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# APPROXIMATIONS OF EULER-PEANO SCHEME FOR REFLECTED STOCHASTIC DIFFERENTIAL EQUATIONS WITH NON-LIPSCHITZ COEFFICIENTS

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Abstract. This article concerns reflected stochastic differential equations with reflecting boundary conditions that were introduced by Lions and Sznitman [5]. We establish a theorem on the existence and uniqueness of the strong solution when the coefficients satisfy non-Lipschitz conditions. We further show that the solutions depend continuously on the initial data. Also we construct a measurable flow of the solution, and prove that the solution is a Markov process. The analytical solution of stochastic differential equations is generally very difficult to obtain, so numerical approximations are important in applications. Their convergence rates are very important for improving efficiency and for designing algorithms. So we characterize the convergence rate of Euler's approximations under some restrictions on the coefficients.

## 1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $\{\mathcal{F}_t\}_{t\geq 0}$ be the right continuous filtration with the property that $\mathcal{F}_t$ contains all null set of $(\Omega, \mathcal{F}, P)$. Let $B = (B_t)$ be an $\{\mathcal{F}_t\}$ Brownian motion taking values in $\mathbb{R}^n$. In this article we are concerned with the reflected stochastic differential equations (RSDEs)

$$\begin{gathered} x(t) = x_0 + \int_0^t \sigma(s, x(s))dB(s) + \int_0^t b(s, x(s))ds + \phi(t), \quad x_0 \in \bar{D}, \\ \phi(t) = \int_0^t \mathbf{n(s)} d\|\phi\|_{[0,s]}, \quad \|\phi\|_{[0,t]} = \int_0^t 1_{(x(s)\in\partial D)} d\|\phi\|_{[0,s]}, \end{gathered} \tag{1.1}$$

where $t \in [0, +\infty)$ and, $d \times n$-dimensional matrix $\sigma(t, x)$ and $d$-vector $b(t, x)$ are continuous in $x$ for any $t$, measurable in $(t, x)$.

By a direct approach based on the Skorokhod problem (SP), Lions and Sznitman [5] proved the existence and uniqueness of the solution of equation (1.1) with reflecting boundary conditions, where the coefficients $\sigma$ and $b$ satisfy the Lipschitz conditions. Aida and Sasaki [1] showed the convergence in $L^p$ of Euler type approximation of equation (1.1). When the coefficients $\sigma$ and $b$ are Lipschitz continuous, the solvability of Itô stochastic equations is obvious. Recently, Ren and Wu [9] studied approximate continuity and the support of reflected stochastic differential equations. When the coefficients $\sigma$ and $b$ do not satisfy the Lipschitz conditions, the solvability of Itô stochastic equations has been studied extensively, see for example [4, 15]. In [15], the authors studied the pathwise uniqueness of solution of vector stochastic differential equations with conditions weaker than Lipschitz conditions on the coefficients. Using Euler's method, Krylov [4] investigated the Itô stochastic equations with the coefficients $\sigma$ and $b$ satisfying one side Lipschitz continuous condition. Based on the Euler scheme, Wang [16] constructed a measurable flow of the solution for stochastic differential equation with non-Lipschitz coefficients, and proved that the solution is a Markov process.

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First, we introduce some notation and conditions on the domain $D$, and the coefficients $\sigma$ and $b$. We set

$$\|\varpi\|_{\infty,[s,t]} = \max_{s\le u\le v\le t} |\varpi(u)-\varpi(v)|,$$
$$\|\varpi\|_{[s,t]} = \sup_{\Delta} \sum_{k=1}^{N} |\varpi(t_k)-\varpi(t_{k-1})|,$$

where $\Delta = \{(t_0,\dots,t_N) : s = t_0 < \dots < t_N = t\}$ is a partition of the interval $[s,t]$. We assume that $D$ is a connected domain in $\mathbb{R}^d$, and define the set $\mathcal{N}_x$ of inward unit normal vectors at the boundary point $x \in \partial D$ by

$$\mathcal{N}_x = \cup_{r>0}\mathcal{N}_{x,r},$$
$$\mathcal{N}_{x,r} = \{\mathbf{n} \in \mathbb{R}^d; |\mathbf{n}| = 1, B(x-r\mathbf{n}, r)\cap D = \emptyset\}.$$

where $B(z,r) = \{y \in \mathbb{R}^d; |y-z| < r\}$, $z \in \mathbb{R}^d$, and $r > 0$.

The reflecting boundary conditions on $D$ were firstly introduced by Lions [5] and Aida [1], as follows:

- (A1) (uniform exterior sphere condition). There exists a constant $r_0 > 0$, such that for any $x \in \partial D$,
$$\mathcal{N}_x = \mathcal{N}_{x,r_0} \neq \emptyset,$$
- (A2) There exist constants $\delta > 0$ and $\beta \ge 1$ satisfying: for any $x \in \partial D$, there exists a unit vector $l_x$, such that
$$(l_x, \mathbf{n}) \ge \frac{1}{\beta},$$
for any $\mathbf{n} \in \cup_{y\in B(x,\delta)\cap\partial D}\mathcal{N}_y$.
- (A3) There exists a $\mathcal{C}_b^2$ function $f$ and a positive constant $\gamma$ such that for any $x \in \partial D$, $y \in \bar{D}$, $\mathbf{n} \in \mathcal{N}_x$, it holds that
$$\langle y-x, \mathbf{n}\rangle + \frac{1}{\gamma}\langle (Df)(x), \mathbf{n}\rangle |x-y|^2 \ge 0. \tag{1.2}$$

**Remark 1.1.** We first observe that when $D$ is a convex domain, condition (A1) holds for arbitrary $r_0 > 0$, while condition $(A3)$ is automatically satisfied when $f \equiv 0$. The admissibility condition on Dintroduced in [5] requires that the domain can be approximated by smooth-boundary domains in a specific sense. While we do not utilize this property in our current work, we refer interested readers to [5] for details.

Notably, Saisho and Tanaka [13] provide an explicit example of a domain satisfying conditions (A1) and (A2). Furthermore, for piecewise smooth bounded domains in $\mathbb{R}^d$ that satisfy (A1) and (A2), [5, Remark 3.1 ] establishes that condition (A3) holds locally.

**Remark 1.2.** $D$ satisfies (A2) if it satisfies the following uniform interior cone condition: There exist $\delta > 0$ and $\alpha \in (0,1)$ with the following property: for any $x \in \partial D$ there exists a unit vector $l_x$ such that

$$C(y, l_x, \alpha) \cap B(x,\delta) \subset \bar{D}, \quad y \in B(x,\delta)\cap\partial D,$$

where $C(y, l_x, \alpha)$ is the convex cone with vertex $y$, defined by

$$C(y, l_x, \alpha) = \{z \in \mathbb{R}^d : \langle z-y, l_x\rangle \ge \alpha|z-y|\}.$$

Let

$$|f(x)| \vee |Df(x)| \vee \|D^2 f(x)\| \vee \|(D_i f(x) D_j f(x))_{i,j}\| \le C,$$
$$c_0 \le \exp(-\frac{2}{\gamma}f(x)) + \exp(-\frac{2}{\gamma}(f(x)+f(y))) \le C_0,$$

where $C$, $c_0$, $C_0$ are positive constants. Fixed $R > 0$ arbitrarily, $K_R : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing concave continuous function satisfying

$$\int_{0+} \frac{du}{K_R(u)} = \infty.$$

Next, we introduce the following assumptions on the coefficients $\sigma$ and $b$.

(A4) Let $R>0$ be fixed arbitrarily. $D_0$ is a positive constant such that $\frac{4C}{\gamma}+2\leq D_0$. For all $t\in[0,\infty)$, $|x|\vee|y|\leq R$, the following locally weak monotonicity condition

$$2\langle x-y, b(t,x)-b(t,y)\rangle + D_0\|\sigma(t,x)-\sigma(t,y)\|^2 \leq g(t)K_R(|x-y|^2) \tag{1.3}$$

is satisfied, where $g$ is a non-negative function such that

$$\int_0^{+\infty} g(s)ds<\infty$$

(A5) For each $0\leq s\leq t<\infty$ and $0<R<+\infty$, there exists a positive function $\lambda(\cdot)$ such that

$$\int_s^t \sup_{|x|\leq R}(|b(u,x)|+\|\sigma(u,x)\|^2)du \leq \int_s^t \lambda(u)du \leq \int_0^{+\infty}\lambda(u)du<+\infty.$$

Motivated by the work of Krylov [4], we introduce the non-Lipschitz continuous coefficients' conditions (A4) and (A5). Our aim of the present paper is to study the solvability of equation (1.1) with coefficients $\sigma$ and $b$ satisfying (A4) and (A5). We also want to explore the properties of the solution of equation (1.1). As we know, if the solution constitutes a stochastic flow of homeomorphisms, many useful properties, such as Markov property, can be constructed by the flow property. In our case, however, the existence of such a well-behaved modification cannot be guaranteed, as the validity of the flow property remains uncertain. Therefore, we investigate the Markov property of solutions to (1.1) under our assumptions.

In the convergence analysis of numerical methods for stochastic differential equations with reflection term (see (**??**) below), the core challenge lies in the circular dependency between the reflection term and the state variables: moment estimation of the state variables $\{x_n\}$ depends on estimates of the reflection term, while the reflection term's estimation reciprocally relies on $\{x_n\}$'s estimates. This bidirectional dependence renders traditional approaches (e.g., directly adopting estimation frameworks from references [4] and [12]) inapplicable. To resolve this fundamental difficulty, it is essential to fully leverage the geometric and analytical properties of the reflection domain $D$ by constructing non-negative functions. Through applying the Itô formula and stopping time techniques, the influence of the reflection term can be transformed into a controllable negative term, effectively decoupling the circular dependency between state variables and reflection terms. Subsequently, Bihari's inequality is employed instead of Gronwall's inequality to handle non-Lipschitz coefficient scenarios, thereby overcoming the key difficulty in the proof.

This article is organized as follows. We recall in Section 2 the existence of solution of Skorohod problem with reflection boundary conditions and the basic results of Skorohod problem. In Section 3, we prove the existence and uniqueness of the solution of equation (1.1) under the assumptions (A1)–(A5). Furthermore, we show that the solution of equation (1.1) is non-explosive and continuous with respect to the initial data. On the other hand, when the domain $D$ and coefficients $\sigma, b$ are bounded, we also obtain the solvability of equation (1.1) under the assumptions (A1), (A2), (A4) and (A5). In Section 4, we present the stability of solution of equation (1.1). In Section 5, we construct a measurable flow of equation (1.1), and also prove that the solution is a Markov process. In Section 6, we assume that the coefficients $b$ and $\sigma$ satisfy some slightly stronger conditions than (A4) and (A5), then we get a characterization of convergence rate of Euler's approximations.

## 2. Skorohod problem

In this section, we recall some basic results of the Skorohod problem with reflection boundary conditions. For details, we refer to Aida and Sasaki [1]. For any fixed $0<T<\infty$, let $\varpi\in\mathcal{C}([0,T];\mathbb{R}^d)$ with $\varpi(0)\in\bar{D}$. The pair of paths $(x,\phi)\in\mathcal{C}([0,T];\mathbb{R}^d)\times\mathcal{C}([0,T];\mathbb{R}^d)$ is said to be a solution of Skorohod problem with reflecting boundary conditions associated with $\varpi$, if:

- $x$ is a continuous function and $x(t)\in\bar{D}$, for all $t\in[0,T]$, with $x_0:=x(0)=\varpi(0)$,
- $x(t)=\varpi(t)+\phi(t)$ for every $t\geq 0$,

- $\phi$ is an $\mathbb{R}^d$ valued continuous function with bounded variations on each finite interval, $\phi(0) = 0$ and

$$\phi(t) = \int_0^t \mathbf{n}(s) d\|\phi\|_{[0,s]}, \tag{2.1}$$

$$\|\phi\|_{[0,t]} = \int_0^t 1_{(x(s)\in\partial D)} d\|\phi\|_{[0,s]}, \tag{2.2}$$

where $\mathbf{n}(t) \in \mathcal{N}_x$ if $x(t) \in \partial D$. We denote this Skorohod problem as $(\varpi, D, \mathbf{n})$. The following lemma is important for our study.

**Lemma 2.1** ([1, Theorem 2.2])**.** *For any fixed $0 < T < \infty$, let $\varpi \in \mathcal{C}([0,T]; \mathbb{R}^d)$ and $x_0 \in \bar{D}$. Then under assumptions* (A1) *and* (A2)*, there exists a unique solution to the Skorohod problem* $(\varpi, D, \mathbf{n})$ *for any continuous path $\varpi$. Moreover, the mapping $\kappa : \varpi \mapsto x$ is continuous in the uniform convergence topology.*

## 3. Solvability of RSDEs

In this section, we study the existence and uniqueness of the solutions of reflected stochastic differential equations driven by Brownian motion under our conditions. To consider the equation (1.1) with initial random variable $x_0 : \Omega \to \bar{D}$ being $\mathcal{F}_0$ measurable. A pair of $\mathcal{F}_t$ adapted continuous processes $(x(t, x_0), \phi(t, x_0))$ is called a solution to (1.1) if the following holds: for

$$W(t, x_0) = x_0 + \int_0^t \sigma(s, x(s, x_0)) dB(s) + \int_0^t b(s, x(s, x_0)) ds. \tag{3.1}$$

the pair $(x(x_0, ., \omega), \phi(x_0, ., \omega))$ is a solution of the Skorohod problem $(W(\cdot, \omega), D, \mathbf{n})$ for almost all $\omega \in \Omega$.

