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CONSTRUCTION OF SINGLE-PEAK SOLUTIONS FOR GRUSHIN EQUATIONS VIA REDUCTION METHOD

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ABSTRACT. In this article, we construct single-peak solutions for a class of Grushin equations by using the Lyapunov-Schmidt reduction method, whose concentration points locate on some special critical points of the potential function.

1. INTRODUCTION

We consider the degenerate elliptic equation

$$\begin{gathered} -\varepsilon^2 \Delta_\gamma u + a(z)u = u^{q-1}, \quad u > 0, \text{ in } \mathbb{R}^{N+l}, \\ u \in H^{1,2}_\gamma(\mathbb{R}^{N+l}), \end{gathered} \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, $2 < q < \frac{2N_\gamma}{N_\gamma - 2}$, and $a(z) \in C^2(\mathbb{R}^{N+l})$ satisfies $0 < a_0 \le a(z) \le a_1$ in $\mathbb{R}^{N+l}$. Moreover, a_0 and a_1 satisfy $0 < a_0 \le 1 \le a_1$, which are the lower and upper bounds of $a(z)$. The definition of $H^{1,2}_\gamma(\mathbb{R}^{N+l})$ can be found in (1.7) below.

Δ_γ is the well-known *Grushin operator* given by

$$\Delta_\gamma u(z) = \Delta_x u(z) + |x|^{2\gamma} \Delta_y u(z). \tag{1.2}$$

Δ_x and Δ_y are the Laplace operators in the variable x and y respectively, with $z = (x, y) \in \mathbb{R}^N \times \mathbb{R}^l = \mathbb{R}^{N+l}$ and $N > 1$, $l > 1$, $N + l \ge 3$. Here, $\gamma \ge 0$ is a real number and

$$N_\gamma = N + (1 + \gamma)l \tag{1.3}$$

is the appropriate homogeneous dimension. It is worth noting that Δ_γ is elliptic for $x \ne 0$ and degenerates on $\{0\} \times \mathbb{R}^l$. When $\gamma > 0$ is an integer, Δ_γ is the Hörmander operator and is hypoelliptic. More about Hörmander condition and hypoellipticity could be found in [13]. Geometrically, Δ_γ comes from a sub-Laplace operator on a nilpotent Lie group of step $\gamma + 1$ by a submersion. Specific explanation of geometric framework can be consulted in [3] by Bauer et al.

In [5], the authors studied the existence of solutions for the perturbed Schrödinger equation

$$\begin{gathered} -\varepsilon^2 \Delta u + V(x)u = u^p, \quad u > 0, \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{gathered} \tag{1.4}$$

where $\varepsilon > 0$ is a parameter, $N \ge 3$, $p \in (2, \frac{2N}{N-2})$, and $V(x)$ is a potential function in $\mathbb{R}^N$. First, they determined the location of the concentration points for solutions of the above elliptic problem by local Pohozaev identities. Second, they demonstrated the existence of peak solutions at the concentration points. Third, they studied the local uniqueness of peak solutions for (1.4) by applying local Pohozaev identities again. Similarly, in our research we consider the first two issues for problem (1.1). In particular, the first issue has been well resolved in [25]. Here, we focus primarily on the second issue for (1.1).

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There are important results on the existence of solutions for problem (1.4). An earlier result is [8], in which Floer and Weinstein obtained a solution concentrated at the global non-degenerate minimum point when ε is small enough and $N = 1, p = 3$. Later, Oh [22] generalized Floer-Weinstein's results to higher dimension with $2 < p < 2N/(N-2)$ and obtained the existence of positive multi-peak solutions concentrated at any given set of non-degenerate critical points of $V(x)$ as $\varepsilon \to 0$. In fact, the results in [8, 22] seem to rely essentially on the non-degeneracy of the critical points. Also, Ambrosetti, Badiale, Cingolani [1] proved that there exists a single-peak solution concentrated at the critical point of $V(x)$ which may be degenerate as $\varepsilon \to 0$ by Lyapunov-Schmidt reduction. On the other hand, by using variational methods, Rabinowitz [23] proved the existence of a positive ground solution to (1.4) under some conditions on $V(x)$. Rencently, Li et al. [20] demonstrated an interesting dichotomy phenomenon for concentrating solutions of the above Schrödinger equation. And one can refer to [6, 7, 10, 12] for further results.

Authors in [14] constructed the multi-peak solutions for the Chern-Simons-Schrödinger system. The crucial step to apply implicit function theorem is to prove linear operator L_ε restricted to $E_{\varepsilon,Y}$ is invertible, where L_ε is the linear operator of Chern-Simons-Schrödinger system

$$\begin{gathered} -\varepsilon^2\Delta u + V(x)u + (A_0 + A_1^2 + A_2^2)u = |u|^{p-2}u, \quad x \in \mathbb{R}^2, \\ \partial_1 A_0 = A_2 u^2, \quad \partial_2 A_0 = -A_1 u^2, \\ \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}|u|^2, \quad \partial_1 A_1 + \partial_2 A_2 = 0, \end{gathered} \tag{1.5}$$

and

$$E_{\varepsilon,Y} := \{\varphi \in H_\varepsilon : \langle \frac{\partial U^i_{\varepsilon,y_i}}{\partial y^i_l}, \varphi\rangle_\varepsilon = 0, i = 1,\dots,k,\ l = 1,2\} \tag{1.6}$$

is the complementary space of the approximate kernel of L_ε. In (1.6),

$$H_\varepsilon = \big\{u \in H^1(\mathbb{R}^2) : \|u\|_\varepsilon^2 = \int_{\mathbb{R}^2} \varepsilon^2|\nabla u|^2 + V(x)|u|^2 dx < \infty\big\}.$$

Thus it is vital to find the kernel or the approximate kernel of linear operator for the corresponding equation. The approximate kernel of linear operator for (1.5) is $K_{\varepsilon,Y} = \operatorname{span}\{\frac{\partial U^i_{\varepsilon,y_i}}{\partial y^i_l}, i = 1,\dots,k, l = 1,2\}$ exactly. This is the result for the Laplacian equation, which has been studied wildly such as in [15, 19]. From [14], the core of finite dimension reduction is that one should first derive the solution in $E_{\varepsilon,Y}$ by using the contraction mapping theorem, and then prove this solution is the solution in whole space H_ε, by using a topological degree theorem, where $H_\varepsilon = E_{\varepsilon,Y} \oplus K_{\varepsilon,Y}$. Existence of such Laplace equations has been widely studied in many literature. One can refer to [9, 11, 16, 17, 19, 22, 24] and the references therein.

However there are few results for degenerate elliptic equations in this aspect. Liu and his collaborators [18] proved that a critical Grushin-type problem has infinitely many positive multi-bubbling solutions with arbitrarily large energy and cylindrical symmetry by mainly applying the Lyapunov-Schmidt reduction argument. In this paper, we consider the existence of solutions for Grushin equation (1.1), which is based on the kernel space of linear operator we have studied in [26]. Now we introduce some notation and preliminaries that will be used later.

Definition 1.1. When $\gamma \geq 0$, we define

$$H^{1,2}_\gamma(\mathbb{R}^{N+l}) = \big\{u \in L^2(\mathbb{R}^{N+l}) \mid \frac{\partial u}{\partial x_i}, |x|^\gamma \frac{\partial u}{\partial y_j} \in L^2(\mathbb{R}^{N+l}), i = 1,\dots,N, j = 1,\dots,l.\big\} \tag{1.7}$$

as a weighted Sobolev space.

If $u \in H^{1,2}_\gamma(\mathbb{R}^{N+l})$, we denote the gradient operator as

$$\nabla_\gamma u = (\nabla_x u, |x|^\gamma \nabla_y u) = (u_{x_1},\dots,u_{x_N}, |x|^\gamma u_{y_1},\dots,|x|^\gamma u_{y_l}), \tag{1.8}$$

and

$$|\nabla_\gamma u|^2 = |\nabla_x u|^2 + |x|^{2\gamma}|\nabla_y u|^2. \tag{1.9}$$

Then $H^{1,2}_\gamma(\mathbb{R}^{N+l})$ is a Hilbert space, endowed with the inner product

$$\langle u,v\rangle_\gamma=\int_{\mathbb{R}^{N+l}}\nabla_\gamma u\cdot\nabla_\gamma v+uvdz, \tag{1.10}$$

and the corresponding norm

$$\|u\|_{H^{1,2}_\gamma(\mathbb{R}^{N+l})}=\Big(\int_{\mathbb{R}^{N+l}}|\nabla_\gamma u|^2+|u|^2dz\Big)^{1/2}. \tag{1.11}$$

Remark 1.2. For studying problem (1.1), we define a new norm in $H^{1,2}_\gamma(\mathbb{R}^{N+l})$ as follows:

$$\|u\|_\varepsilon:=\Big(\int_{\mathbb{R}^{N+l}}\varepsilon^2|\nabla_\gamma u|^2+a(z)u^2dz\Big)^{1/2}, \tag{1.12}$$

which is induced by the inner product

$$\langle u,v\rangle_\varepsilon:=\int_{\mathbb{R}^{N+l}}\varepsilon^2\nabla_\gamma u\cdot\nabla_\gamma v+a(z)uvdz,\quad \forall u,v\in H^{1,2}_\gamma(\mathbb{R}^{N+l}). \tag{1.13}$$

We define

$$\|u\|_1:=\Big(\int_{\mathbb{R}^{N+l}}\varepsilon^2|\nabla_\gamma u|^2+u^2dz\Big)^{1/2}. \tag{1.14}$$

By our assumptions: $0<a_0\le a(z)\le a_1$ and $0<a_0\le 1\le a_1$, we can find that $\|u\|_\varepsilon$ and $\|u\|_1$ are equivalent norms in $H^{1,2}_\gamma(\mathbb{R}^{N+l})$.

Definition 1.3. *Let the following be a new distance on $\mathbb{R}^{N+l}$:*

$$d(z,0)=\Big(\frac{1}{(1+\gamma)^2}|x|^{2+2\gamma}+|y|^2\Big)^{\frac{1}{2+2\gamma}} \tag{1.15}$$

for $z=(x,y)\in\mathbb{R}^{N+l}$, and set

$$\widetilde{B}_r(0):=\{z=(x,y)\in\mathbb{R}^{N+l}|\ d(z,0)<r\}. \tag{1.16}$$

be a ball in the sense of this new distance.

