Electronic Journal of Differential Equations, Vol. 2025 (2025), No. 118, pp. 1–13.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, DOI: 10.58997/ejde.2025.118

TOPOLOGICAL PROPERTIES OF THE SOLUTION SET AND T-CONTROLLABILITY FOR SECOND-ORDER NEUTRAL EVOLUTION EQUATIONS WITH DELAY

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Abstract. In this article we study the topological properties of the mild solution set and T-controllability for a class of second-order neutral evolution equations with finite delay. Firstly, we establish that the set of mild solution is nonempty and compact, via Darbo-Sadovskii's fixed point theorem with sufficient conditions provided to guarantee its R_δ-property. Secondly, we show the trajectory-controllability of the equation, under an invertibility assumption on the control operator. Finally, with an example, we illustrate the applicability of the abstract results.

1. Introduction

The topological properties of solution sets of deterministic problems play an important role in qualitative research. In recent years, the topological structure of differential equation solution sets (including contractibility, acyclic, AR, R_δ-set, etc.) has attracted the attention of many scholars. Among them, R_δ-set has very important research value. In 1890, Peano [33] established that the Cauchy initial value problem

$$x'(t) = f(t, x(t)), \quad t \in [0, T],$$
$$x(0) = x_0,$$

where $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, has local solutions although the uniqueness property does not hold in general. This discovery became a driving force in studying the structure of solution sets. At the same time, Peano also proved that, in the case $n = 1$, the solution set is nonempty, compact and connected in the standard topology of real line, for t in some neighbourhood of 0. In 1942, Aronszajn [3] discovered a more precise characterization of the solution set, and proved that the solution set considered by Peano for the Cauchy problem is an R_δ-set. Since then, the topological structure and geometric structure of solution sets have been widely studied by many scholars [2, 13, 18, 19, 20, 21, 22, 23, 24, 40].

The R_δ-property is a core aspect in the study of the topological structure of solution sets for differential equations, integral equations and differential inclusions. R_δ-set is the intersection of a sequence of decreasing compact sequences. It is nonempty, compact, connected and may not be a single point solution set. From the algebraic topology point of view, an R_δ-set has the same homology group as a space of points.

Chen et al. [13] discussed the R_δ-property of solution sets for nonlinear delay evolution inclusions on compact and noncompact intervals, respectively. By exploiting Krasnoselskii's fixed point Theorem, Hoa et al. [22] investigated the topological structure of mild solution sets of the following fractional order neutral evolution equations

$$^C D^q[x(t) - h(t, x(t), x_t)] = Ax(t) + f(t, x(t), x_t), \quad t > 0,$$
$$x_0(t) = \xi(t), \quad t \in [-r, 0],$$

2020 *Mathematics Subject Classification.* 93B05, 47J35, 93C10.
Key words and phrases. R_δ-set; topological properties; trajectory-controllability; neutral evolution equations.
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Submitted July 9, 2025. Published December 18, 2025.

there ${}^C D^q$ is the Caputo fractional derivative of order $q \in (0,1)$. The functions $f, g : \mathbb{R}^+ \times \mathbb{E} \times C_0$. $x_t : [-r, 0] \to \mathbb{E}$ is defined by $x_t(s) = x(t+s)$ for all $s \in [-r, 0]$.

Recently, Jiang et al. studied the topological properties of the solution set for Riemann-Liouville fractional order evolution equations and Hilfer fractional order evolution equations, and obtained the approximate controllability results of the studied systems under the assumption that the corresponding linear systems are approximately controllable, see articles [23, 24]. In 2025, Yang et al. [40] studied topological properties of solution sets for Caputo fractional order evolution inclusions. For a comprehensive analysis of the topological properties of solution sets, we refer interested readers to [2, 18, 19, 20, 21].

For the study of the equations, we should not only discuss the development characteristics and dynamic properties of the equation itself, such as regularity, stability and so on, but also to study the dynamic behavior of the equations under the action of external factors to achieve the desired state. Thus, controllability is a thought-provoking and interesting problem. As a cornerstone of mathematical control theory, the controllability concept was first formally introduced by Kalman (1963) in his seminal work [26]. Since then, there have been many results about controllability research [39, 37, 1].

Trajectory controllability (T-controllability) was proposed by Chalishajar in [8] to establish a more robust form of control for dynamic systems. In T-controllability problems, we look for a control that steers the system along a prescribed trajectory, rather than a control simply steering a given initial state to desired final state. Hence, T-controllability is a stronger concept of controllability. The research on T-controllability of the system is helpful to reveal the control rule of the system on the specific trajectory. In today's era of swift technological development, many systems such as aerospace systems, robot systems and intelligent transportation systems have put forward higher requirements for precise trajectory control. Therefore, a growing number of scholars have turned their focus to this issue.