We apply the method of Euler approximation to discuss the existence and uniqueness of the solutions of RSDEs. Let $n \in \mathbb{N}$. Let $(x_n(t, x_0), \phi_n(t, x_0))$ $(0 \le t \le T)$ be the solution to the Skorohod problem inductively which is given by $x_n(0, x_0) = x_0$ and

$$x_n(t, x_0) = x_0 + \int_0^t b(s, x_n(\pi_n(s), x_0)) ds + \int_0^t \sigma(s, x_n(\pi_n(s), x_0)) dB(s) + \phi_n(t, x_0), \tag{3.2}$$

where $x_n(0, x_0) = x_0$ and $\pi_n(s) = \frac{k}{2^n}$ for $s \in (\frac{k}{2^n}, \frac{(k+1)}{2^n}]$, $k = 0, 1, \dots, 2^n, \dots$. By Theorem 2.1, we have a unique solution of (3.2). In fact, once $x_n(t, x_0)$ is obtained for $0 \le t \le k2^{-n}$, $x_n(t, x_0)$ for $k2^{-n} \le t \le (k+1)2^{-n}$ is uniquely determined as the solution of the Skorohod equation

$$x_n(t, x_0) = x_n(k2^{-n}, x_0) + \int_{k2^{-n}}^t b(s, x_n(k2^{-n}, x_0)) ds + \int_{k2^{-n}}^t \sigma(s, x_n(k2^{-n}, x_0)) dB(s) + \phi_n(t, x_0).$$

The following theorems are main results in this section.

**Theorem 3.1.** *Assume* (A1)–(A5)*. Let $x_0 : \Omega \to \bar{D}$ be $\mathcal{F}_0$ measurable and such that $E[|x_0|^2] < \infty$. Then the equation* (1.1) *with initial random variable $x_0$ has a solution, which is unique up to indistinguishability and non-explosive.*

**Theorem 3.2.** *Assume* (A1), (A2), (A4), (A5)*. Suppose that $D$ is a bounded domain and $\sigma$, $b$ are bounded functions. Then the equation* (1.1) *with initial random variable$x_0 : \Omega \to \bar{D}$ has a solution, which is unique up to indistinguishability.*

Before proving the above theorems, we introduce the following helpful lemmas.

**Lemma 3.3** (Bihari's inequality [3])**.** *Let $\rho : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be a continuous non-decreasing function. If $h(s)$, $\chi(s)$ are two strictly positive functions on $\mathbb{R}^+$ such that*

$$\chi(t) \le \vartheta + \int_0^t \rho(\chi(s)) h(s) ds, \quad t \ge 0,$$

*then*

$$\chi(t) \le \Phi^{-1}(\Phi(\vartheta) + \int_0^t h(s) ds),$$

*where* $\Phi(x) := -\int_x^a \frac{du}{\rho(u)}$ *is well defined for some* $a > 0$.

**Remark 3.4.** Let $\rho : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be a continuous non-decreasing function. If $p(s)$, $q(s)$ and $\chi(s)$ are strictly positive functions on $\mathbb{R}^+$ such that

$$\begin{aligned}\chi(t) &\le \vartheta + \int_0^t \chi(s)p(s)ds + \int_0^t \rho(\chi(s))q(s)ds\\ &\le \vartheta + \int_0^t (\rho + \mathbb{I})(\chi(s))(p(s)+q(s))ds,\ t \ge 0,\end{aligned}$$

where $\mathbb{I}$ denotes the identity mapping. Then by Bihari's inequality, we have

$$\chi(t) \le \Phi^{-1}(\Phi(\vartheta) + \int_0^t p(s) + q(s)ds),\ t \ge 0,$$

where $\Phi(x) := -\int_x^a \frac{du}{\rho(u)+u}$. In the proof of Proposition 3.7 and Theorem 5.2 we will use this fact.

**Lemma 3.5** ([8, Lemma 3.1.3]]). *Let* $w(t)$ *be a continuous nonnegative,* $\mathbb{R}^+$*-valued,* $\mathcal{F}_t$ *adapted process, and* $\iota$ *be a stopping time. For* $\varepsilon \in (0, +\infty)$*, set*

$$\iota_\varepsilon = \iota \wedge \inf\{t \ge 0 : w(t) \ge \varepsilon\},$$

*where as usual we set* $\inf\{\emptyset\} = \infty$*, then*

$$P\{\sup_{0\le t\le \iota} w(t) \ge \varepsilon\} \le \frac{1}{\varepsilon} E[w(\iota_\varepsilon)].$$

**Lemma 3.6.** *Removing both* (A1) *and* (A2) *in Theorem 3.1, we have*

$$\sup_n E[|x_n(t, x_0)|^2] < \infty,\ t \in [0, +\infty).$$

*Proof.* Denote $\theta_t := |x_n(t, x_0) - x_0|^2$, and $\tau_n^*(R) = \inf\{t > 0 : |x_n(t, x_0)| \ge R\}$, for $R > 0$. For simplicity, we denote $x_n(s, x_0)$ by $x_n(s)$. Applying Itô's formula, one can write

$$z_1(x_n(t \wedge \tau_n^*(R)))\theta_{u\wedge\tau_n^*(R)} = \sum_{i=1}^5 \Lambda_i,$$

where $z_1(x) = \exp(-\frac{2}{\gamma} f(x))$ and

$$\begin{aligned}\Lambda_1 &= 2\int_0^{t\wedge\tau_n^*(R)} z_1(x_n(s))\langle x_n(s) - x_0, \sigma(s, x_n(\pi_n(s)))dB(s)\rangle\\ &\quad - \frac{2}{\gamma}\int_0^{t\wedge\tau_n^*(R)} z_1(x_n(s))\theta_s\langle Df(x_n(s)), \sigma(s, x_n(\pi_n(s)))dB(s)\rangle,\\ \Lambda_2 &= 2\int_0^{t\wedge\tau_n^*(R)} z_1(x_n(s))\langle x_n(s) - x_0, b(s, x_n(\pi_n(s)))\rangle ds\\ &\quad + \int_0^{t\wedge\tau_n^*(R)} \mathrm{tr}(z_1(x_n(s))(\sigma(s, x_n(\pi_n(s))))^T\sigma(s, x_n(\pi_n(s))))ds\\ &\quad - \frac{2}{\gamma}\int_0^{t\wedge\tau_n^*(R)} z_1(x_n(s))\theta_s\langle Df(x_n(s)), b(s, x_n(\pi_n(s)))\rangle ds,\\ \Lambda_3 &= 2\int_0^{t\wedge\tau_n^*(R)} z_1(x_n(s))\langle x_n(s) - x_0, d\phi_n(s)\rangle\rangle\\ &\quad - \frac{2}{\gamma}\int_0^{t\wedge\tau_n^*(R)} z_1(x_n(s))\theta_s\langle Df(x_n(s)), d\phi_n(s)\rangle\rangle,\\ \Lambda_4 &= -\frac{1}{\gamma}\int_0^{t\wedge\tau_n^*(R)} \mathrm{tr}(z_1(x_n(s))\theta_s\sigma^T(x_n(\pi_n(s)))D^2f(x_n(s))\sigma(s, x_n(\pi_n(s)))))ds\\ &\quad + \frac{2}{\gamma^2}\int_0^{t\wedge\tau_n^*(R)} z_1(x_n(s))\theta_s|(\sigma(s, x_n(\pi_n(s))))^T Df(x_n(s))|^2 ds,\end{aligned}$$

$$\Lambda_5 = -\frac{4}{\gamma}\int_0^{t\wedge\tau_n^*(R)} z_1(x_n(s))\langle\sigma^T(x_n(\pi_n(s)))Df(x_n(s)), (\sigma(s,x_n(\pi_n(s))))^T(x_n(s)-x_0)\rangle ds.$$

For the terms $\Lambda_2$ and $\Lambda_4$, we have the following estimate

$$\begin{aligned}\Lambda_2+\Lambda_4 \le& \int_0^{t\wedge\tau_n^*(R)} z_1(x_n(s))(\|\sigma(s,x_n(\pi_n(s)))\|^2+|b(s,x_n(\pi_n(s)))|)ds\\ &+(\frac{2C}{\gamma^2}+\frac{2C}{\gamma}+1)\int_0^{t\wedge\tau_n^*(R)} z_1(x_n(s))\theta_s(\|\sigma(s,x_n(\pi_n(s)))\|^2+|b(s,x_n(\pi_n(s)))|)ds.\end{aligned}$$

It is obvious that the term $\Lambda_3\le 0$ by condition (A3). For the term $\Lambda_5$, we have

$$\begin{aligned}\Lambda_5 &\le \frac{4C}{\gamma}\int_0^{t\wedge\tau_n^*(R)} z_1(x_n(s))\|\sigma^T(x_n(\pi_n(s)))\|^2|x_n(s)-x(0)|ds\\ &\le \frac{2C}{\gamma}\int_0^{t\wedge\tau_n^*(R)} z_1(x_n(s))\theta_s\|\sigma^T(x_n(\pi_n(s)))\|^2ds+\frac{2C}{\gamma}\int_0^{t\wedge\tau_n^*(R)} z_1(x_n(s))\|\sigma^T(x_n(\pi_n(s)))\|^2ds.\end{aligned}$$

With these estimates, taking the expectation both side gives

$$\begin{aligned}E[\theta_{t\wedge\tau_n^*(R)}] \le& \alpha_1 E[\int_0^{t\wedge\tau_n^*(R)}\theta_s(|b(s,x_n(\pi_n(s)))|+\|\sigma(s,x_n(\pi_n(s)))\|^2)ds]\\ &+\alpha_2 E[\int_0^{t\wedge\tau_n^*(R)}(\|\sigma(s,x_n(\pi_n(s)))\|^2+|b(s,x_n(\pi_n(s)))|)ds],\end{aligned}$$

where $\alpha_1=\frac{C_0}{c_0}(\frac{2C}{\gamma^2}+\frac{4C}{\gamma}+1)$ and $\alpha_2=\frac{C_0}{c_0}(\frac{2C}{\gamma}+1)$. Subsequently,

$$\begin{aligned}E[\theta_{t\wedge\tau_n^*(R)}] \le& \alpha_1 E[\int_0^t \theta_{s\wedge\tau^*(R)}\sup_{|x|\le R}(|b(s,x)|+\|\sigma(s,x)\|^2)ds]\\ &+\alpha_2\int_0^{+\infty}\sup_{|x|\le R}(|b(s,x)|+\|\sigma(s,x)\|^2)ds.\end{aligned}$$

By Gronwall's inequality and assumption (A5), we obtain

$$\begin{aligned}E[\theta_{t\wedge\tau_n^*(R)}] &\le \alpha_2\exp\Big[\alpha_1\int_0^t\sup_{|x|\le R}(|b(s,x)|+\|\sigma(s,x)\|^2)ds\Big]\int_0^{+\infty}\sup_{|x|\le R}(\|\sigma(s,x)\|^2+|b(s,x)|)ds\\ &\le \alpha_2\exp\Big[\alpha_1\int_0^t\lambda(s)ds\Big]\int_0^{+\infty}\lambda(s)ds\\ &:= C(t)<\infty,\end{aligned}$$

where $C(t)$ is independent of $R$ and $n$.

Let

$$\liminf_{R\to\infty}\tau_n^*(R)=\eta.$$

Suppose that $P(\eta<\infty)>0$. Then for some $T>0$, $P(\eta<T)>0$. Lemma 3.5, we have

$$\begin{aligned}0 &< P(\eta<T)\\ &\le P(\tau_n^*(R)<T)\\ &= P(\sup_{t\in(0,T]}|x_n(t\wedge\tau_n^*(R),x_0)|\ge R)\\ &\le \frac{E[|x_n(T\wedge\tau_n^*(R),x_0)|^2]}{R^2}\\ &\le \frac{2E[(|x_n(T\wedge\tau_n^*(R),x_0)-x_0|)^2+2E[|x_0|^2]]}{R^2}\\ &= \frac{2E[\theta_{T\wedge\tau_n^*(R)}]+2E[|x_0|^2]}{R^2}\\ &\le \frac{2C(T)+2E[|x_0|^2]}{R^2}\to 0,\quad \text{as } R\to\infty.\end{aligned}$$

Then $\eta = \infty$. By Fatou's Lemma

$$E[\theta_t] \le C(t) < \infty. \tag{3.3}$$

Then the claim follows. □

We denote $p_n(s) = x_n(\pi_n(s), x_0) - x_n(s, x_0)$. Clearly, $x_n(s, x_0)$ satisfies

$$x_n(t, x_0) = x_0 + \int_0^t b(s, x_n(s) + p_n(s))ds + \int_0^t \sigma(s, x_n(s) + p_n(s))dB_s + \phi_n(t), \tag{3.4}$$

where $x_n(s) := x_n(s, x_0)$.

Let $x_0,\ y_0 : \Omega \to \bar{D}$ be $\mathcal{F}_0$ measurable. For $\epsilon_0 > 0$, $R > 0$, we define

$$\begin{gathered}
\tau_{n,m}(R, x_0, y_0) = \inf\{t : |x_n(t, x_0)| \vee |x_m(t, y_0)| \ge R\},\\
\tilde{\tau}_{m,n}(x_0, y_0) = \inf\{t : |x_n(t, x_0) - x_m(t, y_0)|^2 \ge \epsilon_0\},\\
\Gamma_{m,n}(R, x_0, y_0) = \tilde{\tau}_{m,n}(x_0, y_0) \wedge \tau_{n,m}(R, x_0, y_0).
\end{gathered}$$

**Proposition 3.7.** *Assume* (A3)–(A5). *Let $x_0, y_0 : \Omega \to \bar{D}$ be $\mathcal{F}_0$ measurable such that $E[|x_0|^2] < \infty, E[|y_0|^2] < \infty$. Then there exists a positive function $\eta(u)$ on $[0, +\infty)$ which is continuous at origin and $\eta(0) = 0$, such that*

$$\begin{aligned}
&E[|x_n(t \wedge \Gamma_{m,n}(R, x_0, y_0), x_0) - x_m(t \wedge \Gamma_{m,n}(R, x_0, y_0), y_0)|^2]\\
&\le F^{-1}\Big(F\Big(\eta(\frac{1}{2^n}) + \eta(\frac{1}{2^m})\Big) + 3\alpha \int_0^t (g(s) + \lambda(s))ds\Big)\\
&\quad + F^{-1}\Big(F\Big(\frac{C_0}{c_0} E[|x_0 - y_0|^2] + 2\eta(\frac{1}{2^m})\Big) + 3\alpha \int_0^t (g(s) + \lambda(s))ds\Big),
\end{aligned}$$

