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NORMALIZED GROUND STATE SOLUTIONS FOR KIRCHHOFF EQUATION WITH SUBCRITICAL OR CRITICAL PERTURBATION

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ABSTRACT. This article concerns the Kirchhoff equation with subcritical or critical perturbation. We establish the existence of normalized ground state solutions by using Pohozaev manifold and subcritical approximation methods.

1. Introduction and main results

In this article, we study the Kirchhoff equation with subcritical or critical perturbation

$$\begin{gathered} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x)\Delta u+\lambda u=\eta|u|^{p-2}u+|u|^{q-2}u \quad \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3}|u|^2\mathrm{d}x=c, \quad u\in H^1(\mathbb{R}^3), \end{gathered} \tag{1.1}$$

where $a,b>0$ are two constants, $2<p<q\le 2^*=6$, $c,\eta>0$, $\lambda\in\mathbb{R}$. In the previous two decades, a large number of scholars have studied the existence of nontrivial solutions of the following Kirchhoff type equation

$$-(a+b\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x)\Delta u+V(x)u=f(x,u),$$

see [1, 16, 6, 7, 2, 24, 4]. In these references, Alves, Perera and He proved the existence results of the nontrivial solutions by the variational methods. Zhang [24] considered one Kirchhoff equation with critical Sobolev exponent. By Nehari manifold, Chen [4] established the existence of positive solutions to (1.1) for $1<p<2<q<2^*$ in a bounded domain.

Recently, more attention is paid to the existence of solutions to nonlinear Schrödinger (NLS) equation with prescribed mass,

$$\int_{\mathbb{R}^N}|u|^2\mathrm{d}x=c.$$

Such solution is usually called a normalized solution. Soave [17, 18] proved the existence of normalized ground states and properties of ground states for NLS equation

$$\begin{gathered} -\Delta u-\lambda u=\mu|u|^{q-2}u+|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N}|u|^2\mathrm{d}x=a^2, \quad u\in H^1(\mathbb{R}^N), \end{gathered}$$

where $2<q<p\le 2^*$. Yao [21] studied normalized solutions for the Choquard equations with lower critical exponent

$$\begin{gathered} -\Delta u+\lambda u=\gamma(I_\alpha*|u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u+\mu|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N}|u|^2\mathrm{d}x=c, \quad u\in H^1(\mathbb{R}^N), \end{gathered}$$

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where $\alpha \in (0, N),\ 2 < q \leq 2N/(N-2)$. By employing the Sobolev subcritical approximation method, Li, Nie and Zhang [10] studied the problem (1.1) for η sufficiently large and obtained the existence of normalized ground states. Kong and Chen [8] studied (1.1) in for the case $N = 4$. By decomposing Pohozaev manifold and constructing fiber map, they proved the existence of a positive normalized ground state. For more references on normalized solution of Kirchhoff equation and Schrödinger equation, we refer to [5, 9, 11, 12, 22, 23, 25, 26].

Motivated by the above papers, we examine the existence of normalized solutions for Kirchhoff equation in the subcritical case $2 < q < 2^*$ and the critical case $q = 2^*$. For subcritical case, by using the constrained minimization method on a suitable submanifold and using Schwartz symmetrization rearrangements we obtain the existence of normalized ground state solutions. For critical case, the main difficulty in solving the problem (1.1) is to study the estimate of Mountain pass Level and prove $\inf I_q(u)$ is attained in a suitable submanifold. Firstly we compare the zeros of some functions and study the Mountain pass Level, then we construct an increasing sequence to get the properties of $\inf I_q(u)$. Lately in the aid of subcritical approximation method, we prove the existence of normalized ground state solutions for $q = 2^*$ for any $\eta > 0$.

Let $E = H^1(\mathbb{R}^3)$ be the usual Sobolev space equipped with the norm

$$\|u\| = \Big(\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)\mathrm{d}x\Big)^{1/2}.$$

The norm of the space $L^j(\mathbb{R}^3)(1 < j \leq 2^*)$ is defined as

$$\|u\|_j = \Big(\int_{\mathbb{R}^3} |u|^j \mathrm{d}x\Big)^{1/j}.$$

Moreover, $E_r = H^1_{rad}(\mathbb{R}^3)$ denotes the subspace of radial functions

$$E_r = H^1_{rad}(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) : u(x) = u(|x|)\}.$$

From [20] we know that the embedding $E_r \to L^j(\mathbb{R}^3)(2 < j \leq 2^*)$ is continuous and compact for $2 < j < 2^*$.

For $0 \neq u \in D^{1,2}(\mathbb{R}^3)$, we define

$$S_0 = \inf \frac{\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x}{\big(\int_{\mathbb{R}^3} |u|^{2^*} \mathrm{d}x\big)^{2/2^*}}.$$

We now introduce the main results in this article.

Theorem 1.1. *Assume that $c > 0$, $\frac{14}{3} \leq p < q < 2^*$, then problem* (1.1) *admits at least a couple of ground state solution $(u, \lambda) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ for any $\eta > 0$. Moreover, the solution is real-valued positive and radially symmetric non-increasing function with $\lambda > 0$.*

Theorem 1.2. *Assume that $c > 0$, $\frac{14}{3} \leq p < q$, $q = 2^*$, then problem* (1.1) *admits at least a couple of ground state solution $(u, \lambda) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ for any $\eta > 0$. Moreover, the solution is real-valued positive and radially symmetric non-increasing function with $\lambda > 0$.*

We set

$$\gamma_p = \frac{3(p-2)}{2p}, \quad \gamma_q = \frac{3(q-2)}{2q}.$$

Remark 1.3. Direct calculations show that under the assumption of Theorem 1.1 or Theorem 1.2, it results that $p > 2$, $q > 2$, $q\gamma_q > p\gamma_p \geq 4$.

2. Variational framework and technical lemmas

In this section, we assume p, q satisfy the conditions of Theorem 1.1 or Theorem 1.2. By Remark 1.3 we know that $q > p > 2$ and $q\gamma_q > p\gamma_p \geq 4$.

