Electronic Journal of Differential Equations, Vol. 2025 (2025), No. 115, pp. 1–18.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, DOI: 10.58997/ejde.2025.115

DYNAMICS OF A MAY-LEONARD ASYMMETRIC SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

FABIO SCALCO DIAS, REGILENE OLIVEIRA, CLÁUDIA VALLS

Abstract. We study the May-Leonard asymmetric model in $\mathbb{R}^3$ which was introduced in [3, 8]. It is the celebrated classical May-Leonard model incorporating asymmetric competitive effects instead of requiring equal intrinsic growth rates for each competing population. We study this system when it has an invariant of Darboux type and for these values of the parameters we shall describe its global dynamics in the compactification of the sphere, adding its infinity. In particular, we study the dynamics of that system on the invariant planes and we complete the study describing the dynamics at infinity. We also prove that the system is completely integrable and describe the α and ω limits of all the orbits of the system.

1. Introduction and statement of the main results

We study the May-Leonard asymmetric model in $\mathbb{R}^3$ introduced in [3, 8] which is given by

$$\begin{aligned} x' &= x(1 - x - \alpha_1 y - \beta_1 z),\\ y' &= y(1 - y - \beta_2 x - \alpha_2 z),\\ z' &= z(1 - z - \alpha_3 x - \beta_3 y) \end{aligned} \tag{1.1}$$

where $\alpha_i, \beta_i,\ i = 1, 2, 3$ are real parameters. This model describes the competition between three species and depends on six parameters. It is the celebrated classical May-Leonard model, introduced by May and Leonard in 1975 in [6] incorporating asymmetric competitive effects instead of requiring equal intrinsic growth rates for each competing population as it is required in the classical May-Leonard model. In such a way the May-Leonard asymmetric model has a wider range of applications. In contrast with the classical May-Leonard system whose global dynamics is quite known and has been widely investigated, this is not the cases for the asymmetric model for which there are few contributions. In particular system (1.1) was investigated in [2, 3, 8, 10] where the authors study the conditions for the existence and stability of limit cycles, non-periodic oscillations, existence of first integrals, Hopf bifurcations and the stability of steady states. It is well-known that solutions of that system cannot be, in general, written in terms of elementary functions, so the study of qualitative properties of solutions is a well tool to be performed. Since the existence and uniqueness of solutions is obvious, other properties, such as the existence of first integrals (which allow to reduce one dimension of the system) or the knowledge of the qualitative behavior of the orbits (which allow to solve the problem qualitatively) are very desirable.

We note that the May-Leonard asymmetric model has a large number of parameters, so in order to gain insight on the behavior of this model one had to begin by exploring subfamilies depending on less parameters. That is one of the reasons why in this paper we will focus on the May-Leonard asymmetric model when it has an invariant of Darboux type (see Section 2 for the definition). For these subfamilies we shall describe their global dynamics in the compactification of $\mathbb{R}^3$, adding the infinity. Such subfamilies have at most two parameters. The exact knowledge of the dynamics for these values of the parameters provide many information about the dynamics of the May-Leonard asymmetric models when the parameters are sufficient close to the selected ones

2020 *Mathematics Subject Classification.* 34A34, 34D23.
Key words and phrases. Poincaré compactification; phase portraits; dynamics at infinity; integrability.

Submitted August 26, 2025. Published December 16, 2025.

of our study. In particular, we study the dynamics of these models on the invariant planes and we complete the study describing the dynamics at infinity. We also prove that two of these subfamilies are completely integrable and because they have a Darboux invariant we can describe the α- and ω-limits of all their orbits. The main tool will be the Poincaré compactification. Roughly speaking the Poincaré ball is obtained identifying $\mathbb{R}^3$ with the interior of the 3-dimensional ball of radius one centered at the origin of coordinates, and extending analytically the flow of the asymmetric May–Leonard system to the boundary $\S^2$ of this ball, and consequently to the infinity. In this way we can study the behavior of that model in a neighborhood of the infinity, and describe completely the global dynamics of it. For a precise information on the Poincaré compactification see [4, Chapter 5].

The main results of this paper are the following. For a definition of Darboux polynomial, invariant algebraic surface, cofactor, invariant as well as the precise statement of Theorem 2.1, see Section 2. For a precise definition of ω-limit, α-limit and Poincaré sphere, we refer the reader to [4].

Theorem 1.1. *The following holds for system* (1.1)*.*

- (i) *It has a Darboux invariant of the form $I(x,y,z,t) = e^{st} f(x,y,z,t)$, where $f(x,y,z)$ is given by the product of the invariant planes $x = 0$, $y = 0$, $z = 0$ and one of the invariant planes described in Theorem 2.1 if and only if conditions (1)–(6) of Table 1 holds, respectively.*
- (ii) *For any orbit of system* (1.1) *restricted to the conditions of Table 1 we have*
  - (ii.1) *Its ω-limit is contained at infinity in the Poincaré compactification in $\mathbb{R}^3$.*
  - (ii.2) *For each j, from* (1)–(6)*, its α-limit is contained, respectively, in $\Omega_{(j)}$ union with its boundary at infinity in the Poincaré compactification in $\mathbb{R}^3$, where $\Omega_{(j)}$ is equal to $\{(x,y,z) \in \mathbb{R}^3 : x = 0\} \cup \{(x,y,z) \in \mathbb{R}^3 : y = 0\} \cup \{(x,y,z) \in \mathbb{R}^3 : z = 0\} \cup \{(x,y,z) \in \mathbb{R}^3 : F_j = 0\}$.*

Theorem 1.1 is proved in Section 3.

TABLE 1. Conditions on the parameters of system (1.1) to the existence of a Darboux invariant of the form $I(x,y,z,t) = e^{st} f(x,y,z,t)$

	Invariant Planes	Conditions on the parameters: $s = -4$ and
(1)	x, y, z, F_1	$\beta_1 = \alpha_2$, $\alpha_2 = -1/3$, $\beta_2 = -(\alpha_3 + 2)$, $\beta_3 = -(\alpha_1 + 2)$, $\beta_2 \neq 1$
(2)	x, y, z, F_2	$\beta_3 = \alpha_1$, $\alpha_1 = -1/3$, $\beta_1 = -(\alpha_2 + 2)$, $\beta_2 = -(\alpha_3 + 2)$, $\alpha_3 \neq 1$
(3)	x, y, z, F_3	$\beta_2 = \alpha_3$, $\alpha_3 = -1/3$, $\beta_1 = -(\alpha_2 + 2)$, $\beta_3 = -(\alpha_1 + 2)$, $\beta_3 \neq 1$
(4)	x, y, z, F_4	$\beta_1 = \alpha_3 = 1$, $\alpha_2 = -3$, $\beta_2 = -3$, $\beta_3 = -(\alpha_1 + 2)$, $(\alpha_1 - 1)(\beta_3 - 1) \neq 0$
(5)	x, y, z, F_5	$\beta_2 = \alpha_1 = 1$, $\alpha_3 = -3$, $\beta_3 = -3$, $\beta_1 = -(\alpha_2 + 2)$, $(\alpha_2 - 1)(\beta_1 - 1) \neq 0$
(6)	x, y, z, F_6	$\beta_3 = -\frac{(s\alpha_1-1)(\alpha_2-1)(\alpha_3-1)}{(\beta_1-1)(\beta_2-1)} + 1$, $\alpha_1 = -\frac{7+2\alpha_3+2\alpha_2+\alpha_2\alpha_3}{\alpha_2+\alpha_3+2}$, $\beta_1 = -(\alpha_2 + 2)$, $\beta_2 = -(\alpha_3 + 2)$.

We now examine system (1.1) under each of the parameter conditions specified in Table 1. We first note that when restricted to conditions (1), (2), and (3) (or alternatively conditions (4) and (5)) of Table 1, system (1.1) becomes equivalent through the following coordinate transformations:

$$
\begin{aligned}
&(1) \to (2) \quad (x, y, z, \alpha_1, \alpha_3) \to (x, z, y, -(\alpha_2 + 2), \alpha_3 - 2), \\
&(2) \to (3) \quad (x, y, z, \alpha_3, \alpha_2) \to (y, x, z, -(\alpha_1 + 2), -(\alpha_2 + 2)), \\
&(4) \to (5) \quad (x, y, z, \alpha_1) \to (x, z, y, \beta_1).
\end{aligned}
$$

Hence, we shall study only system (1.1) restricted to the conditions (1), (4), and (6) of Table 1. We start with restricting to conditions (1). System (1.1) restricted to conditions (1) of Table 1 is

$$
\begin{aligned}
x' &= x(1 - x - \alpha_1 y + z/3), \\
y' &= y(1 - y + (\alpha_3 + 2)x + z/3), \\
z' &= z(1 - z - \alpha_3 x + (\alpha_1 + 2)y),
\end{aligned} \tag{1.2}
$$

where $\alpha_3 \neq -3, \alpha_1$ are real parameters. Consider $U = \{(\alpha_1, \alpha_3) \in \mathbb{R}^2 : \alpha_3 \neq -3\}$ the parameter space of system (1.2). The next result describes the global dynamics of system (1.2). The definition of Jacobi multiplier is given in Section 2.

Theorem 1.2. *The following statements hold for system* (1.2) *for the parameters in* $\overline{U}$.

- (a) *The phase portraits in the Poincaré sphere are topologically equivalent to one of the phase portraits of Figure 1. Moreover, the ones in the curves $L_{6,7}, L_{8,9}, L_{12,13}$ and for the points P_5 and P_6 in Figure 1 occur only when $\alpha_3 = -3$.*
- (b) *The phase portraits in the Poincaré disc of system* (1.2) *restricted to the invariant planes are topologically equivalent to one of the phase portraits of Figure 2.*
- (c) *It has a Jacobi multiplier.*

Theorem 1.2 is proved in Section 4.