Definition 1.4. *We say that u_ε is a single-peak solution of Equation* (1.1) *concentrated at $z_0=(x_0,y_0)$ if u_ε satisfies*

(i) *$u_{\mathcal{E}}$ has a local maximum point $z_\varepsilon=(x_\varepsilon,y_\varepsilon)\in\mathbb{R}^{N+l}$ such that $z_\varepsilon\to z_0\in\mathbb{R}^{N+l}$,as $\varepsilon\to 0$;*

(ii) *For any given $\tau>0$, there exists $R\gg 1$ such that*

$$|u_\varepsilon(z)|\le\tau\quad \textit{for } z\in\mathbb{R}^{N+l}\backslash\widetilde{B}_{R\varepsilon}(z_\varepsilon);$$

(iii) *There exists $M>0$ such that*

$$u_\varepsilon\le M.$$

Remark 1.5. When performing blow-up analysis for the Grushin operator, the blow up points must lie on the set $\{x=0\}$, otherwise the term $|x|^\gamma$ would change. For example, in [2], for $u\in D^{1,2}_\gamma(\mathbb{R}^{N+l})$ and $\rho>0$, a rescaled sequence of functions of the form $u^{e,\rho}(z):=\rho^{\frac{N_\gamma-2}{2}}u(\rho x,\rho^{1+\gamma}y+e)$ is also defined, where $z=(x,y)\in\mathbb{R}^{N+l}$ and $e\in\mathbb{R}^l$. In which, $D^{1,2}_\gamma(\mathbb{R}^{N+l})$ is the closure of $C^\infty_0(\mathbb{R}^{N+l})$ with respect to the norm $\|v\|_{D^{1,2}_\gamma(\mathbb{R}^{N+l})}:=\big(\int_{\mathbb{R}^{N+l}}|\nabla_\gamma v|^2dz\big)^{1/2}$. Therefore, it is reasonable to assume that the maximum value points of all solutions are located in $\{x=0\}$. So hereafter, we assume that the maximum point is in the form of $z_\varepsilon=(0,y_\varepsilon)$.

Now we state our main result. For (1.1), we want to find a solution of the form

$$u_\varepsilon(z)=U_{\varepsilon,z_\varepsilon}(z)+\omega_\varepsilon(z), \tag{1.17}$$

where $U_{\varepsilon,z_\varepsilon}(z)$ is the solution of

$$\begin{gathered}-\varepsilon^2\Delta_\gamma u+a(z_\varepsilon)u=u^{q-1},\quad u>0,\ \text{in } \mathbb{R}^{N+l},\\ u(z_\varepsilon)=\max_{z\in\mathbb{R}^{N+l}}u(z),\quad u\in H^{1,2}_\gamma(\mathbb{R}^{N+l}),\end{gathered} \tag{1.18}$$

and

$$z_\varepsilon \to z_0, \quad \text{as } \varepsilon \to 0, \quad \text{where } z_\varepsilon = (0, y_\varepsilon),\ z_0 = (0, y_0). \tag{1.19}$$

We regard $U_{\varepsilon,z_\varepsilon}(z)$ as an approximate solution to Equation (1.1) and $\omega_\varepsilon(z)$ is a minor term in the sense that

$$\|\omega_\varepsilon\|_\varepsilon = O(\varepsilon^{\frac{N_\gamma}{2}+1}). \tag{1.20}$$

Then our result is the following.

Theorem 1.6. *Assume that $z_0 = (0, y_0)$ are critical points of $a(z)$ satisfying*

$$\deg\Big(\nabla_y a(z), \widetilde{B}_\delta(z_0), 0\Big) \neq 0. \tag{1.21}$$

Then there exists $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0]$, (1.1) has solutions in the form of

$$u_\varepsilon(z) = U_{\varepsilon,z_\varepsilon}(z) + \omega_\varepsilon(z), \tag{1.22}$$

for some $z_\varepsilon = (0, y_\varepsilon)$ with $z_\varepsilon \to z_0$ and $\|\omega_\varepsilon\|_\varepsilon = O(\varepsilon^{\frac{N_\gamma}{2}+1})$, where $U_{\varepsilon,z_\varepsilon}(z)$ is the solution of (1.18).

The main contributions of this paper are summarized in the following three points. First, we apply the Lyapunov-Schmidt reduction argument to degenerate elliptic equations. Second, we construct solutions of (1.1) in the form of (1.22), where $U_{\varepsilon,z_\varepsilon}(z)$ is obtained by moving plane method basically due to [4] we have studied. Third, we construct solutions of (1.1) in the form of (1.22), where $\omega_\varepsilon(z)$ is obtained by reduction method and is based on the kernel space of linear operator that we have studied in [26].

This article is organized as follows. In Section 2, we obtain the location of concentration points for solutions of (1.1) by using Pohozaev identities generated from translations, which has been studied by us in [25]. In Section 3, we get ready to construct single peak solutions of (1.1), which uses the studies on the properties of main term in [4] and the kernel space in [26]. In Section 4, we solve a finite dimension problem by applying the Lyapunov-Schmidt reduction argument. In Section 5, we prove Theorem 1.6 to derive the existence of single-peak solutions for (1.1) by using the topological degree method.

2. Locating the concentration points

The main tool to determine the location of the concentration points is the local Pohozaev identities generated from translations for solutions of the following degenerate elliptic problem

$$-\Delta_\gamma u = f(z, u), \quad \text{in } \Omega, \tag{2.1}$$

where Ω is a bounded open domain in $\mathbb{R}^{N+l}$ and $f : \mathbb{R}^{N+l} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Let

$$F(z, u) = \int_0^u f(z, s)ds, \tag{2.2}$$

then we have

$$F_{x_i}(z, u) = \frac{\partial F(z, u)}{\partial x_i} + f(z, u)\frac{\partial u}{\partial x_i}, \quad i = 1, \dots, N; \tag{2.3}$$

$$F_{y_j}(z, u) = \frac{\partial F(z, u)}{\partial y_j} + f(z, u)\frac{\partial u}{\partial y_j}, \quad j = 1, \dots, l. \tag{2.4}$$

Let $D \subset\subset \Omega$, we have established two Pohozaev identities generated from translations for solutions of Equation (2.1) in [25] as follows.

Proposition 2.1 ([25, Theorem 1.1])**.** *If $u \in H^{1,2}_\gamma(\Omega) \cap C^2(\Omega)$ is the solution of Equation (2.1), then u satisfies*

$$\begin{aligned} &\frac{1}{2}\int_{\partial D} |\nabla_\gamma u|^2 \nu_x^i dS - \int_{\partial D}\Big(\frac{\partial u}{\partial \nu_x} + |x|^{2\gamma}\frac{\partial u}{\partial \nu_y}\Big)\frac{\partial u}{\partial x_i}dS - \gamma\int_D |\nabla_y u|^2|x|^{2(\gamma-1)}x_i dz \\ &= \int_{\partial D} F(z, u)\nu_x^i dS - \int_D \frac{\partial F(z, u)}{\partial x_i}dz, \quad i = 1, \dots, N, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned}&\frac{1}{2}\int_{\partial D}|\nabla_\gamma u|^2\nu_y^j dS-\int_{\partial D}\Big(\frac{\partial u}{\partial \nu_x}+|x|^{2\gamma}\frac{\partial u}{\partial \nu_y}\Big)\frac{\partial u}{\partial y_j}dS\\&=\int_{\partial D}F(z,u)\nu_y^j dS-\int_D\frac{\partial F(z,u)}{\partial y_j}dz,\quad j=1,\dots,l,\end{aligned}\tag{2.6}$$

where $\nu=(\nu_x,\nu_y)$ *is the unit outward normal of the point of* ∂D*.*

More generally, we consider the degenerate elliptic equation with Grushin type p-sub-Laplacian,

$$-\Delta_\gamma^p u=f(z,u),\quad \text{in }\Omega,\tag{2.7}$$

where Ω is a bounded open domain in $\mathbb{R}^{N+l}$ and $f:\mathbb{R}^{N+l}\times\mathbb{R}\to\mathbb{R}$ is a continuous function. For any $p>1$, Grushin type p-sub-Laplacian is denoted as

$$\Delta_\gamma^p u=div_\gamma(|\nabla_\gamma u|^{p-2}\nabla_\gamma u).\tag{2.8}$$

Different from the method in [25], by applying domain variations we obtain the local Pohozaev identities of translating type for solutions of Equation (2.7) in [27] as follows.

Proposition 2.2 ([27, Theorem 1.1])**.** *If* $u\in W^{1,p}_{\lambda,0}(\Omega)\cap C^1(\Omega)$ *is the solution of equation* (2.7)*, then* u *satisfies*

$$\begin{aligned}&\frac{1}{p}\int_{\partial D}|\nabla_\gamma u|^p\nu_x^i dS-\int_{\partial D}|\nabla_\gamma u|^{p-2}\frac{\partial u}{\partial x_i}\langle\nabla_\gamma u,\nu_\gamma\rangle dS-\int_D|\nabla_\gamma u|^{p-2}\gamma|x|^{2(\gamma-1)}x_i|\nabla_y u|^2dz\\&=\int_{\partial D}F(z,u)\nu_x^i dS-\int_D\frac{\partial F(z,u)}{\partial x_i}dz,\quad i=1,\dots,N,\end{aligned}\tag{2.9}$$

and

$$\begin{aligned}&\frac{1}{p}\int_{\partial D}|\nabla_\gamma u|^p\nu_y^j dS-\int_{\partial D}|\nabla_\gamma u|^{p-2}\frac{\partial u}{\partial y_j}\langle\nabla_\gamma u,\nu_\gamma\rangle dS\\&=\int_{\partial D}F(z,u)\nu_y^j dS-\int_D\frac{\partial F(z,u)}{\partial y_j}(z,u)dz,\quad j=1,\dots,l,\end{aligned}\tag{2.10}$$

where $\nu=(\nu_x,\nu_y)$ *is the unit outward normal of the point of* ∂D *and* $\nu_\gamma=(\nu_x,|x|^\gamma\nu_y)$*.*

Especially, taking $p=2$ in the above proposition, we return to the results in Proposition 2.1. It follows from (2.5) and (2.6) that any solution u of Equation (1.1) in $H^{1,2}_\gamma(\mathbb{R}^{N+l})\cap C^1(\mathbb{R}^{N+l})$ satisfies

$$\begin{aligned}&\frac{1}{2}\varepsilon^2\int_{\partial D}|\nabla_\gamma u|^2\nu_x^i dS-\varepsilon^2\int_{\partial D}\Big(\frac{\partial u}{\partial \nu_x}+|x|^{2\gamma}\frac{\partial u}{\partial \nu_y}\Big)\frac{\partial u}{\partial x_i}dS-\varepsilon^2\gamma\int_D|\nabla_y u|^2|x|^{2(\gamma-1)}x_i dz\\&=\frac{1}{q}\int_{\partial D}u^q\nu_x^i dS-\frac{1}{2}\int_{\partial D}a(z)u^2\nu_x^i dS+\frac{1}{2}\int_D\frac{\partial a(z)}{\partial x_i}u^2dz,\quad i=1,\dots,N,\end{aligned}\tag{2.11}$$

and

$$\begin{aligned}&\frac{1}{2}\varepsilon^2\int_{\partial D}|\nabla_\gamma u|^2\nu_y^j dS-\varepsilon^2\int_{\partial D}\Big(\frac{\partial u}{\partial \nu_x}+|x|^{2\gamma}\frac{\partial u}{\partial \nu_y}\Big)\frac{\partial u}{\partial y_j}dS\\&=\frac{1}{q}\int_{\partial D}u^q\nu_y^j dS-\frac{1}{2}\int_{\partial D}a(z)u^2\nu_y^j dS+\frac{1}{2}\int_D\frac{\partial a(z)}{\partial y_j}u^2dz,\quad j=1,\dots,l,\end{aligned}\tag{2.12}$$

where D is any domain of $\mathbb{R}^{N+l}$.