The functional associated to (1.1) is

$$I_q(u) = \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\Big)^2 - \eta\frac{1}{p}\int_{\mathbb{R}^3} |u|^p \mathrm{d}x - \frac{1}{q}\int_{\mathbb{R}^3} |u|^q \mathrm{d}x$$

$$= \frac{1}{2}A(u) + \frac{1}{4}D(u) - \frac{1}{p}L(u) - \frac{1}{q}K(u)$$

with the constrain

$$S_c = \{u \in E : \int_{\mathbb{R}^3} |u|^2 \mathrm{d}x = c\}.$$

Where

$$A(u) = a\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x, \quad D(u) = b\Big(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\Big)^2,$$
$$L(u) = \eta \int_{\mathbb{R}^3} |u|^p \mathrm{d}x, \ K(u) = \int_{\mathbb{R}^3} |u|^q \mathrm{d}x.$$

The Gagliardo-Nirenberg inequality can be founded in [19]:

$$\|u\|_q \leq C(q)\|\nabla u\|_2^{\gamma_q}\|u\|_2^{1-\gamma_q}, \quad 2 < q < 2^*. \tag{2.1}$$

If $q = 2^*$, by using the definition of S_0 we obtain

$$\int_{\mathbb{R}^3} |u|^{2^*} \mathrm{d}x \leq S_0^{-2^*/2}\|\nabla u\|_2^{2^*}. \tag{2.2}$$

Since the embedding $H^1(\mathbb{R}^3) \to L^j(\mathbb{R}^3)(2 < j \leq 2^*)$ is continuous, then we deduce that $I_q \in C^1(E, R)$. For $u \in E$, we define

$$G_q(u) = A(u) + D(u) - \gamma_p L(u) - \gamma_q K(u).$$

If $u \in E$ is a solution to the first equation of (1.1),

$$-(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x)\Delta u + \lambda u = \eta|u|^{p-2}u + |u|^{q-2}u \quad \text{in } \mathbb{R}^3, \tag{2.3}$$

then

$$A(u) + \lambda B(u) + D(u) = L(u) + K(u),$$

where

$$B(u) = \int_{\mathbb{R}^3} u^2 \mathrm{d}x.$$

Moreover, by using an argument similar to [15, 14] we have the identity

$$\frac{1}{2}A(u) + \lambda\frac{3}{2}B(u) + \frac{1}{2}D(u) - \frac{3}{p}L(u) - \frac{3}{q}K(u) = 0.$$

Hence if $u \in E$ is a solution to (2.3), then u satisfies the Pohozaev equality

$$G_q(u) = A(u) + D(u) - \gamma_p L(u) - \gamma_q K(u) = 0. \tag{2.4}$$

Suppose that $u \in E$, by computations we obtain

$$\langle I_q'(u), \phi\rangle = (a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x)\int_{\mathbb{R}^3} \nabla u \nabla\phi \mathrm{d}x - \eta\int_{\mathbb{R}^3} |u|^{p-2}u\phi \mathrm{d}x - \int_{\mathbb{R}^3} |u|^{q-2}u\phi \mathrm{d}x$$

for every $\phi \in E$. Hence the normalized solutions of problem (1.1) are the critical points of the energy functional I_q under the constrain S_c.

We say that u_c is a ground state of (1.1) on S_c if $(u_c, \lambda_c) \in E \times \mathrm{R}$ is a normalized solution to (1.1) and u_c has the minimal energy among all nontrivial solutions belonging to S_c, that is,

$$I_q(u) = \inf\{I_q(v) : v \in S_c, (I_q\mid_{S_c})'(v) = 0\}.$$

Let $u \in E$, and $u^t = t^{3/2}u(tx),\ t > 0$. By computations we obtain

$$h(t) = I_q(u^t) = \frac{at^2}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{bt^4}{4}\Big(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\Big)^2 - \eta\frac{t^{p\gamma_p}}{p}\int_{\mathbb{R}^3} |u|^p \mathrm{d}x - \frac{t^{q\gamma_q}}{q}\int_{\mathbb{R}^3} |u|^q \mathrm{d}x.$$

Then $I_q(u^t) \to -\infty$ as $t \to +\infty$, I_q is not bounded from below.

Consider the Pohozaev set

$$M_{c,q} = \big\{u \in S_c : G_q(u) = 0\big\}.$$

From (2.4) we deduce that if $u \in E$ is a solution to (1.1), then $u \in M_{c,q}$, that is, any critical points of $I_q |_{S_c}$ stay in $M_{c,q}$. By computations we obtain

$$h'(t) = tA(u) + t^3 D(u) - \gamma_p t^{p\gamma_p - 1} L(u) - \gamma_q t^{q\gamma_q - 1} K(u).$$

It is clear that $G_q(u) = 0$ if and only if $h'(1) = 0$. Next, we claim that $M_{c,q}$ is a natural constraint.

Lemma 2.1. *The following results hold:*

(i) *$M_{c,q}$ is a smooth manifold of codimension 1 in S_c;*
(ii) *If $u \in M_{c,q}$ is a critical point of $I_q |_{M_{c,q}}$, then u is a critical point of $I_q |_{S_c}$.*

Proof. (1) Note that

$$M_{c,q} = \{u \in E : G_q(u) = 0, G_1(u) = 0\},$$

where

$$G_1(u) = \int_{\mathbb{R}^3} |u|^2 \mathrm{d}x - c.$$

Then it is easy to see that $G_q(u)$, $G_1(u)$ are of class C^1 in E. If $G_q'(u) = 0$, then by the Lagrange multipliers rule there exists some $\lambda_1 \in \mathbb{R}$ and $u \in M_{c,q}$ such that

$$G_q'(u) + \lambda_1 {G_1}'(u) = 0,$$

which implies that

$$-2a\Delta u - 4b\Big(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\Big)\Delta u + 2\lambda_1 u = \eta p\gamma_p |u|^{p-2}u + q\gamma_q |u|^{q-2}u.$$

By Pohozaev equality (2.4) we deduce that

$$2A(u) + 4D(u) = p\gamma_p^2 L(u) + q\gamma_q^2 K(u),$$

then

$$-2A(u) + \gamma_p(4 - p\gamma_p)L(u) + \gamma_q(4 - q\gamma_q)K(u) = 0.$$

This contradicts Remark 1.3. Hence we obtain that $G_q'(u) \neq 0$ and $M_{c,q}$ is a smooth manifold of codimension 1 in S_c.

