_5 m(r),\end{aligned} \tag{7.7}$$

where K_5 and ρ_5 are defined in (H15).

From condition (H0) it follows that there exists $r_5 > 0$ such that

$$K_5 + \rho_5 r_5 + M_5 m(r_5) \le r_5.$$

This means that $\mathcal{T}_5x + \mathcal{C}_5y \in B_{r_5}(0)$ for all $x, y \in B_{r_5}(0)$. The relations we have found so far prove the (uniform) equicontinuity of the family $\{\mathcal{C}_5x : x \in B_{r_5}(0)\}$ and so $\mathcal{C}_5$ is a compact mapping. Also, by condition (H15) the operator $\mathcal{T}_5$ is a contraction. By the Krasnosel'skii fixed point theorem the mapping $\mathcal{T}_5 + \mathcal{C}_5$ admits a fixed point in $B_{r_5}(0)$, which is a solution of the original problem. □

8. Existence of solutions of the problem (1.4), (1.5)

In this section we shall apply the results of section 7 to the boundary value problem (1.4), (1.5). First, we write the problem in terms of (1.1), (1.2), (1.3), where, notice that, Θ is the identity 2×2-matrix. To do that we consider the vectors

$$x(t) := \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad \zeta_i := \begin{pmatrix} 0 \\ c_i \end{pmatrix}, \quad i = 0,1, \quad (Nx)(t) := \begin{pmatrix} -f(t, x_1(t), x_2(t)) \\ g(t, x_1(t), x_2(t)) \end{pmatrix}.$$

as well as the matrices

$$A_i := \begin{pmatrix} \tilde{a}_i & 0 \\ 0 & a_i \end{pmatrix}, \quad B_i := \begin{pmatrix} \tilde{b}_i & 0 \\ 0 & b_i \end{pmatrix}, \quad i = 0,1,$$

$$\psi_i(x) := \begin{pmatrix} \tilde{\phi}_i[x_1] & 0 \\ 0 & \phi_i[x_2] \end{pmatrix}, \quad \Psi_i(s) := \begin{pmatrix} \tilde{\Phi}_i(s) & 0 \\ 0 & \Phi_i(s) \end{pmatrix}, \quad i = 0,1,$$

with $\tilde{a}_0 a_0 \tilde{a}_1 a_1 \neq 0$. Since the matrices A_0, A_1 are diagonal, they commute, so Theorem 7.1 may be applicable. To proceed we obtain

$$P_5 := \begin{pmatrix} \tilde{a}_0\tilde{b}_1 + \tilde{a}_0\tilde{a}_1 + \tilde{a}_1\tilde{b}_0 & 0 \\ 0 & a_0b_1 + a_0a_1 + a_1b_0 \end{pmatrix}$$

$$P_5^{-1} := \begin{pmatrix} (\tilde{a}_0\tilde{b}_1 + \tilde{a}_0\tilde{a}_1 + \tilde{a}_1\tilde{b}_0)^{-1} & 0 \\ 0 & (a_0b_1 + a_0a_1 + a_1b_0)^{-1} \end{pmatrix}.$$

Also, we have

$$\begin{gathered} \|P_5^{-1}\|_E = \Big[\big(\tilde{a}_0\tilde{b}_1 + \tilde{a}_0\tilde{a}_1 + \tilde{a}_1\tilde{b}_0\big)^{-2} + \big(a_0b_1 + a_0a_1 + a_1b_0\big)^{-2}\big)\Big]^{1/2}, \\ \|A_0^{-1}B_0\|_E = \Big[\tilde{a}_0^{-2}\tilde{b}_0^2 + a_0^{-2}b_0^2\Big]^{1/2}, \\ \|A_0A_1\|_E = \big[\big(\tilde{a}_0\tilde{a}_1\big)^2 + \big(a_0a_1\big)^2\big]^{1/2}, \\ \|A_0B_1\|_E = \big[\big(\tilde{a}_0\tilde{b}_1\big)^2 + \big(a_0b_1\big)^2\big]^{1/2}, \\ V(A_0\Psi_1) = \int_0^1 d\Big(\big[\big(\tilde{a}_0\tilde{\Phi}_1(s)\big)^2 + \big(a_0\Phi_1(s)\big)^2\big]^{1/2}\Big), \\ V(A_1\Psi_0) = \int_0^1 d\Big(\big[\big(\tilde{a}_1\tilde{\Phi}_0(s)\big)^2 + \big(a_1\Phi_0(s)\big)^2\big]^{1/2}\Big), \\ V(A_0^{-1}\Psi_0) = \int_0^1 d\Big(\big[\big(\tilde{a}_0^{-1}\tilde{\Phi}_0(s)\big)^2 + \big(a_0^{-1}\Phi_1(s)\big)^2\big]^{1/2}\Big), \\ |A_0\zeta_1| = |a_0c_1|, \quad |A_1\zeta_0| = |a_1c_0|, \quad |A_0^{-1}\zeta_0| = |a_0^{-1}c_0|. \end{gathered}$$

Therefore,

$$\begin{aligned} \rho_5 = \Big[\big[\big(\tilde{a}_0\tilde{a}_1\big)^2 + \big(a_0a_1\big)^2\big]^{1/2} + 1\Big]\Big[\big(\tilde{a}_0\tilde{b}_1 + \tilde{a}_0\tilde{a}_1 + \tilde{a}_1\tilde{b}_0\big)^{-2} + \big(a_0b_1 + a_0a_1 \\ + a_1b_0\big)^{-2}\big)\Big]^{1/2}\Big[\int_0^1 d\Big(\big[\big(\tilde{a}_0\tilde{\Phi}_1(s)\big)^2 + \big(a_0\Phi_1(s)\big)^2\big]^{1/2}\Big) + \int_0^1 d\Big(\big[\big(\tilde{a}_1\tilde{\Phi}_0(s)\big)^2 \\ + \big(a_1\Phi_0(s)\big)^2\big]^{1/2}\Big)\Big] + \int_0^1 d\Big(\big[\big(\tilde{a}_0^{-1}\tilde{\Phi}_0(s)\big)^2 + \big(a_0^{-1}\Phi_1(s)\big)^2\big]^{1/2}\Big). \end{aligned}$$