*where $\alpha = 3\frac{C_0}{c_0}(\frac{8C}{\gamma} + \frac{4C}{\gamma^2} + \frac{1}{3})$, and*

$$F(u) = -\int_u^a \frac{dv}{K_R(v) + v}. \tag{3.5}$$

*Proof.* Fix $m, n$ and $R$ temporarily and let $t \in [0, +\infty)$. For simplicity, we denote $x_n(t, x_0)$, $x_m(t, x_0)$ and $\Gamma_{m,n}(R, x_0, y_0)$ by $x_n(t)$, $x_m(t)$ and $\Gamma_{m,n}$, respectively. Set

$$y_t = x_n(t) - x_m(t),\ \zeta_t = |y_t|^2.$$

Let

$$\begin{gathered}
z(x, y) = \exp(-\frac{2}{\gamma}(f(x) + f(y))),\\
z_t = z(x_m(t), x_n(t))\zeta_t.
\end{gathered}$$

By Itô formula, we have

$$z_{t\wedge\Gamma_{m,n}} := \sum_{i=1}^6 I_i,$$

where

$$\begin{aligned}
I_1 &= 2\int_0^{t\wedge\Gamma_{m,n}} z(x_m(s), x_n(s))\langle y_s, (\sigma(s, x_n(\pi_n(s))) - \sigma(s, x_m(\pi_m(s)))dB(s)\rangle\\
&\quad - \frac{2}{\gamma}\int_0^{t\wedge\Gamma_{m,n}} z_s[\langle Df(x_n(s)), (\sigma(s, x_n(\pi_n(s)))dB(s)\rangle\\
&\quad + \langle Df(x_m(s)), \sigma(s, x_m(\pi_m(s)))dB(s)\rangle],\\
I_2 &= 2\int_0^{t\wedge\Gamma_{m,n}} z(x_m(s), x_n(s))\langle y_s, b(s, x_n(\pi_n(s))) - b(s, x_m(\pi_m(s)))\rangle ds\\
&\quad + \int_0^{t\wedge\Gamma_{m,n}} \mathrm{tr}\{z(x_m(s), x_n(s))(\sigma(s, x_n(\pi_n(s))) - \sigma(s, x_m(\pi_m(s)))^T\\
&\quad \times (\sigma(s, x_n(\pi_n(s))) - \sigma(s, x_m(\pi_m(s)))\}ds,
\end{aligned}$$

$$\begin{aligned}
I_3 &= -\frac{2}{\gamma}\int_0^{t\wedge\Gamma_{m,n}} z_s[\langle Df(x_n(s)), b(s,x_n(\pi_n(s)))\rangle \\
&\quad + \langle Df(x_m(s)), b(s,x_m(\pi_m(s)))\rangle]ds,
\end{aligned}$$

$$\begin{aligned}
I_4 &= 2\int_0^{t\wedge\Gamma_{m,n}} z(x_m(s),x_n(s))\langle y_s, d\phi_n(s) - d\phi_m(s)\rangle \\
&\quad - \frac{2}{\gamma}\int_0^{t\wedge\Gamma_{m,n}} z_s[\langle Df(x_n(s)), d\phi_n(s)\rangle + \langle Df(x_m(s)), d\phi_m(s)\rangle],
\end{aligned}$$

$$\begin{aligned}
I_5 &= -\frac{1}{\gamma}\int_0^{t\wedge\Gamma_{m,n}} z_s \operatorname{tr}[\sigma^T(s,x_n(\pi_n(s)))D^2 f(x_n(s))\sigma(s,x_n(\pi_n(s))) \\
&\quad + \sigma^T(s,x_m(\pi_m(s)))D^2 f(x_m(s))\sigma^T(s,x_m(\pi_m(s)))]ds \\
&\quad + \frac{2}{\gamma^2}\int_0^{t\wedge\Gamma_{m,n}} z_s[|\sigma^T(s,x_n(\pi_n(s)))Df(x_n(s))|^2 \\
&\quad + |\sigma^T(s,x_m(\pi_m(s)))(Df(x_m(s)))|^2 \\
&\quad + 2\langle\sigma^T(s,x_n(\pi_n(s)))Df(x_n(s)), \sigma^T(s,x_m(\pi_m(s)))Df(x_m(s))\rangle]ds,
\end{aligned}$$

$$\begin{aligned}
I_6 &= -\frac{4}{\gamma}\int_0^{t\wedge\Gamma_{m,n}} \operatorname{tr}\Big\{z(x_m(s),x_n(s)) \\
&\quad \times \big[\big\langle(\sigma(s,x_n(\pi_n(s))) - \sigma(s,x_m(\pi_m(s))))^T(x_n(s)-x_m(s)), \sigma(s,x_n(\pi_n(s)))Df(x_n(s))\big\rangle \\
&\quad + \big\langle(\sigma(s,x_n(\pi_n(s))) - \sigma(s,x_m(\pi_m(s))))^T(x_n(s)-x_m(s)), \sigma(s,x_m(\pi_m(s)))Df(x_m(s))\big\rangle\big]\Big\}ds.
\end{aligned}$$

For the terms $I_3$ and $I_5$, we have

$$I_3 + I_5 \le \Big(\frac{4C}{\gamma} + \frac{8C}{\gamma^2}\Big)\int_0^{t\wedge\Gamma_{m,n}} z_s \sup_{|x|\le R}(|b(s,x)| + \|\sigma(s,x)\|^2)ds.$$

For the term $I_4$, the condition (A3) implies

$$\begin{aligned}
I_4 &= 2\int_0^{t\wedge\Gamma_{m,n}} z(x_m(s),x_n(s))[\langle y_s, n(x_n(s))\rangle - \frac{1}{\gamma}\zeta_s\langle Df(x_n(s)), n(x_n(s))\rangle]d\|\phi_n\|_{[0,s]} \\
&\quad + 2\int_0^{t\wedge\Gamma_{m,n}} z(x_m(s),x_n(s))[-\langle y_s, n(x_m(s))\rangle - \frac{1}{\gamma}\zeta_s\langle Df(x_m(s)), n(x_m(s))\rangle]d\|\phi_m\|_{[0,s]} \le 0.
\end{aligned}$$

Hence, we can omit this term. We estimate the term $I_6$ as follows,

$$\begin{aligned}
I_6 &\le \frac{4C}{\gamma}\int_0^{t\wedge\Gamma_{m,n}} z(x_m(s),x_n(s))\|\sigma(s,x_n(\pi_n(s))) - \sigma(s,x_m(\pi_m(s)))\| \\
&\quad \times |x_n(s) - x_m(s)|(\|\sigma(s,x_n(\pi_n(s)))\| + \|\sigma(s,x_m(\pi_m(s)))\|)ds \\
&\le \frac{2C}{\gamma}\int_0^{t\wedge\Gamma_{m,n}} z(x_m(s),x_n(s))\|\sigma(s,x_n(\pi_n(s))) - \sigma(s,x_m(\pi_m(s)))\|^2 \\
&\quad + \frac{4C}{\gamma}\int_0^{t\wedge\Gamma_{m,n}} z_s(\|\sigma(s,x_n(\pi_n(s)))\|^2 + \|\sigma(s,x_m(\pi_m(s)))\|^2)ds.
\end{aligned}$$

These estimations lead to

$$\begin{aligned}
z_{t\wedge\Gamma_{m,n}} &\le 2\int_0^{t\wedge\Gamma_{m,n}} z(x_m(s),x_n(s))\langle y_s, b(s,x_n(\pi_n(s)))-b(s,x_m(\pi_m(s)))\rangle ds\\
&\quad + (\frac{2C}{\gamma}+1)\int_0^{t\wedge\Gamma_{m,n}} z(x_m(s),x_n(s))\|\sigma(s,x_n(\pi_n(s)))-\sigma(s,x_m(\pi_m(s)))\|^2 ds\\
&\quad + (\frac{8C}{\gamma}+\frac{8C}{\gamma^2})\int_0^{t\wedge\Gamma_{m,n}} z_s\sup_{|x|\le R}(|b(s,x)|+\|\sigma(s,x)\|^2)ds + I_1\\
&\le 4C_0\int_0^{t\wedge\Gamma_{m,n}}(|p_n(s)|+|p_m(s)|)\sup_{|x|\le R}|b(s,x))|ds\\
&\quad + C_0\int_0^{t\wedge\Gamma_{m,n}} g(s)K_R(|x_n(\pi_n(s))-x_m(\pi_m(s))|^2)ds\\
&\quad + C_0(\frac{8C}{\gamma}+\frac{8C}{\gamma^2})\int_0^{t\wedge\Gamma_{m,n}}\zeta_s\sup_{|x|\le R}(|b(s,x)|+\|\sigma(s,x)\|^2)ds + I_1.
\end{aligned}\tag{3.6}$$

From

$$|x_n(\pi_n(s))-x_m(\pi_m(s))|^2 \le 3(|x_n(s)-x_m(s)|^2+|x_n(\pi_n(s))-x_n(s)|^2+|x_m(\pi_m(s))-x_m(s,x)|^2)$$

and

$$\zeta_s \le |x_n(s)-x_m(s)|^2+|x_n(\pi_n(s))-x_n(s)|^2+|x_m(\pi_m(s))-x_m(s,x)|^2,$$

we obtain

$$\begin{aligned}
&E[\zeta_{t\wedge\Gamma_{m,n}}]\\
&\le 4\frac{C_0}{c_0}E[\int_0^{t\wedge\Gamma_{m,n}}(|p_n(s)|+|p_m(s)|)\sup_{|x|\le R}|b(s,x))|ds]\\
&\quad + \alpha\int_0^t (g(s)+\sup_{|x|\le R}(|b(s,x)|+\|\sigma(s,x)\|^2))\Big\{K_R\Big(3E[|x_n(s\wedge\Gamma_{m,n})-x_m(s\wedge\Gamma_{m,n})|^2]\\
&\quad + 3E[|p_n(s\wedge\Gamma_{m,n})|^2+|p_m(s\wedge\Gamma_{m,n})|^2]\Big)+3E[|x_n(s\wedge\Gamma_{m,n})-x_m(s\wedge\Gamma_{m,n})|^2]\\
&\quad + 3E[|p_n(s\wedge\Gamma_{m,n})|^2+|p_m(s\wedge\Gamma_{m,n})|^2]\Big\}ds,
\end{aligned}$$

where $\alpha = 3\frac{C_0}{c_0}(\frac{8C}{\gamma}+\frac{8C}{\gamma^2}+\frac{1}{3})$. Applying Itô formula as the proof in Lemma 3.6 and by simple calculations, thus

$$\begin{aligned}
E[|p_n(t\wedge\Gamma_{m,n})|^2] &\le \varsigma_1 E[\int_{\pi_n(t)\wedge\Gamma_{m,n}}^{t\wedge\Gamma_{m,n}}|p_n(s)|^2\sup_{|x|\le R}(|b(s,x)|+\|\sigma(s,x)\|^2)ds]\\
&\quad + \varsigma_2 E[\int_{\pi_n(t)\wedge\Gamma_{m,n}}^{t\wedge\Gamma_{m,n}}\sup_{|x|\le R}(\|\sigma(s,x)\|^2+|b(s,x)|)ds]\\
&\le \varsigma_1 E[\int_{\pi_n(t)}^{t}|p_n(s\wedge\Gamma_{m,n})|^2\sup_{|x|\le R}(|b(s,x)|+\|\sigma(s,x)\|^2)ds]\\
&\quad + \varsigma_2\int_{\pi_n(t)}^{t}\sup_{|x|\le R}(\|\sigma(s,x)\|^2+|b(s,x)|)ds\\
&\le \varsigma_1 E[\int_{\pi_n(t)}^{t}|p_n(s\wedge\Gamma_{m,n})|^2\sup_{|x|\le R}(|b(s,x)|+\|\sigma(s,x)\|^2)ds]\\
&\quad + \varsigma_2\sup_{|u-v|\le\frac{1}{2^n}}\int_u^v\sup_{|x|\le R}(\|\sigma(s,x)\|^2+|b(s,x)|)ds,
\end{aligned}$$

where $\varsigma_1 = \frac{C_0}{c_0}(\frac{2C}{\gamma^2} + \frac{3C}{\gamma} + 1)$ and $\varsigma_2 = \frac{C_0}{c_0}(\frac{2C}{\gamma} + 1)$ . Then by Gronwall's inequality and assumption (A5) we rewrite

$$
\begin{aligned}
&E[|p_n(t\wedge\Gamma_{m,n})|^2] \\
&\le \varsigma_2 \exp\Big(\varsigma_1 \int_{\pi_n(t)}^t \sup_{|x|\le R}(|b(s,x)| + \|\sigma(s,x)\|^2)ds\Big) \sup_{|u-v|\le\frac{1}{2^n}} \int_u^v \sup_{|x|\le R}(\|\sigma(s,x)\|^2 + |b(s,x)|)ds \\
&\le \varsigma_2 \exp\Big(\varsigma_1 \int_{\pi_n(t)}^t \lambda(s)ds\Big) \sup_{|u-v|\le\frac{1}{2^n}} \int_u^v \lambda(s)ds \\
&\le \sup_{|u-v|\le\frac{1}{2^n}} \varsigma_2 \exp\Big(\varsigma_1 \int_u^v \lambda(s)ds\Big) \int_u^v \lambda(s)ds.
\end{aligned}
\tag{3.7}
$$

We set

$$
\begin{gathered}
\eta_0(\delta) := \sup_{|u-v|\le\delta} \varsigma_2 \exp(\varsigma_1 \int_u^v \lambda(s)ds) \int_u^v \lambda(s)ds, \\
h_1(t) := 3(\eta_0(\frac{1}{2^n}) + \eta_0(\frac{1}{2^m})) + 3E[|x_n(s\wedge\Gamma_{m,n}) - x_m(s\wedge\Gamma_{m,n})|^2].
\end{gathered}
$$

Then it follows from the assumption (A5) and (3.7) that

$$
\begin{aligned}
h_1(t) &\le 3(\eta_0(\frac{1}{2^n}) + \eta_0(\frac{1}{2^m})) \\
&\quad + 12\frac{C_0}{c_0}E\int_0^t (|p_n(s\wedge\Gamma_{m,n})| + |p_m(s\wedge\Gamma_{m,n})|)\sup_{|x|\le R}|b(s,x)|ds] \\
&\quad + 3\alpha\int_0^t (g(s) + \sup_{|x|\le R}(|b(s,x)| + \|\sigma(s,x)\|^2))(K_R + \mathbb{I})(h_1(s))ds \\
&\le \eta(\frac{1}{2^n}) + \eta(\frac{1}{2^m}) + 3\alpha\int_0^t (g(s) + \sup_{|x|\le R}(|b(s,x)| + \|\sigma(s,x)\|^2))(K_R(h_1(s)) + h_1(s))ds,
\end{aligned}
$$

where

$$
\eta(u) := 3\eta_0(u) + \Big(12\frac{C_0}{c_0}\sqrt{\eta_0(u)}\Big)\int_0^\infty \lambda(s)ds, \quad u \ge 0.
$$