(2) If $u \in M_{c,q}$ is a critical point of $I_q |_{M_{c,q}}$, by the Lagrange multipliers rule there exists some $\lambda, \lambda_2 \in \mathbb{R}$ such that for any $\varphi \in E$,

$$\langle I_q'(u), \varphi\rangle + \lambda \int_{\mathbb{R}^3} u\varphi \mathrm{d}x + \lambda_2 \langle G_q'(u), \varphi\rangle = 0,$$

then u satisfies the equation

$$\begin{aligned} &-(2\lambda_2 + 1)a\Delta u - (4\lambda_2 + 1)b\left(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)\Delta u + \lambda u \\ &= \eta(\lambda_2 p\gamma_p + 1)|u|^{p-2}u + (\lambda_2 q\gamma_q + 1)|u|^{q-2}u. \end{aligned}$$

From Pohozaev equality (2.4) we deduce that

$$(2\lambda_2 + 1)A(u) + (4\lambda_2 + 1)D(u) = \gamma_p(\lambda_2 p\gamma_p + 1)L(u) + \gamma_q(\lambda_2 q\gamma_q + 1)K(u). \tag{2.5}$$

Since $G_q(u) = 0$, then from (2.4) and (2.5) we obtain

$$\lambda_2(-2A(u) - (p\gamma_p - 4)\gamma_p L(u) - (q\gamma_q - 4)\gamma_q K(u)) = 0.$$

Then we obtain that $\lambda_2 = 0$, the proof is complete. □

Lemma 2.2. *For any $u \in S_c$, there exists a unique t_0 such that $u^{t_0} \in M_{c,q}$ and*

$$I_q(u^{t_0}) = \max_{t>0} I_q(u^t).$$

Moreover, if $G_q(u) \leq 0$, then $0 < t_0 \leq 1$.

Proof. Suppose that $u \in S_c$, then

$$
\begin{aligned}
h(t) = I_q(u^t) &= \frac{t^2}{2}A(u) + \frac{t^4}{4}D(u) - \frac{t^{p\gamma_p}}{p}L(u) - \frac{t^{q\gamma_q}}{q}K(u),\\
h'(t) &= tA(u) + t^3D(u) - \gamma_p t^{p\gamma_p-1}L(u) - \gamma_q t^{q\gamma_q-1}K(u),\\
&= t^3(t^{-2}A(u) + D(u) - \gamma_p t^{p\gamma_p-4}L(u) - \gamma_q t^{q\gamma_q-4}K(u)) \quad = t^3 g(t).
\end{aligned}
$$

Then it is clear that $h(0) = 0$ and $h(t) \to -\infty$ as $t \to +\infty$. There exists t_0 such that $h'(t) > 0$ for $t \in (0, t_0)$ and $h'(t) < 0$ for $t \in [t_0, t_1),\ t_1 > t_0 > 0$, it follows that $h(t)$ achieves its local maximum at $t = t_0$ and $h'(t_0) = 0$. Since $p\gamma_p \geq 4$ and $q\gamma_q > 4$, then $g(t)$ is strictly decreasing. On the other hand, we have

$$
\{t > 0 : h'(t) = 0\} = \{t > 0 : g(t) = 0\},
$$

which implies that $h(t)$ has a positive unique critical point which corresponds to its maximum. By computations we have

$$
\begin{aligned}
G_q(u^{t_0}) &= t_0[at_0\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x + bt_0^3\Big(\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\Big)^2 - \gamma_p t_0^{p\gamma_p-1}L(u) - \gamma_q t_0^{q\gamma_q-1}K(u)]\\
&= t_0 h'(t_0).
\end{aligned}
$$

From $h'(t_0) = 0$, we obtain that $G_q(u^{t_0}) = 0$ and $u^{t_0} \in M_{c,q}$.

Suppose $G_q(u) \leq 0$, we claim that $0 < t_0 \leq 1$. Assume by contradiction that $t_0 > 1$, using that $G_q(u^{t_0}) = 0$ we obtain

$$
\begin{aligned}
0 = G_q(u^{t_0}) &= t_0^4[t_0^{-2}A(u) + D(u) - \gamma_p t_0^{p\gamma_p-4}L(u) - \gamma_q t_0^{q\gamma_q-4}K(u)]\\
&< t_0^4[A(u) + D(u) - \gamma_p L(u) - \gamma_q K(u)]\\
&= t_0^4 G_q(u) \leq 0,
\end{aligned}
$$

which is a contradiction, then $0 < t_0 \leq 1$. This completes the proof. $\square$

Let $u \in M_{c,q}$, we obtain

$$
I_q(u) = I_q(u) - \frac{1}{4}G_q(u) = \frac{1}{4}A(u) + (\frac{\gamma_p}{4} - \frac{1}{p})L(u) + (\frac{\gamma_q}{4} - \frac{1}{q})K(u). \tag{2.6}
$$

In view of Remark 1.3, we deduce that $I_q(u) > 0$ and $I_q(u)$ is bounded below on $M_{c,q}$.

Since that $I_q(u)$ is bounded below on $M_{c,q}$, we may define

$$
m(c,q) = \inf_{u \in M_{c,q}} I_q(u).
$$

Lemma 2.3. *There exists $C_1 = C_1(a, p, q, \eta) > 0$ such that $m(c, q) > C_1$.*

Proof. Let $u \in M_{c,q}$. Then

$$
A(u) + D(u) = \gamma_p L(u) + \gamma_q K(u) \geq a_0\|\nabla u\|_2^2, \tag{2.7}
$$

where $a_0 = \min\{1, a\}$. Using (2.1)-(2.2) and (2.7) we can deduce that

$$
\begin{gathered}
a_0 \leq \eta C(p)\gamma_p\|\nabla u\|_2^{p\gamma_p-2}c^{\frac{p(1-\gamma_p)}{2}} + C(q)\gamma_q\|\nabla u\|_2^{q\gamma_q-2}c^{\frac{q(1-\gamma_q)}{2}},\ 2 < q < 2^*,\\
a_0 \leq \eta C(p)\gamma_p\|\nabla u\|_2^{p\gamma_p-2}c^{\frac{p(1-\gamma_p)}{2}} + \gamma_q S_0^{\frac{-2^*}{2}}\|\nabla u\|_2^{2^*-2},\ q = 2^*.
\end{gathered}
$$

Then there exists $C > 0$ such that $\|\nabla u\|_2^2 > C > 0$. It follows from (2.6)-(2.7) that

$$
I_q(u) \geq \frac{1}{4}A(u) = \frac{a}{4}\|\nabla u\|_2^2. \tag{2.8}
$$

Then there exists $C_1 = C_1(a, p, q, \eta) > 0$ such that $m(c, q) > C_1$. $\square$

3. Proof of Theorem 1.1

In this section, we study the subcritical case $\frac{14}{3} \le p < q < 2^*$. From Remark 1.3 we have $q > p > 2$, $q\gamma_q > p\gamma_p \ge 4$. Let $\{u_n\} \subset M_{c,q}$ be a minimizing sequence for $I_q(u)$, that is

$$\lim_{n\to\infty} I_q(u_n) = \inf_{u\in M_{c,q}} I_q(u) = m(c,q).$$

Then we have the following Lemma.