Now we assume that $\rho_5 < 1$ and moreover, that the response functions f, g satisfy

$$|f(t, x_1, x_2)| \leq \alpha_1|x_1|^\mu + \beta_1|x_2|^\nu, \quad |g(t, x_1, x_2)| \leq \alpha_2|x_1|^\nu + \beta_2|x_2|^\mu, \tag{8.1}$$

where the coefficients satisfy $\alpha_i, \beta_i > 0$ and $0 < \mu, \nu < 1$. Obviously, we have

$$m(r) = \Big((\alpha_1 + \beta_1)^2 + (\alpha_2 + \beta_2)^2\Big)^{1/2} r^\xi,$$

where $\xi := \max\{\mu, \nu\}(< 1)$. Therefore

$$\liminf_{r \to +\infty} \frac{m(r)}{r} = 0. \tag{8.2}$$

Thus, assumption (2.7) is true and so all conditions of Theorem 7.1 are satisfied. This implies that there is a solution of the problem (1.4), (1.5).

From the previous special case, we see that, actually, the result depends on the value ρ_5. Thus a question arises: Which value of ρ is better for such a decision, namely, which of the theorems we have presented requires less restrictions. In the sequel we shall show that such a fact depends on the parameters involved in the boundary conditions.

9. Application to a 2-dimensional system

Consider the system of equations (1.4) with f, g satisfying (2.5) with condition (8.2), associated with the boundary conditions

$$A_0x(0) - B_0x'(0) = \lambda\psi_0[x] + \zeta_0, \tag{9.1}$$

$$A_1x(1) + B_1x'(1) = \lambda\psi_1[x] + \zeta_1 \tag{9.2}$$

where λ is a real parameter playing an important role for condition $\rho < 1$. Assume that the 2×2 square matrices A_0, B_0, A_1, B_1 are defined by

$$A_0 := \begin{pmatrix} \mu & 0 \\ 0 & \nu \end{pmatrix}, \quad A_1 = B_0 = B_1 = I_{2\times 2},$$

where μ, ν are nonzero real numbers with $\mu, \nu \neq -\frac{1}{2}$. Then we obtain

$$P_1 := \begin{pmatrix} \frac{2\mu+1}{\mu} & 0 \\ 0 & \frac{2\nu+1}{\nu} \end{pmatrix}, \quad P_2 = P_3 = P_4 = P_5 = \begin{pmatrix} 2\mu+1 & 0 \\ 0 & 2\nu+1 \end{pmatrix},$$
$$\rho_i = Z_i|\lambda|, \quad i = 1,2,3,4,5,$$

where

$$Z_1 := V(A_0^{-1}\Psi_0) + \Big[\big(\frac{\mu+1}{2\mu+1}\big)^2 + \big(\frac{\nu+1}{2\nu+1}\big)^2\Big]^{1/2}\big(V(A_0\Psi_1) + V(\Psi_0)\big),$$
$$Z_2 := (\sqrt{2} + \sqrt{\mu^2+\nu^2})\Big[\frac{1}{(2\mu+1)^2} + \frac{1}{(2\nu+1)^2}\Big]^{1/2}\big(4V(\Psi_0) + V(\Psi_1)\big),$$
$$Z_3 := 2(\sqrt{2}+1)\Big[\frac{1}{(2\mu+1)^2} + \frac{1}{(2\nu+1)^2}\Big]^{1/2}\big(V(A_0\Psi_1) + V(\Psi_0)\big) + V(\Psi_1),$$
$$Z_4 := 2\sqrt{3}\Big[\frac{1}{(2\mu+1)^2} + \frac{1}{(2\nu+1)^2}\Big]^{1/2}\big((\sqrt{\mu^2+\nu^2} + \sqrt{3})V(\Psi_1) + \sqrt{3}V(\Psi_0)\big) + 3V(\Psi_1),$$
$$Z_5 := \Big(2 + \big[\frac{1}{\mu^2} + \frac{1}{\nu^2}\big]^{1/2}\Big)\Big[\frac{1}{(2\mu+1)^2} + \frac{1}{(2\nu+1)^2}\Big]^{1/2}\big(V(A_0\Psi_1) + V(\Psi_0)\big) + V(A_0^{-1}\Psi_0).$$

Once we have found the quantities Z_i, we shall give a comparison of the conditions appeared in theorems 3.1, 4.1, 5.1, 6.1 and 7.1, to see which of them gives better results. And, indeed, in case the condition (8.1) holds, the only requirement which we need to check is the inequality $\rho_j < 1$, for some $j \in \{1,2,3,4,5\}$. For instance, comparing ρ_1 with ρ_5 we see that the inequality $\rho_1 > \rho_5$ may hold independently of the items $\Psi_i, \quad i = 0,1$. Indeed, this inequality holds if and only if $Z_1 > Z_5$, which is equivalent to the inequality

$$\Big[\big(\frac{\mu+1}{2\mu+1}\big)^2 + \big(\frac{\nu+1}{2\nu+1}\big)^2\Big]^{1/2} > \Big(2 + \big[\frac{1}{\mu^2} + \frac{1}{\nu^2}\big]^{1/2}\Big)\Big[\frac{1}{(2\mu+1)^2} + \frac{1}{(2\nu+1)^2}\Big]^{1/2}.$$

If we set

$$Y(\mu,\nu) := \big(\frac{\mu+1}{2\mu+1}\big)^2 + \big(\frac{\nu+1}{2\nu+1}\big)^2 - \Big(2 + \big[\frac{1}{\mu^2} + \frac{1}{\nu^2}\big]^{1/2}\Big)^2\Big[\frac{1}{(2\mu+1)^2} + \frac{1}{(2\nu+1)^2}\Big]$$

we observe that it satisfies

$$\lim_{(\mu,\nu)\to(0,0)} Y(\mu,\nu) = -\infty, \qquad \lim_{(\mu,\nu)\to(\pm\infty,\pm\infty)} Y(\mu,\nu) = \frac{1}{2} > 0.$$

This means that there are values of the parameters μ, ν for which the corresponding quantity ρ_5 is greater than ρ_1. Then we have $(0, Z_5^{-1}) \subseteq (0, Z_1^{-1})$. In this case, from Theorem 3.1, for any λ with $|\lambda| \in [0, Z_1^{-1})$, we result the existence of solutions and this interval is larger than the one obtained from the quantity ρ_5. Also, there are values of the parameters μ, ν for which the corresponding quantity ρ_5 is less than ρ_1. In this case, if $|\lambda| \in [0, Z_5^{-1})$, there are solutions and this interval is larger than the corresponding ρ_1, see Figure (1).