Chalishajar [8] studied the T-controllability in both finite and infinite dimensional abstract nonlinear integro-differential systems. A few years back, Muslim and his team [31] have used Gronwall's inequality to establish the T-controllability of nonlinear fractional order differential equations. The research team led by Chalishajar [9] studied T-controllability of neutral stochastic integro-differential equations with mixed fractional Brownian motion. In 2018, Muslim et al. [32] discussed the exact controllability and T-controllability of the following second-order evolution equations with impulses and deviated arguments

$$
\begin{gathered}
x''(t) = A(t)x(t) + Bu(t) + f(t, x(a(t)), x[h(x(t), t)]), \quad t \in J = [0, T], \quad t \neq t_k, \\
x(0) = x_0, \quad x'(0) = y_0, \quad t \in J, \quad a(t) \le t, \\
\Delta x(t_k) = I_k^1(x(t_k)), \quad \Delta x'(t_k) = I_k^2(x(t_k)), \quad k = 1, 2, \dots m,
\end{gathered}
$$

where $x(t)$ is the state function, the control function $u \in L^2(J, U)$, where U is a Hilbert space known as the control space. A is the infinitesimal generator of a strongly continuous cosine family of linear operators $\{C(t) : t \in \mathbb{R}\}$ on X. The nonlinear function $f : J \times X \times X \to X$.

Recently, Chalishajar [11] discussed the existence and T-controllability of Hilfer fractional order neutral stochastic differential equations with deviated arguments. In [12], by applying the stochastic analysis technique and Banach fixed point Theorem, it is concluded that conformable fractional order stochastic integro-differential systems with infinite delay has a mild solution, in addition, the results of T-controllability of the system are proved. In 2024, Kasinathan et al. [29] obtained the T-controllability for the time-variant impulsive neutral stochastic functional integro-differential equations. For more details on T-controllability readers can refer the articles [7, 10, 14, 15, 16, 28].

Neutral differential equations can describe dynamical systems with memory and delayed feedback. The study of second-order neutral evolution equations can not only deepen the understanding of the nature of delay dynamical systems, but also provide a mathematical tool for solving complex problems in engineering, biological and social sciences. There are many interesting results on neutral evolution equations have been gained in [25, 30, 34, 35].

To the best of our knowledge, the existing literature has scarcely addressed both the topological properties of the mild solution set and T-controllability for second-order neutral evolution equations with delay. Inspired by these previous studies, this paper investigates the following second-order neutral evolution equations in Banach space $\mathbb{X}$,

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}t}[x'(t)-G(t,x_t)] = Ax(t)+\mathbb{B}u(t)+F(t,x_t), \quad t\in J=[0,T], \\ x(t)=\varphi(t), \quad x'(0)=x^0, \quad t\in[-\tau,0],\end{gathered} \tag{1.1}$$

where the state function $x(t)\in\mathbb{X}$, $A:D(A)\subset\mathbb{X}\to\mathbb{X}$ is the infinitesimal generator of strongly continuous cosine family $\{C(t):t\in\mathbb{R}\}$. The control function $u(\cdot)$ takes in Banach space $\mathbb{U}$. $\mathbb{B}$ is a bounded linear operator from $\mathbb{U}$ to $\mathbb{X}$. The nonlinear functions $F,G:J\times C([-\tau,0];\mathbb{X})\to\mathbb{X}$ satisfy certain conditions. $\varphi\in C([-\tau,0];\mathbb{X})$ and $x_t\in C([-\tau,0];\mathbb{X})$ is defined by $x_t(\theta)=x(t+\theta)$ for each $\theta\in[-\tau,0]$.

The main contributions and novelties of this paper are listed below.

- (i) Up to now, there are no results on the topological properties of mild solution sets for second-order neutral evolution equations.
- (ii) There are no papers on the T-controllability of second-order neutral evolution equations with finite delay.
- (iii) Through example, we demonstrate the application of our results.

The rest of this article is organized as follows. In section 2, we introduce notation and preliminaries needed to establish our main results. Section 3 is devoted to the compactness and R_δ-property of the mild solution set for equation (1.1). Results on T-controllability of equation (1.1) are presented in Section 4. Finally, Section 5 provides an example demonstrating the effectiveness of our theoretical results.