Lemma 3.1. *Under the assumption of Theorem 1.1, $m(c,q)$ is attained by a real- valued positive and radially symmetric non-increasing function.*

Proof. Let $v_n = |u_n|^*$ be the Schwartz symmetrization rearrangement of $|u_n|$. Then by [13] we obtain

$$A(v_n) \le A(u_n), \quad D(v_n) \le D(u_n), \quad B(v_n) = B(u_n) = c, \tag{3.1}$$

$$K(v_n) = K(u_n), \quad L(v_n) = L(u_n). \tag{3.2}$$

Therefore,

$$I_q(v_n) \le I_q(u_n), \quad G_q(v_n) \le G_q(u_n) = 0.$$

Since $v_n \in S_c$, then by Lemma 2.2 there exists a unique $0 < t_n \le 1$ such that $\{v_n^{t_n}\} \subset M_{c,q}$, where $v_n^{t_n} = t_n^{3/2} v_n(t_n x)$.

From (3.1) and (3.2) we deduce that

$$\begin{aligned} m(c,q) \le I_q(v_n^{t_n}) &= I_q(v_n^{t_n}) - \frac{1}{p\gamma_p} G_q(v_n^{t_n}) \\ &= (\frac{1}{2} - \frac{1}{p\gamma_p}) t_n^2 A(v_n) + (\frac{1}{4} - \frac{1}{p\gamma_p}) t_n^4 D(v_n) + (\frac{\gamma_q}{p\gamma_p} - \frac{1}{q}) t_n^{q\gamma_q} K(v_n) \\ &\le (\frac{1}{2} - \frac{1}{p\gamma_p}) A(u_n) + (\frac{1}{4} - \frac{1}{p\gamma_p}) D(u_n) + (\frac{\gamma_q}{p\gamma_p} - \frac{1}{q}) K(u_n) \\ &= I_q(u_n) - \frac{1}{p\gamma_p} G_q(u_n) = I_q(u_n) \to m(c,q) \end{aligned}$$

as $n \to \infty$. Then we obtain that

$$I_q(v_n^{t_n}) \to m(c,q), \ n \to \infty.$$

Hence $\{v_n^{t_n}\}$ is a minimizing sequence for $I_q(u)$.

We claim $\{v_n^{t_n}\}$ is bounded in E. Otherwise, if $\|v_n^{t_n}\| \to \infty$ as $n \to \infty$, then by (2.8) we have that $I_q(v_n^{t_n}) \to \infty$, that is a contradiction. Therefore $\{v_n^{t_n}\}$ is bounded in E. We can extract a subsequence(still denoted $\{v_n^{t_n}\}$) and $v \in E$ such that

$$\begin{gathered} v_n^{t_n} \rightharpoonup v \quad \text{in } E_r; \\ v_n^{t_n} \to v \quad \text{in } L^i(\mathbb{R}^3)(2 < i < 2^*), \\ v_n^{t_n} \to v \quad \text{a.e. in } \mathbb{R}^3 \end{gathered}$$

as $n \to \infty$. Therefore we obtain $K(v_n^{t_n}) \to K(v)$, $L(v_n^{t_n}) \to L(v)$. On the other hand, by Lemma 2.3, we have

$$\begin{aligned} &0 < C_1 \\ &< m(c,q) \\ &\le I_q(v_n^{t_n}) \\ &= I_q(v_n^{t_n}) - \frac{1}{2} G_q(v_n^{t_n}) \\ &= -\frac{1}{4} t_n^4 D(v_n^{t_n}) + (\frac{\gamma_p}{2} - \frac{1}{p}) t_n^{p\gamma_p} L(v_n^{t_n}) + (\frac{\gamma_q}{2} - \frac{1}{q}) t_n^{q\gamma_q} K(v_n^{t_n}) \\ &\le (\frac{\gamma_p}{2} - \frac{1}{p}) L(v_n^{t_n}) + (\frac{\gamma_q}{2} - \frac{1}{q}) K(v_n^{t_n}) \end{aligned}$$

$$\to (\frac{\gamma_p}{2}-\frac{1}{p})L(v)+(\frac{\gamma_q}{2}-\frac{1}{q})K(v),$$

which implies that $v\neq 0$. Moreover, by Fatou's Lemma,

$$\begin{aligned} G_q(v)&=A(v)+D(v)-\gamma_pL(v)-\gamma_qK(v)\\ &\le \liminf_{n\to\infty}[A(v_n^{t_n})+D(v_n^{t_n})]-\gamma_p\lim_{n\to\infty}L(v_n^{t_n})-\gamma_q\lim_{n\to\infty}K(v_n^{t_n})\\ &=\liminf_{n\to\infty}G_q(v_n^{t_n})=0, \end{aligned}$$

which implies that

$$G_q(v)\le 0. \tag{3.3}$$

Next we shall prove that

$$\int_{\mathbb{R}^3}|v|^2\mathrm{d}x=c.$$

Set

$$0<\int_{\mathbb{R}^3}|v|^2\mathrm{d}x=c_1\le c,\quad \xi=\frac{c_1}{c},\quad v_1(x)=\xi^{\frac{1}{q-2}}v(\xi^{\frac{q}{3q-6}}x).$$

Note that $2<p<q<6$. Then

$$A(v_1)=\xi^{\frac{6-q}{3q-6}}A(v)\le A(v),\ B(v_1)=c,\ D(v_1)\le D(v), \tag{3.4}$$

$$L(v_1)=\xi^{\frac{p-q}{q-2}}L(v)\ge L(v),\ K(v_1)=K(v). \tag{3.5}$$

It follows from (3.3)-(3.5) that

$$G_q(v_1)\le G_q(v)\le 0.$$

From Lemma 2.2, there exists a unique $0<\tau\le 1$ such that $G_q(v_1^\tau)=0$ and $v_1^\tau\in M_{c,q}$. Then by using (3.4) and (3.5) we deduce that

$$\begin{aligned} m(c,q)&\le I_q(v_1^\tau)=I_q(v_1^\tau)-\frac{1}{p\gamma_p}G(v_1^\tau)\\ &=(\frac12-\frac{1}{p\gamma_p})\tau^2A(v_1)+(\frac14-\frac{1}{p\gamma_p})\tau^4D(v_1)+(\frac{\gamma_q}{p\gamma_p}-\frac1q)\tau^{q\gamma_q}K(v_1)\\ &\le(\frac12-\frac{1}{p\gamma_p})A(v)+(\frac14-\frac{1}{p\gamma_p})D(v)+(\frac{\gamma_q}{p\gamma_p}-\frac1q)K(v)\\ &\le(\frac12-\frac{1}{p\gamma_p})\liminf_{n\to\infty}A(v_n^{t_n})+(\frac14-\frac{1}{p\gamma_p})\liminf_{n\to\infty}D(v_n^{t_n})+(\frac{\gamma_q}{p\gamma_p}-\frac1q)\lim_{n\to\infty}K(v_n^{t_n})\\ &=\liminf_{n\to\infty}[I_q(v_n^{t_n})-\frac{1}{p\gamma_p}G_q(v_n^{t_n})]=m(c,q). \end{aligned} \tag{3.6}$$