Combining this with Bihari's inequality and assumption (A5), we have

$$
h_1(t) \le F^{-1}\Big(F\Big(\eta(\frac{1}{2^n}) + \eta(\frac{1}{2^m})\Big) + 3\alpha\int_0^t (g(s) + \lambda(s))ds\Big). \tag{3.8}
$$

Similarly, one can write

$$
\begin{aligned}
&E[|x_m(t\wedge\Gamma_{m,n}, x_0) - x_m(t\wedge\Gamma_{m,n}, y_0)|^2] \\
&\le \frac{C_0}{c_0}E[|x_0 - y_0|^2] + 4\frac{C_0}{c_0}E\int_0^{t\wedge\Gamma_{m,n}} (|p_m(x_0,s)| + |p_m(y_0,s)|)\sup_{|x|\le R}|b(s,x)|ds] \\
&\quad + \alpha\int_0^t (g(s) + \sup_{|x|\le R}(|b(s,x)| + \|\sigma(s,x)\|^2)) \\
&\quad \times \Big\{K_R\Big(3E[|x_m(s\wedge\Gamma_{m,n}, x_0) - x_m(s\wedge\Gamma_{m,n}, y_0)|^2] \\
&\quad + 3E[|p_m(x_0, s\wedge\Gamma_{m,n})|^2 + |p_m(y_0, s\wedge\Gamma_{m,n})|^2]\Big) \\
&\quad + 3E[|x_m(s\wedge\Gamma_{m,n}, x_0) - x_m(s\wedge\Gamma_{m,n}, y_0)|^2] \\
&\quad + 3E[|p_m(x_0, s\wedge\Gamma_{m,n})|^2 + |p_m(y_0, s\wedge\Gamma_{m,n})|^2]\Big\}ds,
\end{aligned}
$$

where

$$
p_m(x_0,s) = x_m(\pi_n(s), x_0) - x_m(s, x_0),\ \ p_m(y_0,s) = x_m(\pi_n(s), y_0) - x_m(s, y_0).
$$

We denote
$$h_2(t) := 6\eta_0(\frac{1}{2^m}) + 3E[|x_m(s\wedge\Gamma_{m,n},x_0) - x_m(s\wedge\Gamma_{m,n},y_0)|^2.$$
From this and (3.7), we have
$$\begin{aligned}h_2(t) &\le \frac{C_0}{c_0}E[|x_0-y_0|^2] + 2\eta(\frac{1}{2^m})\\&\quad + 3\alpha\int_0^t (g(s) + \sup_{|x|\le R}(|b(s,x)| + \|\sigma(s,x)\|^2))(K_R+\mathbb{I})(h_2(s))ds.\end{aligned}$$
Then Bihari's inequality leads to
$$h_2(t) \le F^{-1}\Big(F\Big(\frac{C_0}{c_0}E[|x_0-y_0|^2] + 2\eta(\frac{1}{2^m})\Big) + 3\alpha\int_0^t (g(s)+\lambda(s))ds\Big). \tag{3.9}$$
The claim follows from (3.8) and (3.9). □

**Corollary 3.8.** *For $E[|x_0|^2] < \infty$, $E[|y_0|^2] < \infty$, $T > 0$ and $\varepsilon > 0$, then we have*
$$\lim_{m,n\to\infty, E[|x_0-y_0|^2]\to 0} P(\sup_{t\le T}|x_n(t,x_0) - x_m(t,y_0)| \ge \varepsilon) = 0. \tag{3.10}$$
*In particular, we have*
$$\lim_{m,n\to\infty} P(\sup_{t\le T}|x_n(t,x_0) - x_m(t,x_0)| \ge \varepsilon) = 0. \tag{3.11}$$

*Proof.* By applying Lemma 3.5, we obtain that for any $\varepsilon > 0$,
$$\begin{aligned}&P(\sup_{s\le t}|x_n(s,x_0) - x_m(s,y_0)|^2 \ge \varepsilon)\\&\le P(\sup_{s\le t\wedge\tau_{n,m}(R,x_0,y_0)}|x_n(s,x_0) - x_m(s,y_0)|^2 \ge \varepsilon) + P(\tau_{n,m}(R,x_0,y_0) \le t)\\&\le P(\sup_{s\le t\wedge\tau_{n,m}(R,x_0,y_0)}|x_n(s,x_0) - x_m(s,y_0)|^2 \ge \varepsilon) + P(|x_n(t\wedge\tau_{n,m}(R,x_0,y_0),x_0)| \ge R)\\&\quad + P(|x_m(t\wedge\tau_{n,m}(R,x_0,y_0),x_0)| \ge R)\\&\le P(\sup_{s\le t\wedge\tau_{n,m}(R,x_0,y_0)}|x_n(s,x_0) - x_m(s,y_0)|^2 \ge \varepsilon) + \frac{E[|x_n(t\wedge\tau_{n,m}(R,x_0,y_0),x_0)|^2]}{R^2}\\&\quad + \frac{E[|x_m(t\wedge\tau_{n,m}(R,x_0,y_0),x_0)|^2]}{R^2}\\&\le \frac{1}{\varepsilon^2}E[|x_n(t\wedge\Gamma_{m,n},x_0) - x_m(t\wedge\Gamma_{m,n},y_0)|^2] + \frac{E[|x_n(t\wedge\tau_{n,m}(R,x_0,y_0),x_0)|^2]}{R^2}\\&\quad + \frac{E[|x_m(t\wedge\tau_{n,m}(R,x_0,y_0),x_0)|^2]}{R^2}.\end{aligned}$$
Furthermore, by Proposition 3.7, one can easily write
$$\begin{aligned}&P(\sup_{s\le t}|x_n(s,x_0) - x_m(s,y_0)|^2 \ge \varepsilon)\\&\le \frac{1}{\varepsilon^2}F^{-1}\Big(F\Big(\eta(\frac{1}{2^n}) + \eta(\frac{1}{2^m})\Big) + 3\alpha\int_0^t (g(s)+\lambda(s))ds\Big)\\&\quad + \frac{1}{\varepsilon^2}F^{-1}\Big(F\Big(\frac{C_0}{c_0}E[|x_0-y_0|^2] + 2\eta(\frac{1}{2^n})\Big) + 3\alpha\int_0^t (g(s)+\lambda(s))ds\Big)\\&\quad + \frac{E[|x_n(t\wedge\tau_{n,m}(R,x_0,y_0),x_0)|^2]}{R^2} + \frac{E[|x_n(t\wedge\tau_{n,m}(R,x_0,y_0),x_0)|^2]}{R^2}.\end{aligned}$$
Therefore, letting $R\to\infty$ and applying Lemma 3.6, we have
$$\begin{aligned}&\sup_{E[|x_0-y_0|^2]\le\delta} P(\sup_{s\le t}|x_n(s,x_0) - x_m(s,y_0)|^2 \ge \varepsilon)\\&\le \frac{1}{\varepsilon^2}F^{-1}\Big(F\Big(\eta(\frac{1}{2^n}) + \eta(\frac{1}{2^m})\Big) + 3\alpha\int_0^t (g(s)+\lambda(s))ds\Big)\end{aligned}$$

$$+\frac{1}{\varepsilon^2}F^{-1}\Big(F\Big(\frac{C_0}{c_0}\delta+2\eta(\frac{1}{2^n})\Big)+3\alpha\int_0^t(g(s)+\lambda(s))ds\Big),$$

which implies

$$\lim_{m,n\to\infty,E[|x_0-y_0|^2]\to 0} P(\sup_{s\le t}|x_n(s,x_0)-x_m(s,y_0)|^2\ge\varepsilon)=0.$$

Considering that $t$ is arbitrary, this completes the proof. $\square$

*Proof of Theorem 3.1.* We denote the P-limit of $\{x_n(t,x_0)\}_n$ by $x(t,x_0)$ in $\mathcal{C}([0,T];\mathbb{R}^d)$. We now can select a subsequence ( denoted by $\{x_{n_k}(t,x_0)\}$), such that

$$\sup_{t\le T}\Big|x_{n_k}(t,x_0)-x(t,x_0)|\xrightarrow{a.s}0,\quad \text{as } k\to\infty. \tag{3.12}$$

Next we check that $x(t,x_0)$ is a solution of equation (1.1).. By equicontinuity of $\{x_{n_k}(t,x_0)\}$ in $t$ and (3.12), we have

$$\sup_{t\le T}|x_{n_k}(\pi_{n_k}(t),x_0)-x(t,x_0)|\xrightarrow{a.s}0,\ \ as\ k\to\infty. \tag{3.13}$$

It is obvious that

$$\begin{aligned} W_{n_k}(t,x_0)&:=\int_0^t b(s,x_{n_k}(\pi_{n_k}(s),x_0))ds+\int_0^t\sigma(s,x_{n_k}(\pi_{n_k}(s),x_0))dB(s)\\ &\to\int_0^t b(s,x(s,x_0))ds+\int_0^t\sigma(s,x(s,x_0))dB(s)=:W(t,x_0),\quad \text{in } L^2(\Omega), \end{aligned}$$

as $k\to\infty$, by the fact that $b$ and $\sigma$ are both continuous with respect to $x$. Then there exists a subsequence $\{n_k\}$ such that

$$\begin{aligned} \sup_{0\le t\le T}\Big|&\int_0^t b(s,x_{n_k}(\pi_{n_k}(s),x_0))ds+\int_0^t\sigma(s,x_{n_k}(\pi_{n_k}(s),x_0))dB(s)\\ &-\int_0^t b(s,x(s,x_0))ds+\int_0^t\sigma(s,x(s,x_0))dB(s)\Big|\to 0,\quad \text{a.s. } t\le T, \end{aligned} \tag{3.14}$$

as $k\to\infty$. Consequently, by Lemma 2.1 we obtain that there exists an $\mathcal{F}_t$ adapted process $\phi(t,x_0)\in\mathcal{C}([0,T];\mathbb{R}^d)$ with the properties (2.1) and (2.2) such that, for a.s. $\omega\in\Omega$, $(x(t,x_0)(\omega),\phi(t,x_0)(\omega))$ is a solution of the Skorohod problem

$$x(t,x_0)(\omega)=W(t,x_0)(\omega)+\phi(t,x_0)(\omega).$$

Since

$$\phi_{n_k}(t,x_0)=W_{n_k}(t,x_0)-x_{n_k}(t,x_0),$$

we apply (3.13) and (3.14) to obtain

$$\sup_{0\le t\le T}|\phi_{n_k}(t,x_0)-\phi(t,x_0)|\to 0\quad \text{a.s.for } t\le T,\ \text{as } k\to\infty,$$

and then

$$x(t,x_0)=\int_0^t b(s,x(s,x_0))ds+\int_0^t\sigma(s,x(s,x_0))dB(s)+\phi(t,x_0),\quad t\le T.$$

The uniqueness and continuous dependence with respect to $x_0$ of the solution of equation (1.1) will be presented in Theorem 3.10. This completes the proof. $\square$

**Remark 3.9.** Now we see that $(x(t,x_0),\phi(t,x_0))$ indeed is a solution of equation (1.1) with initial data $x_0$. By Lemma 2.1 and (3.14), it is easy to see that for any $t\le T$, and fixed $x_0$,

$$\begin{gathered}[0,t]\times\Omega\ni(s,\omega)\mapsto x(s,x_0,\omega),\\ [0,t]\times\Omega\ni(s,\omega)\mapsto\phi(s,x_0,\omega)\end{gathered}$$

are $\mathcal{B}_{[0,t]}\times\mathcal{F}_t$ measurable.

Let $x_0, y_0 : \Omega \to \bar{D}$ be $\mathcal{F}_0$ measurable. Let $x^1(t, x_0)$ and $x^2(t, y_0)$ be the solutions of equation (1.1) with initial conditions $x^1(0, x_0) = x_0$ and $x^2(0, y_0) = y_0$, respectively, and $\epsilon_0 > 0$, $E[|x_0|^2] \vee E[|y_0|^2] < \infty$. We define

$$\begin{aligned}\tau_x(R, x_0, y_0) &= \inf\{t : |x^1(t, x_0)| \vee |x^2(t, y_0)| \geq R\},\\ \tilde{\tau}_x(x_0, y_0) &= \inf\{t : |x^1(t, x_0) - x^2(t, y_0)|^2 \geq \epsilon_0\},\\ \Gamma_x(R, x_0, y_0) &= \tilde{\tau}_x(x_0, y_0) \wedge \tau_x(R, x_0, y_0).\end{aligned}$$

Now we prove that $(x(t), \phi(t))$ is the unique solution. Indeed, we have the following result.