In this paper, our solution is assumed to satisfy Definition 1.4. Next, we determine the location of the concentration point z_0. For this purpose, we have estimated the behavior of the solution u_ε and several surface integrals away from the maximum point z_ε in the following proposition by using maximum principles for unbounded domain in [4] and L^2 estimate for Grushin operator in [21] respectively.

Proposition 2.3 ([25, Lemma 2.1])**.** *Assume that* u_ε *is a solution of Equation* (1.1) *satisfying Definition 1.4 in a subset of* $H^{1,2}_\gamma(\mathbb{R}^{N+l})$*. Then for any* $\alpha\in(0,\inf_{z\in\mathbb{R}^{N+l}}a(z))$*, there exists a constant* $C>0$*, such that*

$$u_\varepsilon(z)\le Ce^{-\frac{\sqrt{\alpha}d(z,z_\varepsilon)}{\varepsilon}},\quad \forall z\in\mathbb{R}^{N+l},\tag{2.13}$$

where $C = (M+1)e^{\sqrt{\alpha}R}$*. Moreover, if we denote*

$$\begin{gathered} J_1 = \int_{\partial \widetilde{B}_\delta(z_\varepsilon)} |\nabla_\gamma u_\varepsilon|^2 \nu_x^i dS, \\ J_2 = \int_{\partial \widetilde{B}_\delta(z_\varepsilon)} \Big(\frac{\partial u_\varepsilon}{\partial \nu_x} + |x|^{2\gamma} \frac{\partial u_\varepsilon}{\partial \nu_y}\Big) \frac{\partial u_\varepsilon}{\partial x_i} dS, \\ J_3 = \int_{\partial \widetilde{B}_\delta(z_\varepsilon)} |\nabla_\gamma u_\varepsilon|^2 \nu_y^j dS, \\ J_4 = \int_{\partial \widetilde{B}_\delta(z_\varepsilon)} \Big(\frac{\partial u_\varepsilon}{\partial \nu_x} + |x|^{2\gamma} \frac{\partial u_\varepsilon}{\partial \nu_y}\Big) \frac{\partial u_\varepsilon}{\partial y_j} dS, \end{gathered}$$

then we have

$$J_i \le \widetilde{C} \int_{\widetilde{B}_\delta(z_\varepsilon)} e^{-\frac{\sqrt{\alpha} d(z, z_\varepsilon)}{\varepsilon}} dz, \quad i = 1, \dots, 4, \tag{2.14}$$

where $\delta > 0$ *and* $\widetilde{C} > 0$.

By using Pohozaev identities (2.11) and (2.12) together with Proposition 2.3, we have proved that the location of the concentration point z_0 is at some special critical points of the potential function $a(z)$ in the following proposition.

Proposition 2.4 ([25, Corollary 1.1]). *Let* $z_{\mathcal{E}}$ *be the local maximum point of the solution* $u_{\mathcal{E}}$ *of Equation* (1.1) *satisfying Definition 1.4 in a subset of* $H_\gamma^{1,2}(\mathbb{R}^{N+l})$*. If* $a(z)$ *has critical points in the form of* $(0, y)$*, then* $z_0 = (x_0, y_0)$ *satisfies* $\nabla a(z_0) = 0$ *and* $x_0 = 0$*.*

By now, we obtain a necessary condition for the concentration point z_0 of the solution satisfying Definition 1.4 for Equation (1.1), which basically due to our researches in [25]. In next section, we will consider the converse of such problem, the existence of such solutions concentrating at the points satisfying the necessary condition.

3. Preparations for constructing single peak solutions

In this section, we shall get ready to construct single peak solutions satisfying Definition 1.4 for Equation (1.1). More specifically, for (1.1), we want to find a solution of the form

$$u_\varepsilon(z) = U_{\varepsilon, z_\varepsilon}(z) + \omega_\varepsilon(z),$$

where $U_{\varepsilon, z_\varepsilon}(z)$ is the solution of (1.18). We regard $U_{\varepsilon, z_\varepsilon}(z)$ as an approximate solution to Equation (1.1) and $\omega_\varepsilon(z)$ is a minor term in the sense that

$$\|\omega_\varepsilon\|_\varepsilon = O(\varepsilon^{\frac{N_\gamma}{2}+1}).$$

Let $w(\widetilde{z})$ be the solution of

$$\begin{aligned} &-\Delta_\gamma w + w = w^{q-1}, \quad w > 0, \text{ in } \mathbb{R}^{N+l}, \\ &w(0) = \max_{\widetilde{z} \in \mathbb{R}^{N+l}} w(\widetilde{z}), \quad w \in H_\gamma^{1,2}(\mathbb{R}^{N+l}). \end{aligned} \tag{3.1}$$

The properties of solutions for this equation is very clear. Actually in [4], we have showed the partial radial symmetry and the decay rate at infinity of solutions to the equation (3.1) in the following proposition.

Proposition 3.1 ([4, Theorem 1.1]). *Let* $w(\widetilde{z}) \in H_\gamma^{1,2}(\mathbb{R}^{N+l}) \cap C^0(\mathbb{R}^{N+l})$ *be a solution to Equation* (3.1) *with* $q > 2$*. Then* $w(\widetilde{z}) = w(\widetilde{x}, \widetilde{y})$ *is radially symmetric with respect to the variable* $\widetilde{y}$ *and has exponential decay at infinity.*

By using scaling transformation, we can easily find that the relationship between $U_{\varepsilon, z_\varepsilon}(z)$ and $w(\widetilde{z})$ is

$$U_{\varepsilon, z_\varepsilon}(z) = \big(a(z_\varepsilon)\big)^{\frac{1}{q-2}} w\Big(\frac{\sqrt{a(z_\varepsilon)}}{\varepsilon} x, \Big(\frac{\sqrt{a(z_\varepsilon)}}{\varepsilon}\Big)^{1+\gamma} (y - y_\varepsilon)\Big). \tag{3.2}$$

Thus, the major term $U_{\varepsilon, z_\varepsilon}(z)$ is founded by Proposition 3.1 and relation (3.2). Next, we should find the minor term $\omega_\varepsilon(z)$ in (1.17).

It is easy to find that $\omega_\varepsilon(z)$ satisfies

$$\begin{gathered} L'_\varepsilon \omega_\varepsilon = l'_\varepsilon + R'_\varepsilon(\omega_\varepsilon), \quad z \in \mathbb{R}^{N+l}, \\ \omega_\varepsilon \in H^{1,2}_\gamma(\mathbb{R}^{N+l}), \end{gathered} \tag{3.3}$$

where

$$L'_\varepsilon \omega_\varepsilon = -\varepsilon^2 \Delta_\gamma \omega_\varepsilon(z) + a(z)\omega_\varepsilon(z) - (q-1)U^{q-2}_{\varepsilon,z_\varepsilon}(z)\omega_\varepsilon(z), \tag{3.4}$$

$$l'_\varepsilon = (a(z_\varepsilon) - a(z))\, U_{\varepsilon,z_\varepsilon}(z), \tag{3.5}$$

$$R'_\varepsilon(\omega_\varepsilon) = (U_{\varepsilon,z_\varepsilon}(z) + \omega_\varepsilon(z))^{q-1} - U^{q-1}_{\varepsilon,z_\varepsilon}(z) - (q-1)U^{q-2}_{\varepsilon,z_\varepsilon}(z)\omega_\varepsilon(z). \tag{3.6}$$

To employ the theory of weighted Sobolev spaces, we write the problem (3.3) in its weak form as

$$\begin{gathered} L_\varepsilon \omega_\varepsilon = l_\varepsilon + R_\varepsilon(\omega_\varepsilon), \quad z \in \mathbb{R}^{N+l}, \\ \omega_\varepsilon \in H^{1,2}_\gamma(\mathbb{R}^{N+l}), \end{gathered} \tag{3.7}$$

where $L_\varepsilon : H^{1,2}_\gamma(\mathbb{R}^{N+l}) \to H^{1,2}_\gamma(\mathbb{R}^{N+l})$ is a bounded linear operator defined by

$$\langle L_\varepsilon \omega_\varepsilon, \psi\rangle_\varepsilon = \int_{\mathbb{R}^{N+l}} \varepsilon^2 \nabla_\gamma \omega_\varepsilon(z) \cdot \nabla_\gamma \psi(z) + a(z)\omega_\varepsilon(z)\psi(z) - (q-1)U^{q-2}_{\varepsilon,z_\varepsilon}(z)\omega_\varepsilon(z)\psi(z)dz, \tag{3.8}$$

$l_\varepsilon \in H^{1,2}_\gamma(\mathbb{R}^{N+l})$ satisfies

$$\langle l_\varepsilon, \psi\rangle_\varepsilon = \int_{\mathbb{R}^{N+l}} (a(z_\varepsilon) - a(z))\, U_{\varepsilon,z_\varepsilon}(z)\psi(z)dz, \tag{3.9}$$

and $R_\varepsilon : H^{1,2}_\gamma(\mathbb{R}^{N+l}) \to H^{1,2}_\gamma(\mathbb{R}^{N+l})$ is a nonlinear map defined by

$$\langle R_\varepsilon(\omega_\varepsilon), \psi\rangle_\varepsilon = \int_{\mathbb{R}^{N+l}} \big[(U_{\varepsilon,z_\varepsilon}(z) + \omega_\varepsilon(z))^{q-1} - U^{q-1}_{\varepsilon,z_\varepsilon}(z) - (q-1)U^{q-2}_{\varepsilon,z_\varepsilon}(z)\omega_\varepsilon(z)\big]\psi dz, \tag{3.10}$$

for any $\psi \in H^{1,2}_\gamma(\mathbb{R}^{N+l})$. And $R_\varepsilon(\omega_\varepsilon)$ satisfies

$$R_\varepsilon(\omega_\varepsilon) = o(\omega_\varepsilon), \quad \text{as } \omega_\varepsilon \to 0. \tag{3.11}$$

Linear operator L_ε in (3.8) is not always invertible, so we cannot directly use contraction mapping theorem to derive ω_ε from (3.7). To construct a single peak solution for Equation (1.1), we need to find out a kernel space or an approximate kernel of L'_ε.