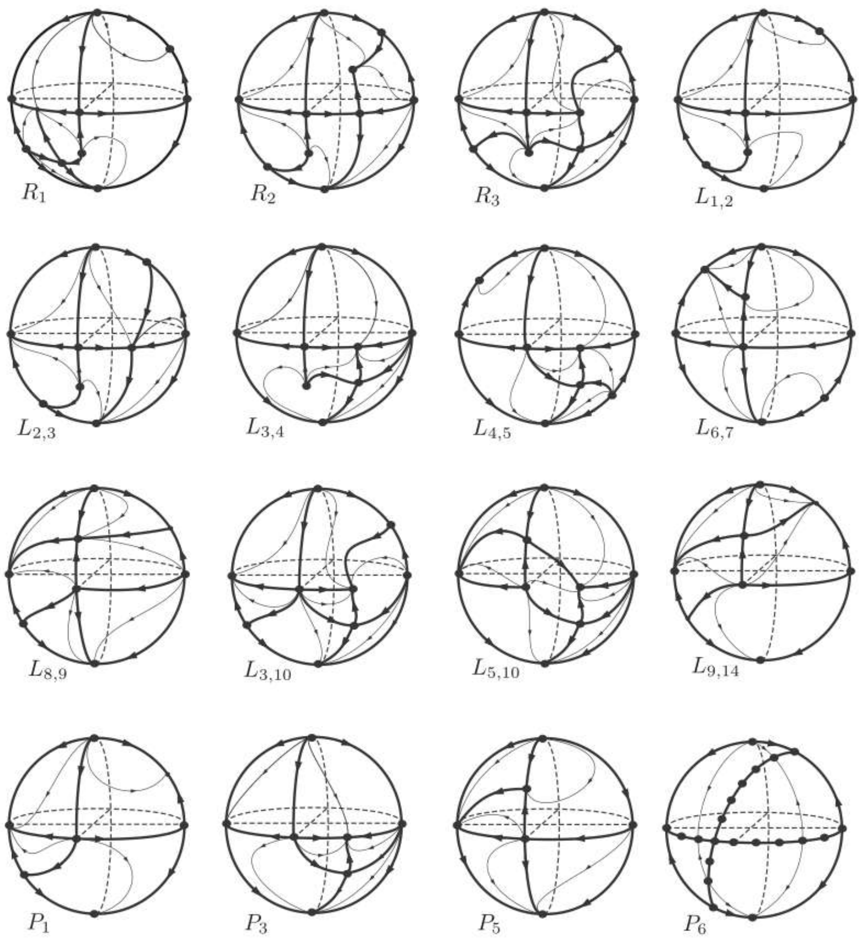

FIGURE 1. Phase portrait of system (1.2) in $\overline{U}$ on the Poincaré sphere. Phase portraits for parameters in $L_{6,7}, L_{12,13}, P_5$ and P_6 occur only in $\overline{U} \setminus U$.

In the next theorem we study system (1.1) restricted to the conditions (4) of Table 1. The definition of completely integrability is given in Section 2. System (1.1) restricted to conditions (4) of Table 1 is given by

$$x' = x(1 - x - z - \alpha_1 y), \quad y' = y(1 - y + 3x + 3z), \quad z' = z(1 - z - x + (\alpha_1 - 2)y), \tag{1.3}$$

where $\alpha_1 \neq \{1, -3\}$ is a real parameter. Consider $U = \{\alpha_1 \in \mathbb{R} : \alpha_1 \neq 1, -3\}$ the parameter space of system (1.3). The next result describes the global dynamics of system (1.3).

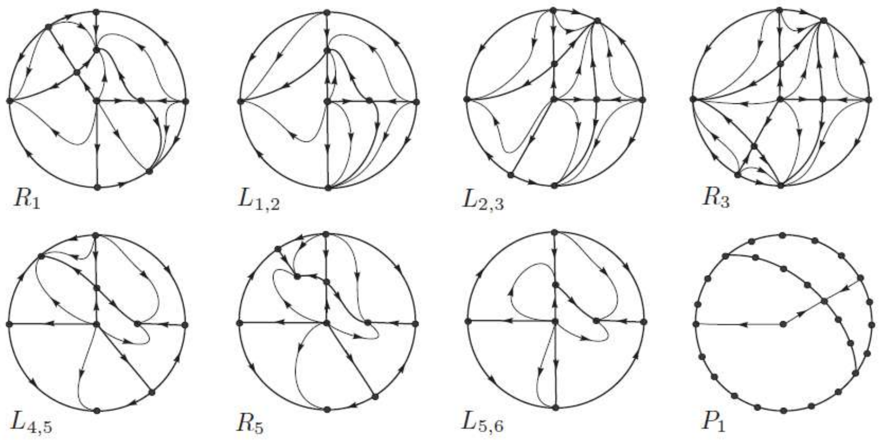

FIGURE 2. Phase portraits of system (4.5) with parameters in $\overline{U}$ on the Poincaré disc.

Theorem 1.3. *The following statements hold for system* (1.3) *for the parameters in* $\overline{U}$.

(a) *The phase portraits in the Poincaré sphere are topologically equivalent to one of the phase portraits of Figure 3.*
(b) *The phase portraits in the Poincaré disc of system* (1.3) *restricted to the invariant planes are topologically equivalent to one of the phase portraits of Figure 2.*
(c) *It is completely integrable (see the proof for a precise definition).*

The proof of Theorem 1.3 is given in Section 5.

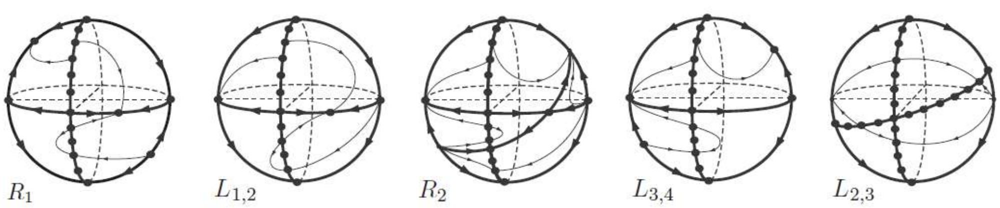

FIGURE 3. Phase portrait of system (1.3), with parameters in $\overline{U}$, on the Poincaré sphere.

We study system (1.1) restricted to the conditions (6) of Table 1, that is, we consider system

$$
\begin{aligned}
x' &= x\Big(1 - x + \frac{(\alpha_2\alpha_3 + 2\alpha_2 + \alpha_3 + 7)}{(\alpha_2 + \alpha_3 + 2)}y + (\alpha_2 + 2)z\Big),\\
y' &= y(1 + (\alpha_3 + 2)x - y - \alpha_2 z),\\
z' &= z\Big(1 - \alpha_3 x - \frac{(\alpha_2\alpha_3 + 3)}{(\alpha_2 + \alpha_3 + 2)}y - z\Big),
\end{aligned} \tag{1.4}
$$

where $\alpha_2 \neq -3$, $\alpha_3 \neq -3$ and $\alpha_2 + \alpha_3 + 2 \neq 0$. Consider $U = \{(\alpha_2, \alpha_3) \in \mathbb{R}^2 : \alpha_2 \neq -3, \alpha_3 \neq -3, \text{ and } \alpha_2 + \alpha_3 + 2 \neq 0\}$ the parameter space of system (1.4). The next result describes the global dynamics of system (1.4).

Theorem 1.4. *The following statements hold for system* (1.4) *for the parameters in* $\overline{U}$.

(a) *The phase portraits in the Poincaré sphere are topologically equivalent to one of the phase portraits of Figure 4.*
(b) *The phase portraits in the Poincaré disc of system* (1.4) *restricted to the invariant planes are topologically equivalent to one of the phase portraits of Figure 2.*

(c) *In region R_1 (see Figure 4), the boundary of the infinity of the second octant is a heteroclinic cycle formed by three equilibrium points coming from the ones located at the end of the x-negative half-axis, and the y and z positive half-axes, and three orbits connecting these equilibria, each one coming from the orbit at the end of every plane of coordinates. In the interior of the infinity of the second octant there is an attractor whose orbits fill completely this interior. The same happens for the fourth octant in the region L_{34} and in the third octant in the region L_{56}. In region R_3 (see again Figure 4) the heteroclinic cycle is formed by the orbits connecting the origin with the point p_1, the point p_1 with the point p_2 and the point p_2 with the origin instead of the three orbits connecting the equilibria at the end of the z-negative half axis, and the x and y positive half-axes.*

(d) *It is completely integrable.*

The proof of Theorem 1.4 is given in Section 6. We have also introduced a section, Section 2, with the notions and results that will be used to prove Theorems 1.1–1.4.

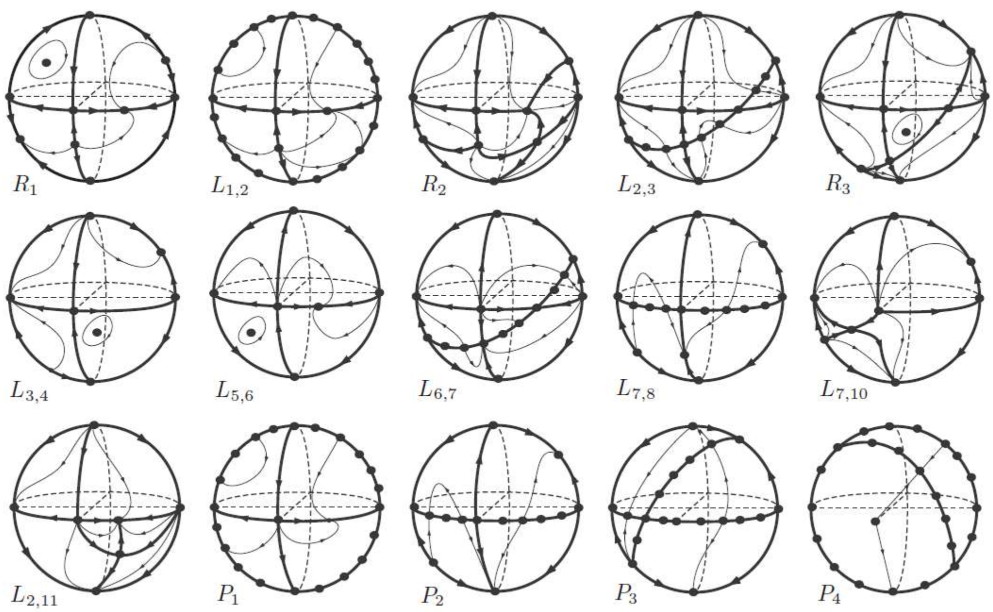

FIGURE 4. Phase portraits of system (1.4) on the Poincaré sphere.

2. PRELIMINARIES

We introduce some notion and results that will be used in the proof of the main theorems given in the introduction.

Let $\mathbb{R}[x, y, z]$ be the ring of the real polynomials in the variables x, y and z. We say that $F = F(x, y, z) \in \mathbb{R}[x, y, z]$ is a *Darboux polynomial* or a *invariant algebraic surface* of the system (1.1) if there exist $K \in \mathbb{C}[x, y, z]$ such that $\nabla F \cdot (P, Q, R) = KF$. K is called *cofactor* of the invariant surface $F(x, y, z) = 0$. When $K \equiv 0$ then F is a *first integral* of system (1.1).