Which implies that $I_q(v_1^\tau)=m(c,q)$. We claim that $\tau=1,\ \xi=1$. Otherwise, if $0<\tau<1$ or $0<\xi<1$, we replace $\le$ with $<$ in (3.6) and get a contradiction. Therefore we deduce that

$$I_q(v_1^\tau)=m(c,q),\ \tau=1,\ v_1=v,\ c=c_1.$$

We deduce that $m(c,q)$ is attained by a real-valued nonnegative and radially symmetric non-increasing function. By strong maximum principle $v>0$, then v is positive. □

Proof of Theorem 1.1. From Lemma 3.1 we know that v is a critical point of $I_q|_{M_{c,q}}$ and $v\in M_{c,q}$ is a real-valued positive and radially symmetric non-increasing function. It follows from Lemma 2.1 that v is a critical point of $I_q|_{S_c}$. By using the Lagrange multipliers rule there exists some $\lambda\in\mathbb{R}$ such that for any $\varphi\in E$

$$\langle I_q'(v),\varphi\rangle+\lambda\int_{\mathbb{R}^3}v\varphi\mathrm{d}x=0,$$

then v satisfies the equation

$$-(a+b\int_{\mathbb{R}^3}|\nabla v|^2\mathrm{d}x)\Delta v+\lambda v=\eta|v|^{p-2}v+|v|^{q-2}v.$$

Since $\gamma_p < 1, \gamma_q < 1$, then from Pohozaev equality (2.4) we obtain

$$\lambda \int_{\mathbb{R}^3} v^2 \mathrm{d}x = (1-\gamma_p)L(v) + (1-\gamma_q)K(v) > 0,$$

consequently, $\lambda > 0$. The proof is complete. $\square$

4. Proof of Theorem 1.2

Now we turn to the critical case $\frac{14}{3} \le p < q,\ q = 2^*$. In this case, we have

$$m(c, 2^*) = \inf_{u \in M_{c,2^*}} I_{2^*}(u).$$

We define

$$q_n = 2^* - \frac{1}{n} > 2, \quad n = 1, 2...,$$

then $\{q_n\}$ is increasing and $q_n \to 2^*$ as $n \to \infty$. We first give the following properties of $m(c, q_n)$.

Lemma 4.1. *The following results hold:*

(i) $\liminf_{n\to\infty} m(c, q_n) > 0$;
(ii) $\limsup_{n\to\infty} m(c, q_n) \le m(c, 2^*)$.

Proof. Item (i) follows from Lemma 2.3.

(ii) From the definition of $m(c, 2^*)$ there exists $u \in M_{c,2^*}$ such that

$$I_{2^*}(u) < m(c, 2^*) + \varepsilon, \quad 0 < \varepsilon < 1. \tag{4.1}$$

Let $u^t = t^{3/2}u(tx)$, $t > 0$. It is clear that

$$h_1(t) = I_{q_n}(u^t) = \frac{t^2}{2}A(u) + \frac{t^4}{4}D(u) - \frac{t^{p\gamma_p}}{p}L(u) - \frac{t^{q_n\gamma_{q_n}}}{q_n}K_n(u),$$

where

$$K_n(u) = \int_{\mathbb{R}^3} |u|^{q_n} \mathrm{d}x.$$

Note that $|u|^{q_n} \le |u|^2 + |u|^{2^*}$ and $|u|^{q_n} \to |u|^{2^*}$ as $n \to \infty$, then from Lebesgue dominated convergence theorem we obtain

$$\frac{t^{q_n\gamma_{q_n}}}{q_n}\int_{\mathbb{R}^3} |u|^{q_n} \mathrm{d}x = \frac{t^{\frac{3q_n-6}{2}}}{q_n}\int_{\mathbb{R}^3} |u|^{q_n} \mathrm{d}x \to \frac{t^{2^*}}{2^*}\int_{\mathbb{R}^3} |u|^{2^*} \mathrm{d}x.$$

Which implies that

$$|I_{q_n}(u^t) - I_{2^*}(u^t)| < \varepsilon. \tag{4.2}$$

On the other hand, it is easy to see that $h_1(0) = 0$ and $h_1(t) \to -\infty$ as $t \to +\infty$. By the proof of Lemma 2.2, we deduce that $h_1(t)$ achieves its unique maximum at $t = t_0$ for $t_0 \in [0, t_1),\ t_1 > t_0 > 0$. Moreover, we have $u^{t_0} \in M_{c,q_n}$. Since $u \in M_{c,2^*}$, it follows that

$$I_{2^*}(u) = \max_{t>0} I_{2^*}(u^t).$$

From (4.1) and (4.2), we obtain

$$m(c, q_n) \le I_{q_n}(u^{t_0}) \le I_{2^*}(u^{t_0}) + \varepsilon \le I_{2^*}(u) + \varepsilon < m(c, 2^*) + 2\varepsilon,$$

which implies that

$$\limsup_{n\to\infty} m(c, q_n) \le m(c, 2^*).$$

The proof is complete. $\square$

Let $B_\delta(0)$ be a ball centered at the origin with radius $\delta > 0$. Let $\rho(x) \in C_0^\infty(\mathbb{R}^3)$ be a radial cut-off function such that $\rho(x) = 1$ for $|x| \le \delta$ and $\rho(x) = 0$ for $|x| \ge 2\delta$, where $|\nabla\rho(x)| \le C, B_{2\delta}(0) \subset \mathbb{R}^3$. Let $0 < \varepsilon < 1$, we choose the function $u_\varepsilon(x)$ as follows

$$u_\varepsilon(x) = \rho(x)U_\varepsilon(x),$$

where

$$U_\varepsilon(x)=\frac{(3\varepsilon^2)^{1/4}}{(\varepsilon^2+|x|^2)^{1/2}}.$$

If $\varepsilon>0$ is small enough, by using [14] we have that the function $u_\varepsilon(x)$ satisfies

$$\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2\mathrm{d}x=S_0^{3/2}+O(\varepsilon); \tag{4.3}$$

$$\int_{\mathbb{R}^3}|u_\varepsilon|^{2^*}\mathrm{d}x=S_0^{3/2}+O(\varepsilon^3); \tag{4.4}$$

$$\int_{\mathbb{R}^3}|u_\varepsilon|^2\mathrm{d}x=C_2\varepsilon+O(\varepsilon^2). \tag{4.5}$$