Fortunately in [26], we have showed that the approximate kernel K_ε of L'_ε in $H^{1,2}_\gamma(\mathbb{R}^{N+l})$ is given by

$$K_\varepsilon := \operatorname{span}\Big\{\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_1}, \frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_2}, \dots, \frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_l}\Big\}, \tag{3.12}$$

which is the kernel of linear operator

$$\widetilde{L}_\varepsilon \omega_\varepsilon := -\varepsilon^2 \Delta_\gamma \omega_\varepsilon + a(z_\varepsilon)\omega_\varepsilon - (q-1)U^{q-2}_{\varepsilon,z_\varepsilon}\omega_\varepsilon. \tag{3.13}$$

We define

$$E_\varepsilon = K_\varepsilon^\perp := \{\omega \in H^{1,2}_\gamma(\mathbb{R}^{N+l}) : \langle \omega, \frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}\rangle_\varepsilon = 0, \quad j = 1, \dots, l\}. \tag{3.14}$$

In [26], we have proved that the linear operator $L_{\mathcal{E}}$ is invertible when restricted to the space E_ε in the following proposition.

Proposition 3.2 ([4, Theorem 1.3]). *Let L_ε be the linear operator defined in* (3.8)*, K_ε be the kernel space of linear operator* (3.13) *defined in* (3.12)*, E_ε be the the complement space of K_ε defined in* (3.14)*, Q_ε be the projection from $H^{1,2}_\gamma(\mathbb{R}^{N+l})$ to E_ε as follows*

$$Q_\varepsilon u = u - \sum_{j=1}^{l} b_j \frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}, \tag{3.15}$$

where $U_{\varepsilon,z_\varepsilon}(z)$ is defined in (3.2) *and b_j satisfies*

$$\langle Q_\varepsilon u, \frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_k}\rangle_\varepsilon = 0, \quad \textit{for } k = 1, \dots, l. \tag{3.16}$$

Then there exist $\varepsilon_0 > 0$, $\theta_0 > 0$ and $\rho > 0$, such that for any $\varepsilon \in (0, \varepsilon_0]$ and $z_\varepsilon \in \widetilde{B}_{\theta_0}(z_0)$, $Q_\varepsilon L_\varepsilon$ is a bijective mapping in E_ε, moreover

$$\|Q_\varepsilon L_\varepsilon \omega_\varepsilon\|_\varepsilon \geq \rho \|\omega_\varepsilon\|_\varepsilon, \quad \forall \omega_\varepsilon \in E_\varepsilon. \tag{3.17}$$

The above proposition plays an essential role in carrying out the reduction argument. In next section, we will carry out the reduction for equation (3.7).

4. Reduction argument

In this section, we are now ready to carry out the reduction for (3.7). That is , we first consider the equation (3.7) restricted to E_ε to obtain the solution, and then prove that this solution holds on the whole space $H^{1,2}_\gamma(\mathbb{R}^{N+l})$. To use the the contraction mapping theorem to carry out the reduction, we now estimate $\|l_\varepsilon\|_\varepsilon$ and $\|R_\varepsilon(\omega_\varepsilon)\|_\varepsilon$.

Lemma 4.1. *Under the assumption*

$$a_0 \leq \min\{1, a(z)\}, \tag{4.1}$$

we have

$$\|l_\varepsilon\|_\varepsilon = O\Big(\varepsilon^{\frac{N_\gamma}{2}} \left(\varepsilon|\nabla_x a(z_\varepsilon)| + \varepsilon^{1+\gamma}|\nabla_y a(z_\varepsilon)| + \varepsilon^2\right)\Big). \tag{4.2}$$

Proof. According to Remark 1.2, the assumption (4.1) indicates $\|\eta\|_1$ and $\|\eta\|_\varepsilon$ are equivalent norms in $H^{1,2}_\gamma(\mathbb{R}^{N+l})$. For convenience, we denote them all as $\|\eta\|_\varepsilon$. Recall that

$$\langle l_\varepsilon, \eta\rangle_\varepsilon = \int_{\mathbb{R}^{N+l}} \big(a(z_\varepsilon) - a(z)\big) U_{\varepsilon,z_\varepsilon}(z)\eta(z)dz. \tag{4.3}$$

Thus we have

$$\begin{aligned}
&|\langle l_\varepsilon, \eta\rangle_\varepsilon| \\
&= \Big|\int_{\mathbb{R}^{N+l}} \big(a(z_\varepsilon) - a(z)\big) U_{\varepsilon,z_\varepsilon}(z)\eta(z)dz\Big| \\
&\leq \Big(\int_{\mathbb{R}^{N+l}} \big(a(z_\varepsilon) - a(z)\big)^2 U^2_{\varepsilon,z_\varepsilon}(z)dz\Big)^{1/2} \Big(\int_{\mathbb{R}^{N+l}} \eta^2(z)dz\Big)^{1/2} \\
&\leq \Big(\int_{\mathbb{R}^{N+l}} \big(a(z_\varepsilon) - a(z)\big)^2 U^2_{\varepsilon,z_\varepsilon}(z)dz\Big)^{1/2} \Big(\int_{\mathbb{R}^{N+l}} \varepsilon^2|\nabla_\gamma \eta(z)|^2 + \eta^2(z)dz\Big)^{1/2} \\
&\overset{(4.1)}{=} \Big(\int_{\mathbb{R}^{N+l}} \big(a(z_\varepsilon) - a(z)\big)^2 U^2_{\varepsilon,z_\varepsilon}(z)dz\Big)^{1/2} \|\eta\|_\varepsilon
\end{aligned} \tag{4.4}$$

On the other hand, according to (3.2), we let

$$\widetilde{x} = \frac{\sqrt{a(z_\varepsilon)}}{\varepsilon} x, \quad \widetilde{y} = \Big(\frac{\sqrt{a(z_\varepsilon)}}{\varepsilon}\Big)^{1+\gamma}(y - y_\varepsilon), \tag{4.5}$$

then we derive

$$U_{\varepsilon,z_\varepsilon}(z) = \big(a(z_\varepsilon)\big)^{\frac{1}{q-2}} w(\widetilde{x}, \widetilde{y}), \tag{4.6}$$

$$a(z) = a\Big(\frac{\varepsilon}{\sqrt{a(z_\varepsilon)}}\widetilde{x}, \big(\frac{\varepsilon}{\sqrt{a(z_\varepsilon)}}\big)^{1+\gamma}\widetilde{y} + y_\varepsilon\Big). \tag{4.7}$$

By Taylor expansion,

$$\begin{aligned}
a(z) - a(z_\varepsilon) &= \nabla a(z_\varepsilon)\cdot(z - z_\varepsilon) + O\left(\left(d(z, z_\varepsilon)\right)^2\right) \\
&= \nabla_x a(z_\varepsilon)\cdot \frac{\varepsilon}{\sqrt{a(z_\varepsilon)}}\widetilde{x} + \nabla_y a(z_\varepsilon)\cdot \big(\frac{\varepsilon}{\sqrt{a(z_\varepsilon)}}\big)^{1+\gamma}\widetilde{y} + O\Big(\frac{\varepsilon^2}{a(z_\varepsilon)}(d(\widetilde{z}, 0))^2\Big).
\end{aligned} \tag{4.8}$$

Since $\omega(\widetilde{z})$ has exponential decay at infinity by Proposition 3.1, for any $\alpha > 0$, it holds that

$$\int_{\mathbb{R}^{N+l}} \left(d(\widetilde{z}, 0)\right)^\alpha w^2(\widetilde{z})d\widetilde{z} \leq C < +\infty. \tag{4.9}$$

Continuing with the last row of (4.4), we have

$$
\begin{aligned}
&|\langle l_\varepsilon, \eta\rangle_\varepsilon| \\
&\le \Big(\int_{\mathbb{R}^{N+l}} \big(a(z_\varepsilon)-a(z)\big)^2 U^2_{\varepsilon,z_\varepsilon}(z)dz\Big)^{1/2}\|\eta\|_\varepsilon \\
&\le \Big(\int_{\mathbb{R}^{N+l}} \Big(|\nabla_x a(z_\varepsilon)|\frac{\varepsilon}{\sqrt{a(z_\varepsilon)}}|\widetilde{x}| + |\nabla_y a(z_\varepsilon)|(\frac{\varepsilon}{\sqrt{a(z_\varepsilon)}})^{1+\gamma}|\widetilde{y}| + \frac{\varepsilon^2}{a(z_\varepsilon)}(d(\widetilde{z},0))^2\Big)^2 \\
&\quad \times \big(a(z_\varepsilon)\big)^{\frac{2}{q-2}} w^2(\widetilde{z})\Big(\frac{\varepsilon}{\sqrt{a(z_\varepsilon)}}\Big)^{N_\gamma} d\widetilde{z}\Big)^{1/2}\|\eta\|_\varepsilon \\
&\le C\varepsilon^{\frac{N_\gamma}{2}}\Big(\varepsilon|\nabla_x a(z_\varepsilon)| + \varepsilon^{1+\gamma}|\nabla_y a(z_\varepsilon)| + \varepsilon^2\Big)\|\eta\|_\varepsilon.
\end{aligned} \tag{4.10}
$$

Therefore, (4.2) holds. □

Lemma 4.2. *We have the estimate*

$$
\|R_\varepsilon(\omega_\varepsilon)\|_\varepsilon = O\left(\varepsilon^{N_\gamma(1-\frac{min\{2,q-1\}+1}{2})}\|\omega_\varepsilon\|_\varepsilon^{min\{2,q-1\}}\right). \tag{4.11}
$$

Proof. Recall that

$$
\langle R_\varepsilon(\omega_\varepsilon), \eta\rangle_\varepsilon = \int_{\mathbb{R}^{N+l}} \Big[\big(U_{\varepsilon,z_\varepsilon}(z)+\omega_\varepsilon(z)\big)^{q-1} - U^{q-1}_{\varepsilon,z_\varepsilon}(z) - (q-1)U^{q-2}_{\varepsilon,z_\varepsilon}(z)\omega_\varepsilon(z)\Big]\eta dz. \tag{4.12}
$$

We denote

$$
\widehat{R_\varepsilon}(\omega_\varepsilon) = \big(U_{\varepsilon,z_\varepsilon}(z)+\omega_\varepsilon(z)\big)^{q-1} - U^{q-1}_{\varepsilon,z_\varepsilon}(z) - (q-1)U^{q-2}_{\varepsilon,z_\varepsilon}(z)\omega_\varepsilon(z). \tag{4.13}
$$