To obtain a more clear picture about the values of the parameter λ we consider the case when $\Psi_i(s) := s$. Then we obtain

$$V(\Psi_i) = 1, \quad V(A_0^{-1}\Psi_i) = \sqrt{\frac{1}{\mu^2} + \frac{1}{\nu^2}}, \quad V(A_0\Psi_i) := \sqrt{\mu^2+\nu^2}, \quad i = 0,1$$

and so

$$Z_1 := \sqrt{\frac{1}{\mu^2} + \frac{1}{\nu^2}} + \Big[\big(\frac{\mu+1}{2\mu+1}\big)^2 + \big(\frac{\nu+1}{2\nu+1}\big)^2\Big]^{1/2}\big(\sqrt{\mu^2+\nu^2} + 1\big),$$

$$Z_2 := 5(\sqrt{2} + \sqrt{\mu^2 + \nu^2})\Big[\frac{1}{(2\mu+1)^2} + \frac{1}{(2\nu+1)^2}\Big]^{1/2},$$
$$Z_3 := 2(\sqrt{2}+1)\Big[\frac{1}{(2\mu+1)^2} + \frac{1}{(2\nu+1)^2}\Big]^{1/2}\big(\sqrt{\mu^2+\nu^2}+1\big)+1,$$
$$Z_4 := (2+\sqrt{2})\Big[\frac{1}{(2\mu+1)^2} + \frac{1}{(2\nu+1)^2}\Big]^{1/2}\big(\sqrt{\mu^2+\nu^2}+2\sqrt{2}\big)+2,$$
$$Z_5 := \Big(2+\big[\frac{1}{\mu^2}+\frac{1}{\nu^2}\big]^{1/2}\Big)\Big[\frac{1}{(2\mu+1)^2} + \frac{1}{(2\nu+1)^2}\Big]^{1/2}\big(\sqrt{\mu^2+\nu^2}+1\big)+\sqrt{\frac{1}{\mu^2}+\frac{1}{\nu^2}}.$$

Therefore the existence of solutions is guaranteed if we assume that $|\lambda| \in [0, Z_1^{-1})$ in case we apply Theorem 3.1, $|\lambda| \in [0, Z_2^{-1})$ in case we apply Theorem 3.1, $\lambda \in [0, Z_3^{-1})$ in case we apply Theorem 5.1, $|\lambda| \in [0, Z_4^{-1})$ in case we apply Theorem 6.1 and $|\lambda| \in [0, Z_5^{-1})$ in case we apply Theorem 7.1.

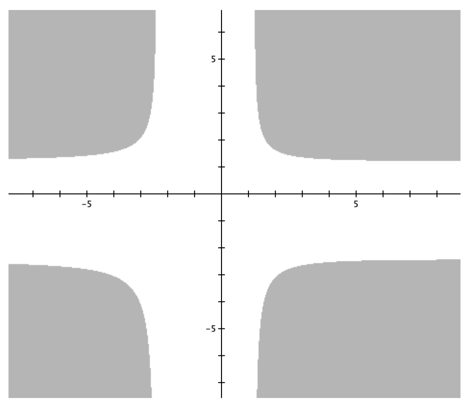

FIGURE 1. The shadow area represent the set of values of (μ, ν) for which the inequality $\rho_1 > \rho_5$ holds.

9.1. **A special numerical case.** Assume that $(\mu, \nu) = (1, 1)$ and consider the cases $\|A_0 A_1\|_E = \|A_0 B_1\|_E = \|A_0^{-1} B_0\|_E = \sqrt{2}$, $\Psi_0(s) := \Psi_1(s) := (\frac{s^2}{2} + s) I_{2\times 2}$. Then we obtain $V(A_0\Psi_1) = V(A_1\Psi_0) = V(A_0^{-1}\Psi_0) = \frac{3\sqrt{2}}{2}\lambda$, $P_i = 3I_{2\times 2}$, for $i = 1, 2, 3, 4, 5$, as well as

$$Z_1 = \frac{3\sqrt{2}+8}{2}, \quad Z_2 = \frac{20}{3}, \quad Z_3 = \frac{11\sqrt{2}+4}{2}, \quad Z_4 = 4+\sqrt{2}, \quad Z_5 = \frac{7\sqrt{2}+8}{2}.$$

These facts imply that
for $|\lambda| < \frac{2}{3\sqrt{2}+8} \approx 0.16336$, we have $\rho_1 < 1$,
for $|\lambda| < \frac{3}{20} \approx 0.15$, we have $\rho_2 < 1$,
for $|\lambda| < \frac{2}{11\sqrt{2}+4} \approx 0.10226$, we have $\rho_3 < 1$,
for $|\lambda| < \frac{1}{4+\sqrt{2}} \approx 0.18469903$, we have $\rho_4 < 1$, and
for $|\lambda| < \frac{2}{7\sqrt{2}+8} \approx 0.11173$, we have $\rho_5 < 1$.

Obviously, the best upper bound of the parameter $|\lambda|$ is 0.18469903 and so for such values of $|\lambda|$ in the interval $[0, 0.18469903)$, the conditions of Theorem 6.1 are satisfied, and so for all these values of the parameter λ there is a solution of the boundary value problem (1.4)-(1.5).