Where C_2 is a positive constant independent of ε. By computations we have

$$\int_\Omega|u_\varepsilon|^p\mathrm{d}x\geq\begin{cases}C_3\varepsilon^{3-\frac{p}{2}}, & p>3,\\ C_3\varepsilon^{3/2}|\ln\varepsilon|, & p=3,\\ C_3\varepsilon^{p/2}, & p<3,\end{cases} \tag{4.6}$$

where C_3 is a positive constant independent of ε. We set

$$0<\int_{\mathbb{R}^3}|u_\varepsilon|^2\mathrm{d}x=c_2,\quad \xi_1=\frac{c_2}{c},\ v_\varepsilon(x)=\xi_1^{1/4}u_\varepsilon(\xi_1^{1/2}x).$$

By computations we have

$$A(v_\varepsilon)=A(u_\varepsilon),\quad B(v_\varepsilon)=c,\quad D(v_\varepsilon)=D(u_\varepsilon), \tag{4.7}$$

$$L(v_\varepsilon)=\xi_1^{\frac{p-6}{4}}L(u_\varepsilon),\quad K(v_\varepsilon)=K(u_\varepsilon). \tag{4.8}$$

Lemma 4.2. *Under the assumption of Theorem 1.2, we have*

$$0<m(c,2^*)<(\frac{1}{2}-\frac{1}{2^*})ay_1+(\frac{1}{4}-\frac{1}{2^*})by_1^2=c_*,$$

where y_1 is given in the following discussion.

Proof. From Lemma 2.3, we obtain $m(c,2^*)>0$. We will show $m(c,2^*)<c_*$. Since $B(v_\varepsilon)=c$, from Lemma 2.2, there exists a unique τ_0 such that $v_\varepsilon^{\tau_0}\in M_{c,2^*}$ and

$$I_{2^*}(v_\varepsilon^{\tau_0})=\max_{t>0}I_{2^*}(v_\varepsilon^t).$$

Consider the function

$$H(t)=I_{2^*}(v_\varepsilon^t)=\frac{at^2}{2}\int_{\mathbb{R}^3}|\nabla v_\varepsilon|^2\mathrm{d}x+\frac{bt^4}{4}\Big(\int_{\mathbb{R}^3}|\nabla v_\varepsilon|^2\mathrm{d}x\Big)^2-\eta\frac{t^{p\gamma_p}}{p}\int_{\mathbb{R}^3}|v_\varepsilon|^p\mathrm{d}x-\frac{t^{2^*}}{2^*}\int_{\mathbb{R}^3}|v_\varepsilon|^{2^*}\mathrm{d}x.$$

By using (4.7) and (4.8) we deduce

$$H(t)=\frac{t^2}{2}A(u_\varepsilon)+\frac{t^4}{4}D(u_\varepsilon)-\frac{t^{p\gamma_p}}{p}\xi_1^{\frac{p-6}{4}}L(u_\varepsilon)-\frac{t^{2^*}}{2^*}K(u_\varepsilon).$$

We set

$$H_1(t)=\frac{1}{2}at^2\|\nabla u_\varepsilon\|_2^2+\frac{bt^4}{4}\|\nabla u_\varepsilon\|_2^4-\frac{t^{2^*}}{2^*}\int_{\mathbb{R}^3}|u_\varepsilon|^{2^*}\mathrm{d}x.$$

Then from (4.3) and (4.4) we have

$$H_1(t)\leq H_2(t)+C_3(t)\varepsilon^3+C_4(t)\varepsilon, \tag{4.9}$$

and

$$H_2(t)=\frac{a}{2}t^2S_0^\beta+\frac{bt^4}{4}S_0^{2\beta}-\frac{t^{2^*}}{2^*}S_0^\beta,\ \beta=\frac{3}{2}.$$

By the same argument as in Lemma 2.2, we deduce that $H_2(t)$ achieves its maximum at $t=t_2>0$ and

$$aS_0^\beta+bt_2^2S_0^{2\beta}-t_2^{2^*-2}S_0^\beta=0. \tag{4.10}$$

We consider the function

$$F(t)=a+bt-S_0^{\frac{-2^*}{2}}t^{\frac{2^*-2}{2}},t>0.$$

Suppose y_1 is the unique zero of $F(t)$, then by $(\beta-1)\frac{2^*}{2}=\beta$, we know that y_1 and $t_2^2S^\beta$ are both zeros of the function $F(t)$. Since the zero of the function $F(t)$ is unique for some $t>0$, then we deduce that $y_1=t_2^2S_0^\beta$. From (4.10) we obtain

$$\begin{aligned}H_2(t)\le H_2(t_2)&=\frac{a}{2}t_2^2S_0^\beta+\frac{bt_2^4}{4}S_0^{2\beta}-\frac{t_2^{2^*}}{2^*}S_0^\beta\\&=(\frac{1}{2}-\frac{1}{2^*})at_2^2S_0^\beta+(\frac{1}{4}-\frac{1}{2^*})bt_2^4S_0^{2\beta}\\&=(\frac{1}{2}-\frac{1}{2^*})ay_1+(\frac{1}{4}-\frac{1}{2^*})by_1^2=c_*.\end{aligned}\tag{4.11}$$

By using (4.5), (4.6) and (4.9)-(4.11) we deduce that

$$\begin{aligned}H(t)&\le H_2(t_2)+C_4\varepsilon^3+C_4\varepsilon-C_5\tau_0^{p\gamma_p}\xi_1^{\frac{p-6}{4}}L(u_\varepsilon)\\&\le c_*+C_4\varepsilon^3+C_4\varepsilon-C_5\eta\tau_0^{p\gamma_p}c^{\frac{6-p}{4}}c_2^{\frac{p-6}{4}}L(u_\varepsilon)\\&\le c_*+C_4\varepsilon^3+C_4\varepsilon-C_5\eta\tau_0^{p\gamma_p}c^{\frac{6-p}{4}}\varepsilon^{\frac{6-p}{4}},\end{aligned}\tag{4.12}$$

where C_i $(i=4,5)$ are positive constants independent of ε.