2. Preliminaries

In this section, we first recall some useful definitions and provide preliminary results. Let $(\mathbb{X},\|\cdot\|)$ and $(\mathbb{K},\|\cdot\|_{\mathbb{K}})$ be two Banach spaces. $L^2(J,\mathbb{X})$ denote the Banach space of all square integrable functions from J into $\mathbb{X}$ with the norm $\|x\|_{L^2}=(\int_0^T\|x(t)\|^2\mathrm{d}t)^{\frac{1}{2}}$ and $C(J,\mathbb{X})$ denote the Banach space of all continuous functions from J into $\mathbb{X}$ with the super norm $\|x\|_C=\sup_{t\in J}\|x(t)\|$. Set $C_t=C([-\tau,t];\mathbb{X})$ represent the Banach space of all continuous functions from $[-\tau,t]$ into $\mathbb{X}$ with the super norm $\|x\|_{C_t}=\sup_{\theta\in[-\tau,t]}\|x(\theta)\|$, $t\in J$. The space of all bounded linear operators from $\mathbb{X}$ into $\mathbb{K}$ is denoted by $\mathcal{L}(\mathbb{X},\mathbb{K})$. We use the same notation $\|\cdot\|$ for the norm of $\mathcal{L}(\mathbb{X},\mathbb{K})$. Particulary, $\mathcal{L}(\mathbb{X})$ will denote $\mathcal{L}(\mathbb{X},\mathbb{X})$.

Let $\mathbb{V}$ and $\mathbb{Z}$ be two metric spaces. For convenience, we introduce following symbols.

$$\begin{gathered}\mathcal{P}(\mathbb{V})=\{D\subseteq\mathbb{V}:D\text{ is nonempty}\}, \\ \mathcal{P}_b(\mathbb{V})=\{D\subseteq\mathbb{V}:D\text{ is nonempty and bounded}\}, \\ \mathcal{P}_f(\mathbb{V})=\{D\subseteq\mathbb{V}:D\text{ is nonempty and closed}\}, \\ \mathcal{K}(\mathbb{V})=\{D\subseteq\mathbb{V}:D\text{ is nonempty and compact (convex)}\}.\end{gathered}$$

The graph of set-valued map $f:\mathbb{V}\to\mathcal{P}_f(\mathbb{Z})$ is represented by $\mathbb{G}(f)=\{(v,z)\in\mathbb{V}\times\mathbb{Z}:z\in f(v)\}$. If Δ is subset of $\mathbb{Z}$, then we denote the complete pre-image of Δ by $f^{-1}(\Delta)=\{v\in\mathbb{V}:f(v)\cap\Delta\neq\emptyset\}$. If $\mathbb{G}(f)$ is closed in $\mathbb{V}\times\mathbb{Z}$, then f is referred to as closed.

Definition 2.1 ([27]). Let $f:\mathbb{V}\to\mathcal{P}_f(\mathbb{Z})$ be a set-valued map.

- (i) If $D\subset\mathbb{V}$ is compact, $f(D)$ is relatively compact, then f is said to be quasicompact.
- (ii) For $x_0\in\mathbb{V}$, if for any neighborhood $O(f(x_0))$ of $f(x_0)$, there exists a neighborhood $O(x_0)$ of x_0 such that $f(x)\subset O(f(x_0))$ for all $x\in O(x_0)$, then f is called upper semicontinuous (u.s.c. for short).

If $\mathbb{V}\cap\mathbb{Z}\neq\emptyset$, then $v\in\mathbb{V}\cap\mathbb{Z}$ is said to be a fixed point of f if $v\in f(v)$. The set of all fixed points of f is denoted by $\mathrm{Fix}(f)$.

Definition 2.2 ([17]). $D \subset \mathcal{P}(\mathbb{V})$ is referred to as contractible if there exist a point $x_0 \in D$ and a continuous map $f : D \times [0,1] \to D$ such that $f(x,0) = x$ and $f(x,1) = x_0$, for every $x \in D$.

Definition 2.3 ([17]). $D \subset \mathcal{P}(\mathbb{V})$ is called to be an R_δ-set if $D = \cap_{n=1}^{\infty} D_n$, where $\{D_n\}_{n=1}^{\infty}$ is a decreasing sequence of nonempty and compact contractible sets.

Lemma 2.4 ([6]). *Let $\hbar : \mathbb{V} \to \mathbb{X}$ be continuous. Assume that $\hbar$ is a proper map, that is to say for $D \subset \mathcal{K}(\mathbb{X})$, $\hbar^{-1}(D)$ is compact. Moreover, suppose that there exists a sequence $\{\hbar_n\}_{n=1}^{\infty}$ of mapping from $\mathbb{V}$ into $\mathbb{X}$ such that*

- (i) *$\{\hbar_n\}_{n=1}^{\infty}$ converges uniformly to $\hbar$ in $\mathbb{V}$ and $\hbar_n$ is proper.*
- (ii) *For fixed $y_0 \in \mathbb{X}$ and any $y \in O(y_0)$, then equation $\hbar_n(x) = y$ admits exactly only solution, where $O(y_0)$ is the neighborhood of y_0.*

Then, $\hbar^{-1}(y_0)$ is an R_δ-set.