10. APPLICATION TO A 3-DIMENSIONAL SYSTEM

Consider the system of differential equations

$$\begin{aligned} x_1''(t) &= f_1(t, x_1, x_2, x_3), \\ x_2''(t) &= f_2(t, x_1, x_2, x_3), \\ x_3''(t) &= f_3(t, x_1, x_2, x_3), \end{aligned} \tag{10.1}$$

for $t \in I$, where for each $i = 1, 2, 3$ the function f_i maps the set $I \times C^2(I, \mathbb{R})^3$ into $C(I, \mathbb{R})$. Also, for any $x \in C(I, \mathbb{R}^3)$ we assume that the operator

$$(Nx)(t) := \begin{pmatrix} f_1(t, x_1, x_2, x_3) \\ f_2(t, x_1, x_2, x_3) \\ f_3(t, x_1, x_2, x_3) \end{pmatrix}$$

satisfies a condition specified latter. We associate system (10.1) with the following nonlocal boundary value conditions:

$$\begin{gathered} 2x_1(0) + x_3(0) - x_1'(0) = \lambda \int_0^1 x_1(s)ds + 1, \\ x_1(0) + 2x_2(0) - x_2'(0) = \lambda \int_0^1 (x_1(s) + x_3(s))ds, \\ x_2(0) + x_3(0) - x_3'(0) = \lambda \int_0^1 x_2(s)ds - 1 \\ x_1(1) + x_3(0) + 2x_3(1) + x_1'(1) = \lambda \int_0^1 x_2(s)ds + 1, \\ 3x_1(1) + x_2(1) + x_3(1) + x_2'(1) = \lambda \int_0^1 x_1(s)ds, \\ x_1(1) + 2x_2(1) - x_3(1) + x_3'(1) = \lambda \int_0^1 x_3(s)ds - 1. \end{gathered} \tag{10.2}$$

The parameter λ is a real number which plays an important role in the existence of solutions.

To write the problem in the general form, we formulate the nonsingular matrices

$$A_0 = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 1 & 2 & -1 \end{pmatrix}, \quad B_0 = B_1 = \Theta(s) = I_{3\times 3}, \quad s \in I$$

and

$$\zeta_0 := \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \zeta_1 := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \Psi_0(s) := \lambda \begin{pmatrix} s & 0 & 1 \\ s & 1 & s \\ 0 & s & 1 \end{pmatrix}, \quad \Psi_1(s) := \lambda \begin{pmatrix} 1 & s & 0 \\ s & 0 & 1 \\ 2 & 0 & s \end{pmatrix}.$$

It is obvious that due to the values of ζ_0, ζ_1, the zero function is not a solution.

Since the matrices satisfy $\det(A_0) = 5$, $\det(A_1) = 11$ and $\det(B_0) = \det(B_1) = 1$, they are nonsingular, and since

$$A_0 A_1 = \begin{pmatrix} 3 & 4 & 3 \\ 7 & 3 & 4 \\ 4 & 3 & 0 \end{pmatrix} = A_1 A_0,$$

the matrices A_0, A_1 commute. These facts mean that all Theorems 3.1-7.1 are applicable. Before we continue we must calculate some items needed in the sequel. Now we have

$$A_0^{-1} = \frac{1}{5} \begin{pmatrix} 2 & 1 & -2 \\ -1 & 2 & 1 \\ 1 & -2 & 4 \end{pmatrix}, \quad A_1^{-1} = \frac{1}{11} \begin{pmatrix} -3 & 5 & 1 \\ 4 & -3 & 5 \\ 5 & -1 & -2 \end{pmatrix},$$

with norms $\|\Theta(t)\|_E = \sqrt{3}$, $t \in I$, $\|A_0\|_E = \sqrt{12}$, $\|A_1\|_E = \sqrt{22}$, $\|A_1^{-1}\|_E = \frac{1}{11}\sqrt{115}$, and $\|{A_0}^{-1} B_0\| = \|{A_0}^{-1}\| = \frac{6}{5}$, $\|A_0 A_1\| = \sqrt{133}$. Also, we have

$$V(\Psi_0) = \frac{1}{11}|\lambda| \int_0^1 \Big\| d \begin{pmatrix} s & 0 & 1 \\ s & 1 & s \\ 0 & s & 1 \end{pmatrix} \Big\| = 2|\lambda|,$$

$$V(\Psi_1) = \frac{1}{11}|\lambda| \int_0^1 \Big\| d \begin{pmatrix} 1 & s & 0 \\ s & 0 & 1 \\ 2 & 0 & s \end{pmatrix} \Big\| = \sqrt{3}|\lambda|,$$

$$V(A_0\Psi_1) = |\lambda| \int_0^1 \Big\| d \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & s & 0 \\ s & 0 & 1 \\ 2 & 0 & s \end{pmatrix} \Big\| = \sqrt{12}|\lambda|,$$

$$V(A_1\Psi_0) = |\lambda| \int_0^1 \Big\| d \begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} s & 0 & 1 \\ s & 1 & s \\ 0 & s & 1 \end{pmatrix} \Big\| = \sqrt{41}|\lambda|,$$

$$V(A_0^{-1}\Psi_0) = |\lambda|\frac{1}{5} \int_0^1 \Big\| d \begin{pmatrix} 2 & 1 & -2 \\ -1 & 2 & 1 \\ 1 & -2 & 4 \end{pmatrix} \begin{pmatrix} s & 0 & 1 \\ s & 1 & s \\ 0 & s & 1 \end{pmatrix} \Big\| = \frac{1}{5}\sqrt{41}|\lambda|,$$

$$V(A_1^{-1}\Psi_1) = \frac{1}{11}|\lambda| \int_0^1 \Big\| d \begin{pmatrix} -3 & 5 & 1 \\ 4 & -3 & 5 \\ 5 & -1 & -2 \end{pmatrix} \begin{pmatrix} 1 & s & 0 \\ s & 0 & 1 \\ 2 & 0 & s \end{pmatrix} \Big\| = \frac{1}{11}\sqrt{134}|\lambda|,$$

$$V(A_0A_1^{-1}\Psi_1) = \frac{1}{11}|\lambda| \int_0^1 \Big\| d = \begin{pmatrix} -1 & 9 & 0 \\ 5 & -1 & 11 \\ 9 & -4 & 3 \end{pmatrix} \begin{pmatrix} 1 & s & 0 \\ s & 0 & 1 \\ 2 & 0 & s \end{pmatrix} \Big\| = \frac{1}{11}\sqrt{335}|\lambda|.$$