We claim that there exists a constant M_1 independent of ε such that $\tau_0\le M_1$. If $\tau_0\to\infty$ as $\varepsilon\to 0$, then $H(\tau_0)\to-\infty$ and $m(c,2^*)\le 0$, which contradicts $m(c,2^*)>0$. On the other hand, since 0 is a local minimum of $H(t)$, there exists a constant M_2 independent of ε such that $H(\tau_0)\ge M_2>0$. This implies that there exists a constant M_3 independent of ε such that $\tau_0\ge M_3>0$.

Note that $p>2$, then

$$\frac{6-p}{4}<1.\tag{4.13}$$

By (4.12) and (4.13) we deduce that

$$m(c,2^*)\le\sup_{t>0}H(t)<c_*.$$

The proof is complete. □

The proof of the following Lemma can be founded in [3].

Lemma 4.3. *Let $N\ge 3$ and $1\le t<+\infty$. If $u\in L^t(\mathbb{R}^N)$ is a radial non-increasing function, then one has*

$$|u(x)|\le|x|^{-N/t}(\frac{N}{|S^{N-1}|})\|u\|_t,\quad x\ne 0,$$

where $|S^{N-1}|$ is the area of the unit sphere in $\mathbb{R}^N$.

Lemma 4.4. *Under the assumption of Theorem 1.2, $m(c,2^*)$ is attained by a real-valued positive and radially symmetric non-increasing function.*

Proof. For $q_n=2^*-\frac{1}{n}$, by Lemma 3.1, $m(c,q_n)$ is attained by a sequence of real-valued positive and radially symmetric non-increasing functions $\{u_n\}$, and $\{u_n\}\subset M_{c,q_n}$, $I_{q_n}(u_n)=m(c,q_n)$. From the proof of Theorem 1.1 we know that there exists $\lambda_n>0$ such that u_n satisfies

$$-(a+b\int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x)\Delta u_n+\lambda_nu_n=\eta|u_n|^{p-2}u_n+|u_n|^{q_n-2}u_n.\tag{4.14}$$

From Pohozaev equality (2.4) we obtain

$$\lambda_n\int_{\mathbb{R}^3}u_n^2\mathrm{d}x=(1-\gamma_p)L(u_n)+(1-\gamma_{q_n})K(u_n),$$

which implies that $\{\lambda_n\}$ is bounded. Then there exists $\lambda>0$ such that up to a subsequence, $\lim_{n\to\infty}\lambda_n=\lambda$.

From the proof of Lemma 3.1 we deduce that $\{u_n\}$ is bounded in E_r. We can extract a subsequence (still denoted u_n) and $w \in E_r$ such that

$$\begin{gathered} u_n \rightharpoonup w \quad \text{in } E_r, \\ u_n \to w \quad \text{in } L^i(\mathbb{R}^3)(2 < i < 2^*), \\ u_n \to w \quad \text{a.e. in } \mathbb{R}^3 a \end{gathered}$$

as $n \to \infty$. Then $\{|u_n|^{p-2}u_n\}$ is bounded in $L^{\frac{p}{p-1}}(\mathbb{R}^3)$, which implies that

$$\int_{\mathbb{R}^3} |u_n|^{p-2}u_n\varphi \mathrm{d}x \to \int_{\mathbb{R}^3} |u|^{p-2}u\varphi \mathrm{d}x$$

for $\varphi \in C_0^\infty(\mathbb{R}^3)$.

Note that For $\phi \in L^t(\mathbb{R}^3), t > 1$, by the Young inequality, the Hölder inequality and Lemma 4.3 there exists a constant $C > 0$ independent of n such that

$$\begin{aligned} ||u_n|^{q_n-2}u_n\phi| &\le C(|u_n|^{2-1}|\phi| + |u_n|^{2^*-1}|\phi|) \\ &\le C(|x|^{-\frac{1}{2}}|\phi| + |x|^{\frac{-(2^*-1)}{2}}|\phi|) \in L^1(\mathbb{R}^3). \end{aligned}$$

By using the Lebesgue dominated convergence theorem and passing to the limit in (4.14) we obtain for $\varphi \in C_0^\infty(\mathbb{R}^3)$,

$$0 = \langle I'_{q_n}(u_n), \varphi\rangle \to \langle I'_{2^*}(w), \varphi\rangle.$$

Therefore w is a solution of the equation

$$-(a + b\int_{\mathbb{R}^3} |\nabla w|^2 \mathrm{d}x)\Delta w + \lambda w = \eta|w|^{p-2}u + |w|^{2^*-2}w.$$

Then $G_{2^*}(w) = 0$.

We claim that $w \neq 0$. Otherwise, we assume that $u_n \rightharpoonup w = 0$, then from (3.3) we have

$$L(u_n) \to 0. \tag{4.15}$$

From Young inequality we obtain

$$|u_n|^{q_n} \le \frac{2^* - q_n}{2^* - \theta}|u_n|^\theta + \frac{q_n - \theta}{2^* - \theta}|u_n|^{2^*}, \quad 2 < \theta < q_n < 2^*.$$

On the other hand, by using the definition of S_0 we obtain

$$\int_{\mathbb{R}^3} |u|^{2^*}\mathrm{d}x \le S_0^{\frac{-2^*}{2}}\Big(\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x\Big)^{2^*/2}.$$

Hence we deduce that

$$K_n(u_n) = \int_{\mathbb{R}^3} |u_n|^{q_n}\mathrm{d}x \le \frac{q_n - \theta}{2^* - \theta} S_0^{\frac{-2^*}{2}}\Big(\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x\Big)^{2^*/2} + o(1). \tag{4.16}$$

Note that $u_n \in M_{c,q_n}$, then

$$0 = G_{q_n}(u_n) = A(u_n) + D(u_n) - \gamma_p L(u_n) - \gamma_{q_n} K_n(u_n). \tag{4.17}$$

From the proof of Lemma 2.3, there exists a constant $C > 0$ such that $\|\nabla u_n\|_2^2 \ge C > 0$, then we may therefore assume that

$$\liminf_{n\to\infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x = l,\ l > 0.$$