**Theorem 3.10.** *Let $x^1(t, x_0)$ and $x^2(t, y_0)$ be the solutions of (1.1) with initial conditions $x^1(0, x_0) = x_0$ and $x^2(0, y_0) = y_0$, respectively. Then for any $\varepsilon > 0$, we have*

$$P(\sup_{s\leq t} |x^1(s, x_0) - x^2(s, y_0)|^2 \geq \varepsilon) \leq \frac{1}{\varepsilon} F^{-1}\Big(F(\frac{C_0}{c_0} E[|x_0 - y_0|^2]) + \tilde{\alpha} \int_0^t (g(s) + \lambda(s))ds\Big), \quad (3.15)$$

*where the function $F$ is defined by (3.5) and $\tilde{\alpha} = \frac{C_0}{c_0}(\frac{8C}{\gamma} + \frac{8C}{\gamma^2})$.*

*Proof.* Using Itô formula for the processes

$$Z_t := \exp(-\frac{2}{\gamma}(f(x(t, x_0)) + f(x(t, y_0)))|x^1(t, x_0) - x^2(t, y_0)|^2.$$

and estimating as (3.6), there is a positive constant $C$ such that

$$\begin{aligned}&Z_{t\wedge\Gamma_x(R,x_0,y_0)}\\ &\leq C_0|x_0 - y_0|^2 + C_0 \int_0^{t\wedge\Gamma_x(R,x_0,y_0)} g(s)K_R(|x^1(s, x_0) - x^2(s, y_0)|^2)ds\\ &\quad + C_0(\frac{8C}{\gamma} + \frac{8C}{\gamma^2}) \int_0^{t\wedge\Gamma_x(R,x_0,y_0)} |x^1(s, x_0) - x^2(s, y_0)|^2 \sup_{|x|\leq R} (|b(s, x)| + \|\sigma(s, x)\|^2)ds + R_1(t),\end{aligned}$$

where $R_1(t)$ is a martingale with zero expectation. From the fact that $\exp(-\frac{2}{\gamma}(f(x) + f(y)) \geq c_0$ and the function $K_R$ being concave, we obtain

$$\begin{aligned}&E[|x^1(t \wedge \Gamma_x(R, x_0, y_0), x_0) - x^2(t \wedge \Gamma_x(R, x_0, y_0), y_0)|^2]\\ &\leq \frac{C_0}{c_0} E[|x_0 - y_0|^2] + \frac{C_0}{c_0} \int_0^{t\wedge\Gamma_x(R,x_0,y_0)} g(s)K_R(E[|x^1(s, x_0) - x^2(s, y_0)|^2])ds\\ &\quad + \tilde{\alpha} \int_0^{t\wedge\Gamma_x(R,x_0,y_0)} E[|x^1(s, x_0) - x^2(s, y_0)|^2] \sup_{|x|\leq R} (|b(s, x)| + \|\sigma(s, x)\|^2)ds\\ &\leq \frac{C_0}{c_0} E[|x_0 - y_0|^2]\\ &\quad + \frac{C_0}{c_0} \int_0^t g(s)K_R(E[|x^1(s \wedge \wedge\Gamma_x(R, x_0, y_0), x_0) - x^2(s \wedge \wedge\Gamma_x(R, x_0, y_0), y_0)|^2])ds\\ &\quad + \tilde{\alpha} \int_0^t E[|x^1(s \wedge \wedge\Gamma_x(R, x_0, y_0), x_0) - x^2(s \wedge \wedge\Gamma_x(R, x_0, y_0), y_0)|^2]\\ &\quad \times \sup_{|x|\leq R} (|b(s, x)| + \|\sigma(s, x)\|^2)ds\end{aligned}$$

By remark 3.4, we have

$$\begin{aligned}&E[|x^1(t \wedge \Gamma_x(R, x_0, y_0), x_0) - x^2(t \wedge \Gamma_x(R, x_0, y_0), y_0)|^2]\\ &\leq F^{-1}\Big(F(\frac{C_0}{c_0} E[|x_0 - y_0|^2]) + \tilde{\alpha} \int_0^t (g(s) + \lambda(s))ds\Big)\end{aligned}$$

By Lemma 3.5, for any $\varepsilon > 0$, we write

$$P(\sup_{s\leq t} |x^1(s, x_0) - x^2(s, y_0)|^2 \geq \varepsilon)$$

$$
\begin{aligned}
&\le P(\sup_{s\le t\wedge\tau_x(R,x_0,y_0)}|x^1(x_0,s)-x^2(y_0,s)|^2\ge\varepsilon)+P(\tau_x(R,x_0,y_0)\le t)\\
&\le \frac{E[|x^1(t\wedge\Gamma_x(R,x_0,y_0),x_0)-x^2(t\wedge\Gamma_x(R,x_0,y_0),y_0)|^2]}{\varepsilon}+\frac{E[|x_n^1(t\wedge\tau_x(R,x_0,y_0),x_0)|^2]}{R^2}\\
&\quad+\frac{E[|x_n^2(t\wedge\tau_x(R,x_0,y_0),y_0)|^2]}{R^2}\\
&\le \frac{1}{\varepsilon}F^{-1}\Big(F(\frac{C_0}{c_0}E[|x_0-y_0|^2])+\tilde\alpha\int_0^t(g(s)+\lambda(s))ds\Big)+\frac{E[|x_n^1(t\wedge\tau_x(R,x_0,y_0),x_0)|^2]}{R^2}\\
&\quad+\frac{E[|x_n^2(t\wedge\tau_x(R,x_0,y_0),y_0)|^2]}{R^2}.
\end{aligned}
$$

Letting $R\to\infty$, we have

$$
P(\sup_{s\le t}|x^1(s,x_0)-x^2(s,y_0)|^2\ge\varepsilon)\le\frac{1}{\varepsilon}F^{-1}\Big(F(\frac{C_0}{c_0}E[|x_0-y_0|^2])+\tilde\alpha\int_0^t(g(s)+\lambda(s))ds\Big).
$$

This completes the proof. □

Theorem 3.10 tells us that the solution of equation (1.1) continuously depends on the initial random variable. That is to say we have the following result.

**Corollary 3.11.** *Let $x_0:\Omega\to\bar D$ and $x_n(0):\to\bar D$ be $\mathcal{F}_0$ measurable random variables and assume that $E[|x_0|^2]<\infty$ and $\sup_n E[|x_n(0)|^2]<\infty$ and that $E[|x_n(0)-x_0|^2]\to 0$, as $n\to\infty$. Assume that $(x(t,x_0),\phi(t,x_0))$ and $(x_n(t,x_n(0)),\phi_n(t,x_n(0)))$ are the solutions of equation* (1.1) *with initial random variables $x_0$ and $x_n(0)$, respectively. Then*

$$
P\text{-}\lim_{n\to\infty}\sup_{t\le T}|x_n(t,x_n(0))-x(t,x_0)|=0, \tag{3.16}
$$

*where $P$-lim means a limit in probability.*

To prove Theorem 3.2, we introduce the following condition and lemma.

(A6) There exist positive constants $\delta'$ and $\gamma$ such that for each $x_0\in\partial D$ we can find a function $f\in\mathcal{C}^2(\mathbb{R}^d)$ satisfying (1.2) for any $x\in B(x_0,\delta')\cap\partial D$, $y\in B(x_0,\delta')\cap\bar D$ and n $\in\mathcal{N}_x$.

**Lemma 3.12** ([12, Lemma 5.3])**.** *Assume* (A1) *and* (A2)*. Then the domain $D$ satisfies* (A6) *with $\gamma=2r_0\beta^{-1}$.*

*Proof of Theorem 3.2.* We use the techniques in Pardoux and Rascanu [7, Proposition 3.38 ]. By a standard compactness argument, there exists a colection of sets $\{U_i:U_i=B(x_i,\delta'),\ i=1,2,\dots,k\}$ covering $\partial D$ and there exists an open set $U_0\subset D$ such that $\bar D\subset\cup_{i=0}^k\{U_i\}$. For $\mathcal{F}_0$ measurable random variable $x_0\in\bar D$ a.s., we consider the equation (1.1) with initial value $x_0^i:=x_0I_{U_i}(x_0)$ (where $I_U$ is the indicator function of set $U$). We see that Euler approximation (3.2) which is well-posed by [12, Theorem 4.1]. We denote

$$
\begin{gathered}
\hat\tau^i_{n,m}=\inf\{t:x_n(t,x_0^i)\notin\bar D\cap U_i\}\wedge\inf\{t:x_m(t,x_0^i)\notin\bar D\cap U_i\},\quad i=0,1,\dots,k,\\
\bar\iota^i_{m,n}=\inf\{t:|x_n(t,x_0^i)-x_m(t,x_0^i)|^2\ge\epsilon_0\},\quad i=0,1,\dots,k.
\end{gathered}
$$

Let

$$
\begin{gathered}
\hat\tau^i=\liminf_{m,n\to\infty}\hat\tau^i_{n,m},\\
\hat\tau(\omega)=\min\{\hat\tau^i(\omega)\ne 0:i=0,1,\dots,k\},\quad \omega\in\Omega,\\
\hat\Gamma_{m,n}=\bar\iota_{m,n}\wedge\hat\tau.
\end{gathered}
$$

We may assume that $\hat\tau<\infty$ a.s.; otherwise, consider $\hat\tau\wedge N$. Applying Lemma 3.12 and the proof of Proposition 3.7, we can write

$$
E[|x_n(t\wedge\hat\Gamma_{m,n},x_0^i)-x_m(t\wedge\hat\Gamma_{m,n},x_0^i)|^2]\le F^{-1}\Big(F\Big(\eta(\frac{1}{2^n})+\eta(\frac{1}{2^m})\Big)+3\alpha\int_0^t(g(s)+\lambda(s))ds\Big),
$$

where $\alpha$, $F(u)$ and $\eta(u)$ are defined as in Proposition 3.7. According to Lemma 3.5 and the proof of Corollary 3.8, we have for any $\varepsilon > 0$,

$$
\begin{aligned}
P(\sup_{s\le t\wedge\hat\tau}|x_n(s,x_0^i)-x_m(s,x_0^i)|^2\ge\varepsilon) &\le \frac{E[|x_n(t\wedge\hat\Gamma_{m,n},x_0^i)-x_m(t\wedge\hat\Gamma_{m,n},x_0^i)|^2]}{\varepsilon}\\
&\le \frac{1}{\varepsilon}F^{-1}\Big(F\Big(\eta(\frac{1}{2^n})+\eta(\frac{1}{2^m})\Big)+3\alpha\int_0^t(g(s)+\lambda(s))ds\Big).
\end{aligned}
$$

Hence,

$$
\lim_{m,n\to\infty}P(\sup_{0\le s\le t\wedge\hat\tau}|x_n(s,x_0^i)-x_m(s,x_0^i)|\ge\varepsilon)=0. \tag{3.17}
$$

The proof of Theorem 3.1 says that there exists a unique strong solution $(\hat x^i(t),\hat\phi^i(t))$ $(t\le\hat\tau)$ of equation (1.1) with initial random variable $x_0^i$.

We denote $\alpha(x)=\sum_{i=0}^k I_{U_i}(x)$, $x\in\bar D$, and

$$
\begin{aligned}
\hat x(t)&=\sum_{i=0}^k(\alpha(x_0))^{-1}\hat x^i(t)I_{U_i}(x_0),\quad t\le\hat\tau,\\
\hat\phi(t)&=\sum_{i=0}^k(\alpha(x_0))^{-1}\hat\phi^i(t)I_{U_i}(x_0),\quad t\le\hat\tau.
\end{aligned}
$$

Then $(\hat x(t),\hat\phi(t))$ $(t\le\hat\tau)$ is the unique strong solution of equation (1.1) with initial random variable $x_0$. Indeed, by the fact

$$
\begin{aligned}
\hat x^i(t)&=\hat x(t)I_{U_i}(x_0),\\
\hat\phi^i(t)&=\hat\phi(t)I_{U_i}(x_0),
\end{aligned}
$$

for $t\le\hat\tau$, we have

$$
\begin{aligned}
\hat x(t)&=\sum_{i=0}^k(\alpha(x_0))^{-1}\hat x^i(t)I_{U_i}(x_0)\\
&=\sum_{i=0}^k(\alpha(x_0))^{-1}x_0^i+\int_0^t\sum_{i=0}^k(\alpha(x_0))^{-1}\sigma(s,\hat x(t)I_{U_i}(x_0))I_{U_i}(x_0)dB(s)\\
&\quad+\int_0^t\sum_{i=0}^k(\alpha(x_0))^{-1}b(s,\hat x(t)I_{U_i}(x_0))I_{U_i}(x_0)ds+\sum_{i=0}^k(\alpha(x_0))^{-1}\hat\phi^i(t)I_{U_i}(x_0)\\
&=\sum_{i=0}^k(\alpha(x_0))^{-1}x_0^i+\int_0^t\sum_{i=0}^k(\alpha(x_0))^{-1}\sigma(s,\hat x(t))I_{U_i}(x_0)dB(s)\\
&\quad+\int_0^t\sum_{i=0}^k(\alpha(x_0))^{-1}b(s,\hat x(t))I_{U_i}(x_0)ds+\hat\phi(t)\\
&=x_0+\int_0^t\sigma(s,\hat x(t))dB(s)+\int_0^t b(s,\hat x(t))ds+\hat\phi(t).
\end{aligned}
$$

Assume that there exists a stopping time $\hat\tau_l$ such that $x(t)$ $(0\le t\le\hat\tau_l)$ is the unique solution of equation (1.1) with initial random variable $x_0$. We may assume $\hat\tau_l<\infty$ a.s.; otherwise, consider $\hat\tau_l\wedge N$. Next, define $\hat B(u)=B(\hat\tau_l+u)-B(\tau_l)$ and $\hat{\mathcal F}_u=\mathcal F_{\hat\tau_l+u}=\{\Lambda\in\mathcal F:\Lambda\cap\{\hat\tau_l+u\le t\}\in\mathcal F_t,\forall t\ge0\}$, $u\ge0$. Then it is easy to see that $\hat B$ is an $(\hat{\mathcal F}_u)_{u\ge0}$ Brownain motion and $\bar\tau_l=:t-\hat\tau_l\wedge t$ is an $(\hat{\mathcal F}_u)_{u\ge0}$ stopping time for each fixed $t\ge0$. Thus we consider the SDE

$$
\hat Y(t)=x(\hat\tau_l)+\int_0^t\sigma(t,\hat Y(s))d\hat B(t)+\int_0^t b(t,\hat Y(s))dt+\hat\phi(t). \tag{3.18}
$$

As proved as above, there exist a stopping time $\hat{\tau}_{l+1}$ and a unique strong solution $\hat{Y}(t)$ $(t \le \hat{\tau}_{l+1})$ of (3.18) with initial value $x(\hat{\tau}_l)$. Let

$$x(t) = \begin{cases} \hat{x}(t), & \text{if } t \le \hat{\tau}_l, \\ \hat{Y}(t - \hat{\tau}_l), & \text{if } \hat{\tau}_l \le t \le \hat{\tau}_{l+1}. \end{cases}$$

Then $x(t)$ $(0 \le t \le \tau_{l+1})$ is the unique strong solution of (1.1) with initial value $x_0$. This completes the proof. □

## 4. Stability results

In this section, we consider the stability of the solution of equation (1.1). Suppose that $\sigma_n(t,x)$ and $b_n(t,x)$ are functions satisfying the assumptions (A4) and (A5), where the function $\lambda$ does not depend on $n$ and for any $0 < R < +\infty$ and $0 < T < +\infty$,

$$\int_0^T \sup_{|x|\le R} (\|\sigma_n(t,x) - \sigma(t,x)\|^2 + |b_n(t,x) - b(t,x)|)dt \to 0. \tag{4.1}$$

Let $0 \le \chi \in C_c^\infty(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \chi(d)dx = 1$ and $\chi_n(x) = n^d \chi(nx)$. If $b$ and $\sigma$ satisfying the assumptions (A4) and (A5), $b_n := \chi_n * b(t,\cdot)$ and $\sigma_n := \chi_n * \sigma(t,\cdot)$ satisfy the condition (4.1).