Then we obtain

$$P_1 = \frac{1}{5}\begin{pmatrix} 13 & 4 & 17 \\ 21 & 13 & 4 \\ 4 & 13 & -4 \end{pmatrix}, \quad P_1^{-1} = \frac{1}{14025}\begin{pmatrix} -104 & 237 & -205 \\ 100 & -80 & 305 \\ -136 & -153 & 85 \end{pmatrix}.$$

Also we have

$$(A_0^{-1} + I_{3\times 3})P_1^{-1} = \frac{1}{70125}\begin{pmatrix} -356 & 1885 & -1300 \\ 668 & -950 & 2425 \\ -1528 & -980 & -50 \end{pmatrix}$$

and therefore

$$\|(A_0^{-1} + I_{3\times 3})P_1^{-1}\|_E = 3987, 10345991.$$

To apply Theorem 4.1 we need the quantity

$$P_2 = A_1 + A_1A_0 + A_0 = \begin{pmatrix} 5 & 5 & 6 \\ 11 & 6 & 5 \\ 5 & 6 & 0 \end{pmatrix}, \quad P_2^{-1} = \frac{1}{191}\begin{pmatrix} -30 & 36 & -11 \\ 25 & -30 & 41 \\ 36 & -5 & -25 \end{pmatrix}$$

and so $\|P_2^{-1}\|_E = 0, 45226633$. To apply Theorem 5.1 we need

$$P_3 = A_0 + A_0A_1^{-1} + I = \frac{1}{11}\begin{pmatrix} -3 & 5 & 1 \\ 4 & -3 & 5 \\ 5 & -1 & -2 \end{pmatrix}, \quad P_3^{-1} = \frac{1}{771}\begin{pmatrix} 42 & -30 & 97 \\ -81 & -3 & -16 \\ 36 & 87 & -50 \end{pmatrix}$$

with norm $\|P_3^{-1}\|_E = 0, 17830562$. To apply Theorem 6.1 we need P_4 which equals to P_2 and so we have $\|P_4^{-1}\|_E = 0, 45226633$.

For Theorem 7.1 we have

$$P_5 = \begin{pmatrix} 6 & 5 & 6 \\ 11 & 6 & 5 \\ 5 & 8 & 0 \end{pmatrix}, \quad P_5^{-1} = \frac{1}{233}\begin{pmatrix} -40 & 48 & -11 \\ 25 & -30 & 36 \\ 58 & -23 & 19 \end{pmatrix},$$

with norm $\|P_5^{-1}\| = 0, 4078$.

Now we assume that the response function satisfies a condition like

$$|f_i(t, x_1(t), x_2(t), x_3(t))| \le \alpha_i|x_1(\frac{t}{2})|^{\mu_1} + \beta_i|x_2(t)|^{\mu_2} + \gamma_i|x_3(\sin(t))|^{\mu_3}, \quad i = 1, 2, 3, \tag{10.3}$$

where $\alpha_i, \beta_i, \gamma_i$ are positive real numbers and μ_i are such that $\mu := \max_i \mu_i < 1$. In this case the truth of the Theorems 3.1–7.1 depends on the values of the parameters $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5$, only, since conditions (2.5) and (8.2) are satisfied. So, in order to apply the previous theorems we must obtain the basic parameters $\rho_1, \dots, \rho_5$, which correspond to the five cases above.

We can easily obtain estimates of these parameters as $\rho_1 = 1, 9858304|\lambda|$, $\rho_2 = 32, 27093226|\lambda|$, $\rho_3 = 4, 78066942|\lambda|$, $\rho_4 = 12.85258214|\lambda|$, $\rho_5 = 22, 52419992|\lambda|$. Therefore, the best value for λ, which satisfies $\rho < 1$, is given by ρ_1 and it is $|\lambda| < 0,50356768$. Hence for any such λ, Theorem 3.1 guarantees the existence of solutions of the problem, in the ball $B(0,r) \subseteq \mathbb{R}^3$, where r is any large positive real number satisfying the inequality

$$\frac{K_1}{r} + \rho_1 + M_1 \frac{m(r)}{r} < 1,$$

where

$$m(r) := \sqrt{\sum_i \alpha_i^2 + \beta_i^2 + \gamma_i^2 r^\mu}.$$

Now we assume that the response function satisfies the condition

$$|F(t,x)| \leq \alpha|x| + \beta, \tag{10.4}$$

for some nonnegative real numbers α, β. In this case condition (2.6) becomes

$$K + \rho r + Mar + Mb < r,$$

which is satisfied only if

$$1 - \rho - Ma > 0. \tag{10.5}$$

To see what happens when applying all Theorems 3.1-7.1, we need to calculate the parameters M_1, M_2, M_3, M_4, M_5. Indeed we calculate these numbers and obtain the following estimated values: $M_1 = 1,92816175$, $M_2 = 32,93510197$, $M_3 = 6,57065738$, $M_4 = 23,17394972$, $M_5 = 41,8775797$. Then in the various cases inequality (10.5) becomes

$$\begin{aligned}
\sigma_1(\lambda,a) &:= 1 - 1,9858304\|\lambda| - 1,92816175a > 0,\\
\sigma_2(\lambda,a) &:= 1 - 32,27093226|\lambda| - 38,32050810a > 0,\\
\sigma_3(\lambda,a) &:= 1 - 4,78066942|\lambda| - 6,57065738a > 0,\\
\sigma_4(\lambda,a) &:= 1 - 31,87116522|\lambda| - 12.85258214a > 0,\\
\sigma_5(\lambda,a) &:= 1 - 22,52419992|\lambda| - 41,8775797a > 0.
\end{aligned}$$