We say that a nonconstant C^1 function $I(x, y, z, t)$ is an *invariant* of the differential system (1.1) on an open subset U of $\mathbb{R}^3$ if $dI/dt = 0$ on the trajectories of the system contained in U, i.e.

$$x(1 - x - \alpha_1 y - \beta_1 z)\frac{\partial I}{\partial x} + y(1 - y - \beta_2 x - \alpha_2 z)\frac{\partial I}{\partial y} + z(1 - z - \alpha_3 x - \beta_3 y)\frac{\partial I}{\partial z} + \frac{\partial I}{\partial t} = 0. \quad (2.1)$$

Then a first integral is an invariant of the system that is independent of the time t. When a system has a Darboux polynomial F with a constant factor $k = k_0 \in \mathbb{R}$, then the function $I(x, y, z, t) = F(x, y, z)e^{-k_0 t}$ is called a *Darboux invariant* of that system, see [4, Chapter 8].

The following result was proved in [8].

Theorem 2.1. *System* (1.1) *has an invariant plane $F_i = 0$ different from the planes $x = 0, y = 0$ and $z = 0$ passing through the origin if one of the conditions (1)–(6) of Table 2 holds on the parameter space.*

Table 2. Invariant algebraic surfaces and cofactors according to conditions on the parameters.

Invariant Plane	Cofactor	Parameters
$F_1 = (\beta_2 - 1)x + (1 - \alpha_1)y$	$K = 1 - x - y - \beta_1 z,$	$\alpha_2 = \beta_1, \beta_2 \neq 1$
$F_2 = (\alpha_3 - 1)x + (1 - \beta_1)z$	$K = 1 - x - z - \beta_3 y,$	$\alpha_1 = \beta_3, \alpha_3 \neq 1$
$F_3 = (\beta_3 - 1)y + (1 - \alpha_2)z$	$K = 1 - y - z - \alpha_3 x,$	$\alpha_3 = \beta_2, \beta_3 \neq 1$
$F_4 = -(1 - \beta_2)(1 - \beta_3)x$ $+(1 - \alpha_1)(1 - \beta_3)y$ $-(1 - \alpha_1)(1 - \alpha_2)z$	$K = 1 - x - y - z$	$\beta_1 = \alpha_3 = 1, (\alpha_1 - 1)(\beta_3 - 1) \neq 0,$
$F_5 = -(1 - \alpha_2)(1 - \alpha_3)x$ $-(1 - \beta_1)(1 - \beta_3)y$ $+(1 - \alpha_2)(1 - \beta_1)z,$	$K = 1 - x - y - z,$	$\beta_2 = \alpha_1 = 1, (\alpha_2 - 1)(\beta_1 - 1) \neq 0$
$F_6 = -(1 - \alpha_3)(1 - \beta_2)x$ $+(1 - \alpha_1)(1 - \alpha_3)y$ $+(1 - \beta_1)(1 - \beta_2)z$	$K = 1 - x - y - z,$	$\beta_3 = -\frac{(\alpha_1-1)(\alpha_2-1)(\alpha_3-1)}{(\beta_1-1)(\beta_2-1)} + 1$

Let $J = J(x, y, z)$ be a non-negative C^1 function non-identically zero on any open subset whose domain of definition is an open and dense subset of $\mathbb{R}^3$. Then J is a *Jacobi multiplier* of the differential system (1.1) if

$$\int_\Omega J(x, y, z)\, dx\, dy\, dz = \int_{\phi_t(\Omega)} J(x, y, z)\, dx\, dy\, dz,$$

where Ω is any open subset of $\mathbb{R}^3$ and ϕ_t is the flow defined by the differential system (1.1). In general Jacobi multipliers are very difficult to detect but we have the following result from [9] that provides a good way for obtaining them.

Proposition 2.2. *Let $J = J(x, y, z)$ be a non–negative C^1 function non-identically zero on any open subset of $\mathbb{R}^3$ whose domain of definition is dense in $\mathbb{R}^3$. Then J is a Jacobi multiplier of the differential system* (1.1) *if and only if the divergence of the below differential system* (2.2) *is zero*

$$\begin{aligned} x' &= J(x, y, z)x(1 - x - \alpha_1 y - \beta_1 z), \\ y' &= J(x, y, z)y(1 - y - \beta_2 x - \alpha_2 z), \\ z' &= J(x, y, z)z(1 - z - \alpha_3 x - \beta_3 y), \end{aligned} \tag{2.2}$$

Let $H_i \colon U_i \to \mathbb{R}$ for $i = 1, 2$ be two first integrals of the differential system (1.1). We say they are *independent* in $U_1 \cap U_2$ if their gradients are independent in all the points of $U_1 \cap U_2$ except perhaps in a zero Lebesgue measure set. System (1.1) is *completely integrable* in $\mathbb{R}^3$ if it has two independent first integrals.

We have the following result, which goes back to Jacobi (for a proof see [5, Theorem 2.7], that is a good tool to determine when a differential is completely integrable.

Theorem 2.3. *Consider the differential system* (1.1) *in $\mathbb{R}^3$, and assume that it admits a Jacobi multiplier $J = J(x, y, z)$ and one first integral H. Then the system admits an additional first integral functionally independent with the previous one, i.e. the differential system* (1.1) *is completely integrable in $\mathbb{R}^3$.*

3. Proof of Theorem 1.1

We separate the proof of the two statements.

Proof of Theorem 1.1(i). We present only the proof of the first case as the proof of the other cases are similar and hence we have omitted them here. Taking into account that $\alpha_2 = \beta_1$ and $\beta_2 \neq 1$, it follows from Theorem 2.1 that system (1.1) has $F_1(x, y, z) = 0$, $x = 0, y = 0$ and $z = 0$ as invariant planes passing through the origin whose cofactors are given in Table 2.

System (1.1) admits a Darboux invariant of the form $I(x, y, z, t) = e^{st} f(x, y, z)$, where $f(x, y, z)$ is given by the product of invariant planes $F_1(x, y, z) = 0$, $x = 0, y = 0$ and $z = 0$, if and only if, equation (2.1) is satisfied for a real $s \neq 0$. Doing so we get

$$s = -4, \quad \alpha_2 = -1/3, \quad \beta_2 = -\alpha_3 - 2, \quad \beta_3 = -\alpha_1 - 2.$$

This concludes the proof of statement (i) of the theorem.

The Proof of Theorem 1.1(ii) follows directly from [7, Proposition 6] in the case $s < 0$.

4. Proof of Theorem 1.2

We separate the proof of each of the statements of the theorem in different subsections.

Proof of Theorem 1.2 (a). We present the Poincaré compactification of system (1.2) in the local charts U_i, V_i for $i = 1, 2, 3$ in order to understand the global behavior of the solutions near infinity. See [1] for the definition of these charts and more details about them.

The expression of the Poincaré compactification of system (1.2) in the local chart U_1 is given by

$$\begin{aligned} z_1' &= z_1(3 + \alpha_3 + \alpha_1 z_1 - z_1), \\ z_2' &= z_2(1 - \alpha_3 + 2\alpha_1 z_1 + 2\, z_1 - 4\, z_2/3), \\ z_3' &= z_3(1 - z_3 + \alpha_1 z_1 - z_2/3). \end{aligned} \tag{4.1}$$

For $z_3 = 0$ (which correspond to the points on the sphere $\mathbb{S}^2$ at infinity) system (4.1) becomes

$$z_1' = z_1(3 + \alpha_3 + \alpha_1 z_1 - z_1), \quad z_2' = z_2(1 - \alpha_3 + 2\alpha_1 z_1 + 2\, z_1 - 4\, z_2/3). \tag{4.2}$$

System (4.2) has the following possible equilibrium points.

$$p_0 = (0, 0), \quad p_1 = \Big(-\frac{\alpha_3 + 3}{\alpha_1 - 1}, 0 \Big), \quad p_2 = \Big(0, -\frac{3}{4}(\alpha_3 - 1) \Big),$$
$$p_3 = \Big(-\frac{\alpha_3 + 3}{\alpha_1 - 1}, -\frac{3}{4} \frac{(3\alpha_1\alpha_3 + 5\alpha_1 + \alpha_3 + 7)}{\alpha_1 - 1} \Big).$$

The eigenvalues of the Jacobian matrix evaluated in each of the equilibria are $\alpha_3 + 3$ and $1 - \alpha_3$ for p_0, $-(3 + \alpha_3)$ and $-(3\alpha_1\alpha_3 + 5\alpha_1 + \alpha_3 + 7)/(\alpha_1 - 1)$ for p_1, $(\alpha_3 + 3)$ and $\alpha_3 - 1$ for p_2 and $-(3 + \alpha_3)$ and $(3\alpha_1\alpha_3 + 5\alpha_1 + \alpha_3 + 7)/(\alpha_1 - 1)$ for p_3.

When $\alpha_3 = 1$, p_2 coalesce with p_0 and have a saddle-node. When $\alpha_1 = \frac{-(7+\alpha_3)}{(5+3\alpha_3)}$, p_3 coalesce with p_1 and have a saddle-node. When $(\alpha_1 - 1)^2 + (\alpha_3 + 3)^2 = 0$ system (4.2) has two lines filled up of equilibria and it is topologically equivalent to the phase portrait P_6 in Figure 1.

The flow in the local chart V_1 is the same as the one in U_1 because the compactified vector field $p(X)$ in V_1 coincides with the compactified vector field in U_1 multiplied by -1. Hence the phase portrait on V_1 is the same as the one in U_1 reserving the sense of the flow.

The expression of the Poincaré compactification of system (1.2) in the local chart U_2 restricted to infinity (that is with $z_3 = 0$) becomes

$$z_1' = -z_1(-1 + \alpha_1 + \alpha_3 z_1 + 3z_1), \quad z_2' = -z_2(-3 - \alpha_1 + 2\alpha_3 z_1 + 2z_1 + 4z_2/3). \tag{4.3}$$

In this case system (4.3) has the following possible equilibrium points.

$$q_0 = (0, 0), \quad q_1 = \Big(0, \frac{3\alpha_1 + 9}{4} \Big), \quad q_2 = \Big(-\frac{\alpha_1 - 1}{\alpha_3 + 3}, 0 \Big),$$
$$q_3 = \Big(-\frac{(\alpha_1 - 1)}{(\alpha_3 + 3)}, \frac{3}{4} \frac{(3\alpha_1\alpha_3 + \alpha_3 + 5\alpha_1 + 7)}{(\alpha_3 + 3)} \Big).$$

The equilibria q_2 and q_3 were studied in the local chart U_1. The eigenvalues of the equilibria q_0 are $1 - \alpha_1$ and $3 + \alpha_1$ and the eigenvalues of the equilibria q_1 are $1 - \alpha_1$ and $-(\alpha_1 + 3)$.

The flow in V_2 is the same as the one in U_2 reserving the sense of the flow.