First, we consider the case of $2 < q \le 3$. Note that for any $a > 0$ and $b > 0$, if $p < 1$, one has

$$
(a+b)^p < a^p + b^p. \tag{4.14}
$$

By the mean value theorem and the above inequality, we have

$$
\begin{aligned}
\widehat{R_\varepsilon}(\omega_\varepsilon) &= (q-1)\big(U_{\varepsilon,z_\varepsilon}(z)+\theta\omega_\varepsilon(z)\big)^{q-2}\omega_\varepsilon(z) - (q-1)U^{q-2}_{\varepsilon,z_\varepsilon}(z)\omega_\varepsilon(z) \\
&\le (q-1)\big(U^{q-2}_{\varepsilon,z_\varepsilon}(z)+\theta^{q-2}\omega_\varepsilon^{q-2}(z)\big)\omega_\varepsilon(z) - (q-1)U^{q-2}_{\varepsilon,z_\varepsilon}(z)\omega_\varepsilon(z) \\
&= (q-1)\theta^{q-2}\omega_\varepsilon^{q-1}(z) \\
&= O(\omega_\varepsilon^{q-1}(z)),
\end{aligned} \tag{4.15}
$$

where $\theta \in (0,1)$. Thus we obtain

$$
\begin{aligned}
|\langle R_\varepsilon(\omega_\varepsilon), \eta\rangle_\varepsilon| &= \Big|\int_{\mathbb{R}^{N+l}} \widehat{R_\varepsilon}(\omega_\varepsilon)\eta(z)dz\Big| \\
&\le C\int_{\mathbb{R}^{N+l}} |\omega_\varepsilon(z)|^{q-1}|\eta(z)|dz \\
&\le C\Big(\int_{\mathbb{R}^{N+l}} |\omega_\varepsilon(z)|^q dz\Big)^{\frac{q-1}{q}}\Big(\int_{\mathbb{R}^{N+l}} |\eta(z)|^q dz\Big)^{1/q}
\end{aligned} \tag{4.16}
$$

Next according to the fact that the embedding $H^{1,2}_\gamma(\mathbb{R}^{N+l}) \hookrightarrow L^p(\mathbb{R}^{N+l})$ is continue for every $p \in [2, 2^*_\gamma]$ and by blow up, we can obtain the relationship between norm $\|\eta\|_{L^p(\mathbb{R}^{N+l})}$ and norm $\|\eta\|_1$ (we still denote $\|\eta\|_1$ by $\|\eta\|_\varepsilon$ for convenience). Let

$$
\widetilde{x} = \frac{x}{\varepsilon}, \quad \widetilde{y} = \frac{y-y_\varepsilon}{\varepsilon^{1+\gamma}}, \tag{4.17}
$$

and

$$
b(\widetilde{z}) = \eta(\varepsilon\widetilde{x}, \varepsilon^{1+\gamma}\widetilde{y}+y_\varepsilon).
$$

Then we derive

$$\begin{aligned}\int_{\mathbb{R}^{N+l}}|\eta(z)|^q dz &= \varepsilon^{N_\gamma}\int_{\mathbb{R}^{N+l}}|b(\widetilde{z})|^q d\widetilde{z}\\ &\le C\varepsilon^{N_\gamma}\Big(\int_{\mathbb{R}^{N+l}}|\nabla_\gamma b(\widetilde{z})|^2+|b(\widetilde{z})|^2 d\widetilde{z}\Big)^{q/2}\\ &= C\varepsilon^{N_\gamma}\Big(\int_{\mathbb{R}^{N+l}}(\varepsilon^2|\nabla_\gamma\eta(z)|^2+|\eta(z)|^2)\frac{1}{\varepsilon^{N_\gamma}}dz\Big)^{q/2}\\ &= C\varepsilon^{N_\gamma(1-\frac{q}{2})}\Big(\int_{\mathbb{R}^{N+l}}\varepsilon^2|\nabla_\gamma\eta(z)|^2+|\eta(z)|^2 dz\Big)^{q/2}.\end{aligned}\tag{4.18}$$

Continuing with the last row of (4.16), we have

$$\begin{aligned}|\langle R_\varepsilon(\omega_\varepsilon),\eta\rangle_\varepsilon| &\le C\Big(\int_{\mathbb{R}^{N+l}}|\omega_\varepsilon(z)|^q dz\Big)^{\frac{q-1}{q}}\Big(\int_{\mathbb{R}^{N+l}}|\eta(z)|^q dz\Big)^{1/q}\\ &\overset{(4.18)}{\le} C\varepsilon^{N_\gamma(1-\frac{q}{2})\frac{q-1}{q}}\Big(\int_{\mathbb{R}^{N+l}}\varepsilon^2|\nabla_\gamma\omega_\varepsilon(z)|^2+|\omega_\varepsilon(z)|^2 dz\Big)^{\frac{q}{2}\cdot\frac{q-1}{q}}\\ &\quad\times\varepsilon^{N_\gamma(1-\frac{q}{2})\frac{1}{q}}\Big(\int_{\mathbb{R}^{N+l}}\varepsilon^2|\nabla_\gamma\eta(z)|^2+|\eta(z)|^2 dz\Big)^{\frac{q}{2}\cdot\frac{1}{q}}\\ &= C\varepsilon^{N_\gamma(1-\frac{q}{2})}\|\omega_\varepsilon\|_\varepsilon^{q-1}\|\eta\|_\varepsilon.\end{aligned}\tag{4.19}$$

Now, we consider the case of $q>3$. Note that for any $a>0$ and $b>0$, if $p>2$, one has

$$(a+b)^p = a^p + pa^{p-1}b + \frac{p(p-1)}{2}a^{p-2}b^2 + O(b^p).\tag{4.20}$$

According to (4.20), we have

$$\begin{aligned}\big(U_{\varepsilon,z_\varepsilon}(z)+\omega_\varepsilon(z)\big)^{q-1} &= U_{\varepsilon,z_\varepsilon}^{q-1}(z)+(q-1)U_{\varepsilon,z_\varepsilon}^{q-2}(z)\omega_\varepsilon(z)\\ &\quad+\frac{(q-1)(q-2)}{2}U_{\varepsilon,z_\varepsilon}^{q-3}(z)\omega_\varepsilon^2(z)+O\left(\omega_\varepsilon^{q-1}(z)\right).\end{aligned}\tag{4.21}$$

Then

$$\widehat{R_\varepsilon}(\omega_\varepsilon)\le CU_{\varepsilon,z_\varepsilon}^{q-3}(z)\omega_\varepsilon^2(z)+C\omega_\varepsilon^{q-1}(z).\tag{4.22}$$

By using the Hölder's inequality and (4.18), we obtain

$$\begin{aligned}&\Big|\int_{\mathbb{R}^{N+l}}U_{\varepsilon,z_\varepsilon}^{q-3}(z)\omega_\varepsilon^2(z)\eta(z)dz\Big|\\ &\le\Big(\int_{\mathbb{R}^{N+l}}U_{\varepsilon,z_\varepsilon}^{q}(z)dz\Big)^{\frac{q-3}{q}}\Big(\int_{\mathbb{R}^{N+l}}|\omega_\varepsilon(z)|^q dz\Big)^{2/q}\Big(\int_{\mathbb{R}^{N+l}}|\eta(z)|^q dz\Big)^{1/q}\\ &\le C\varepsilon^{N_\gamma(1-\frac{q}{2})\frac{q-3}{q}}\|U_{\varepsilon,z_\varepsilon}\|_\varepsilon^{q-3}\cdot\varepsilon^{N_\gamma(1-\frac{q}{2})\frac{2}{q}}\|\omega_\varepsilon\|_\varepsilon^2\cdot\varepsilon^{N_\gamma(1-\frac{q}{2})\frac{1}{q}}\|\eta\|_\varepsilon\\ &= C\varepsilon^{-\frac{N_\gamma}{2}}\|\omega_\varepsilon\|_\varepsilon^2\|\eta\|_\varepsilon,\end{aligned}\tag{4.23}$$

which uses

$$\|U_{\varepsilon,z_\varepsilon}\|_\varepsilon = O\Big(\varepsilon^{\frac{N_\gamma}{2}}\Big).\tag{4.24}$$

As same as (4.19), we can obtain

$$\Big|\int_{\mathbb{R}^{N+l}}\omega_\varepsilon^{q-1}(z)\eta(z)dz\Big|\le C\varepsilon^{N_\gamma(1-\frac{q}{2})}\|\omega_\varepsilon\|_\varepsilon^{q-1}\|\eta\|_\varepsilon,\tag{4.25}$$

and it can be absorbed in (4.23).

By combining (4.19) and (4.23), we complete the proof. □

Now we use the contraction mapping theorem to carry out the reduction. Namely, we solve the problem

$$Q_\varepsilon L_\varepsilon\omega_\varepsilon = Q_\varepsilon l_\varepsilon + Q_\varepsilon R_\varepsilon(\omega_\varepsilon),\quad \omega_\varepsilon\in E_\varepsilon.\tag{4.26}$$

Lemma 4.3. *There exists $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0]$, and any z_ε with $z_\varepsilon \to z_0$, there is a unique solution $\omega_\varepsilon \in E_\varepsilon$ satisfying (4.26). In addition, we have the estimate*

$$\|\omega_\varepsilon\|_\varepsilon \le C\|l_\varepsilon\|_\varepsilon \le C\varepsilon^{\frac{N_\gamma}{2}} \left(\varepsilon|\nabla_x a(z_\varepsilon)| + \varepsilon^{1+\gamma}|\nabla_y a(z_\varepsilon)| + \varepsilon^2\right). \tag{4.27}$$

Proof. According to Proposition 3.2, $Q_\varepsilon L_\varepsilon$ is invertible on E_ε, we can rewrite (4.26) as

$$\omega_\varepsilon = (Q_\varepsilon L_\varepsilon)^{-1} Q_\varepsilon l_\varepsilon + (Q_\varepsilon L_\varepsilon)^{-1} Q_\varepsilon R_\varepsilon(\omega_\varepsilon) := A\omega_\varepsilon. \tag{4.28}$$

It follows from Proposition 3.2 and Lemma 4.1 that

$$\|(Q_\varepsilon L_\varepsilon)^{-1} Q_\varepsilon l_\varepsilon\|_\varepsilon \le C\|l_\varepsilon\|_\varepsilon \le C\varepsilon^{\frac{N_\gamma}{2}+1}. \tag{4.29}$$

Let

$$B := \{\omega_\varepsilon \in E_\varepsilon : \|\omega_\varepsilon\|_\varepsilon \le \varepsilon^{\frac{N_\gamma}{2}+1-\mu}\}, \tag{4.30}$$

where $\mu > 0$ is a fixed small constant.

We will apply the contraction mapping theorem in ball B.