Each of these relations guarantee the existence of solutions, for instance when apply Theorem 7.1, we conclude that a solution of the problem exists in the ball $B(0,r_5) \subseteq \mathbb{R}^3$, where r_5 satisfies

$$r_5 > \frac{K_5 + M_5 b}{\sigma_5(\lambda,a)} = \frac{14,08835163 + 41,8775797b}{1 - 22,52419992|\lambda| - 41,8775797a},$$

where $K_5 = 14,08835163$. Notice that between these five cases the first one gives better results, because, as we can, easily, see it holds

$$\{(\lambda,a) : \sigma_i(\lambda,a) > 0\} \subseteq \{(\lambda,a) : \sigma_3(\lambda,a) > 0\} \subseteq \{(\lambda,a) : \sigma_1(\lambda,a) > 0\}, \quad i = 2,4,5.$$

11. Positive solutions

We say that a vector $y := (y_1, y_2, \dots, y_n)^T$ is positive (nonnegative) if all its coordinates are positive (nonnegative) real numbers. Then we write $y > 0 (y \geq 0)$, or $0 < y$ $(y \leq 0)$. Also, we write $y < w$ $(y \leq w)$, if $y - w < 0$, $(y - w \leq 0)$. Analogous things we have for matrices. It is clear that for any vector $x > 0$ and matrices $A, B > 0$ we have $Ax > 0$, $A + B > 0$ and $AB > 0$.

In this section we shall be concerned with the existence of positive solutions of the original problem. To this direction we will show the following results.

Theorem 11.1. *Assume that rm (H13) and the following conditions hold:*

- (H16) *For each* $t \in I$ *the matrix* $(A_0^{-1}B_0 + \int_0^t \Theta(s)ds\Theta(0))P_5^{-1}$ *is positive.*
- (H17) $\Psi_0(s) = \Psi_1(s) = 0, \quad s \in I.$
- (H18) *The operator* N *maps the set* $C(I,\mathbb{R}^n)$ *into* $C(I,\mathbb{R}^n_-)$.
- (H19) *The inequality* $A_0\zeta_1 - A_1\zeta_0 > 0$ *holds.*
- (H20) *For any* $x \in C(I,\mathbb{R}^n)$ *and* $t \in I$ *it holds*

$$A_0^{-1}\zeta_0 + \int_0^t \Theta^{-1}(s) \int_0^s (N(x)(u)\,du\,ds > 0.$$

Then there is a solution of the problem and such a solution is positive.

Proof. Since $\Psi_0 = \Psi_1 = 0$, it follows that $\rho_5 = 0(< 1)$. Therefore Theorem 7.1 is applicable. From relation (7.3) we see that any solution is expressed in the form

$$\begin{aligned} x(t) = (A_0^{-1}B_0 + \int_0^t \Theta(s)^{-1} ds \Theta(0)) P_5^{-1}\Big[&[A_0\zeta_1 - A_1\zeta_0] \\ &+ A_0A_1 \int_0^1 \Theta(s)^{-1} \int_0^s (-Nx)(u)\,du\,ds + A_0B_1\Theta(1)^{-1}\int_0^1 (-Nx)(u)du\Big] \\ &+ A_0^{-1}\zeta_0 - \int_0^t \Theta(s)^{-1}\int_0^s (-Nx)(u)\,du\,ds. \end{aligned} \tag{11.1}$$

Here, all the factors are positive, so $x(t) > 0$, for all $t \in I$.. □

11.1. **An application.** Consider the system of equations

$$x_i''(t) = f_i(t, x_1, x_2, \dots, x_n)), \quad i = 1, 2, \dots, n$$

where $x_1, x_2, \dots x_n \in C^2(I, \mathbb{R}^2)$, associated with the boundary value conditions

$$\begin{gathered} \kappa_i x_i(0) - x_i'(0) = \zeta_0^i, \quad i = 1, 2, \dots, n, \\ x_i(1) + x_i'(1) = \zeta_i^1, \quad i = 1, 2, \dots, n, \end{gathered}$$

where κ_i are positive real numbers, ζ_0^i, ζ_1^i, are nonzero reals and such that

$$\kappa_i\zeta_1^i - \zeta_0^i > 0, \quad i = 1, 2, \dots, n.$$

Assume, also, that the functions f_i satisfy the conditions

$$-\min\big\{\nu_i, \sum_{j=1}^n \alpha_{i,j}|x_j(\gamma_{i,j}(t))|^{\mu_{i,j}}\big\} \le f_i(t, x_1, x_2, \dots, n) \le 0, \quad i = 1, 2, \dots, n$$

for all $x_i \in C^2(I, \mathbb{R}^2)$. The coefficients $\nu_i, \mu_{i,j}$, are positive reals with $\mu_{i,j} < 1$ and $2\zeta_0^i \ge \kappa_i\nu_i$, $i = 1, 2, \dots, n$. The arguments $\gamma_{i,j} : I \to I$ are continuous functions. Then conditions (2.5) and (8.2) are satisfied. Here we have $A_1 = B_0 = B_1 = I_{n\times n}$ and

$$A_0 := \begin{pmatrix} \kappa_1 & 0 & 0 & \dots 0 \\ 0 & \kappa_2 & 0 & \dots 0 \\ \vdots & & & \\ 0 & 0 & 0 & \dots \kappa_n \end{pmatrix}, \quad P_5 := \begin{pmatrix} 2\kappa_1 + 1 & 0 & 0 & \dots 0 \\ 0 & 2\kappa_2 + 1 & 0 & \dots 0 \\ \vdots & & & \\ 0 & 0 & 0 & \dots 2\kappa_n + 1 \end{pmatrix}$$

and so the matrix in condition (H16) is equal to

$$\begin{pmatrix} \frac{1}{k_1} + 1 + \frac{1}{2k_1+1} & 0 & 0 & \dots 0 \\ 0 & \frac{1}{k_2} + 1 + \frac{1}{2k_2+1} & 0 & \dots 0 \\ \vdots & & & \\ 0 & 0 & 0 & \dots \frac{1}{k_n} + 1 + \frac{1}{2k_n+1} \end{pmatrix} =: \Xi,$$

which is positive. Also we have

$$(Nx)(t) := \begin{pmatrix} f_1(t, x_1, x_2, \dots, x_n) \\ f_2(t, x_1, x_2, \dots, x_n) \\ \vdots \\ f_n(t, x_1, x_2, \dots, x_n) \end{pmatrix}$$