For every $n \in \mathbb{N}$, let $(y_n(t, y_n(0)), \phi_n(t, y_n(0)))$ be the solution of the Skorohod problem,

$$y_n(t, y_n(0)) = y_n(0) + \int_0^t \sigma_n(s, y_n(s, y_n(0)))dB(s) + \int_0^t b_n(s, y_n(s, y_n(0)))ds + \phi_n(t, y_n(0)). \tag{4.2}$$

Our main object in this section is to prove the following theorem.

**Theorem 4.1.** *Assume* (A1)–(A3) *hold. Suppose that* $E[|y_n(0) - x_0|^2] \to 0$ *as* $n \to \infty$. *Then* $y_n(t, x_0) \to x(t, x_0)$ *uniformly on* $[0,T]$ *in probability, where* $x(t,x_0)$ *satisfies the equation* (1.1) *with initial random variable* $x_0$.

*Proof.* We denote $y_n(t, y_n(0))$ and $x(t,x_0)$ by $y_n(t)$ and $x(t)$, respectively. We define

$$\hat{\varsigma}_n = \inf\{t > 0 : |y_n(t)| \vee |x(t)| \ge R\}.$$

Let

$$v(t) = y_n(t) - x(t), \quad \zeta_t = |v(t)|^2, \quad z_t = z(y_n(t), x(t))\zeta_t,$$
$$\hat{\lambda}_n := \inf\{t > 0, \zeta_t \ge \epsilon_0\}, \quad \Lambda_n = \hat{\lambda}_n \wedge \hat{\varsigma}_n,$$

where $z(x,y)$ is defined in the proof of Proposition 3.7.

Using Itô's formula for $z_{t\wedge\Lambda_n}$, and following the same as proof in Proposition 3.7, we have

$$\begin{aligned} z_{t\wedge\Lambda_n} &\le z_0 + 2\int_0^{t\wedge\Lambda_n} z(y_n(s), x(s))\langle v_s, b_n(s, y_n(s)) - b(s, x(s)))\rangle ds \\ &\quad + (\frac{2C}{\gamma} + 1)\int_0^{t\wedge\Gamma_n(R)} z(y_n(s), x(s))\|\sigma_n(s, y_n(s)) - \sigma(s, x(s))\|^2 ds \\ &\quad + (\frac{6C}{\gamma} + \frac{4C}{\gamma^2})\int_0^{t\wedge\Lambda_n} z_s[\sup_{|x|\le R}(|b(s,x)| + \|\sigma(s,x)\|^2) \\ &\quad + \sup_{|x|\le R}(|b_n(s,x)| + \|\sigma_n(s,x)\|^2)]ds + I_1' \\ &= z_0 + 2\int_0^{t\wedge\Lambda_n} z(y_n(s), x(s))\langle v_s, b_n(s, y_n(s)) - b_n(s, x(s)))\rangle ds \\ &\quad + (\frac{4C}{\gamma} + 2)\int_0^{t\wedge\Lambda_n} z(y_n(s), x(s))\|\sigma_n(s, y_n(s)) - \sigma_n(s, x(s))\|^2 ds \\ &\quad + 2\int_0^{t\wedge\Lambda_n} z(y_n(s), x(s))\langle v_s, b_n(s, x(s)) - b(s, x(s)))\rangle ds \end{aligned}$$

$$
\begin{aligned}
&+(\frac{4C}{\gamma}+2)\int_0^{t\wedge\Lambda_n} z(y_n(s),x(s))\|\sigma_n(s,x(s))-\sigma(s,x(s))\|^2ds\\
&+(\frac{6C}{\gamma}+\frac{4C}{\gamma^2})\int_0^{t\wedge\tau_n} z_s[\sup_{|x|\le R}(|b(s,x)|+\|\sigma(s,x)\|^2)\\
&+\sup_{|x|\le R}(|b_n(s,x)|+\|\sigma_n(s,x)\|^2)]ds+I_1'\\
\le{}& z_0+\int_0^{t\wedge\Lambda_n} z(y_n(s),x(s))g(s)K_R(\zeta_s)ds\\
&+\int_0^{t\wedge\Lambda_n} z(y_n(s),x(s))|b_n(s,y_n(s))-b(s,x(s))|ds\\
&+(\frac{4C}{\gamma}+2)\int_0^{t\wedge\Lambda_n} z(y_n(s),x(s))\|\sigma_n(s,y_n(s))-\sigma(s,x(s))\|^2ds\\
&+(\frac{6C}{\gamma}+\frac{4C}{\gamma^2}+1)\int_0^{t\wedge\Lambda_n} z_s[\sup_{|x|\le R}(|b(s,x)|+\|\sigma(s,x)\|^2)\\
&+\sup_{|x|\le R}(|b_n(s,x)|+\|\sigma_n(s,x)\|^2)]ds+I_1',
\end{aligned}
$$

where $I_1'$ is a martingale with $E[I_1']=0$. Thus taking expectation on both side, hence

$$
\begin{aligned}
E[\zeta_{t\wedge\Lambda_n}]\le{}&\beta_2\int_0^t\sup_{|x|\le R}(\|\sigma_n(s,x)-\sigma(s,x)\|^2+|b_n(s,x)-b(s,x)|)ds\\
&+\beta_3E[\int_0^t(K_R+\mathbb{I})(\zeta_{s\wedge\Lambda_n})[\sup_{|x|\le R}(|b(s,x)|+\|\sigma(s,x)\|^2)\\
&+\sup_{|x|\le R}(|b_n(s,x)|+\|\sigma_n(s,x)\|^2)+g(s)]ds]+\frac{C_0}{c_0}E[|y_n(0)-x_0|^2],
\end{aligned}
$$

where $\beta_2=\frac{C_0}{c_0}(\frac{8C}{\gamma}+2)$ and $\beta_3=\frac{C_0}{c_0}(\frac{6C}{\gamma}+\frac{4C}{\gamma^2}+1)$. By Bihari's inequality, we obtain

$$
\begin{aligned}
E[\zeta_{t\wedge\Lambda_n}]\le{}&F^{-1}\Big(F\Big(\beta_2\int_0^t\sup_{|x|\le R}(\|\sigma_n(s,x)-\sigma(s,x)\|^2+|b_n(s,x)-b(s,x)|)ds\\
&+\frac{C_0}{c_0}E[|y_n(0)-x_0|^2]\Big)+\beta_3\int_0^t[\sup_{|x|\le R}(|b(s,x)|+\|\sigma(s,x)\|^2)\\
&+\sup_{|x|\le R}(|b^n(s,x)|+\|\sigma^n(s,x)\|^2)+g(s)]ds\Big)\\
\le{}&F^{-1}\Big(F\Big(\beta_1\int_0^t\sup_{|x|\le R}(\|\sigma_n(s,x)-\sigma(s,x)\|^2+|b_n(s,x)-b(s,x)|)ds\\
&+\frac{C_0}{c_0}E[|y_n(0)-x_0|^2]\Big)+\beta_3\int_0^t[2\lambda(s)+g(s)]ds\Big),
\end{aligned}
$$

where $F$ is defined by (3.5).

As proof in Corollary 3.8, we claim that

$$
P(\sup_{s\le T}\zeta_s\ge\varepsilon)\to 0,\quad \text{as } n\to\infty,
$$

and the theorem follows. □

## 5. Regular version

For $0\le s<T<+\infty$, $x_0\in\bar{D}$ being constant, under the conditions of Theorem 3.1, the equation (1.1) admits a unique solution on $[s,T]$ with the initial value $x(s)=x_0$. We will denote it by $(x(s,x_0;t,\omega),\phi(s,x_0;t,\omega))$. In this section, we will explore the Markov property of the solution of equation (1.1).

We will say that a function $f:\mathbb{R}^+\times\mathbb{R}^d\times\Omega\mapsto\mathbb{R}^d$ is progressively measurable if for any $t>0$, restricted on $[0,t]\times\mathbb{R}^d\times\Omega$ it is $\mathcal{B}([0,t])\times\mathcal{B}^d\times\mathcal{F}_t$-measurable.

Now, the canonical space is used such that it holds $B(t)(\omega)=\omega(t)$. To obtain the Markov property of the solution $x(s,x_0;t,\omega)$, we hope that for $s\le s_1<s_2\le t$, the identity

$$x(s_1,x_0;t,\omega)=x(s_2,x(s_1,x_0;s_2,\omega);t,\theta_{s_2}\omega)$$

holds, where $\theta_s\omega(t)=\omega(s+t)-\omega(s)$. Next we will show the following lemma.

**Lemma 5.1.** *Let $g_1(s,x,\omega)$ and $g_2(s,x,\omega)$ be progressively measurable functions and almost surely continuous with respect to $x$ for any $t$ satisfying*

$$E[\int_0^{+\infty}\sup_{|x|\le R}(|g_2(s,x)|+|g_1(s,x)|^2)ds]<\infty,\quad R>0. \tag{5.1}$$

*Then*

(i) *there exists a progressively measurable function $G(x,t,\omega)$ such that*

$$G(x,t)=\int_0^t g_1(s,x)dB(s)+\int_0^t g_2(s,x)ds,\ \forall\ x\in\mathbb{R}^d;$$

(ii) *if $\eta$ is independent of $(B(t))_{t\ge0}$, then*

$$G(\eta,t,\omega)=\int_0^t g_1(s,\eta,\omega)dB(s)+\int_0^t g_2(s,\eta,\omega)ds.$$

*Proof.* For simplicity we assume that $g_2=0$ because of the ordinary integral term produces no difficulty. Set $y^k:=2^{-k}[2^k y]$. For $x_0=(x_1,\dots,x_d)\in\mathbb{R}^d$, set $x^k=(x_1^k,\dots,x_d^k)$. We denote

$$G_n(t,x)=\int_0^t g_1(s,x^n)dB(s).$$

According to the Burkholder-Davis-Gundy inequality, we have

$$E[\sup_{0\le t\le T}|G_n(x,t)-\int_0^t g_1(s,x)dB(s)|^2]\le E[\int_0^T|g_1(s,x^n)-g_1(s,x)|^2ds].$$

Combining this with

$$g_1(s,x^n)\to g_1(s,x),\ a.e.,\quad \text{as }\to\infty,$$

and (5.1), we have

$$\begin{aligned}\limsup_{n\to+\infty}E[\int_0^T|g_1(s,x^n)-g_1(s,x)|^2ds]&\le E[\int_0^T\limsup_{n\to+\infty}|g_1(s,x^n)-g_1(s,x)|^2ds]\\&=E[\int_0^T\lim_{n\to+\infty}|g_1(s,x^n)-g_1(s,x)|^2ds]=0,\end{aligned}$$

which leads to

$$\lim_{n\to+\infty}E[\int_0^T|g_1(s,x^n)-g_1(s,x)|^2ds]=0. \tag{5.2}$$

Then, there exists a subsequence $\{G_{n_k}\}$ (still denote $\{G_n\}$), such that

$$\lim_{n\to\infty}G_n(t,x)=\int_0^t g_1(s,x)dB(s)\quad\text{a.s. in }\mathcal{C}([0,T]),\ \forall x\in\mathbb{R}^d.$$

Consequently we define

$$G(x,\cdot)=\begin{cases}\lim_{n\to\infty}G_n(.,x), & \text{if the limit exists,}\\ 0, & \text{otherwise.}\end{cases}$$

Then $G$ satisfies (i).

To prove (ii), we first assume that $s \mapsto g_1(s,x,\omega)$ is almost surely continuous. Denote $t_k := (k/2^n) \wedge t$, $k = 1, 2, \dots$. Then we have (see [11, Proposition IV 2.13])

$$G(t,x) = P - \lim_{n\to\infty} \sum_{k=1}^{\infty} g_1(t_k,x)(B(t_{k+1}) - B(t_k)),$$
$$\int_0^t g_1(s,\eta)dB(s) = P - \lim_{n\to\infty} \sum_{k=1}^{\infty} g_1(t_k,\eta)(B(t_{k+1}) - B(t_k)).$$

Then there exists a subsequence of $\{n\}$ (we still denote by $\{n\}$) such that

$$\sum_{k=1}^{\infty} g_1(t_k,\eta)(B(t_{k+1}) - B(t_k)) \to \int_0^t g_1(s,\eta)dB(s), \quad \text{a.s., as } n \to \infty.$$

Because of the independence of $\eta$ and $(B(t))_{t\geq 0}$, for $k = 1, 2, \dots$, we have

$$\begin{aligned}
P(G(t,\eta) &= \int_0^t g_1(s,\eta)dB(s)) \\
&= P(G(t,\eta) = \lim_{n\to\infty} \sum_{k=1}^{\infty} g_1(t_k,\eta)(B(t_{k+1}) - B(t_k))) \\
&= \int_{\mathbb{R}^d} P(G(t,\eta) = \lim_{n\to\infty} \sum_{k=1}^{\infty} g_1(t_k,\eta)(B(t_{k+1}) - B(t_k))|\eta = x)P(\eta \in dx) \\
&= \int_{\mathbb{R}^d} P(G(t,x) = \lim_{n\to\infty} \sum_{k=1}^{\infty} g_1(t_k,x)(B(t_{k+1}) - B(t_k))|\eta = x)P(\eta \in dx) = 1.
\end{aligned}$$