Let us recall the definition and properties of the measure of noncompactness. Set D be a bounded subset of $\mathbb{X}$. The Hausdorff measure of noncompactness is expressed as

$$\alpha(D) = \inf\{\varepsilon : D \text{ has a finite } \varepsilon\text{-net}\}.$$

Lemma 2.5 ([4]). *Let $D, D_1 \subset \mathbb{X}$ be bounded, then the following properties are satisfied:*

- (i) *D is precompact if and only if $\alpha(D) = 0$;*
- (ii) *$\alpha(D) = \alpha(\bar{D}) = \alpha(coD)$, where $\bar{D}$ and coD are the closure and the convex hull of D, respectively;*
- (iii) *$\alpha(D) \le \alpha(D_1)$ when $D \subset D_1$;*
- (iv) *$\alpha(D + D_1) \le \alpha(D) + \alpha(D_1)$, where $D + D_1 = \{x + y : x \in D, y \in D_1\}$;*
- (v) *$\alpha(D \cup D_1) \le \max\{\alpha(D), \alpha(D_1)\}$;*
- (vi) *$\alpha(\lambda D) \le |\lambda|\alpha(D)$ for any $\lambda \in \mathbb{R}$.*

For each $W \subset C(J,\mathbb{X})$, we define $\int_0^t W(s)\mathrm{d}s = \int_0^t v(s)\mathrm{d}s$, where $W(s) = \{v(s) : v \in W\} \subset \mathbb{X}$.

Lemma 2.6 ([4]). *If $W \subset C(J,\mathbb{X})$ is bounded and equicontinuous, then $t \to \alpha(W(t))$ is continuous on J and*

$$\alpha(W) = \max_{t\in J} \alpha(W(t)), \quad \alpha(\int_0^t W(s)\mathrm{d}s) \le \int_0^t \alpha(W(s))\mathrm{d}s, \quad \textit{for all } t \in J.$$

Lemma 2.7 ([4]). *Let $\{v_n\}_{n=1}^{\infty}$ be a sequence of Bochner integrable functions from J to $\mathbb{X}$ with $\|v_n(t)\| \le m(t)$ for almost all $t \in J$ and every $n \ge 1$, where $m(t) \in L(J,\mathbb{R}^+)$, then the function $\psi(t) = \alpha(\{v_n\}_{n=1}^{\infty}) \in L(J,\mathbb{R}^+)$ and satisfies*

$$\alpha(\{\int_0^t v_n(s)\mathrm{d}s\}) \le 2\int_0^t \psi(s)\mathrm{d}s.$$

Remark 2.8 ([41]). If $W \subset C(J,\mathbb{X})$ is bounded and equicontinuous, then $coW \subset C(J,\mathbb{X})$ is also bounded and equicontinuous, where coW stands for the convex hull of W.

Lemma 2.9 ([5]). *For every bounded subset D of $\mathbb{X}$ and $\varepsilon > 0$, there exists a sequence $\{w_n\}_{n=1}^{\infty} \subset D$ satisfying*

$$\alpha(D) \le 2\alpha(\{w_n\}_{n=1}^{\infty}) + \varepsilon.$$

Definition 2.10 ([4]). For $D^* \subset D$, if there exists a positive constant $\gamma < 1$ such that $\alpha(f(D^*)) \le \gamma\alpha(D^*)$, then $f : D \subset \mathbb{X} \to \mathbb{X}$ is said that α-contractive.

Lemma 2.11 ([41]). *Supposed that D be a closed and convex subset of Banach space $\mathbb{X}$, $\mathcal{F} : D \to D$ be a continuous map and $\mathcal{F}(D)$ be bounded. For every bounded subset $D_0 \subset D$, set*

$$\mathcal{F}^1(D_0) = \mathcal{F}(D_0), \quad \mathcal{F}^n(D_0) = \mathcal{F}(\overline{co}\mathcal{F}^{n-1}(D_0)), \quad n = 2, 3, \dots.$$

If there are a constant $0 \le \gamma < 1$ and a positive integer n_0 to such that, for any bounded subset $D_0 \subset D$, $\alpha(\mathcal{F}^{n_0}(D_0)) \le \gamma\alpha(D_0)$, then there exists a $D^* \subset D$ satisfying $\alpha(\mathcal{F}(D^*)) = 0$.

Lemma 2.12 (Darbo-Sadovskii's fixed point Theorem [4])**.** *Supposed that D be a bounded, closed and convex subset of $\mathbb{X}$, and the continuous map $\mathcal{F}: D \to D$ be a α-contraction. Then $Fix(\mathcal{F})$ is nonempty on D.*

Next, we review some facts about strongly continuous cosine family.

Definition 2.13 ([36])**.** A one parameter family $\{C(t) : t \in \mathbb{R}\} \subset \mathcal{L}(\mathbb{X})$ is called a strongly continuous cosine family if

(i) $C(s+t) + C(s-t) = 2C(s)C(t)$, for all $s, t \in \mathbb{R}$;
(ii) $C(0) = I$;
(iii) $C(t)x$ is continuous in t on $\mathbb{R}$ for each fixed $x \in \mathbb{X}$.