Finally, the expression of the Poincaré compactification in the local chart U_3 is

$$\begin{aligned} z_1' &= z_1(4/3 + \alpha_3 z_1 - z_1 - 2z_2 - 2\alpha_1 z_2), \\ z_2' &= z_2(4/3 + 2\alpha_3 z_1 + 2z_1 - 3z_2 - \alpha_1 z_2), \\ z_3' &= z_3(1 + \alpha_3 z_1 - 2z_2 - \alpha_1 z_2 - z_3). \end{aligned} \tag{4.4}$$

System (4.4) restricted to infinity (that is with $z_3 = 0$) becomes

$$\begin{aligned} z_1' &= z_1(4/3 + \alpha_3 z_1 - z_1 - 2z_2 - 2\alpha_1 z_2), \\ z_2' &= z_2(4/3 + 2\alpha_3 z_1 + 2z_1 - 3z_2 - \alpha_1 z_2). \end{aligned}$$

Now the only point of the local chart U_3 which is not covered by the local charts U_1, V_1, U_2 and V_2 is the origin of coordinates of U_3. The eigenvalues of the origin of U_3 are $4/3$ and $4/3$.

The flow at infinity in the local chart V_3 is the same as the flow in the local chart U_3 reversing the time. Accordingly, we define the following bifurcation curves and regions, as in Figure 5.

$$\begin{gathered} L_{1,2} = \big\{(\alpha_1,\alpha_3) \in \mathbb{R}^2 : \alpha_1 = 1, \alpha_3 > 1\big\}, \quad L_{2,3} = \big\{(\alpha_1,\alpha_3) \in \mathbb{R}^2 : \alpha_3 > 1, \mathcal{L} = 0\big\}, \\ L_{3,4} = \big\{(\alpha_1,\alpha_3) \in \mathbb{R}^2 : \alpha_1 = -3, \alpha_3 > 1\big\}, \quad L_{4,5} = \big\{(\alpha_1,\alpha_3) \in \mathbb{R}^2 : \alpha_3 = 1, \alpha_1 < -3\big\}, \\ L_{5,6} = \big\{(\alpha_1,\alpha_3) \in \mathbb{R}^2 : \alpha_1 < -3, \mathcal{L} = 0\big\}, L_{6,7} = \big\{(\alpha_1,\alpha_3) \in \mathbb{R}^2 : \alpha_3 = -3, \alpha_1 < -3\big\}, \\ L_{6,9} = \big\{(\alpha_1,\alpha_3) \in \mathbb{R}^2 : \alpha_1 = -3, -3 < \alpha_3 < -1\big\}, \quad L_{7,8} = \big\{(\alpha_1,\alpha_3) \in \mathbb{R}^2 : \alpha_1 = -3, \alpha_3 < -3\big\}, \\ L_{8,9} = \big\{(\alpha_1,\alpha_3) \in \mathbb{R}^2 : \alpha_3 = -3, -3 < \alpha_1 < 1\big\}, \quad L_{9,10} = \big\{(\alpha_1,\alpha_3) \in \mathbb{R}^2 : -3 < \alpha_1 < -1, \mathcal{L} = 0\big\}, \\ L_{3,10} = \big\{(\alpha_1,\alpha_3) \in \mathbb{R}^2 : \alpha_3 = 1, -3 < \alpha_1 < -1\big\}, \\ L_{5,10} = \big\{(\alpha_1,\alpha_3) \in \mathbb{R}^2 : \alpha_1 = -3, -3 < \alpha_3 < -1\big\}, \\ L_{2,9} = \big\{(\alpha_1,\alpha_3) \in \mathbb{R}^2 : \alpha_3 = 1, -1 < \alpha_1 < 1\big\}, \quad L_{8,11} = \big\{(\alpha_1,\alpha_3) \in \mathbb{R}^2 : \alpha_3 < -1, \mathcal{L} = 0\big\}, \\ L_{11,12} = \big\{(\alpha_1,\alpha_3) \in \mathbb{R}^2 : \alpha_1 = 1, \alpha_3 < -3\big\}, \quad L_{12,13} = \big\{(\alpha_1,\alpha_3) \in \mathbb{R}^2 : \alpha_3 = -3, \alpha_1 > 1\big\}, \\ L_{13,14} = \big\{(\alpha_1,\alpha_3) \in \mathbb{R}^2 : \alpha_1 > 1, \mathcal{L} = 0\big\}, \quad L_{9,14} = \big\{(\alpha_1,\alpha_3) \in \mathbb{R}^2 : \alpha_1 = 1, -3 < \alpha_3 < 1\big\}, \\ L_{1,14} = \big\{(\alpha_1,\alpha_3) \in \mathbb{R}^2 : \alpha_3 = 1, \alpha_1 > 1\big\} \end{gathered}$$

where $\mathcal{L} = 3\alpha_1\alpha_3 + \alpha_3 + 5\alpha_1 + 7$.

In Table 3 we provide the description of the topological type of each equilibria (p_i and $q_j, i = 0,1,2,3$ and $j = 0,2$) according to the values of the parameters α_1 and α_3 in each the local charts U_1 U_2 and U_3, respectively. This table use the following notations: S (saddle), UN (unstable node), SN (stable node), S-N (saddle-node) and PP (phase portrait). The corresponding phase portraits are given in Figure 1.

Proof of Theorem 1.2 (b). In this section we study the dynamics of system (1.2) restricted to the invariants planes.

Restriction of system (1.2) *to the invariant plane* $z = 0$. On the invariant plane $z = 0$ system 1.2 becomes

$$x' = x(1 - x - \alpha_1 y), \quad y' = y(1 - y + (\alpha_3 + 2)x). \tag{4.5}$$

The equilibrium points of systems (4.5) are

$$r_0 = (0,0), \quad r_1 = (0,1), \quad r_2 = (1,0), \quad r_3 = \Big(-\frac{\alpha_1 - 1}{\alpha_1\alpha_3 + 2\alpha_1 + 1}, \frac{\alpha_3 + 3}{\alpha_1\alpha_3 + 2\alpha_1 + 1}\Big).$$

The eigenvalues of the Jacobian matrix evaluated at each of the equilibria are $1, 1$ for r_0; $1-\alpha_1, -1$ for r_1; $-1, \alpha_3 + 3$ for r_2; and $-1, \frac{(\alpha_3+3)(\alpha_1-1)}{\alpha_1\alpha_3+2\alpha_1+1}$ for r_3. When $\alpha_1 = 1$, r_3 coalesce with r_1 and have a saddle-node. Finally, when $\alpha_1 = 1$ and $\alpha_3 = -3$ system (4.5) becomes

$$x' = x(1 - x - y), \quad y' = y(1 - x - y). \tag{4.6}$$

The phase portrait of system (4.6) has a line of equilibria and it is given in Figure 2.

To study the infinite equilibrium points of system (4.5) we use the Poincaré compactification for a polynomial vector field in $\mathbb{R}^2$ which is described in [4, Chapter 5].

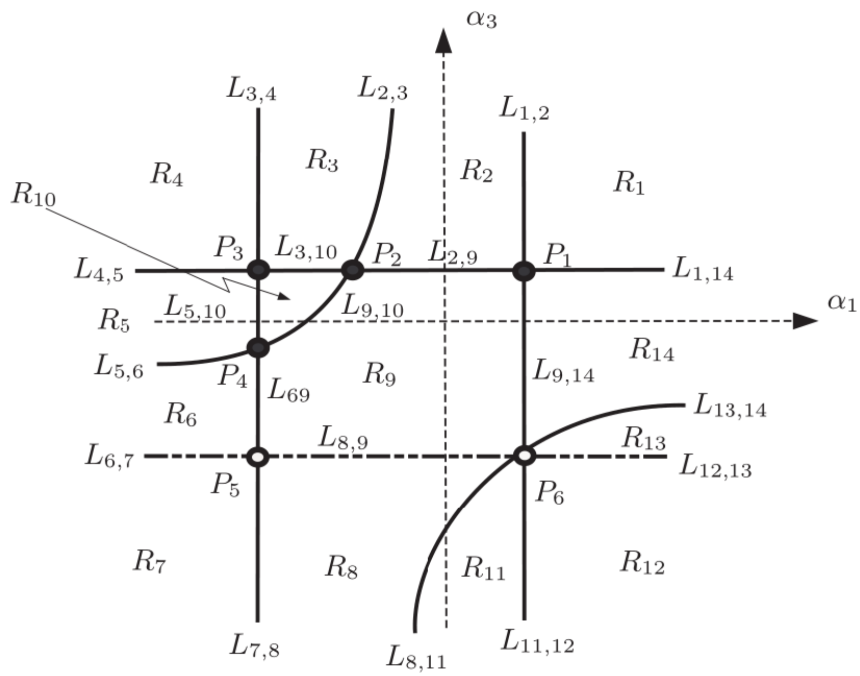

FIGURE 5. Bifurcation diagram of system (1.2) for the parameters in $\overline{U}$. Note that the transitions $L_{6,7}, L_{8,9}, L_{12,13}$ and the points P_5 and P_6 belong to $\alpha_3 = -3$.

The Poincaré compactification of system (4.5) in the local chart U_1 is given by

$$u' = u((\alpha_1 - 1)u + \alpha_3 + 3), \quad v' = v(1 + \alpha_1 u - v).$$

So, there are two equilibria on $v = 0$, namely $s_0 = (0, 0)$ and $s_1 = \big(-(\alpha_3 + 3), (\alpha_1 - 1)\big)$. The eigenvalues of the Jacobian matrix evaluated at these two equilibria are $1, (\alpha_3 + 3)$ for s_0 and $-(\alpha_3 + 1), -(\alpha_1\alpha_3 + 2\alpha_1 + 1)/(\alpha_1 - 1)$ for s_1.

The Poincaré compactification of system (4.5) in the local chart U_2 is given by

$$u' = u((\alpha_3 + 3)u + \alpha_1 - 1), \quad v' = -v(v - 1 + (\alpha_3 + 2)u).$$

In this case the origin is an equilibrium point whose eigenvalues of its Jacobian matrix are $1, \alpha_3+3$.