(i) A maps from B to B. According to Lemma 4.1 and Lemma 4.2, for any $\omega_\varepsilon \in B$, it holds

$$\begin{aligned}
\|A\omega_\varepsilon\|_\varepsilon &\le C\|l_\varepsilon\|_\varepsilon + C\|R_\varepsilon(\omega_\varepsilon)\|_\varepsilon \\
&\le C\varepsilon^{\frac{N_\gamma}{2}+1} + C\varepsilon^{N_\gamma(1-\frac{min\{2,q-1\}+1}{2})}\|\omega_\varepsilon\|_\varepsilon^{min\{2,q-1\}} \\
&\le C\varepsilon^{\frac{N_\gamma}{2}+1} + C\varepsilon^{N_\gamma(1-\frac{min\{2,q-1\}+1}{2})}\varepsilon^{(\frac{N_\gamma}{2}+1-\mu)min\{2,q-1\}} \\
&= C\varepsilon^{\frac{N_\gamma}{2}+1} + C\varepsilon^{\frac{N_\gamma}{2}+(1-\mu)min\{2,q-1\}} \\
&\le \frac{1}{2}\varepsilon^{\frac{N_\gamma}{2}+1-\mu} + \frac{1}{2}\varepsilon^{\frac{N_\gamma}{2}+1-\mu} \\
&= \varepsilon^{\frac{N_\gamma}{2}+1-\mu}.
\end{aligned} \tag{4.31}$$

(ii) A is a contraction map. For any $\omega_{\varepsilon,1}$, $\omega_{\varepsilon,2} \in B$,

$$\|A\omega_{\varepsilon,1} - A\omega_{\varepsilon,2}\|_\varepsilon \le C\|R_\varepsilon(\omega_{\varepsilon,1}) - R_\varepsilon(\omega_{\varepsilon,2})\|_\varepsilon. \tag{4.32}$$

By (4.12), we have

$$\langle R_\varepsilon(\omega_{\varepsilon,1}) - R_\varepsilon(\omega_{\varepsilon,2}), \eta\rangle_\varepsilon = \int_{\mathbb{R}^{N+l}} \left(\widehat{R_\varepsilon}(\omega_{\varepsilon,1}) - \widehat{R_\varepsilon}(\omega_{\varepsilon,2})\right)\eta(z)dz, \tag{4.33}$$

where

$$\begin{aligned}
\widehat{R_\varepsilon}(\omega_{\varepsilon,1}) - \widehat{R_\varepsilon}(\omega_{\varepsilon,2}) &= \big(U_{\varepsilon,z_\varepsilon}(z) + \omega_{\varepsilon,1}(z)\big)^{q-1} - \big(U_{\varepsilon,z_\varepsilon}(z) + \omega_{\varepsilon,2}(z)\big)^{q-1} \\
&\quad - (q-1)U_{\varepsilon,z_\varepsilon}^{q-2}(z)\big(\omega_{\varepsilon,1}(z) - \omega_{\varepsilon,2}(z)\big).
\end{aligned} \tag{4.34}$$

First, we consider the case of $2 < q \le 3$. By the mean value theorem and the inequality (4.14), we have

$$\begin{aligned}
&\widehat{R_\varepsilon}(\omega_{\varepsilon,1}) - \widehat{R_\varepsilon}(\omega_{\varepsilon,2}) \\
&= (q-1)\Big[\Big(U_{\varepsilon,z_\varepsilon}(z) + \omega_{\varepsilon,2}(z) + \theta\big(\omega_{\varepsilon,1}(z) - \omega_{\varepsilon,2}(z)\big)\Big)^{q-2} - U_{\varepsilon,z_\varepsilon}^{q-2}(z)\Big]\big(\omega_{\varepsilon,1} - \omega_{\varepsilon,2}\big) \\
&\le (q-1)\Big[(\theta\omega_{\varepsilon,1}(z))^{q-2} + ((1-\theta)\omega_{\varepsilon,2}(z))^{q-2}\Big]\big(\omega_{\varepsilon,1}(z) - \omega_{\varepsilon,2}(z)\big) \\
&\le (q-1)\left(\omega_{\varepsilon,1}^{q-2}(z) + \omega_{\varepsilon,2}^{q-2}(z)\right)\big(\omega_{\varepsilon,1}(z) - \omega_{\varepsilon,2}(z)\big),
\end{aligned} \tag{4.35}$$

where $\theta \in (0,1)$. Thus we have

$$
\begin{aligned}
&|\langle R_\varepsilon(\omega_{\varepsilon,1}) - R_\varepsilon(\omega_{\varepsilon,2}), \eta\rangle_\varepsilon| \\
&\le (q-1)\int_{\mathbb{R}^{N+l}} \big(|\omega_{\varepsilon,1}(z)|^{q-2} + |\omega_{\varepsilon,2}(z)|^{q-2}\big)\, |\omega_{\varepsilon,1}(z) - \omega_{\varepsilon,2}(z)||\eta(z)|dz \\
&\overset{\text{Hólder}}{\le} (q-1)\Big(\Big(\int_{\mathbb{R}^{N+l}} |\omega_{\varepsilon,1}(z)|^q dz\Big)^{\frac{q-2}{q}} + \Big(\int_{\mathbb{R}^{N+l}} |\omega_{\varepsilon,2}(z)|^q dz\Big)^{\frac{q-2}{q}}\Big) \\
&\quad \times \Big(\int_{\mathbb{R}^{N+l}} |\omega_{\varepsilon,1}(z) - \omega_{\varepsilon,2}(z)|^q dz\Big)^{1/q} \Big(\int_{\mathbb{R}^{N+l}} |\eta(z)|^q dz\Big)^{1/q} \\
&\overset{(4.18)}{\le} C\varepsilon^{N_\gamma(1-\frac{q}{2})} \big(\|\omega_{\varepsilon,1}(z)\|_\varepsilon^{q-2} + \|\omega_{\varepsilon,2}(z)\|_\varepsilon^{q-2}\big)\, \|\omega_{\varepsilon,1}(z) - \omega_{\varepsilon,2}(z)\|_\varepsilon \|\eta(z)\|_\varepsilon \\
&\overset{(4.30)}{\le} C\varepsilon^{N_\gamma(1-\frac{q}{2})} \varepsilon^{(\frac{N_\gamma}{2}+1-\mu)(q-2)} \|\omega_{\varepsilon,1}(z) - \omega_{\varepsilon,2}(z)\|_\varepsilon \|\eta(z)\|_\varepsilon \\
&= C\varepsilon^{(1-\mu)(q-2)} \|\omega_{\varepsilon,1}(z) - \omega_{\varepsilon,2}(z)\|_\varepsilon \|\eta(z)\|_\varepsilon \\
&\le \frac{1}{2}\|\omega_{\varepsilon,1}(z) - \omega_{\varepsilon,2}(z)\|_\varepsilon \|\eta(z)\|_\varepsilon.
\end{aligned} \tag{4.36}
$$

Next, we consider the case of $q > 3$. Applying (4.21) and according to (4.34), we can obtain

$$
\widehat{R_\varepsilon}(\omega_{\varepsilon,1}) - \widehat{R_\varepsilon}(\omega_{\varepsilon,2}) \le \frac{(q-1)(q-2)}{2} U_{\varepsilon.z_\varepsilon}^{q-3}(z)\big(\omega_{\varepsilon,1}^2(z) - \omega_{\varepsilon,2}^2(z)\big) + C\left(\omega_{\varepsilon,1}^{q-1}(z) - \omega_{\varepsilon,2}^{q-1}(z)\right). \tag{4.37}
$$

Moreover, it holds that

$$
\begin{aligned}
|\widehat{R_\varepsilon}(\omega_{\varepsilon,1}) - \widehat{R_\varepsilon}(\omega_{\varepsilon,2})| &\le CU_{\varepsilon.z_\varepsilon}^{q-3}(z)\big(|\omega_{\varepsilon,1}(z)| + |\omega_{\varepsilon,2}(z)|\big)|\omega_{\varepsilon,1}(z) - \omega_{\varepsilon,2}(z)| \\
&\quad + C\left(|\omega_{\varepsilon,1}(z)|^{q-2} + |\omega_{\varepsilon,2}(z)|^{q-2}\right) |\omega_{\varepsilon,1}(z) - \omega_{\varepsilon,2}(z)|.
\end{aligned} \tag{4.38}
$$

Because the second term of (4.38) is the same as in (4.36), we estimate the first term of (4.38).

$$
\begin{aligned}
&\int_{\mathbb{R}^{N+l}} U_{\varepsilon.z_\varepsilon}^{q-3}(z)\big(|\omega_{\varepsilon,1}(z)| + |\omega_{\varepsilon,2}(z)|\big)|\omega_{\varepsilon,1}(z) - \omega_{\varepsilon,2}(z)||\eta(z)|dz \\
&\overset{\text{Hölder}}{\le} \Big(\int_{\mathbb{R}^{N+l}} U_{\varepsilon.z_\varepsilon}^{q}(z)dz\Big)^{\frac{q-3}{q}} \Big(\Big(\int_{\mathbb{R}^{N+l}} |\omega_{\varepsilon,1}(z)|^q\Big)^{1/q} + \Big(\int_{\mathbb{R}^{N+l}} |\omega_{\varepsilon,2}(z)|^q\Big)^{1/q}\Big) \\
&\quad \times \Big(\int_{\mathbb{R}^{N+l}} |\omega_{\varepsilon,1}(z) - \omega_{\varepsilon,2}(z)|^q dz\Big)^{1/q} \Big(\int_{\mathbb{R}^{N+l}} |\eta(z)|^q dz\Big)^{1/q} \\
&\overset{(4.18)}{\le} C\varepsilon^{-\frac{N_\gamma}{2}} \left(\|\omega_{\varepsilon,1}(z)\|_\varepsilon + \|\omega_{\varepsilon,2}(z)\|_\varepsilon\right) \|\omega_{\varepsilon,1}(z) - \omega_{\varepsilon,2}(z)\|_\varepsilon \|\eta(z)\|_\varepsilon. \\
&\overset{(4.30)}{\le} C\varepsilon^{1-\mu} \|\omega_{\varepsilon,1}(z) - \omega_{\varepsilon,2}(z)\|_\varepsilon \|\eta(z)\|_\varepsilon \\
&\le \frac{1}{2}\|\omega_{\varepsilon,1}(z) - \omega_{\varepsilon,2}(z)\|_\varepsilon \|\eta(z)\|_\varepsilon,
\end{aligned} \tag{4.39}
$$

which uses (4.24).