Define the associated sine family $\{S(t) : t \in \mathbb{R}\}$ by

$$S(t)x = \int_0^t C(s)x\mathrm{d}s, \quad x \in \mathbb{X}, \quad t \in \mathbb{R}.$$

The infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ is the operator $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ defined by

$$Ax = \frac{\mathrm{d}^2}{\mathrm{d}t^2} C(t)x \mid_{t=0}, \quad x \in D(A),$$

where $D(A) = \{x \in \mathbb{X} : C(\cdot)x \in C^2(\mathbb{R}, \mathbb{X})\}$. Define $E = \{x \in \mathbb{X} : C(\cdot)x \in C^1(\mathbb{R}, \mathbb{X})\}$. The properties of the strong continuous cosine family are given below.

Lemma 2.14 ([36])**.** *If $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is the infinitesimal generator of strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ and $\{S(t) : t \in \mathbb{R}\}$ is the associated sine family family, then*

(i) *there exist constants $M \geq 0$ and $\mu \geq 0$, such that $\|C(t)\| \leq Me^{\mu|t|}$, and $\|S(t)\| \leq Me^{\mu|t|}$;*
(ii) *if $x \in E$, then $S(t)x \in D(A)$ and $\frac{\mathrm{d}}{\mathrm{d}t} C(t)x = AS(t)x$;*
(iii) *if $x \in D(A)$, then $C(t)x \in D(A)$ and $\frac{\mathrm{d}^2}{\mathrm{d}t^2} C(t)x = AC(t)x = C(t)Ax$;*
(iv) *if $x \in E$, then $S(t)x \in D(A)$ and $\frac{\mathrm{d}^2}{\mathrm{d}t^2} S(t)x = AS(t)x$.*

We know from the Lemma 2.14(i) that $\{C(t) : t \in J\}$ and $\{S(t) : t \in J\}$ are uniformly bounded, that is, there are positive numbers M_1, M_2 such that

$$\|C(t)\| \leq M_1, \quad \|S(t)\| \leq M_2, \quad t \in J.$$

Now, we give the concept of mild solution of equation (1.1) and the definition of T-controllability of equation (1.1).

Definition 2.15. A function $x \in C_T$ is said to be a mild solution of equation (1.1) if for $u \in L^2(J, \mathbb{U})$ it satisfies

$$\begin{aligned} x(t) = C(t)\varphi(0) + S(t)(x^0 - G(0, \varphi)) + \int_0^t C(t-s)G(s, x_s)\mathrm{d}s \\ + \int_0^t S(t-s)(\mathbb{B}u(s) + F(s, x_s))\mathrm{d}s, \quad t \in J, \end{aligned} \tag{2.1}$$

and $x(t) = \varphi(t), \quad t \in [-\tau, 0]$.

Let Φ be the set of all functions $z(\cdot)$ defined on J such that $z(0) = \varphi(0)$, $z(T) = x_T$, $z'(0) = x^0$, and $z(\cdot)$ is twice continuously differentiable, where x_T represents desired final state. The set of all feasible trajectories for equation (1.1) is denoted by Φ.

Definition 2.16. The equation (1.1) is said to be trajectory controllable (T-controllable) on J if for any $z \in \Phi$, there exists a control function $u \in L^2(J, \mathbb{U})$ such that the mild solution $x(\cdot)$ of equation (1.1) satisfies $x(t) = z(t)$ a.e. on J.

3. Topological properties of the mild solution set

To obtain our main results, we need to impose the following hypotheses:

(H1) $\mathbb{B} : L^2(J, \mathbb{U}) \to L^2(J, \mathbb{X})$ is a bounded linear operator, that is, there is a positive constant b such that $\|\mathbb{B}\| \le b$.

(H2) The nonlinear function $F : J \times C_0 \to \mathbb{X}$ satisfies the following conditions:
- (i) $F(t, \cdot) : C_0 \to \mathbb{X}$ is continuous for $t \in J$ and $F(\cdot, x) : J \to \mathbb{X}$ is measurable for each $x \in C_0$.
- (ii) There exist a function $m_F \in L^2(J, \mathbb{R}^+)$ and a constant $m_1 > 0$ such that
$$\|F(t, x)\| \le m_F(t) + m_1\|x\|_{C_0},$$
for all $x \in C_0$, a.e. $t \in J$.
- (iii) There exists a constant $L_1 > 0$ such that for any bounded subset $D \subset C_0$ and a.e. $t \in J$,
$$\alpha(F(t, D)) \le L_1 \sup_{\theta\in[-\tau,0]} \alpha(D(\theta)).$$