We present the bifurcation diagram and the phase portraits of system (4.5) in Figures 6 and 2, respectively. In Figure 6 we have the following curves:

$$L_{1,2} = \big\{(\alpha_1, \alpha_3) \in \mathbb{R}^2 : \alpha_1 = 1, \alpha_3 > -3\big\}, \quad L_{2,3} = \big\{(\alpha_1, \alpha_3) \in \mathbb{R}^2 : \alpha_1 < 0, \mathcal{L} = 0\big\},$$
$$L_{2,4} = \big\{(\alpha_1, \alpha_3) \in \mathbb{R}^2 : \alpha_3 = -3, \alpha_1 < 1\big\}, \quad L_{4,5} = \big\{(\alpha_1, \alpha_3) \in \mathbb{R}^2 : 0 < \alpha_1 < 1, \mathcal{L} = 0\big\},$$
$$L_{5,6} = \big\{(\alpha_1, \alpha_3) \in \mathbb{R}^2 : \alpha_1 = 1, \alpha_3 < -3\big\}, \quad L_{6,7} = \big\{(\alpha_1, \alpha_3) \in \mathbb{R}^2 : \alpha_3 = -3, \alpha_1 > 1\big\},$$
$$L_{1,7} = \big\{(\alpha_1, \alpha_3) \in \mathbb{R}^2 : \alpha_1 > 1, \mathcal{L} = 0\big\}$$

where $\mathcal{L} = \alpha_1\alpha_3 + 2\alpha_1 + 1 = 0$. The topological type of each equilibrium point of system (4.5) is given in Table 4.

Restriction of system (1.2) *to the invariant planes* $x = 0$, $y = 0$ *and* $F_1(x, y) = 0$. System (1.2) restricted to the planes $x = 0$ and $y = 0$ is equivalent to system 4.5 by the change of coordinates $(x, y, \alpha_1, \alpha_3) \to (y, z, -1/3, \alpha_1)$ and $(x, y, \alpha_1, \alpha_3) \to (x, z, -1/3, -(\alpha_3 + 2))$, respectively. On the other hand, system (1.2) restricted to the plane $F_1(x, y) = 0$ is equivalent to system 4.5 by the change of coordinates

$$(x, y, \alpha_1, \alpha_3) \to \Big(\frac{(\alpha_1\alpha_3 + 2\alpha_1 + 1)}{(\alpha_3 + 3)} y, z, -1/3, \frac{\alpha_3 + 2\alpha_1\alpha_3 + 3\alpha_1 + 6}{1 + 2\alpha_1 + \alpha_1\alpha_3} - 2\Big).$$

TABLE 3. Topological type of each equilibria of system (1.2) on the Poincaré sphere according to the values $(\alpha_1, \alpha_3) \in \overline{U}$.

	p_0	p_1	p_2	p_3	q_1	q_2	w_0	PP
R_1	S	SN	UN	S	S	SN	UN	R_1
R_2	S	S	UN	SN	UN	S	UN	R_2
R_3	S	SN	UN	S	UN	S	UN	R_3
R_4	S	SN	UN	S	S	UN	UN	R_1
R_5	UN	SN	S	S	S	UN	UN	R_3
R_6	UN	S	S	SN	S	UN	UN	R_1
R_7	S	UN	SN	S	S	UN	UN	R_1
R_8	S	UN	SN	S	UN	S	UN	R_3
R_9	UN	S	S	SN	UN	S	UN	R_3
R_{10}	UN	SN	S	S	UN	S	UN	R_3
R_{11}	S	S	SN	UN	UN	S	UN	R_2
R_{12}	S	UN	SN	S	S	SN	UN	R_1
R_{13}	UN	S	S	SN	S	SN	UN	R_1
R_{14}	UN	SN	S	S	S	SN	UN	R_3
P_1	S-N	$\nexists$	p_0	$\nexists$	S-N	S-N	UN	P_1
P_2	S-N	S-N	p_0	p_1	UN	S	UN	$L_{8,9}$
P_3	S-N	SN	p_0	S	S-N	q_1	UN	P_3
P_4	UN	S-N	S	p_1	S-N	q_1	UN	P_3
P_5	S-N	p_0	S-N	p_2	S-N	q_1	UN	P_5
P_6	-	-	-	-	-	-	-	P_6

	p_0	p_1	p_2	p_3	q_1	q_2	w_0	PP
$L_{1,2}$	S	$\nexists$	UN	$\nexists$	S-N	S-N	UN	$L_{1,2}$
$L_{2,3}$	S	S-N	UN	p_1	UN	S	UN	$L_{2,3}$
$L_{3,4}$	S	SN	UN	S	S-N	q_1	UN	$L_{3,4}$
$L_{4,5}$	S-N	SN	p_0	S	S	UN	UN	$L_{4,5}$
$L_{5,6}$	UN	S-N	S	p_1	S	UN	UN	$L_{4,5}$
$L_{6,7}$	S-N	p_0	S-N	p_2	S	UN	UN	$L_{6,7}$
$L_{7,8}$	UN	S	S	SN	S-N	q_1	UN	$L_{3,4}$
$L_{6,9}$	UN	SN	S	S	S-N	q_1	UN	$L_{3,4}$
$L_{8,9}$	S-N	p_0	S-N	p_2	UN	S	UN	$L_{8,9}$
$L_{9,10}$	UN	S-N	S	p_1	UN	S	UN	$L_{3,10}$
$L_{5,10}$	UN	SN	S	S	S-N	q_1	UN	$L_{5,10}$
$L_{8,11}$	S	S-N	S	p_1	UN	S	UN	$L_{2,3}$
$L_{11,12}$	S	$\nexists$	SN	$\nexists$	S-N	S-N	UN	$L_{1,2}$
$L_{12,13}$	S-N	p_0	S-N	p_2	S	SN	UN	$L_{6,7}$
$L_{13,14}$	UN	S-N	S	p_1	S	SN	UN	$L_{4,5}$
$L_{9,14}$	UN	$\nexists$	S	$\nexists$	S-N	S-N	UN	$L_{9,14}$
$L_{2,9}$	S-N	S	p_0	SN	UN	S	UN	$L_{2,3}$
$L_{3,10}$	S-N	SN	p_0	S	UN	S	UN	$L_{3,10}$
$L_{1,14}$	S-N	SN	p_0	S	S	SN	UN	$L_{4,5}$

Therefore, this new system has only the previously studied phase portraits except when the parameters α_1 and α_3 belong to the curve $\mathcal{L}_1$ given by $2\alpha_1 + \alpha_1\alpha_3 + 1 = 0$. This system for the parameters α_1 and α_3 on this curve is given by

$$y' = y\Big(\frac{z}{3} + 1\Big), \quad z' = z(1 - z + Ay), \tag{4.7}$$

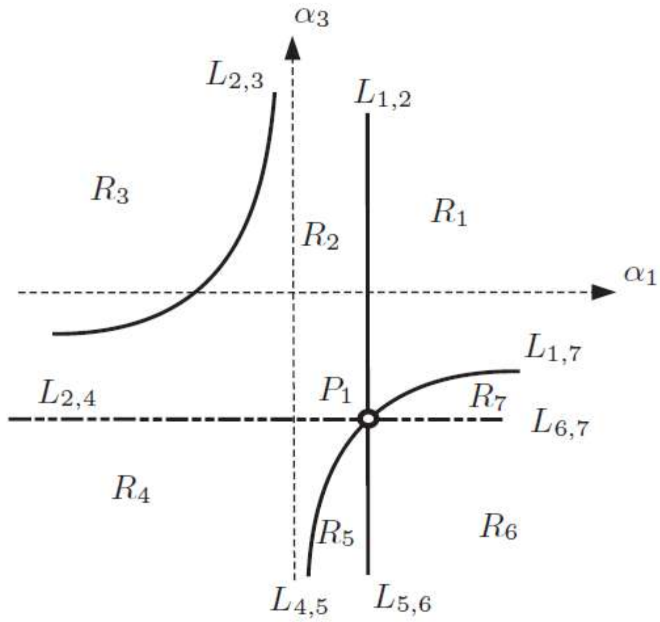

FIGURE 6. Bifurcation diagram of system (4.5). Note that transitions $L_{2,4}, L_{6,7}$ and the point P_1 belong to the line $\alpha_3 = -3$.

TABLE 4. Topological type of each equilibria of system (4.5) on the Poincaré disc according to the values $(\alpha_1, \alpha_3) \in \overline{U}$.

Regions	r_0	r_1	r_2	r_3	s_0	s_1	w_0	PP
R_1	UN	SN	S	S	UN	SN	S	R_1
R_2	UN	S	S	SN	UN	S	UN	R_1
R_3	UN	S	S	S	UN	SN	UN	R_3
R_4	UN	S	SN	S	S	UN	UN	R_1
R_5	UN	S	SN	SN	S	S	UN	R_5
R_6	UN	SN	SN	S	S	UN	S	R_5
R_7	UN	SN	S	SN	UN	S	S	R_5
$L_{1,2}$	UN	S-N	S	r_1	UN	$\nexists$	S-N	$L_{1,2}$
$L_{2,3}$	UN	S	S	$\nexists$	UN	S-N	UN	$L_{2,3}$
$L_{2,4}$	UN	S	S-N	r_2	S-N	s_0	UN	$L_{1,2}$
$L_{4,5}$	UN	S	SN	$\nexists$	S	S-N	UN	$L_{4,5}$
$L_{5,6}$	UN	S-N	SN	r_1	S	$\nexists$	S-N	$L_{5,6}$
$L_{6,7}$	UN	SN	S-N	r_2	S-N	s_1	S	$L_{5,6}$
$L_{1,7}$	UN	SN	S	$\nexists$	UN	S-N	S	$L_{4,5}$
P_1	-	-	-	-	-	-	-	P_1

where $A = (\alpha_3 + 3)/(\alpha_3 + 2)$. Note $\alpha_3 \neq -2$, otherwise $(\alpha_1, -2) \notin \mathcal{L}_1$. Since $\alpha_3 \neq -3$, system (4.7) is equivalent to

$$y' = y(\frac{z}{3} + 1), \quad z' = z(1 - z + y). \tag{4.8}$$

The phase portrait of system (4.8) is topologically equivalent to $L_{2,3}$ of Figure 2.

Proof of Theorem 1.2 (c). System (1.1) becomes system (1.2) and so system (2.2) becomes

$$\begin{aligned} x' &= J(x,y,z)x(1-x-\alpha_1 y+z/3),\\ y' &= J(x,y,z)y(1-y+(\alpha_3+2)x+z/3),\\ z' &= J(x,y,z)z(1-z-\alpha_3 x+(\alpha_1+2)y). \end{aligned} \tag{4.9}$$

We define

$$J = x^{\frac{7\alpha_1+9}{4(\alpha_1-1)}} y^{\frac{7\alpha_3+5}{4(\alpha_3+3)}} z^{7/4}((1-\alpha_1)y-(3+\alpha_3)x)^{-\frac{9\alpha_1\alpha_3+11\alpha_1+7\alpha_3+37}{4(\alpha_1-1)(\alpha_3+3)}}.$$

It is easy to see that J is a Jacobi multiplier of system (1.2) because one can check directly that the divergence of system (4.9) is zero. This completes the proof of Theorem 1.2(c) and so the proof of Theorem 1.2.

5. Proof of Theorem 1.3

We separate the proof of each of the statements of the theorem in different subsections.