Let $n \to \infty$, from (4.15)-(4.17) we have

$$a + bl - S_0^{\frac{-2^*}{2}} l^{\frac{2^*-2}{2}} \le 0,\ l > 0. \tag{4.18}$$

We set

$$F(y) = a + by - S_0^{\frac{-2^*}{2}} y^{\frac{2^*-2}{2}},\ y > 0,$$

where $F(y)$ has been defined in the proof of Lemma 4.2. From $F'(y) = 0$ we obtain $y = y_0 > 0$ and $F(0) = a > 0$. it is easy to see that $F(y) \to -\infty$ as $y \to +\infty$. Moreover we have $F'(y) > 0$

for $y \in [0, y_0)$ and $F'(y) < 0$ for $y \in [y_0, \infty)$. It follows that $F(y)$ achieves its maximum at $y = y_0$. If $y > 0$, then $F(y)$ have only one zero point $y = y_1 > 0$. Since $F(y_1) = 0$, we deduce that

$$a + by_1 - S_0^{\frac{-2^*}{2}} y_1^{\frac{2^*-2}{2}} = 0. \tag{4.19}$$

From (4.18) and (4.19) we deduce that $l \geq y_1$.

By using Lemma 4.1 and (4.15) we obtain

$$\begin{aligned}
m(c, 2^*) &\geq \limsup_{n\to\infty} m(c, q_n) \\
&= \limsup_{n\to\infty} I_{q_n}(u_n) \\
&= \limsup_{n\to\infty} [I_{q_n}(u_n) - \frac{1}{q_n\gamma_{q_n}} G_{q_n}(u_n)] \\
&= \limsup_{n\to\infty} [(\frac{1}{2} - \frac{1}{q_n\gamma_{q_n}})A(u_n) + (\frac{1}{4} - \frac{1}{q_n\gamma_{q_n}})D(u_n)] \\
&\geq \liminf_{n\to\infty} [(\frac{1}{2} - \frac{1}{q_n\gamma_{q_n}})A(u_n) + (\frac{1}{4} - \frac{1}{q_n\gamma_{q_n}})D(u_n)] \\
&= a(\frac{1}{2} - \frac{1}{2^*})l + b(\frac{1}{4} - \frac{1}{2^*})l^2 \\
&\geq a(\frac{1}{2} - \frac{1}{2^*})y_1 + b(\frac{1}{4} - \frac{1}{2^*})y_1^2 = c_*.
\end{aligned}$$

Which contradicts Lemma 4.2, so we obtain $w \neq 0$.

We set

$$0 < \int_{\mathbb{R}^3} |w|^2 \mathrm{d}x = c_3 \leq c, \quad \xi_2 = \frac{c_3}{c}, \quad w_1(x) = \xi_2^{1/4} w(\xi_2^{1/2} x).$$

Then by computation we have

$$A(w_1) = A(w), \quad B(w_1) = c, \ D(w_1) = D(w). \tag{4.20}$$

$$L(w_1) = \xi_2^{\frac{p-6}{4}} L(w) \geq L(w), \ K(w_1) = K(w), \tag{4.21}$$

which implies that

$$G_{2^*}(w_1) \leq G_{2^*}(w) \leq 0.$$

From Lemma 2.2, there exists a unique $0 < \sigma \leq 1$ such that $G_{2^*}(w_1^\sigma) = 0$ and $w_1^\sigma \in M_{c,2^*}$. Then by Lemma 4.1, (4.20), and (4.21), we deduce that

$$\begin{aligned}
m(c, 2^*) &\leq I_{2^*}(w_1^\sigma) \\
&= I_{2^*}(w_1^\sigma) - \frac{1}{p\gamma_p} G_{2^*}(w_1^\sigma) \\
&= (\frac{1}{2} - \frac{1}{p\gamma_p})\sigma^2 A(w_1) + (\frac{1}{4} - \frac{1}{p\gamma_p})\sigma^4 D(w_1) + (\frac{\gamma_{2^*}}{p\gamma_p} - \frac{1}{2^*})\sigma^{2^*} \int_{\mathbb{R}^3} |w_1|^{2^*} \mathrm{d}x \\
&\leq (\frac{1}{2} - \frac{1}{p\gamma_p})A(w) + (\frac{1}{4} - \frac{1}{p\gamma_p})D(w) + (\frac{\gamma_{2^*}}{p\gamma_p} - \frac{1}{2^*}) \int_{\mathbb{R}^3} |w|^{2^*} \mathrm{d}x \\
&\leq \liminf_{n\to\infty} [(\frac{1}{2} - \frac{1}{p\gamma_p})A(u_n) + (\frac{1}{4} - \frac{1}{p\gamma_p})D(u_n) + (\frac{\gamma_{q_n}}{p\gamma_p} - \frac{1}{q_n}) \int_{\mathbb{R}^3} |u_n|^{q_n} \mathrm{d}x] \\
&\leq \liminf_{n\to\infty} [I_{q_n}(u_n) - \frac{1}{p\gamma_p} G_{q_n}(u_n)] = \liminf_{n\to\infty} I_{q_n}(u_n) \\
&= \liminf_{n\to\infty} m(c, q_n) \leq \limsup_{n\to\infty} m(c, q_n) \leq m(c, 2^*).
\end{aligned}$$

By a similar method to that of Lemma 3.1, we have

$$I_{2^*} w_1^\sigma) = m(c, 2^*), \quad \sigma = 1, \quad w_1 = w, \quad c = c_3,$$

then we deduce that $m(c, 2^*)$ is attained by a real-valued nonnegative and radially symmetric non-increasing function. By strong maximum principle $v > 0$, then v is positive. $\square$

Proof of Theorem 1.2. From Lemma 4.4 we know that w is a critical point of $I_{2^*}\,|_{M_{c,2^*}}$ and $w \in M_{c,2^*}$ is a real-valued positive and radially symmetric non-increasing function. It follows from Lemma 2.1 that w is a critical point of $I_{2^*}\,|_{S_c}$. By using the Lagrange multipliers rule there exists some $\lambda \in \mathbb{R}$ such that for any $\varphi \in E$,

$$\langle I'_{2^*}(w), \varphi\rangle + \lambda \int_{\mathbb{R}^3} w\varphi \mathrm{d}x = 0.$$

Then w satisfies

$$-(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x)\Delta u + \lambda u = \eta|u|^{p-2}u + |u|^{2^*-2}u.$$

Similarly to the proof of Theorem 1.1, we have $\lambda > 0$, The proof is complete. $\square$

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