Now, a solution of the problem exists according to Theorem (7.1) and it has an expression of the form (11.1), which is positive, because it can be written as

$$x(t) = \Xi\Big[\begin{pmatrix} k_1\zeta_1^1 - \zeta_0^1 \\ k_2\zeta_1^2 - \zeta_0^2 \\ \vdots \\ k_n\zeta_1^n - \zeta_0^n \end{pmatrix} + A_0A_1\int_0^1 \Theta(s)^{-1}\int_0^s (-Nx)(u)\,du\,ds$$

$$+ A_0 B_1 \Theta(1)^{-1} \int_0^1 (-Nx)(u)du\Big] + A_0^{-1}\zeta_0 + \int_0^t \Theta(s)^{-1} \int_0^s (Nx)(u)\,du\,ds$$

$$\geq \Xi\Big[\begin{pmatrix} k_1\zeta_1^1 - \zeta_0^1 \\ k_2\zeta_1^2 - \zeta_0^2 \\ \vdots \\ k_n\zeta_1^n - \zeta_0^n \end{pmatrix} + (\text{positive quantity})\Big] + \begin{pmatrix} \frac{\zeta_0^1}{\kappa_1} - \frac{\nu_1}{2} \\ \frac{\zeta_0^2}{\kappa_2} - \frac{\nu_2}{2} \\ \vdots \\ \frac{\zeta_0^n}{\kappa_n} - \frac{\nu_n}{2} \end{pmatrix} > 0, \quad t \in I.$$

The proof is complete.

References

[1] Aleksejs Antonuks; Existence of a positive solution with concave and convex components for a system of boundary value problems, *Non. Analysis: Modelling and Control,* Vol. 30(2), (2025), 333-345. zbl DE080095489.
[2] G. E. Abduragimov; On the Existence and Uniqueness of a Positive Solution to a Boundary Value Problem with Integral Boundary Conditions for a Second-Order Nonlinear Functional Differential Equation, *Russian Mathematics*, 2024, Vol. 68, No. 12, pp. 33–39.
[3] A. V. Bitsadze; On the theory of nonlocal boundary value problems, *Soviet Math. Dock.* 30 (1964), 8-10.
[4] A. V. Bitsadze, A. A. Samarskii; Some elementary generalizations of linear elliptic boundary value problems, *Dokl. Akad. Nauk SSSR* 185 (1969), 739-740.
[5] Gabriele Bonanno, Giuseppa Riccobono; Multiplicity results for Sturm-Liouville boundary value problems, *Applied Mathematics and Computation*, 210 (2009), 294-297.
[6] Abdelkader Boucherif; Second-order boundary value problems with integral boundary conditions, *Nonlinear Analysis*, 70 (2006), 364-371.
[7] J. Chandra, V. Lakshmikantham, S. Leela; Comparison principle and theory of nonlinear boundary value problems, in *Proc. International Conf. on Nonlinear Phenomena in Mathematical Sciences,* Academic Press, New York, 1982, pp. 241-248.
[8] J. Chandra, V. Lakshmikantham, A. R. Mitchell; Existence of solutions of boundary value problems for nonlinear second order systems in a Banach space, *Nonlinear Analysis, Theory, Methods and Applications* 2(2) (1978) 157-168.
[9] Donald S. Cohen; Singular perturbation of nonlinear two-point boundary-value problems, *J. Mathematical Analysis and Applications*, 43 (1973), 151-160.
[10] Julio G. Dix, George L. Karakostas,; A fixed-point theorem for S-type operators on Banach spaces and its applications to boundary-value problems, *Nonlinear Analysis* 71 (2009) ,3872-3880.
[11] J. Dugundji, A. Granas; *Fixed Point Theory,* Vol. I, Monographie Matematyczne, Warsaw, 1982.
[12] L. H. Erbe, S. Hu, H. Wang; Multiple positive solutions of some boundary value problems, *J. Math. Anal. Appl.*, 184 (1994), 640-648.
[13] L. H. Erbe, H. Wang; On existence of positive solutions of ordinary differential equations, *Proc. Amer. Math. Soc.*, 120 (1994), 743-748.
[14] W. Feng; On a m-point nonlinear boundary-value problem, *Nonlinear Anal.*, 30 (1997) 5369-5374.
[15] W. Feng, J. R. L. Webb; Solvability of three point boundary value problems at resonance, *Nonlinear Anal.*, 30 (1997), 3227-3238.
[16] G. N. Galanis, A. P. Palamides; Positive solutions of three-point boundary-value problems for p-Laplacian singular differential equations, *Electron. J. Differential Equations*, 2005 (2005) No. 106, 1-18.
[17] J. A. Gatica, Vladimir Oliker, Paul Waltman; Singular nonlinear boundary value problems for second order ordinary differential equations, *J. Differential equations* 79, (1989), 62-78.
[18] X. He, W. Ge; Triple solutions for second-order three-point boundary-value problems, *J. Math. Anal. Appl.*, 268 (2002) 256-265.
[19] D. Guo, V. Lakshmikantham; *Nonlinear Problems in Abstract Cone,* Academic Press, New York, 1988.
[20] C. Gupta; Solvability of a generalized multi-point boundary value problem of mixed type for second order ordinary differential equations, *Proc. Dynamic Systems and Applications*, **2** (1995), 215-222.
[21] C. Gupta; Solvability of a three-point boundary-value problem for a second order ordinary differential equation, *J. Math. Anal. Appl.*, 168 (1992), 540-551.
[22] C. Gupta, S. N. Ntouyas, P. Ch. Tsamatos; Existence results for m-point boundary value problems, *Diff. Eq. Dynam. Systems*, 2 (1994), 289-298.
[23] Johnny Henderson, Rodica Luca; On a system of second-order multi-point boundary value problems, *Applied Mathematics Letters*, (2012), DOI: 10.1016/j.aml.2012.05.005.
[24] V. Il'in,d E. Moiseev; Nonlocal boundary value problems of the second kind for a Sturm-Liouville operator, *Differential Equations*, 23 (1987), 979-987.
[25] V. Il'in, E. Moiseev; Nonlocal boundary value problems of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects, *Differential Equations*, 23 (1988), 803 810.
[26] G. Infante, G. Mascali, J. Rodriguez-Lopez; A hybrid Krasnoselskii-Schauder fixed point theorem for systems, *Nonlinear Anal., Real World Appl.*, Vol. 80:104165 (2024), DOI: 10.1016/j.nonrwa.2024.104165.