Proof of Theorem 1.3(a). The Poincaré compactification of system (1.3) in the local chart U_1 is given by

$$\begin{aligned} z_1' &= z_1(4+(\alpha_1-1)z_1+4z_2),\\ z_2' &= 2(\alpha_1+1)z_1z_2,\\ z_3' &= z_3(-z_3+1+\alpha_1 z_1+z_2). \end{aligned} \tag{5.1}$$

For $z_3=0$ system (5.1) becomes

$$z_1' = z_1(4+(\alpha_1-1)z_1+4z_2), \quad z_2' = 2(\alpha_1+1)z_1z_2. \tag{5.2}$$

So $z_1=0$ is a line of equilibrium points. After a convenient rescaling of the time we eliminate the common factor z_1 and we conclude that system (5.2) admits a unique equilibrium point (with $\alpha_1 \neq 1$) described by $p_0 = (-4/(\alpha_1-1), 0)$. The eigenvalues of the Jacobian matrix evaluated at p_0 are -4 and $-8(\alpha_1+1)/(\alpha_1-1)$. If $\alpha_1 = 1$, system (5.2) has no equilibrium points. Moreover, when $\alpha_1=-1$ system (5.1) becomes

$$z_1' = 2z_1(2-z_1+2z_2), \quad z_2' = 0.$$

The Poincaré compactification of system (1.3) in U_2 restricted to the infinity becomes

$$z_1' = -z_1(\alpha_1-1+4(z_1+z_2)), \quad z_2' = -z_2(-\alpha_1-3+4(z_1+z_2)). \tag{5.3}$$

In this case system (5.3) (with $\alpha_1 \neq -1$) has the following equilibrium points

$$q_0 = (0,0), \quad q_1 = \Big(0, \frac{\alpha_1+3}{4}\Big), \quad q_2 = \Big(-\frac{\alpha_1-1}{4}, 0\Big).$$

The equilibria q_2 has already been studied in the local chart U_1. The eigenvalues of the equilibria q_0 are $1-\alpha_1$ and $3+\alpha_1$ and for the equilibria q_1 we have the following eigenvalues $-2(1+\alpha_1)$ and $-(\alpha_1+3)$. When $\alpha_1=-1$ system (5.3) becomes

$$z_1' = 2z_1(1-2(z_1+z_2)), \quad z_2' = 2z_2(1-2(z_1+z_2)).$$

So $2(z_1+z_2)=1$ is a line of equilibrium points. The Poincaré compactification in the local chart U_3 restricted to the infinity is

$$z_1' = -2(\alpha_1+1)z_1z_2, \quad z_2' = z_2(4+4z_1-\alpha_1 z_2-3z_2). \tag{5.4}$$

So $z_2=0$ is a line of equilibrium points, that after a convenient rescaling of time, it can be eliminated and the new resulting system does not has any distinguished equilibrium point in the local chart U_3. When $\alpha_1=-1$ system (5.4) becomes

$$z_1' = 0, \quad z_2' = 2z_2(2z_1-z_2+2).$$

In short we have Table 5. This table uses the following notations: S (saddle), UN (unstable node), SN (stable node), S-N (saddle-node) and PP (phase portrait). The corresponding phase portraits are given in Figure 3.

TABLE 5. Topological type of each equilibria of system (1.3) on the Poincaré sphere according to the values $(\alpha_1, \alpha_3) \in \overline{U}$.

Regions	p_0	q_0	q_1	PP
$\alpha_1 < -3$	SN	S	UN	R_1
$\alpha_1 = -3$	SN	S-N	q_0	$L_{1,2}$
$-3 < \alpha_1 < -1$	SN	UN	S	R_1
$\alpha_1 = -1$	-	-	-	$L_{2,3}$
$-1 < \alpha_1 < 1$	S	UN	SN	R_2
$\alpha_1 = 1$	$\nexists$	S-N	SN	$L_{3,4}$
$\alpha_1 > 1$	SN	S	SN	R_1

Proof of Theorem 1.3(b). System (1.3) restricted to the planes $x = 0$, $y = 0$ and $z = 0$ is equivalent to system (4.5) by the change of coordinates $(x, y, \alpha_1, \alpha_3) \to (y, z, -3, \alpha_1)$, $(x, y, \alpha_1, \alpha_3) \to (x, z, 1, -3)$ and $(x, y, \alpha_1, \alpha_3) \to (x, y, \alpha_1, 1)$, respectively. On $F_4 = 0$ system (1.3) becomes

$$x' = x\Big(1 - 2\frac{(\alpha_1 + 1)}{(\alpha_1 - 1)}x - \frac{(5\alpha_1 + 3)}{4}y\Big), \quad y' = y\Big(1 - 6\frac{(\alpha_1 + 1)}{(\alpha_1 - 1)}x + \frac{(3\alpha_1 + 5)}{4}y\Big). \tag{5.5}$$

System (5.5) is equivalent to system (4.5) by the change of coordinates

$$(x, y, \alpha_1, \alpha_3) \to \Big(2\Big(\frac{\alpha_1 + 1}{\alpha_1 - 1}\Big)x, -\frac{(3\alpha_1 + 5)}{4}y, -\frac{5\alpha_1 + 3}{3\alpha_1 + 5}, -5\Big).$$

This new system has only the previously studied phase portraits except when the parameter $\alpha_1 \in \{-1, 1, -5/3, -3/5\}$. When $\alpha_1 = -1, -5/3, -3/5$, the phase portraits are topologically equivalent to $P_1, L_{2,3}, R_1$ of Figure 2, respectively and system (5.5) is not defined for $\alpha_1 = 1$.

Proof of Theorem 1.3(c). System (1.1) becomes (1.3). It is easy to see that

$$H = x^{\alpha_1+3} z^{\alpha_1-1}(-4(\alpha_1 + 3)x - (\alpha_1 - 1)((\alpha_1 + 3)y - 4z))^{-2(\alpha_1+1)}$$

is a first integral of system (1.3). Since now we are working with system (1.3), system (2.2) becomes

$$\begin{aligned} x' &= J(x, y, z)x(1 - x - z - \alpha_1 y),\\ y' &= J(x, y, z)y(1 - y + 3x + 3z),\\ z' &= J(x, y, z)z(1 - z - x + (\alpha_1 - 2)y). \end{aligned} \tag{5.6}$$

We define

$$J = y^{\frac{3}{4}} z^{\frac{3}{\alpha_1+3}}(-4(\alpha_1 + 3)x - (\alpha_1 - 1)((\alpha_1 + 3)y - 4z))^{\frac{9}{4} - \frac{3}{\alpha_1+3}}.$$

It is easy to see that J is a Jacobi multiplier of system (1.3) because one can check directly that the divergence of system (5.6) is zero. Now, since system (1.3) has the first integral H and a Jacobi multiplier, by Theorem 2.3, it is completely integrable. This completes the proof of Theorem 1.3(c) and so the proof of Theorem 1.3.

6. Proof of Theorem 1.4

We separate the proof of each of the statements in different subsections.

Proof of Theorem 1.4 (a). The Poincaré compactification of system (1.4) in the local chart U_1 restricted to infinity becomes

$$\begin{aligned} z_1' &= z_1\Big(-\frac{(\alpha_3 + 3)(\alpha_2 + 3)}{\alpha_2 + \alpha_3 + 2}z_1 - 2(\alpha_2 + 1)z_2 + \alpha_3 + 3\Big),\\ z_2' &= z_2\Big(-2\,\frac{(\alpha_3 + \alpha_2\alpha_3 + \alpha_2 + 5)}{\alpha_2 + \alpha_3 + 2}z_1 - (\alpha_2 + 3)z_2 - \alpha_3 + 1\Big). \end{aligned} \tag{6.1}$$

The equilibrium points on the local chart U_1 are

$$p_0 = (0, 0), \quad p_1 = \Big(\frac{\alpha_2 + \alpha_3 + 2}{\alpha_2 + 3}, 0\Big), \quad p_2 = \Big(0, -\frac{\alpha_3 - 1}{\alpha_2 + 3}\Big),$$

$$p_3 = \Big(-\frac{\alpha_2+\alpha_3+2}{\alpha_2-1}, \frac{\alpha_3+3}{\alpha_2-1}\Big).$$

The eigenvalues of the Jacobian matrix evaluated in each of the equilibria are α_3+3 and $1-\alpha_3$ for p_0, $-(3+\alpha_3)$ and $-(3\alpha_2\alpha_3+5\alpha_3+\alpha_2+7)/(\alpha_2+3)$ for p_1, $(3\alpha_2\alpha_3+5\alpha_3+\alpha_2+7)/(\alpha_2+3)$ and (α_3-1) for p_2 and $\pm\sqrt{-(\alpha_2-1)(\alpha_3+3)(3\alpha_2\alpha_3+5\alpha_3+\alpha_2+7)}/(\alpha_2-1)$ for p_3.

To study the local behavior of p_3, we must follow the following steps: shift p_3 to the origin; diagonalize the linear part (with a linear change of coordinates) and apply [4, Theorem 8.15 (iii)]. From this, we conclude that the point p_3 is a center. The stability for each of these equilibria is presented in Table 6.

We observe that when $3\alpha_2\alpha_3+5\alpha_3+\alpha_2+7=0$ system (6.1) becomes

$$\begin{aligned} z_1' &= z_1(\alpha_3+3)\Big(\frac{-4}{(3\,\alpha_3+5)}z_1+\frac{4}{(3\alpha_3+1)}z_2+1\Big),\\ z_2' &= -z_2(\alpha_3-1)\Big(\frac{-4}{(3\,\alpha_3+5)}z_1+\frac{4}{(3\alpha_3+1)}z_2+1\Big). \end{aligned} \tag{6.2}$$

For system (6.2), we have the origin and the line

$$z_1 = \frac{(3\,\alpha_3+5)(4\,z_2+3\,\alpha_3+1)}{4(3\,\alpha_3+1)}$$

as equilibrium points. Finally, when $\alpha_3=-3$ system (6.2) has a line $z_2=0$ filled up of equilibria.