(H3) The nonlinear function $G : J \times C_0 \to \mathbb{X}$ satisfies the following conditions:
- (i) $G(t, \cdot) : C_0 \to \mathbb{X}$ is continuous for $t \in J$ and $G(\cdot, x) : J \to \mathbb{X}$ is measurable for each $x \in C_0$.
- (ii) There exist a non-negative function $m_G \in L^2(J, \mathbb{R})$ and a constant $m_2 > 0$ such that
$$\|G(t, x)\| \le m_G(t) + m_2\|x\|_{C_0},$$
for all $x \in C_0$, a.e. $t \in J$. In particular, when $t = 0$, $m_G(t) = 0$.
- (iii) There exists a constant $L_2 > 0$ such that for any bounded subset $D \subset C_0$ and a.e. $t \in J$,
$$\alpha(G(t, D)) \le L_2 \sup_{\theta\in[-\tau,0]} \alpha(D(\theta)).$$

(H4) $\{C(t) : t \in J\}$ and $\{S(t) : t \in J\}$ are uniformly continuous.

For the sake of simplicity, we denote $\Xi(u) = \{x \in C(J, \mathbb{X})\colon x \text{ satisfies } (2.1)\}$.

Theorem 3.1. *If hypotheses* (H1)–(H4) *hold, then for $u \in L^2(J, \mathbb{U})$, $\Xi(u)$ is nonempty and compact provided*
$$(M_1m_2 + M_2m_1)T < 1.$$

Proof. We define the operator $\Gamma : C(J, \mathbb{X}) \to C(J, \mathbb{X})$ by
$$\begin{aligned}(\Gamma x)(t) = C(t)\varphi(0) + S(t)(x^0 - G(0, \varphi)) + \int_0^t C(t - s)G(s, x_s)\mathrm{d}s \\ + \int_0^t S(t - s)(\mathbb{B}u(s) + F(s, x_s))\mathrm{d}s, \quad t \in J.\end{aligned} \tag{3.1}$$

It is clear to see that $x \in \Xi(u)$ is equivalent to $x \in \mathrm{Fix}(\Gamma)$. The proof is divided into 4 steps.

Step 1. There exists $r > 0$ such that Γ maps the set $B_r := \{x \in C(J, \mathbb{X}) : \|x\|_C \le r\}$ to itself. In fact, for any $x \in B_r$ and $\rho \in J$, one has
$$\begin{aligned}\|x\|_{C_0} &= \sup_{\theta\in[-\tau,0]} \|x(\rho + \theta)\| \\ &\le \sup_{s\in[-\tau,0]} \|x(s)\| + \sup_{s\in[0,\rho]} \|x(s)\| \\ &\le \|\varphi\|_{C_0} + \sup_{s\in[0,\rho]} \|x(s)\|.\end{aligned} \tag{3.2}$$

In addition, for any $x \in B_r$ and $t \in J$, from hypotheses (H1)–(H3), (3.1), (3.2) and well-known Hölder inequality, we have

$$\begin{aligned}
\|(\Gamma x)(t)\| &\leq M_1\|\varphi\|_{C_0} + M_2(\|x^0\| + \|G(0,\varphi)\|) + M_1\int_0^t \|G(s,x_s)\|\mathrm{d}s \\
&\quad + M_2\int_0^t \|\mathbb{B}u(s)\|\mathrm{d}s + M_2\int_0^t \|F(s,x_s)\|\mathrm{d}s \\
&\leq M_1\|\varphi\|_{C_0} + M_2\|x^0\| + M_2 m_2\|\varphi\|_{C_0} + M_1\sqrt{T}\|m_G\|_{L^2(J,\mathbb{R})} \\
&\quad + M_2 b\sqrt{T}\|u\|_{L^2(J,\mathbb{U})} + M_2\sqrt{T}\|m_F\|_{L^2(J,\mathbb{R}^+)} + (M_1 m_2 + M_2 m_1)T\|\varphi\|_{C_0} \\
&\quad + (M_1 m_2 + M_2 m_1)Tr \leq r.
\end{aligned} \tag{3.3}$$

This shows that $\Gamma(B_r) \subseteq \Gamma(B_r)$.

Step 2. The map Γ is continuous on B_r. Let $\{x_n\}_{n=1}^{\infty} \subseteq B_r$ and $\lim_{n\to\infty} x_n = x$ in B_r. According to hypotheses (H2)(i) and (H3)(i), we can infer that

$$\lim_{n\to\infty} F(t,(x_n)_t) = F(t,x_t), \quad \lim_{n\to\infty} G(t,(x_n)_t) = G(t,x_t), \quad t \in J.$$

Consequently, for $t \in J$, from hypotheses (H2)(ii), (H3)(ii) and the Lebesgue dominated convergence Theorem, we can prove that

$$\|(\Gamma x_n)(t) - (\Gamma x)(t)\| \leq M_1\int_0^t \|G(s,(x_n)_s) - G(s,x_s)\|\mathrm{d}s + M_2\int_0^t \|F(s,(x_n)_s) - F(s,x_s)\|\mathrm{d}s \to 0,$$

as $n \to \infty$. Therefore, we can affirm that the map Γ is continuous on B_r.