[27] Jiqiang Jiang, Lishan Liu; Positive solutions for nonlinear second-order m-point boundary value problems, *Electronic Journal of Differential Equations*, Vol. 2009(2009), No. 110, pp. 1–12.

[28] R. A. Khan; The generalized quasilinearization technique for a second order differential equation with separate boundary conditions, *Math. Comput. Modelling*, 43(2006) 727-742.

[29] M. A. Krasnoselskii; *Some Problems of Nonlinear Analysis*, AMS Translations, Ser. 2 Vol. 10, Providence, RI, 1958.

[30] V. Lakshmikantham; *Abstract boundary value problems, in Nonlinear Equations in Abstract Spaces,* Academic Press, New York, 1978.

[31] L. Liu, B. Liu, Y. Wu; Positive solutions of singular boundary-value problems for nonlinear differential systems, *Appl. Math. Comput.*, 186 (2007) 1163-1172.

[32] Zhaoli Liu, Fuyi Li; Multiple positive solutions of nonlinear two-point boundary value problems, *Journal of Mathematical Analysis and Applications*, 203 (1996), 620-625.

[33] Hongyu Li, Jingxian Sun; Positive solutions of sublinear Sturm-Liouville problems with changing sign nonlinearity, *Computers and Mathematics with Applications*, 58 (2009), 1808-1815.

[34] A. G. Lomtatidze; A nonlocal boundary value problem for second-order linear ordinary differential equations, *Differential Equations*, 31 (1995), 411-420.

[35] R. Ma; Positive solutions of a nonlinear three-point boundary-value problem, *Electron. J. Differential Equations* 1999 (1999) No. 34, 1-8.

[36] R. Ma; Multiple positive solutions for nonlinear m-point boundary-value problems, *Appl. Math. Comput.* 148 (2004) 249-262.

[37] Jean Mawhin, Katarzyna Szymanska-Debowska; Convexity topology and nonlinear differential systems with nonlocal boundary conditions: a survey, *Rend. Istit. Mat. Univ. Trieste*, Volume 51 (2019), 125-166. DOI: 10.13137/2464-8728/29333.

[38] Klaus Schmitt; A nonlinear Boundary value problem, *Journal of Differential equations*, 7(1970), 527-537.

[39] B. L. Shekhter; On the existence and zeros of solutions of a nonlinear two-point boundary value problem, *Journal of Mathematical Analysis and Applications*, 97 (1983), 1-20.

[40] Szymanska-Debowska, Katarzyna; On two systems of non-resonant nonlocal boundary value problems, *An. St. Univ. Ovidius Constanta* , Vol. 21(3), 2013, 257-267.

[41] Szymanska-Debowska, Katarzyna; On the existence of solutions for nonlocal boundary value problems, *Georgian Mathematical Journal,* Vol. 22, no. 2, 2015, pp. 273-279. DOI: 10.1515/gmj-2015-0005.

[42] Y. Sun; Positive solutions of nonlinear second-order m-point boundary-value problem, *Nonlinear Anal.*, 61 (2005), 1283-1294.

[43] J. R. Ward; *Semilinear boundary value problems in Banach Spaces, in Nonlinear Equations in Abstract Spaces,* Academic Press, New York, 1978.

[44] J. R. L. Webb, Gennaro Infante; Positive solutions of nonlocal boundary value problems: A unified approach, *J. London Math. Soc.*, 74(2), (2006), 673-693.

[45] J. R. L. Webb; A class of positive linear operators and applications to nonlinear boundary value problems, *Topological Methods in Nonlinear Analysis, Journal of the Juliusz Schauder University Centre*, 39(2012), 221-242.

[46] F. H. Wong, T. G. Chen, S. P. Wang; Existence of positive solutions for various boundary value problems, *Computers and Mathematics with Applications* 56 (2006), 953-958.

[47] Shouliang Xi, Mei Jia, Huipeng Ji; Multiple nonnegative solutions for second-order boundary value problems with sign-changing nonlinearities, *Electron. J. Differential Equations*, 2009(2009), No. 66. pp. 1-10.

[48] D. Yuyun; On solvability of second-order Sturm-Liouville boundary value problems at resonance, *Proc. Amer. Math. Soc.*, 126 (1998), 145-152.

[49] G. Zhang, J. Sun; Multiple positive solutions of singular second-order m-point boundary-value problems, *J. Math. Anal. Appl.*, 317 (2006) 442-447.

[50] Xuemei Zhang, Weigao Ge; Existence of positive solutions for a class of m-point boundary value problems, *Advances in Difference Equations*, Vol. 2008, Art. ID 845121, 9 pages.

[51] Xuemei Zhang, Meiqiang Feng, Weigao Ge; Multiple positive solutions for a class of m-point boundary value problems, *Applied Mathematics Letters*, 22(2009), 12-18.

[52] Xinguang Zhang, Lishan Liu; A necessary and sufficient condition of positive solutions for nonlinear singular differential systems with four-point boundary conditions, *Applied Mathematics and Computation* 215 (2010) 3501-3508.

George L. Karakostas
Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece
Email address: gkarako@uoi.gr, gkarako@hotmail.com