The Poincaré compactification of system (1.4) in the local chart U_2 restricted to infinity becomes

$$\begin{aligned} z_1' &= z_1\Big(-(\alpha_3+3)z_1+2(\alpha_2+1)z_2+\frac{(\alpha_3+3)(\alpha_2+3)}{\alpha_2+\alpha_3+2}\Big),\\ z_2' &= z_2\Big(-2(\alpha_3+1)z_1+(\alpha_2-1)z_2-\frac{(\alpha_3-1)(\alpha_2-1)}{\alpha_2+\alpha_3+2}\Big). \end{aligned} \tag{6.3}$$

So, in the local chart U_2, there are two equilibrium points not yet studied on the chart U_1

$$q_0=(0,0),\quad q_1=\Big(0,\frac{\alpha_3-1}{\alpha_2+\alpha_3+2}\Big).$$

The eigenvalues of the equilibria q_0 are $(\alpha_3+3)(\alpha_2+3)/(\alpha_2+\alpha_3+2)$ and $(\alpha_3-1)(\alpha_2-1)/(\alpha_2+\alpha_3+2)$ and the eigenvalues of the equilibria q_1 are $(\alpha_2+3\alpha_2\alpha_3+5\alpha_3+7)/(\alpha_2+\alpha_3+2)$ and $(\alpha_3-1)(\alpha_2-1)/(\alpha_2+\alpha_3+2)$. Again, when $3\alpha_2\alpha_3+5\alpha_3+\alpha_2+7=0$ and $\alpha_2+\alpha_3+2\neq 0$ system (6.3) becomes

$$\begin{aligned} z_1' &= -z_1(\alpha_3+3)\Big(z_1+\frac{4}{(3\alpha_3+1)}z_2-\frac{4}{3\alpha_3+5}\Big),\\ z_2' &= -2z_2(\alpha_3+1)\Big(z_1+\frac{4}{(3\alpha_3+1)}z_2-\frac{4}{3\alpha_3+5}\Big). \end{aligned} \tag{6.4}$$

In this case, system (6.4) has as equilibrium points the origin and the line $z_1=-\frac{4}{(3\alpha_3+1)}z_2+\frac{4}{3\alpha_3+5}$.

In the chart U_3 the origin is an equilibrium point whose eigenvalues of the Jacobian matrix are α_2+3 and $1-\alpha_2$.

In summary, we present the bifurcation diagram and the topological type of each equilibrium point of system (1.4) in Figure 7 and Table 4, respectively. Consider the following curve $\mathcal{L}=3\alpha_2\alpha_3+5\alpha_3+\alpha_2+7=0$ in Figure 7.

$$L_{1,2}=\{(\alpha_2,\alpha_3)\in\mathbb{R}^2:\alpha_2=1,\alpha_3>1\},\quad L_{2,3}=\{(\alpha_2,\alpha_3)\in\mathbb{R}^2:\alpha_3>1,\mathcal{L}=0\},$$
$$L_{3,4}=\{(\alpha_2,\alpha_3)\in\mathbb{R}^2:\alpha_2=-3,\alpha_3>1\},\quad L_{5,6}=\{(\alpha_2,\alpha_3)\in\mathbb{R}^2:\alpha_3=1,\alpha_2<-3\},$$
$$L_{6,7}=\{(\alpha_2,\alpha_3)\in\mathbb{R}^2:\alpha_2<-3,\mathcal{L}=0\},\quad L_{7,8}=\{(\alpha_2,\alpha_3)\in\mathbb{R}^2:\alpha_3=-3,\alpha_2<-3\},$$
$$L_{8,9}=\{(\alpha_2,\alpha_3)\in\mathbb{R}^2:\alpha_2=-3,\alpha_3<-3\},\quad L_{9,10}=\{(\alpha_2,\alpha_3)\in\mathbb{R}^2:\alpha_3=-3,-3<\alpha_2<-1\},$$
$$L_{9,14}=\{(\alpha_2,\alpha_3)\in\mathbb{R}^2:\alpha_3<-3,\mathcal{L}=0\},\quad L_{13,14}=\{(\alpha_2,\alpha_3)\in\mathbb{R}^2:\alpha_3=-3,-1<\alpha_2<1\},$$
$$L_{10,13}=\{(\alpha_2,\alpha_3)\in\mathbb{R}^2:-3<\alpha_3<-5/3,\mathcal{L}=0\},$$
$$L_{11,12}=\{(\alpha_2,\alpha_3)\in\mathbb{R}^2:-5/3<\alpha_3<-1,\mathcal{L}=0\},$$

$$L_{2,11} = \{(\alpha_2,\alpha_3) \in \mathbb{R}^2 : \alpha_3 = 1, -3 < \alpha_2 < 1\}, \quad L_{11,18} = \{(\alpha_2,\alpha_3) \in \mathbb{R}^2 : \alpha_2 = 1, -1 < \alpha_3 < 1\},$$
$$L_{12,17} = \{(\alpha_2,\alpha_3) \in \mathbb{R}^2 : \alpha_2 = 1, -3 < \alpha_3 < -1\}, \quad L_{14,15} = \{(\alpha_2,\alpha_3) \in \mathbb{R}^2 : \alpha_2 = 1, \alpha_3 < -3\},$$
$$L_{16,17} = \{(\alpha_2,\alpha_3) \in \mathbb{R}^2 : \alpha_3 = -3, \alpha_1 > 1\}, \quad L_{17,18} = \{(\alpha_2,\alpha_3) \in \mathbb{R}^2 : \alpha_2 > 1, \mathcal{L} = 0\},$$
$$L_{1,18} = \{(\alpha_2,\alpha_3) \in \mathbb{R}^2 : \alpha_3 = 1, \alpha_2 > 1\}.$$

The phase portraits of system (1.4) are presented in Figure 4.

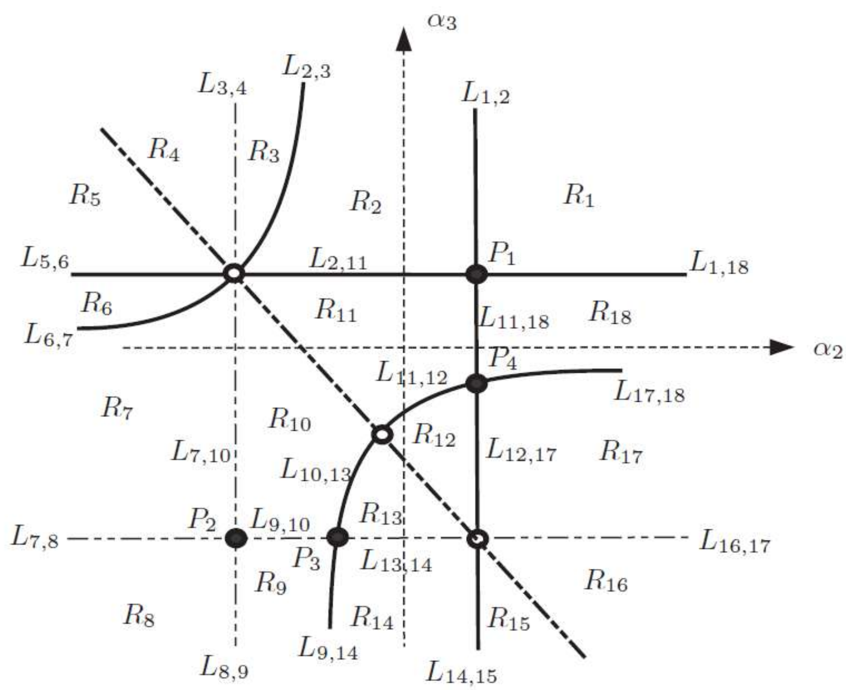

FIGURE 7. Bifurcation diagram of system (1.4). Note that transitions $L_{3,4}, L_{7,10}$ and $L_{8,9}$ belong to the line $\alpha_2 = -3$, while transitions $L_{7,8}, L_{9,10}, L_{13,14}$ and $L_{16,17}$ belong to the line $\alpha_3 = -3$.

Proof of Theorem 1.4 (b). System (1.4) restricted to the planes $x = 0$, $y = 0$ and $z = 0$ is equivalent to system (4.5) by the change of coordinates $(x, y, \alpha_1, \alpha_3) \to (y, z, \alpha_2, \gamma)$, $(x, y, \alpha_1, \alpha_3) \to (x, y, \gamma, \alpha_3)$ and $(x, y, \alpha_1, \alpha_3) \to (x, z, -(\alpha_2+2), -(\alpha_3+2))$, respectively, where $\gamma = -\frac{\alpha_2\alpha_3+3}{\alpha_2+\alpha_3+2} - 2$

On the invariant plane $F_6 = 0$ system (1.4) becomes

$$\begin{aligned} x' &= x\Big(\frac{(2\,\alpha_2\alpha_3 + 4\,\alpha_3 + \alpha_2 + 5)}{\alpha_2 + \alpha_3 + 2} y - \frac{(1 + \alpha_2\alpha_3 + 2\,\alpha_3)}{\alpha_2 + 3} x + 1\Big), \\ y' &= y\Big(-\frac{(\alpha_2\alpha_3 + \alpha_3 + 2)}{\alpha_2 + \alpha_3 + 2} y + \frac{(3\,\alpha_3 + 2\,\alpha_2\alpha_3 + \alpha_2 + 6)}{\alpha_2 + 3} x + 1\Big). \end{aligned} \tag{6.5}$$

System (6.5) is equivalent to (4.5) by the change of coordinates $(x, y, \alpha_1, \alpha_3) \to (Ax, By, C, D)$, where

$$A = \frac{\alpha_2\alpha_3 + 2\,\alpha_3 + 1}{\alpha_2 + 3}, \quad B = \frac{\alpha_3 + \alpha_2\alpha_3 + 2}{\alpha_2 + \alpha_3 + 2},$$
$$C = -\frac{2\alpha_2\alpha_3 + 4\,\alpha_3 + \alpha_2 + 5}{\alpha_3 + \alpha_2\alpha_3 + 2}, \quad D = \frac{\alpha_2 + 6 + 2\,\alpha_2\alpha_3 + 3\,\alpha_3}{1 + \alpha_2\alpha_3 + 2\,\alpha_3}.$$

Therefore, system (6.5) has only the previously studied phase portraits except when the parameters α_2 and α_3 belong the curve $\mathcal{L}_1 = \alpha_2\alpha_3 + 2\,\alpha_3 + 1 = 0$ or $\mathcal{L}_2 = \alpha_3 + \alpha_2\alpha_3 + 2 = 0$.

Table 6. Topological type of each equilibria of system (1.4) on the Poincaré sphere according to the values of α_2 and α_3.