Step 3. We prove that Γ is a compact operator. To this goal, let $0 \leq t_1 < t_2 \leq T$ and for any $x \in B_r$, from hypotheses (H2)(ii), (H3)(ii), (H4) and Hölder inequality, we have

$$\begin{aligned}
\|(\Gamma x)(t_2) - (\Gamma x)(t_1)\| &\leq \|[C(t_2) - C(t_1)]\varphi(0)\| + \|[S(t_2) - S(t_1)](x^0 - G(0,\varphi))\| \\
&\quad + \|\int_{t_1}^{t_2} C(t_2 - s)G(s,x_s)\mathrm{d}s\| + \|\int_{t_1}^{t_2} S(t_2 - s)F(s,x_s)\mathrm{d}s\| \\
&\quad + \|\int_0^{t_1} [C(t_2 - s) - C(t_1 - s)]G(s,x_s)\mathrm{d}s\| \\
&\quad + \|\int_0^{t_1} [S(t_2 - s) - S(t_1 - s)]F(s,x_s)\mathrm{d}s\| \\
&\quad + \|\int_{t_1}^{t_2} S(t_2 - s)\mathbb{B}u(s)\mathrm{d}s\| + \|\int_0^{t_1} [S(t_2 - s) - S(t_1 - s)]\mathbb{B}u(s)\mathrm{d}s\| \\
&\leq \|[C(t_2) - C(t_1)]\varphi(0)\| + \|[S(t_2) - S(t_1)](x^0 - G(0,\varphi))\| \\
&\quad + (t_2 - t_1)M_1 m_2\|x_s\|_{C_0} + \sqrt{t_2 - t_1}M_1\|m_G\|_{L^2(J,\mathbb{R})} \\
&\quad + (t_2 - t_1)M_2 m_1\|x_s\|_{C_0} + \sqrt{t_2 - t_1}M_2\|m_F\|_{L^2(J,\mathbb{R}^+)} \\
&\quad + \int_0^{t_1} \|[C(t_2 - s) - C(t_1 - s)]G(s,x_s)\|\mathrm{d}s \\
&\quad + \int_0^{t_1} \|[S(t_2 - s) - S(t_1 - s)]F(s,x_s)\|\mathrm{d}s \\
&\quad + \sqrt{t_2 - t_1}M_2 b\|u\|_{L^2(J,\mathbb{U})} + b\int_0^{t_1} \|[S(t_2 - s) - S(t_1 - s)]u(s)\|\mathrm{d}s \\
&:= \sum_{i=1}^{10} I_i.
\end{aligned}$$

In fact, it suffices to prove that when $t_2 \to t_1$, $I_i \to 0$ $(i = 1, 2, \ldots, 10)$. By the strong continuity of $\{C(t)\}$ and $\{S(t)\}$, we can infer that when $t_2 \to t_1$, $I_1 \to 0$ and $I_2 \to 0$. In addition, $\lim_{t_2\to t_1} I_i = 0$ $(i = 3, 4, 5, 6, 9)$. According to condition (H4), we can obtain that when $t_2 \to t_1$, $I_7 \to 0$, $I_8 \to 0$

and $I_{10} \to 0$. Therefore, when $t_2 \to t_1$, it can be concluded that $\|(\Gamma x)(t_2) - (\Gamma x)(t_1)\| \to 0$, then $\Gamma(B_r)$ is equicontinuous.

Let $\Omega = \overline{co}\Gamma(B_r)$ be a bound, closed and convex set. It follows from Remark 2.8 that $\Omega = \overline{co}\Gamma(B_r)$ is also equicontinuous. Thus, the continuity and boundedness of $\Gamma : \Omega \to \Omega$ can be easily demonstrated. For any bounded subset $\Omega_0 \subseteq \Omega$, set $\Gamma^1(\Omega_0) = \Gamma(\Omega_0)$, $\Gamma^n(\Omega_0) = \Gamma(\overline{co}\Gamma^{n-1}(\Omega_0))$, $n = 2, 3, \dots$.