Combining the expressions above, we have proved

$$
\|A\omega_{\varepsilon,1} - A\omega_{\varepsilon,2}\|_\varepsilon \le \frac{1}{2}\|\omega_{\varepsilon,1}(z) - \omega_{\varepsilon,2}(z)\|_\varepsilon, \quad \forall \omega_{\varepsilon,1},\ \omega_{\varepsilon,2} \in B. \tag{4.40}
$$

By contraction mapping theorem, we conclude that there exists $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0]$, and any z_ε with $z_\varepsilon \to z_0$, there is a unique $\omega_\varepsilon \in E_\varepsilon$, which depend ε and z_ε, satisfying

$$
Q_\varepsilon L_\varepsilon \omega_\varepsilon = Q_\varepsilon l_\varepsilon + Q_\varepsilon R_\varepsilon(\omega_\varepsilon). \tag{4.41}
$$

At last, similar to (4.31), we obtain that

$$\begin{aligned}\|\omega_\varepsilon\|_\varepsilon &= \|A\omega_\varepsilon\|_\varepsilon\\ &\le C\|l_\varepsilon\|_\varepsilon + C\|R_\varepsilon(\omega_\varepsilon)\|_\varepsilon\\ &\le C\|l_\varepsilon\|_\varepsilon + C\varepsilon^{N_\gamma(1-\frac{min\{2,q-1\}+1}{2})}\|\omega_\varepsilon\|_\varepsilon^{min\{2,q-1\}}\\ &\overset{(4.30)}{\le} C\|l_\varepsilon\|_\varepsilon + C\varepsilon^{N_\gamma(1-\frac{min\{2,q-1\}+1}{2})}\left(\varepsilon^{\frac{N_\gamma}{2}+1-\mu}\right)^{min\{2,q-1\}-1}\|\omega_\varepsilon\|_\varepsilon\\ &= C\|l_\varepsilon\|_\varepsilon + C\varepsilon^{(1-\mu)(min\{2,q-1\}-1)}\|\omega_\varepsilon\|_\varepsilon.\end{aligned} \tag{4.42}$$

Taking $\varepsilon > 0$ small enough, such that $C\varepsilon^{(1-\mu)(min\{2,q-1\}-1)} < \frac{1}{2}$, gives that

$$\|\omega_\varepsilon\|_\varepsilon \le C\|l_\varepsilon\|_\varepsilon \le C\varepsilon^{\frac{N_\gamma}{2}}\left(\varepsilon|\nabla_x a(z_\varepsilon)| + \varepsilon^{1+\gamma}|\nabla_y a(z_\varepsilon)| + \varepsilon^2\right). \qquad \square$$

5. Existence of single peak solutions

In this section, we are committed to proving Theorem 1.6 to obtain a solution in the form of (1.17) for equation (1.1). Lemma 4.3 tells us that

$$Q_\varepsilon\left(L_\varepsilon\omega_\varepsilon - l_\varepsilon - R_\varepsilon(\omega_\varepsilon)\right) = 0, \tag{5.1}$$

i.e., for some constants a_j,

$$L_\varepsilon\omega_\varepsilon - l_\varepsilon - R_\varepsilon(\omega_\varepsilon) = \sum_{j=1}^{l} a_j \frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}. \tag{5.2}$$

In next step, we need to choose the appropriate $z_\varepsilon = (0, y_\varepsilon)$, such that $a_j = 0$, which is dependent in z_ε. Therefore, if we take z_ε such that

$$\langle L_\varepsilon\omega_\varepsilon - l_\varepsilon - R_\varepsilon(\omega_\varepsilon), \frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}\rangle_\varepsilon = 0, \quad \text{for } j = 1, \dots, l, \tag{5.3}$$

the right-hand side of (5.2) must be equal to 0; then we achieve our goal. Therefore, if z_ε satisfies

$$\int_{\mathbb{R}^{N+l}} \varepsilon^2\nabla_\gamma u_\varepsilon(z)\cdot\nabla_\gamma\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j} + a(z)u_\varepsilon(z)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j} - u_\varepsilon^{q-1}(z)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}dz = 0, \tag{5.4}$$

for $j = 1, \dots, l$, then we have $a_j = 0$, $j = 1, \dots, l$.

Proof of Theorem 1.6. We only need to solve z_ε from (5.4) by using degree theorem. The main idea is that we simply the LHS of (5.4) and find major term with $U_{\varepsilon,z_\varepsilon}(z)$, then we prove that the influence of $\omega_\varepsilon(z)$ is negligible and will not destroy the major term without $\omega_\varepsilon(z)$. Now we insert $U_{\varepsilon,z_\varepsilon}(z)$ into the LHS of (5.4), then we have

$$\begin{aligned}&\int_{\mathbb{R}^{N+l}} \varepsilon^2\nabla_\gamma U_{\varepsilon,z_\varepsilon}(z)\cdot\nabla_\gamma\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j} + a(z)U_{\varepsilon,z_\varepsilon}(z)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j} - U_{\varepsilon,z_\varepsilon}^{q-1}(z)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}dz\\ &:= I_1 + I_2 - I_3.\end{aligned} \tag{5.5}$$

For first term, since $U_{\varepsilon,z_\varepsilon}(z)$ is the solution of (1.18), we have

$$-\varepsilon^2\Delta_\gamma U_{\varepsilon,z_\varepsilon}(z) + a(z_\varepsilon)U_{\varepsilon,z_\varepsilon}(z) = U_{\varepsilon,z_\varepsilon}^{q-1}(z). \tag{5.6}$$

Taking the derivative with respect to y_j on both sides of (5.6) gives

$$-\varepsilon^2\Delta_\gamma\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j} + a(z_\varepsilon)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j} = (q-1)U_{\varepsilon,z_\varepsilon}^{q-2}(z)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}. \tag{5.7}$$

From (5.7), the symmetry of $U_{\varepsilon,z_\varepsilon}(z)$ with respect to the second variable and let

$$\widetilde{y} = (\frac{\sqrt{a(0)}}{\varepsilon})^{1+\gamma}y, \quad r = |\widetilde{y}|, \tag{5.8}$$

we obtain

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}^{N+l}} \varepsilon^2 \nabla_\gamma U_{\varepsilon,z_\varepsilon}(z)\cdot \nabla_\gamma \frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j} dz \\
&= \int_{\mathbb{R}^{N+l}} (q-1)U^{q-1}_{\varepsilon,z_\varepsilon}(z)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j} - a(z_\varepsilon)U_{\varepsilon,z_\varepsilon}(z)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j} dz \\
&= \int_{\mathbb{R}^{N+l}} \Big[(q-1)\Big(\big(a(0)\big)^{\frac{1}{q-2}} w(\frac{\sqrt{a(0)}}{\varepsilon}x,r)\Big)^{q-1} - \big(a(0)\big)^{\frac{q-1}{q-2}} w(\frac{\sqrt{a(0)}}{\varepsilon}x,r)\Big] \\
&\quad \times C\frac{\partial w(\frac{\sqrt{a(0)}}{\varepsilon}x,r)}{\partial r}\frac{y_j}{r}\varepsilon^{(1+\gamma)(l-1)}\,dx\,dy = 0,
\end{aligned} \tag{5.9}
$$

where $C = C(a(0))$ and we use the fact that $\int_{\mathbb{R}^{N+l}} f(x,|y|)y_j dz = 0$, $f(x,|y|)$ is radially symmetric with respect to the variable y.

Similarly, for the third term, we obtain

$$
\begin{aligned}
I_3 &= \int_{\mathbb{R}^{N+l}} U^{q-1}_{\varepsilon,z_\varepsilon}(z)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j} dz \\
&= \int_{\mathbb{R}^{N+l}} \Big(\big(a(0)\big)^{\frac{1}{q-2}} w(\frac{\sqrt{a(0)}}{\varepsilon}x,r)\Big)^{q-1} \\
&\quad \times \big(a(0)\big)^{\frac{1}{q-2}} \frac{\partial w(\frac{\sqrt{a(0)}}{\varepsilon}x,r)}{\partial r}(\frac{\sqrt{a(0)}}{\varepsilon})^{1+\gamma}\frac{y_j}{r}\Big(\frac{\varepsilon}{\sqrt{a(0)}}\Big)^{(1+\gamma)l}\,dx\,dy = 0.
\end{aligned} \tag{5.10}
$$

For the second term, let (4.5), and denote $b(\widetilde{z}) := a(\frac{\varepsilon}{\sqrt{a(z_\varepsilon)}}\widetilde{x}, (\frac{\varepsilon}{\sqrt{a(z_\varepsilon)}})^{1+\gamma}\widetilde{y} + y_\varepsilon)$, we derive

$$
\begin{aligned}
I_2 &= \int_{\mathbb{R}^{N+l}} a(z)U_{\varepsilon,z_\varepsilon}(z)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j} dz \\
&= \frac{1}{2}\int_{\mathbb{R}^{N+l}} a(z)\frac{\partial U^2_{\varepsilon,z_\varepsilon}(z)}{\partial y_j} dz \\
&= -\frac{1}{2}\int_{\mathbb{R}^{N+l}} \frac{\partial a(z)}{\partial y_j} U^2_{\varepsilon,z_\varepsilon}(z) dz + O(e^{-\frac{\tau}{\varepsilon}}) \\
&\overset{(4.5)}{=} -\frac{1}{2}\int_{\mathbb{R}^{N+l}} \frac{\partial b(\widetilde{z})}{\partial \widetilde{y}_j}\Big(\frac{\sqrt{a(z_\varepsilon)}}{\varepsilon}\Big)^{1+\gamma}\big(a(z_\varepsilon)\big)^{\frac{2}{q-2}} w^2(\widetilde{z})\Big(\frac{\varepsilon}{\sqrt{a(z_\varepsilon)}}\Big)^{N_\gamma} d\widetilde{z} + O(e^{-\frac{\tau}{\varepsilon}}) \\
&= -\frac{1}{2}\Big(a(z_\varepsilon)\Big)^{\alpha}\varepsilon^{N_\gamma-1-\gamma}\int_{\mathbb{R}^{N+l}} \frac{\partial b(\widetilde{z})}{\partial \widetilde{y}_j} w^2(\widetilde{z}) d\widetilde{z} + O(e^{-\frac{\tau}{\varepsilon}}) \\
&\overset{\text{Taylor}}{=} -\frac{1}{2}\left(a(z_\varepsilon)\right)^{\alpha}\varepsilon^{N_\gamma-1-\gamma}\frac{\partial a(z)}{\partial y_j}\Big|_{z=z_\varepsilon}\int_{\mathbb{R}^{N+l}} w^2(\widetilde{z}) d\widetilde{z} + O\left(\varepsilon^{N_\gamma-\gamma}\right),
\end{aligned} \tag{5.11}
$$

where

$$
\begin{gathered}
\int_{\mathbb{R}^{N+l}} w^2(\widetilde{z}) d\widetilde{z} \le C < +\infty, \\
\alpha = \frac{2}{q-2} + \frac{1+\gamma-N_\gamma}{2}.
\end{gathered}
$$