For the general case, we set $g_1(s,x,\omega) = 0$ for $t < 0$ and define

$$g_1^n(t,x) = n\int_{-1/n}^{1/n} \rho(ns)g_1(t - s - \frac{1}{n}, x)ds,$$

where $\rho(t) \in C^\infty$ is a positive function with support in $[-1,1]$ and $\int_{-1}^1 \rho(s)ds = 1$. It is clear that $g_1^n(t,x)$ is almost surely continuous with respect to $t$. Furthermore, by Minkowski's inequality,

$$\begin{aligned}
&\int_0^T |g_1^n(t,y) - g_1^n(t,x)|^2 dt \\
&= \Big[\Big(\int_0^T |\int_{-\frac{1}{n}}^{1/n} n\rho(ns)(g_1(t-s-\frac{1}{n},y) - g_1(t-s-\frac{1}{n},x))ds|^2 dt\Big)^{1/2}\Big]^2 \\
&\leq \Big[\int_{-\frac{1}{n}}^{1/n} \Big(\int_0^T |n\rho(ns)(g_1(t-s-\frac{1}{n},y) - g_1(t-s-\frac{1}{n},x))|^2 dt\Big)^{1/2} ds\Big]^2 \\
&= \Big[\int_{-\frac{1}{n}}^{1/n} n\rho(ns)\Big(\int_0^T |(g_1(t-s-\frac{1}{n},y) - g_1(t-s-\frac{1}{n},x))|^2 dt\Big)^{1/2} ds\Big]^2 \\
&\leq \Big(\int_{-\frac{1}{n}}^{1/n} n\rho(ns)ds\Big)^2 \int_0^T |(g_1(t,y) - g_1(t,x))|^2 dt.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
E[\int_0^T |g_1^n(t,y) - g_1^n(t,x)|^2 dt] &\leq \Big(n\int_{-\frac{1}{n}}^{1/n} \rho(nu)du\Big)^2 E[\int_0^T |g_1(t,y) - g_1(t,x)|^2 dt] \\
&= E[\int_0^T |g_1(t,y) - g_1(t,x)|^2 dt].
\end{aligned} \tag{5.3}$$

Then, for any $n$, $g_1^n(t,x,\omega)$ is almost surely continuous with respect to $t$ for any $x$ and almost surely continuous with respect to $x$ for any $t$. By the proof above there exists a progressively measurable function $\mathcal{G}_n$ such that

$$\mathcal{G}_n(t,x) = \int_0^t g_1^n(s,x)dB(s),$$
$$\mathcal{G}_n(t,\eta) = \int_0^t g_1^n(s,\eta)dB(s).$$

We define $B_r = \{x \in \mathbb{R}^d : |x_i| < r, i = 1,2,\dots,d\}$ for $r > 0$ and

$$\Sigma_n(r) = \{(rk_1 2^{-n}, rk_2 2^{-n}, \dots, rk_d 2^{-n}) : k_i = -2^n, \dots, 2^n, i = 1,2,\dots,d\}.$$

From the definition of $g_1^n$, we have for any $x$,

$$\lim_{n\to\infty} \int_0^T |g_1^n(s,x) - g_1(s,x)|^2 ds = 0, \quad \text{a.s..}$$

Then applying the Burkholder-Davis-Gundy inequality and the lemma of Fatou, we can write

$$\begin{aligned}
&\limsup_{n\to\infty} E[\sup_{0\le t\le T} |\mathcal{G}_n(t,x) - \int_0^t g_1(s,x)dB(s)|^2] \\
&\le \limsup_{n\to\infty} E[\int_0^T |g_1^n(s,x) - g_1(s,x)|^2 ds] \\
&\le E[\limsup_{n\to\infty} \int_0^T |g_1^n(s,x) - g_1(s,x)|^2 ds] = 0.
\end{aligned}$$

For any $m$, we take $n_m \ge m$ such that

$$\max_{x\in\Sigma_{n_m}(r)} E[\sup_{0\le t\le T} |\mathcal{G}_{n_m}(t,x) - \int_0^t g_1(s,x)dB(s)|^2] \le 2^{-2m}. \tag{5.4}$$

For any $i$, by (5.2), we can take $m_i \ge i$ and $x_{m_i} \in \Sigma_{m_i}(r)$ with $|x_{m_i} - x| \le \frac{r}{2^{m_i}}$, such that

$$E[\int_0^T |g_1(t,x_{m_i}) - g_1(t,x)|^2 ds] \le 2^{-2i}. \tag{5.5}$$

Combining with (5.3), (5.4) and (5.5), we then have for any $x \in B_r$,

$$\begin{aligned}
&E[\sup_{0\le t\le T} |\mathcal{G}_{n_{m_i}}(t,x) - \int_0^t g_1(s,x)dB(s)|^2] \\
&\le 3E[\sup_{0\le t\le T} |\mathcal{G}_{n_{m_i}}(t,x) - \mathcal{G}_{n_{m_i}}(t,x_{m_i})|^2] + 3E[\sup_{0\le t\le T} |\mathcal{G}_{n_{m_i}}(t,x_{m_i}) - \int_0^t g_1(x_{s,m_i})dB(s)|^2] \\
&\quad + 3E[\sup_{0\le t\le T} |\int_0^t g_1(x_{s,m_i}) - g_1(s,x)dB(s)|^2] \\
&\le 6E[\int_0^T |g_1(s,x_{m_i}) - g_1(s,x)|^2 ds] + 3\times 2^{-2i} \\
&\le 9\times 2^{-2i}.
\end{aligned}$$

By Chebyshev's inequality, thus

$$\sum_{i=1}^\infty P(\sup_{0\le t\le T} |\mathcal{G}_{n_{m_i}}(t,x) - \int_0^t g_1(s,x)dB(s)| > \frac{1}{2^i}) < \infty.$$

Then, from the Borel-Cantelli lemma, we have

$$\lim_{i\to\infty} \mathcal{G}_{n_{m_i}}(t,x) = \int_0^t g_1(s,x)dB(s) \text{ in } \mathcal{C}([0,T]), \quad \text{a.s., for all } x \in B_r. \tag{5.6}$$

We can see that (5.6) still holds for $|x| < \infty$. We define

$$F(\cdot,x) = \begin{cases} \lim_{i\to\infty} \mathcal{G}_{n_{m_i}}(.,x), & \text{if the limit exists,} \\ 0, & \text{otherwise.} \end{cases}$$

Then $F$ is progressively measurable. We claim that

$$F(\cdot,\eta)=\int_0^{\cdot} g_1(s,\eta)dB(s).$$

Indeed,

$$\begin{aligned}
&E[\sup_{0\le t\le T}|F(t,\eta)-\int_0^t g_1(s,\eta)dB(s)|^2]\\
&\le \lim_{n\to\infty}E[\sup_{0\le t\le T}|F(t,\eta)-\mathcal{G}_n(t,\eta)|^2]+\lim_{n\to\infty}E[\sup_{0\le t\le T}|\mathcal{G}_n(t,\eta)-\int_0^t g_1^n(s,\eta)dB(s)|^2]\\
&\quad+\lim_{n\to\infty}E[\sup_{0\le t\le T}|\int_0^t g_1^n(s,\eta)dB(s)-\int_0^t g_1(s,\eta)dB(s)|^2].
\end{aligned}$$

Applying the Burkholder-Davis-Gundy inequality, the dominated convergence theorem and that

$$E[\sup_{0\le t\le T}|\mathcal{G}_n(t,\eta)-\int_0^t g_1^n(s,\eta)dB(s)|^2]=0,$$

we have

$$\begin{aligned}
&E[\sup_{0\le t\le T}|F(t,\eta)-\int_0^t g_1(s,\eta)dB(s)|^2]\\
&\le \lim_{n\to\infty}\int_{\mathbb{R}^d}P(\eta\in dx)E[\sup_{0\le t\le T}|F(t,x)-\mathcal{G}_n(t,x)|^2]+\lim_{n\to\infty}E[\int_0^T|g_1^n(s,\eta)-g_1(s,\eta)|^2ds]\\
&\le \lim_{n\to\infty}\int_{\mathbb{R}^d}P(\eta\in dx)E[\sup_{0\le t\le T}|F(t,x)-\mathcal{G}_n(t,x)|^2]\\
&\quad+\lim_{n\to\infty}\int_{\mathbb{R}^d}P(\eta\in dx)E[\int_0^T|g_1^n(s,x)-g_1(s,x)|^2ds]=0.
\end{aligned}$$

Thus the claim follows, and this completes the proof. □

We consider the equation

$$x(s,x_0;t)=x_0+\int_s^t\sigma(s,x(s))dB(s)+\int_s^t b(s,x(s))ds+\phi(s,x_0;t),\quad x_0\in\bar{D} \tag{5.7}$$

The following theorem is the main result in this section. We denote the Borel $\sigma$-algebra of the set $\bar{\mathcal{D}}$ by $\mathcal{B}_{\bar{D}}$.

**Theorem 5.2.** *Assume* (A1–(A5). *Then, for any $s\ge 0$, there exists a function $(s,x_0,\omega)\mapsto x(s,x_0;t,\omega)$ such that*

(i) *for any $s\le t$, $\bar{D}\times[s,t]\times\Omega\ni(x_0,u,\omega)\mapsto x(s,x_0;u,\omega)$ is $\mathcal{B}_{\bar{D}}\times\mathcal{B}_{[s,t]}\times\mathcal{F}_{[s,t]}/\mathcal{B}^d$ measurable, also $\mathcal{B}_{\bar{D}}\times[s,t]\times\Omega\ni(x_0,u,\omega)\mapsto\phi(s,x_0;u,\omega)$ is $\mathcal{B}^d\times\mathcal{B}_{[s,t]}\times\mathcal{F}_{[s,t]}/\mathcal{B}^d$ measurable.*

(ii) *for any $x_0\in\bar{D}$, $\{(x(s,x_0;t,\omega),\phi(s,x_0;u,\omega)),s\le t\}$ is a unique solution of equation* (5.7) *with initial value $x_0$.*

(iii)
$$x(s,x_0;t,\omega)=x(u,x(s,x_0;u,\omega);t,\theta_u\omega),\quad s\le u\le t.$$
*Here, for $t\ge s\ge 0$, $\theta_s\omega(t):=\omega(s+t)-\omega(s)$ , and for $s\le t$,*
$$\mathcal{F}_{[s,t]}:=\sigma\{\omega(u)-\omega(v):s\le v\le u\le t\}.$$

*Proof.* For $y\in\mathbb{R}$, set $y^k:=2^{-k}[2^ky]$. For $x_0=(x_1,\dots,x_d)\in\mathbb{R}^d$, set $x^k=(x_1^k,\dots,x_d^k)$.

We take the subsequence $\{n_k\}_k$ of $\{n\}$, such that

$$F^{-1}\Big(\frac{C_0}{c_0}F(2^{-n_k})+3\alpha\int_0^T(g(s)+\lambda(s))ds\Big)<\frac{1}{2^{2k}},$$

and take the subsequence $\{n_{k_i}\}_i$ of $\{n_k\}_k$, such that

$$|x^{n_{k_i}}-x_0|\le\frac{1}{2^{n_{k_i}}}.$$

Then by the proof of Theorem 3.1, we see that $(x(s,x^k;u,\omega),\phi(s,x^k;u,\omega))$ is a unique solution to equation (1.1) with initial value $x^k$. By Theorem 3.10, we obtain

$$\sum_{k=1}^{\infty} P(\sup_{u\le T}|x(x^k,s;u,\cdot)-x(x_0,s;u,.)|^2\ge \frac{1}{2^k})\le \sum_{k=1}^{\infty}2^k\frac{1}{2^{2k}}\le \sum_{k=1}^{\infty}\frac{1}{2^k}<\infty.$$

Then by the Borel-Cantelli lemma we obtain

$$\lim_{k\to\infty} x(s,x^k;u,.)=x(s,x_0;u,.),\quad \text{uniformly on } [s,T]\ a.s..$$

Consequently

$$\lim_{k\to\infty} \phi(s,x^k;u,.)=\phi(s,x_0;u,.),\quad \text{uniformly on } [s,T] \text{ a.s..}$$

We define

$$\tilde{x}(s,x_0;u,.)=\begin{cases}\lim_{k\to\infty} x(s,x^k;u,.), & \text{if the limit exists}\\ 0, & \text{otherwise.}\end{cases}$$

$$\tilde{\phi}(s,x_0;u,.)=\begin{cases}\lim_{k\to\infty} \phi(s,x^k;u,.), & \text{if the limit exists,}\\ 0, & \text{otherwise.}\end{cases}$$

By Remark 3.9, it is clear that for each $k$, $[s,t]\times\Omega\mapsto x(s,x^k;\cdot,\cdot)$ and $[s,t]\times\Omega\mapsto \phi(s,x^k;\cdot,\cdot)$ are $\mathcal{B}_{[s,t]}\times\mathcal{F}_{[s,t]}$ measurable with respect to $(u,\omega)$. Since $x(s,x^k;u,\cdot)$ and $\phi(s,x^k;u,\cdot)$ are piecewise constant in $x_0$ for every fixed $(u,\omega)$, then they are $\mathcal{B}^d\times\mathcal{B}_{[s,t]}\times\mathcal{F}_{[s,t]}/\mathcal{B}^d$ measurable in $(x_0,u,\omega)$. Thus, the limits of $x(s,x^k;u,\cdot)$ and $\phi(s,x^k;u,\cdot)$ have the same measurability. This shows (i).

(ii) is obvious because

$$x(s,x_0;u,.)=\tilde{x}(s,x_0;u,.),\ a.s., \text{for every } x_0,$$

and

$$\phi(s,x_0;u,\cdot)=\tilde{\phi}(s,x_0;u,\cdot),\quad \text{a.s., for every }\ x_0.$$

Form now on we denote $\tilde{x}(s,x_0;u,\cdot)$ by $x(s,x_0;u,\cdot)$.