Region	p_0	p_1	p_2	p_3	q_0	q_1	w_0	PP
R_1	S	SN	UN	C	S	UN	S	R_1
R_2	S	SN	UN	S	UN	S	UN	R_2
R_3	S	S	S	C	UN	SN	UN	R_3
R_4	S	SN	UN	C	S	SN	S	R_1
R_5	S	SN	UN	C	S	UN	S	R_1
R_6	UN	SN	S	C	UN	S	S	R_1
R_7	UN	S	SN	S	UN	SN	S	R_2
R_8	S	UN	SN	C	S	SN	S	R_1
R_9	S	S	S	C	UN	SN	UN	R_3
R_{10}	UN	SN	S	S	S	SN	UN	R_2
R_{11}	UN	SN	S	S	S	UN	UN	R_2
R_{12}	UN	S	SN	C	S	S	UN	R_1
R_{13}	UN	S	SN	C	S	S	UN	R_1
R_{14}	S	UN	SN	S	UN	S	UN	R_2
R_{15}	S	UN	SN	C	S	UN	S	R_1
R_{16}	S	UN	SN	C	S	SN	S	R_1
R_{17}	UN	S	SN	S	UN	SN	S	R_2
R_{18}	UN	SN	S	C	UN	S	S	R_1
P_1	S-N	SN	p_0	$\nexists$	-	-	-	P_1
P_2	-	-	-	-	$\nexists$	SN	S-N	P_2
P_3	-	-	-	-	$\nexists$	$\nexists$	UN	P_3
P_4	UN	-	-	-	-	-	-	P_4

Region	p_0	p_1	p_2	p_3	q_0	q_1	w_0	PP
$L_{1,2}$	S	SN	UN	$\nexists$	-	-	-	$L_{1,2}$
$L_{2,3}$	S	-	-	-	UN	$\nexists$	UN	$L_{2,3}$
$L_{3,4}$	S	$\nexists$	$\nexists$	C	S-N	SN	S-N	$L_{3,4}$
$L_{5,6}$	S-N	SN	p_0	C	S-N	q_0	S	$L_{5,6}$
$L_{2,11}$	S-N	SN	p_0	S	S-N	q_0	UN	$L_{2,11}$
$L_{1,18}$	S-N	SN	p_0	C	S-N	q_0	S	$L_{5,6}$
$L_{6,7}$	UN	-	-	-	UN	$\nexists$	S	$L_{2,3}$
$L_{9,14}$	S	-	-	-	UN	$\nexists$	UN	$L_{2,3}$
$L_{10,13}$	UN	-	-	-	S	$\nexists$	UN	$L_{2,3}$
$L_{11,12}$	UN	-	-	-	S	$\nexists$	UN	$L_{2,3}$
$L_{17,18}$	UN	-	-	-	UN	$\nexists$	S	$L_{2,3}$
$L_{7,10}$	UN	$\nexists$	$\nexists$	S	S-N	SN	S-N	$L_{7,10}$
$L_{8,9}$	S	$\nexists$	$\nexists$	C	S-N	SN	S-N	$L_{3,4}$
$L_{7,8}$	-	-	SN	-	$\nexists$	SN	S	$L_{7,8}$
$L_{9,10}$	-	-	S	-	$\nexists$	SN	UN	$L_{2,3}$
$L_{13,14}$	-	-	SN	-	$\nexists$	S	UN	$L_{7,8}$
$L_{16,17}$	-	-	SN	-	$\nexists$	SN	S	$L_{7,8}$
$L_{11,18}$	UN	SN	S	$\nexists$	-	-	-	$L_{1,2}$
$L_{12,17}$	UN	S	SN	$\nexists$	-	-	-	$L_{1,2}$
$L_{14,15}$	S	UN	SN	$\nexists$	-	-	-	$L_{1,2}$

System (6.5) for the parameters α_2 and α_3 on the curve $\mathcal{L}_1 = 0$, after a trivial change of coordinates, is given by

$$x' = x(1+y), \quad y' = y\Big(1 + x - \frac{1}{(\alpha_2+2)}y\Big). \tag{6.6}$$

When $\alpha_2 < -3$ or $-3 < \alpha_2 < -2$, the phase portrait is topologically equivalent to $L_{4,5}$ of Figure 2 and when $\alpha_2 > -2$, it is topologically equivalent to $L_{2,3}$ of Figure 2.

Finally, system (6.5) for the parameters α_2 and α_3 on the curve $\mathcal{L}_2 = 0$, after a trivial change of coordinates (considering $\alpha_2 \neq 1$), is given by

$$x' = x(1+y), \quad y' = y\Big(1 + x + \frac{1}{\alpha_2}y\Big). \tag{6.7}$$

System (6.7) is equivalent to system (6.6). If $\alpha_2 = 1$ and $\alpha_3 = -1$, system (6.5) is given by

$$x' = x\Big(1 + \frac{x}{2}\Big), \quad y' = y\Big(1 + \frac{x}{2}\Big). \tag{6.8}$$

The phase portrait of system (6.8) is topologically equivalent to P_1 of Figure 2.

Proof of Theorem 1.4 (c). We only prove the case when the parameters belong to the region R_1 since the other cases can be done in the same manner. In that case, the open arc of the infinity of the second octant in the local chart U_1 corresponding to the end of the plane $z = 0$ is formed by an orbit having as ω-limit the equilibrium $(0, 0, 0)$. The open arc of the infinity of the second octant corresponding to the end of the plane $y = 0$ is formed by an orbit having as α-limit the equilibrium $(0, 0, 0)$. The orbit on the x-axis near of $(0, 0, 0)$ has this equilibrium as its α-limit. Similar studies can be done for the equilibria located at the origin of the local charts U_2 and U_3 that is, at the end of the y- and z-axes, respectively. Hence the boundary of the infinity of the second octant is formed by a heteroclinic cycle formed by three equilibria coming from the ones located at the end of the negative x half-axis and the positive y and z half-axes, and the three orbits living on the three open arcs connecting these three points and contained in the boundary of the infinity of the second octant. From the study of the global dynamics on the Poincaré sphere we see that there is an additional equilibria in the interior of the heteroclinic cycle that is an stable attractor and it is the α-limit set of each point on the interior of the heteroclinic cycle.

Proof of Theorem 1.4 (d). System (1.1) becomes system (1.4). It is easy to see that

$$H = x^{(\alpha_2-1)(\alpha_3-1)}y^{(\alpha_2+3)(\alpha_3-1)}z^{(\alpha_2+3)(\alpha_3+3)}T(x,y,z)^{-\alpha_2(3\alpha_3+1)-5\alpha_3-7},$$

where $T(x,y,z) = (\alpha_3-1)(\alpha_2+\alpha_3+2)x + (\alpha_2+3)\big((1-\alpha_3)y + (\alpha_2+\alpha_3+2)z\big)$ is a first integral of system (1.4). Since now we are working with system (1.4), system (2.2) becomes

$$\begin{aligned} x' &= J(x,y,z)x\Big(1 - x + \frac{(\alpha_2\alpha_3 + 2\alpha_2 + \alpha_3 + 7)}{(\alpha_2+\alpha_3+2)}y + (\alpha_2+2)z\Big), \\ y' &= J(x,y,z)y(1 + (\alpha_3+2)x - y - \alpha_2 z), \\ z' &= J(x,y,z)z\Big(1 - \alpha_3 x - \frac{(\alpha_2\alpha_3+3)}{(\alpha_2+\alpha_3+2)}y - z\Big). \end{aligned} \tag{6.9}$$

We define

$$J = y^{-\frac{3}{\alpha_2-1}}z^{-\frac{3(\alpha_2+\alpha_3+2)}{(\alpha_2-1)(\alpha_3-1)}}R(x,y,z)^{\frac{3(\alpha_2\alpha_3+\alpha_3+2)}{(\alpha_2-1)(\alpha_3-1)}},$$

where $R(x,y,z) = (\alpha_3-1)(\alpha_2+\alpha_3+2)x - (\alpha_2+3)(\alpha_3-1)y + (\alpha_2+3)(\alpha_2+\alpha_3+2)z$. It is easy to see that J is a Jacobi multiplier of system (1.4) because one can check directly that the divergence of system (6.9) is zero. Now, since system (1.4) has the first integral H and a Jacobi multiplier, by Theorem 2.3, it is completely integrable. This completes the proof of Theorem 1.4(d) and so the proof of Theorem 1.4.

Acknowledgements. F. S. Dias was supported by FAPEMIG grant APQ-00628-24. R. Oliveira was supported by CNPq grants 310857/2023-6 and 407454/2023-3, DSYREKI- HORIZON-MSCA-2023-SE-01 101183111, and FAPESP grant 2019/21181-0. C. Valls was supported by FCT/Portugal through CAMGSD, IST-ID by UIDB/04459/2020 and UIDP/04459/2020.

References

[1] J. Alvavez-Ramírez, G. Blé, V. Castellanos, J. Llibre; *On the global flow of a 3-dimensional Lotka–Volterra system*, Nonlinear Anal., **75** (2012), 4114–4125.

[2] V. Antonov, D. Dolicanin, V. G. Romanovski, J. Tóth; *Invariant planes and periodic oscillations in the May–Leonard asymmetric model.* MATCH. Commun. Math. Comput. Chem., **76** (2016), 455–474.

[3] C.H. Chi, S.B. Hsu, L.T. Wu; *On the asymmetric May–Leonard model of three competing species*, SIAM J. Appl. Math., **58** (1998), 211–226.

[4] F. Dumortier, J. Llibre, J. C. Artés; *Qualitative Theory of Planar Differential Systems, Springer–Verlag, Berlin, 2006.*

[5] A. Goriely; Integrability and nonintegrability of dynamical systems*, Advanced Series in Nonlinear Dynamics 19, World Scientific Publishing Co., Inc., River Edge, NJ, 2001.*

[6] R. M. May, W. J. Leonard; Nonlinear aspects of competition between three species*, SIAM J. Appl. Math.,* ***29*** *(1975), 243–253.*

[7] R. D. Oliveira, C. Valls; Global Dynamics od the May-Leonard System with a Darboux Invariant*, Electron. J. Differ. Equ.,* ***2020*** *(2020), no. 55 pp. 1-19.*

[8] Q. van der Hoff, J. C. Greeff, T. H. Fay; Defining a stability boundary for three species competition models*, Ecol. Model.,* ***220*** *(2009), 2640–2645.*

[9] E. T. Whittaker; A treatise on the analytic dynamics of particles and rigid bodies*, Dover, New York, 1944.*

[10] M. L. Zeeman; Hopf bifurcations in competitive three dimensional Lotka-Volterra systems*, Dyn. Stab. Syst.* ***8*** *(1993), 189–216.*

Fabio Scalco Dias
Instituto de Matemática e Computação, Universidade Federal de Itajubá, Avenida BPS 1303, Pinheirinho, CEP 37.500-903, Itajubá, MG, Brazil
Email address: scalco@unifei.edu.br

Regilene Oliveira
Departamento de Matemática, ICMC-Universidade de São Paulo, Avenida Trabalhador São-carlense, 400 - 13566-590, São Carlos, SP, Brazil
Email address: regilene@icmc.usp.br

Cláudia Valls
Departamento de Matemática, Instituto SuperiorTécnico, Universidade Técnica de Lisboa, Av. Rovisco Pais 1049-001, Lisboa, Portugal
Email address: claudia.valls@tecnico.pt