By combining (5.9), (5.10), and (5.11), we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^{N+l}} \varepsilon^2 \nabla_\gamma U_{\varepsilon,z_\varepsilon}(z)\cdot \nabla_\gamma \frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j} + a(z)U_{\varepsilon,z_\varepsilon}(z)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j} - U^{q-1}_{\varepsilon,z_\varepsilon}(z)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j} dz \\
&= -\frac{1}{2}\left(a(z_\varepsilon)\right)^{\alpha}\varepsilon^{N_\gamma-1-\gamma}\frac{\partial a(z)}{\partial y_j}\Big|_{z=z_\varepsilon}\int_{\mathbb{R}^{N+l}} w^2(\widetilde{z}) d\widetilde{z} + O\left(\varepsilon^{N_\gamma-\gamma}\right).
\end{aligned} \tag{5.12}
$$

Next we prove that the influence of $\omega_\varepsilon(z)$ is negligible and will not destroy the major term without $\omega_\varepsilon(z)$. We insert $U_{\varepsilon,z_\varepsilon}(z)+\omega_\varepsilon(z)$ into the LHS of (5.4), then we have

$$\begin{aligned}
&\int_{\mathbb{R}^{N+l}} \varepsilon^2\nabla_\gamma\big(U_{\varepsilon,z_\varepsilon}(z)+\omega_\varepsilon(z)\big)\cdot\nabla_\gamma\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}+a(z)\big(U_{\varepsilon,z_\varepsilon}(z)+\omega_\varepsilon(z)\big)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}\\
&-\big(U_{\varepsilon,z_\varepsilon}(z)+\omega_\varepsilon(z)\big)^{q-1}\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}dz\\
&:=J_1+J_2-J_3.
\end{aligned}\tag{5.13}$$

For the first two terms, we have

$$\begin{aligned}
&J_1+J_2\\
&=\int_{\mathbb{R}^{N+l}} \varepsilon^2\nabla_\gamma\big(U_{\varepsilon,z_\varepsilon}(z)+\omega_\varepsilon(z)\big)\cdot\nabla_\gamma\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}+a(z)\big(U_{\varepsilon,z_\varepsilon}(z)+\omega_\varepsilon(z)\big)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}\\
&=\int_{\mathbb{R}^{N+l}} \varepsilon^2\nabla_\gamma U_{\varepsilon,z_\varepsilon}(z)\cdot\nabla_\gamma\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}+a(z)U_{\varepsilon,z_\varepsilon}(z)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}\\
&\quad+\int_{\mathbb{R}^{N+l}} \varepsilon^2\nabla_\gamma\omega_\varepsilon(z)\cdot\nabla_\gamma\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}+a(z)\omega_\varepsilon(z)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}\\
&\overset{(5.9)}{=}0+\int_{\mathbb{R}^{N+l}} a(z)U_{\varepsilon,z_\varepsilon}(z)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}dz+\langle\omega_\varepsilon(z),\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}\rangle_\varepsilon\\
&=\int_{\mathbb{R}^{N+l}} a(z)U_{\varepsilon,z_\varepsilon}(z)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}dz\\
&\overset{(5.11)}{=}-\frac12\left(a(z_\varepsilon)\right)^\alpha\varepsilon^{N_\gamma-1-\gamma}\frac{\partial a(z)}{\partial y_j}\Big|_{z=z_\varepsilon}\int_{\mathbb{R}^{N+l}} w^2(\tilde z)d\tilde z+O\left(\varepsilon^{N_\gamma-\gamma}\right),
\end{aligned}\tag{5.14}$$

which we have used the fact $\langle\omega_\varepsilon(z),\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}\rangle_\varepsilon=0$ since $\omega_\varepsilon\in E_\varepsilon$ and $\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}\in K_\varepsilon$.

Finally, we estimate the last term

$$\begin{aligned}
J_3&=\int_{\mathbb{R}^{N+l}}\left(U_{\varepsilon,z_\varepsilon}(z)+\omega_\varepsilon(z)\right)^{q-1}\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}dz\\
&=\int_{\mathbb{R}^{N+l}} U^{q-1}_{\varepsilon,z_\varepsilon}(z)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}+(q-1)U^{q-2}_{\varepsilon,z_\varepsilon}(z)\omega_\varepsilon(z)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}dz\\
&\quad+\begin{cases}O\Big(\int_{\mathbb{R}^{N+l}}\omega_\varepsilon^{q-1}(z)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}dz\Big), & 2<q\le 3,\\
\int_{\mathbb{R}^{N+l}} U^{q-3}_{\varepsilon,z_\varepsilon}(z)\omega_\varepsilon^2(z)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}dz+O\Big(\int_{\mathbb{R}^{N+l}}\omega_\varepsilon^{q-1}(z)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}dz\Big), & q>3.\end{cases}
\end{aligned}\tag{5.15}$$

In which as for (5.10),

$$\int_{\mathbb{R}^{N+l}} U^{q-1}_{\varepsilon,z_\varepsilon}(z)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}dz=0.\tag{5.16}$$

Similar to (4.10) and by using (1.20), we have

$$\begin{aligned}
&(q-1)\int_{\mathbb{R}^{N+l}} U^{q-2}_{\varepsilon,z_\varepsilon}(z)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}\omega_\varepsilon(z)dz\\
&\overset{(5.7)}{=}\int_{\mathbb{R}^{N+l}}\varepsilon^2\nabla_\gamma\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}\cdot\nabla_\gamma\omega_\varepsilon(z)+a(z_\varepsilon)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}\omega_\varepsilon(z)dz\\
&=\langle\omega_\varepsilon(z),\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}\rangle_\varepsilon+\int_{\mathbb{R}^{N+l}}\left(a(z_\varepsilon)-a(z)\right)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}\omega_\varepsilon(z)dz\\
&=\int_{\mathbb{R}^{N+l}}\left(a(z_\varepsilon)-a(z)\right)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}\omega_\varepsilon(z)dz\\
&=O\Big(\varepsilon^{\frac{N_\gamma}{2}-1-\gamma}\left(\varepsilon|\nabla_x a(z_\varepsilon)|+\varepsilon^{1+\gamma}|\nabla_y a(z_\varepsilon)|+\varepsilon^2\right)\|\omega_\varepsilon\|_\varepsilon\Big)\\
&\overset{(1.20)}{=}O\left(\varepsilon^{N_\gamma-\gamma}\left(\varepsilon|\nabla_x a(z_\varepsilon)|+\varepsilon^{1+\gamma}|\nabla_y a(z_\varepsilon)|+\varepsilon^2\right)\right).
\end{aligned}\tag{5.17}$$

For the other terms,

$$\begin{aligned}\int_{\mathbb{R}^{N+l}} \frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}\omega_\varepsilon^{q-1}(z)dz &\overset{Hölder}{\leq} \Big(\int_{\mathbb{R}^{N+l}} \Big(\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}\Big)^q dz\Big)^{1/q}\Big(\int_{\mathbb{R}^{N+l}} \omega_\varepsilon^q(z)dz\Big)^{\frac{q-1}{q}} \\ &\overset{(4.18)}{\leq} C\varepsilon^{N_\gamma(1-\frac{q}{2})}\|\frac{\partial U_{\varepsilon,z_\varepsilon}}{\partial y_j}\|_\varepsilon\|\omega_\varepsilon\|_\varepsilon^{q-1} \\ &\leq C\varepsilon^{N_\gamma-\gamma+q-2},\end{aligned} \tag{5.18}$$

which uses (1.20) and

$$\|\frac{\partial U_{\varepsilon,z_\varepsilon}}{\partial y_j}\|_\varepsilon = O\Big(\varepsilon^{\frac{N_\gamma}{2}-1-\gamma}\Big). \tag{5.19}$$

This equation is proved in [26, Proposition 4.1]. Similarly, we conclude that

$$\begin{aligned}&\int_{\mathbb{R}^{N+l}} U_{\varepsilon,z_\varepsilon}^{q-3}(z)\omega_\varepsilon^2(z)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}dz \\ &\overset{Hölder}{\leq} \Big(\int_{\mathbb{R}^{N+l}} U_{\varepsilon,z_\varepsilon}^q(z)dz\Big)^{\frac{q-3}{q}}\Big(\int_{\mathbb{R}^{N+l}} \omega_\varepsilon^q(z)dz\Big)^{2/q}\Big(\int_{\mathbb{R}^{N+l}} \Big(\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}\Big)^q dz\Big)^{1/q} \\ &\overset{(4.18)}{\leq} C\varepsilon^{N_\gamma(1-\frac{q}{2})}\|U_{\varepsilon,z_\varepsilon}\|_\varepsilon^{q-3}\|\omega_\varepsilon\|_\varepsilon^2\|\frac{\partial U_{\varepsilon,z_\varepsilon}}{\partial y_j}\|_\varepsilon \\ &\leq C\varepsilon^{N_\gamma-\gamma+1},\end{aligned} \tag{5.20}$$

which uses (1.20), (4.24) and (5.19).

From (5.14)-(5.20), we have proved that

$$\begin{aligned}&\int_{\mathbb{R}^{N+l}} \varepsilon^2\nabla_\gamma\big(U_{\varepsilon,z_\varepsilon}(z)+\omega_\varepsilon(z)\big)\nabla_\gamma\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j} + a(z)\big(U_{\varepsilon,z_\varepsilon}(z)+\omega_\varepsilon(z)\big)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j} \\ &- \big(U_{\varepsilon,z_\varepsilon}(z)+\omega_\varepsilon(z)\big)^{q-1}\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}dz. \\ &= \int_{\mathbb{R}^{N+l}} \varepsilon^2\nabla_\gamma U_{\varepsilon,z_\varepsilon}(z)\nabla_\gamma\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j} + a(z)U_{\varepsilon,z_\varepsilon}(z)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j} - U_{\varepsilon,z_\varepsilon}^{q-1}(z)\frac{\partial U_{\varepsilon,z_\varepsilon}(z)}{\partial y_j}dz \\ &\quad + \varepsilon^{N_\gamma-\gamma-1}O(\varepsilon^{min\{1,q-1\}}).\end{aligned} \tag{5.21}$$

By (5.4), (5.12), and (5.21), we know that

$$\nabla_y a(z_\varepsilon) = O(\varepsilon^{min\{1,q-1\}}). \tag{5.22}$$

According to our assumption $\deg\big(\nabla_y a(z), \widetilde{B}_\delta(z_0), 0\big) \neq 0$, there exists such a z_ε which satisfies (5.22). Moreover, $d(z_\varepsilon, z_0) = O(\varepsilon^{min\{1,q-1\}})$. This completes the proof. □

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