Now we start to prove (iii). By the Lemma 5.1, there exists a progressively measurable function $\mathcal{H}$ such that

$$\mathcal{H}(s_2,y_0;t,\theta_{s_2}(\omega))=y_0+\int_{s_2}^t \sigma(u,x(s_2,y_0;u))dB(u)+\int_{s_2}^t b(u,x(s_2,y_0;u))du,\quad \forall y_0.$$

Then, since $\{x(s_2,y_0;u),\phi(s_2,y_0;u),u\ge s_2\}$ satisfies the equation (5.7), we have

$$\mathcal{H}(s_2,y_0;t,.)=x(s_2,y_0;t,.)-\phi(s_2,y_0;u),\quad \forall y_0.$$

Both $\mathcal{H}(s_2,y_0;t,\cdot)$ and $x(s_2,y_0;t,.)-\phi(s_2,y_0;t,.)$ are independent of $B_{t\wedge s_2}$, because of both $\mathcal{H}(s_2,y_0;t,.)$ and $x(s_2,y_0;t,.)-\phi(s_2,y_0;t,.)$ depending on $\omega$ though $\theta_{s_2}(\omega)$. Form $(i)$, $x(s_2,y_0;t,.)-\phi(s_2,y_0;t,.)$ is progressively measurable. Therefore we have for any $\mu\in\mathcal{F}_{s_2}$,

$$\begin{aligned}&P(\mathcal{H}(s_2,\mu;t,.)=x(s_2,\mu;t,.)-\phi(s_2,\mu;t,\cdot))\\&=\int P(\mu\in dy)P(\mathcal{H}(s_2,y;t,.)=x(s_2,y;t,.)-\phi(s_2,y;t,.))=1.\end{aligned}$$

Taking $\mu=x(s_1,x_0;s_2,.)$, thus

$$\begin{aligned}&x(s_2,x(s_1,x_0;s_2,\omega);t,\theta_{s_2})-\phi(s_2,x(s_1,x_0;s_2,\omega);t,\theta_{s_2})\\&=\mathcal{H}(s_2,x(s_1,x_0;s_2,\omega);t,\theta_{s_2})\\&=x(s_1,x_0;s_2)+\int_{s_2}^t\sigma(u,x(s_2;u))dB_u+\int_{s_2}^t b(u,x(s_2;u))du.\end{aligned}$$

But $x(s_1,x_0;t)$ satisfies the same equation. Therefore, we have by the uniqueness

$$x(s_1,x_0;t,\omega)=x(s_2,x(s_1,x_0;s_2,\omega);t,\theta_{s_2}(\omega)).$$

Then the proof is complete. □

**Corollary 5.3.** *Under the conditions of Theorem 5.2, $x(s,x_0;t,\omega)$ is a strong Markov process.*

*Proof.* First, we know that $x(s,x_0;t,\omega)$ is a Markov process. Indeed, for any $x_0$, any $s_1 < s_2 < t$, and any bounded Borel function $f$ we have

$$\begin{aligned} E[f(x(s_1,x_0;t))|\mathcal{F}_{s_2}] &= E[f(x(s_2,x(s_1,x_0;s_2);t))|\mathcal{F}_{s_2}] \\ &= E[f(x(s_2,y;t))]|_{y=x(s_1,x_0;s_2)} \\ &= E[f(x(s_2,x(s_1,x_0;s_2);t))|x(s_1,x_0;s_2)] \\ &= E[f(x(s_1,x_0;t))|x(s_1,x_0;s_2)]. \end{aligned}$$

Replacing $s_2$ by a stopping time and using the fact that the Brownian motion starts afresh at a stopping time, one can show that $x(s,x_0;t,\omega)$ is a strong Markov process by the same arguments. □

## 6. Rate of convergence of Euler's approximations

In this section, we investigate the characterization of convergence rate of Euler's approximations of equation (1.1) under more restrictions on the coefficients $b$ and $\sigma$. For simplicity, in this section we let $C_1$ stand for a positive constant depending only on some determinate parameters and its value may be different in different appearance.

First, we introduce the following assumptions:

(A7) Suppose that $\sigma$ and $b$ are bounded and satisfy

$$\begin{gathered} |b(t,x)-b(t,y)| \le |x-y|\tilde{K}(|x-y|), \\ \|\sigma(t,x)-\sigma(t,y)\|^2 \le |x-y|^2\tilde{K}(|x-y|), \end{gathered}$$

where $\tilde{K}(x) = \hat{C}(\log(\frac{1}{x}) \vee \tilde{C})^{\frac{1}{\hat{\beta}}}$ for some $\hat{\beta} > 2$ and the constants $\hat{C}, \tilde{C} > 0$.

The assumptions above arise in the study of Brownian motion on the group of diffeomorphisms of the circle (cf. [2, 6]). For $0 < \hat{\eta} < \frac{1}{e}$, we define a concave function

$$\varrho_{\hat{\eta}}(x) = \begin{cases} x\log x^{-1}, & x \le \hat{\eta}, \\ (\log \hat{\eta}^{-1} - 1)x + \hat{\eta}, & x > \hat{\eta}. \end{cases}$$

**Remark 6.1** ([10, Example 2.2])**.** Taking the function $\rho(x) = \varrho_{\hat{\eta}}(x)$ in Lemma 3.3, we can describe the Bihari's inequality as follows: If $h(s)$, $\chi(s)$ are two strictly positive functions on $\mathbb{R}^+$ such that

$$\chi(t) \le \vartheta + \int_0^t \varrho_{\hat{\eta}}(\chi(s))h(s)ds, \quad t \ge 0,$$

then

$$\chi(t) \le \vartheta^{\exp(-\int_0^t h(s)ds)}. \tag{6.1}$$

**Lemma 6.2.** *Assume that $b$ and $\sigma$ are bounded. Then for any $p \ge 1$ and for any $T \in [0,+\infty)$, there is a constant $C_1 > 0$ such that for all $n \in \mathbb{N}$ and $s,t \in [0,T]$,*

$$E[\|x_n\|_{\infty,[s,t]}|^{2p}] + E[\|x\|_{\infty,[s,t]}|^{2p}] \le C_1|t-s|^p.$$

*Proof.* If $\sigma$ is bounded, then the quadratic variation of $M(t) := \int_0^t \sigma(s,x(s))dB(s)$ is bounded and $M^n(t) := \int_0^t \sigma(s,x_n(\pi_n(s)))dB(s)$ is a martingale whose quadratic variation is uniformly bounded for $n$. Thus we see that

$$\begin{gathered} E[\exp(\max_{0\le t\le T}|M(t)|)] < \infty, \\ \sup_n E[\exp(\hat{a}\max_{0\le t\le T}|M^n(t)|)] < \infty, \end{gathered}$$

for all constant $\hat{a} > 0$. The assertion follows from [1, Lemmas 2.3 and 2.7] and the Burkholder-Davis-Gundy inequality. □

**Theorem 6.3.** *Assume* (A1)–(A3) *and* (A7). *Then for any* $T \in [0, +\infty)$, *there exist constants* $C_1 > 0$ *and* $\hat{\delta} > 1$ *such that*

$$E[\sup_{0\le t\le T} |x_n(t, x_0) - x(t, x_0)|^4] \le C_1 \hat{\delta}^{-n},$$

*where* $n \ge N_0$ *is sufficiently large.*

*Proof.* In the proof of Proposition 3.7, if we take $\Gamma_{m,n}(R) = \infty$, then we have

$$z_t := \sum_{i=1}^{6} I_i,$$

and $I_4 \le 0$. Thus

$$\sup_{0\le s\le t} |x_n(s) - x_m(s)|^4 \le C_1 \sum_{i=1, i\ne 4}^{6} \sup_{0\le s\le t} I_i^2.$$

By the boundness of $\sigma$ and $b$ and Burkholder-Davis-Gundy inequality, we have the following estimates

$$\begin{aligned}
&E[\sup_{0\le s\le t} I_1^2 + \sup_{0\le s\le t} I_3^2 + \sup_{0\le s\le t} I_5^2] \\
&\le C_1 E\Big[\int_0^t |x_n(s) - x_m(s)|^2 \|\sigma(s, x_n(\pi_n(s))) - \sigma(s, x_m(\pi_m(s)))\|^2 + |x_n(s) - x_m(s)|^4 ds\Big] \\
&\le C_1 E\Big[\int_0^t \|\sigma(s, x_n(\pi_n(s))) - \sigma(s, x_m(\pi_m(s)))\|^4 + |x_n(s) - x_m(s)|^4 ds\Big],
\end{aligned}$$

$$\begin{aligned}
E[\sup_{0\le s\le t} I_2^2] &\le C_1 E\Big[\int_0^t |x_n(s) - x_m(s)|^2 |b(s, x_n(\pi_n(s))) - b(s, x_m(\pi_m(s)))|^2 \\
&\quad + \|\sigma(s, x_n(\pi_n(s))) - \sigma(s, x_m(\pi_m(s)))\|^4 ds\Big],
\end{aligned}$$

$$\begin{aligned}
E[\sup_{0\le s\le t} I_6^2] &\le C_1 E\Big[(\int_0^t \|\sigma(s, x_n(\pi_n(s))) - \sigma(s, x_m(\pi_m(s)))\| |x_n(s) - x_m(s)| ds)^2\Big] \\
&\le C_1 E\Big[\int_0^t |x_n(s) - x_m(s)|^4 + \|\sigma(s, x_n(\pi_n(s))) - \sigma(s, x_m(\pi_m(s)))\|^4 ds\Big].
\end{aligned}$$

Combining this with (A7), we can write

$$\begin{aligned}
E[\sup_{0\le s\le t} |x_n(s) - x_m(s)|^4] &\le C_1 \sum_{i=1, i\ne 4}^{6} E[\sup_{0\le s\le t} I_i^2] \\
&\le C_1 E\Big[\int_0^t |x_n(s) - x_m(s)|^4 + \|\sigma(s, x_n(\pi_n(s))) - \sigma(s, x_m(\pi_m(s)))\|^4 \\
&\quad + |x_n(t) - x_m(t)|^2 |b(s, x_n(\pi_n(s))) - b(s, x_m(\pi_m(s)))|^2 ds\Big] \\
&\le C_1 E\Big[\int_0^t |x_n(s) - x_m(s)|^4 + \|\sigma(s, x_n(s)) - \sigma(s, x_m(s))\|^4 \\
&\quad + \|\sigma(s, x_n(s)) - \sigma(s, x_n(\pi_n(s)))\|^4 + \|\sigma(s, x_m(s)) - \sigma(s, x_m(\pi_m(s)))\|^4 \\
&\quad + |x_n(s) - x_m(s)|^2 |b(s, x_n(s)) - b(s, x_m(s))|^2 \\
&\quad + \|b(s, x_n(s)) - b(s, x_n(\pi_n(s)))\|^4 + \|b(s, x_m(s)) - b(s, x_m(\pi_m(s)))\|^4 ds\Big] \\
&\le C_1 E\Big[\int_0^t |x_n(s) - x_m(s)|^4 \tilde{K}^2(|x_n(s) - x_m(s)|) \\
&\quad + |x_m(s) - x_m(\pi_m(s))|^4 \tilde{K}^2(|x_m(s) - x_m(\pi_m(s))|) \\
&\quad + |x_n(s) - x_n(\pi_n(s))|^4 \tilde{K}^2(|x_n(s) - x_n(\pi_n(s))|) \\
&\quad + |x_m(s) - x_m(\pi_m(s))|^4 \tilde{K}^4(|x_m(s) - x_m(\pi_m(s))|)
\end{aligned}$$

$$
\begin{aligned}
&+|x_n(s)-x_n(\pi_n(s))|^4\tilde{K}^4(|x_n(s)-x_n(\pi_n(s))|)\\
&+|x_n(s)-x_m(s)|^4ds\Big].
\end{aligned}
$$

Since

$$
\lim_{x\downarrow 0}\frac{C_p(x^4\tilde{K}^2(x)+x^4)}{x^4\ln x^{-4}}=0,\quad\text{and}\quad \lim_{x\downarrow 0}\frac{C_px^4\tilde{K}^4(x)}{(x^2\ln x^{-2})^2}=0,
$$

there exists a constant $\hat{\eta}>0$ such that

$$
\begin{aligned}
C_px^4&\le \varrho_{\hat{\eta}}(x^4),\\
C_px^4\tilde{K}^4(x)&\le \varrho^2_{\hat{\eta}}(x^2),\\
C_px^4\tilde{K}^2(x)&\le \varrho_{\hat{\eta}}(x^4).
\end{aligned}
$$

Furthermore, by Jensen's inequality and Lemma 6.2, we have

$$
\begin{aligned}
E[\sup_{0\le s\le t}|x_n(s)-x_m(s)|^4]&\le C_1E\Big[\int_0^t \varrho_{\hat{\eta}}(|x_n(s)-x_m(s)|^4)+\varrho_{\hat{\eta}}(|x_n(s)-x_n(\pi_n(s))|^4)\\
&\quad+\varrho_{\hat{\eta}}(|x_m(s)-x_m(\pi_m(s))|^4)+\varrho^2_{\hat{\eta}}(|x_n(s)-x_n(\pi_n(s))|^2)\\
&\quad+\varrho^2_{\hat{\eta}}(|x_m(s)-x_m(\pi_m(s))|^2)ds\Big]\\
&\le C_1\int_0^t \varrho_{\hat{\eta}}(E[|x_n(s)-x_n(\pi_n(s))|^4])+\varrho_{\hat{\eta}}(E[|x_m(s)-x_m(\pi_m(s))|^4])\\
&\quad+E[\varrho^2_{\hat{\eta}}(|x_n(s)-x_n(\pi_n(s))|^2)]+E[\varrho^2_{\hat{\eta}}(|x_m(s)-x_m(\pi_m(s))|^2)]\\
&\quad+\varrho_{\hat{\eta}}(E[\sup_{0\le u\le s}|x_n(u)-x_m(u)|^4])ds\\
&\le C_1(2^{-2\hat{\alpha}n}T+2^{-2\hat{\alpha}m}T+\int_0^t\varrho_{\hat{\eta}}(E[\sup_{0\le u\le s}|x_n(u)-x_m(u)|^4])ds),
\end{aligned}
$$

where $0<\hat{\alpha}<1$ and $n,m$ are sufficiently large. The last inequality above is due to $\varrho_{\hat{\eta}}(x)\le C_1(x^{\hat{\alpha}}+x)$, for $x>0$. It follows from Remark 6.1 that

$$
E[\sup_{0\le s\le t}|x_n(s)-x_m(s)|^4]\le C_1(2^{-2\alpha n}T+2^{-2\alpha m}T)^{\exp(-C_1T)}. \tag{6.2}
$$

Hence, $x_n(t)$ converges in probability, as $n\to\infty$, to a continuous stochastic process $x(t)$, uniformly in $t\in[0,T]$. Clearly

$$
\sup_m E[\sup_{0\le s\le t}|x_n(s)-x_m(s)|^4]\le C_1(2^{-2\alpha n}T+T)^{\exp(-C_1T)}<\infty.
$$

Thus, letting $m\to\infty$ in (6.2) completes the proof. $\square$

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