By Lemma 2.6, Lemma 2.7, Lemma 2.9, (H2)(iii) and (H3)(iii), we can deduce that, for any $\varepsilon > 0$, there is a set $\Omega_1 = \{z_n\}_{n=1}^{\infty} \subseteq \Omega_0$ satisfying

$$\begin{aligned} \alpha(\Gamma^1(\Omega_0)(t)) &= \alpha(\Gamma(\Omega_0)(t)) \le 2\alpha(\Gamma(\Omega_1)(t)) + \varepsilon \\ &\le 4M_1 \int_0^t L_2 \sup_{\theta\in[-\tau,0]} \alpha(\Omega_1(\theta + s))\mathrm{d}s + 4M_2 \int_0^t L_1 \sup_{\theta\in[-\tau,0]} \alpha(\Omega_1(\theta + s))\mathrm{d}s + \varepsilon \\ &\le 4(M_1L_2 + M_2L_1)t\alpha(\Omega_0) + \varepsilon. \end{aligned} \tag{3.4}$$

Since $\varepsilon > 0$ is arbitrary, one has

$$\alpha(\Gamma^1(\Omega_0)(t)) \le 4(M_1L_2 + M_2L_1)t\alpha(\Omega_0).$$

Similarly, for any $\varepsilon > 0$, the sequence $\Omega_2 = \{v_n\}_{n=1}^{\infty} \subseteq \overline{co}\Gamma^1(\Omega_0)$ can be found and for all $t \in J$,

$$\begin{aligned} &\alpha(\Gamma^2(\Omega_0)(t)) \\ &= \alpha(\Gamma(\overline{co}\Gamma^1(\Omega_0))(t)) \le 2\alpha(\Gamma(\Omega_2)(t)) + \varepsilon \\ &\le 4M_1 \int_0^t L_2 \sup_{\theta\in[-\tau,0]} \alpha(\Omega_2(\theta + s))\mathrm{d}s + 4M_2 \int_0^t L_1 \sup_{\theta\in[-\tau,0]} \alpha(\Omega_2(\theta + s))\mathrm{d}s + \varepsilon \\ &\le 4(M_1L_2 + M_2L_1) \int_0^t \sup_{\sigma\in[0,s]} \alpha((\overline{co}\Gamma^1(\Omega_0))(\sigma))\mathrm{d}s + \varepsilon. \\ &\le 4(M_1L_2 + M_2L_1) \int_0^t 4(M_1L_2 + M_2L_1)s\alpha(\Omega_0)\mathrm{d}s + \varepsilon. \\ &\le \frac{[4(M_1L_2 + M_2L_1)t]^2}{2!}\alpha(\Omega_0) + \varepsilon. \end{aligned} \tag{3.5}$$

Since $\varepsilon > 0$ is arbitrary, we can get

$$\alpha(\Gamma^2(\Omega_0)(t)) \le \frac{[4(M_1L_2 + M_2L_1)t]^2}{2!}\alpha(\Omega_0).$$

Using mathematical induction, one has

$$\alpha(\Gamma^n(\Omega_0)(t)) \le \frac{[4(M_1L_2 + M_2L_1)t]^n}{n!}\alpha(\Omega_0) \le \frac{[4(M_1L_2 + M_2L_1)T]^n}{n!}\alpha(\Omega_0). \tag{3.6}$$

By the Stirling formula, there is a constant $\varrho \in (0, 1)$ satisfying

$$n! = \sqrt{2\pi n}(\frac{n}{e})^n e^{\frac{1}{12n+\varrho}}.$$

This and (3.6) imply that

$$\lim_{n\to\infty} \frac{[4(M_1L_2 + M_2L_1)T]^n}{n!} = \lim_{n\to\infty} \frac{[4(M_1L_2 + M_2L_1)T]^n}{\sqrt{2\pi n}(\frac{n}{e})^n e^{\frac{1}{12n+\varrho}}} = 0.$$

Therefore, there are $\gamma \in (0, 1)$ and $n \ge 1$ such that

$$\alpha(\Gamma^n(\Omega_0)(t)) \le \gamma\alpha(\Omega_0).$$

Moreover, we can obtain

$$\alpha(\Gamma^n(\Omega_0)) = \max_{t\in J} \alpha(\Gamma^n(\Omega_0)(t)) \le \gamma\alpha(\Omega_0).$$

It follows from Lemma 2.11 that there is a $\widehat{\Omega} \subset \Omega \subset B_r$ such that

$$\alpha(\Gamma(\widehat{\Omega})) = 0.$$

Employing the generalized Ascoli-Arzela Theorem, we can obtain the compactness of Γ. Therefore, Lemma 2.12 shows that Γ has a fixed point in B_r, that is, $\Xi(u)$ is nonempty.

Step 4. The fixed point set $\text{Fix}(\Gamma)$ is compact. By Step 1, we can acquire the boundedness of $\text{Fix}(\Gamma)$. Thus, the compactness of Γ ensures that $\Gamma(\text{Fix}(\Gamma))$ is relatively compact. In addition, in view of continuity Γ, we can easily see that $\text{Fix}(\Gamma)$ is closed. Hence, from $\text{Fix}(\Gamma) \subseteq \Gamma(\text{Fix}(\Gamma))$, we can obtain that $\text{Fix}(\Gamma)$ is relatively compact, that is, $\Xi(u)$ is compact. This completes the proof. □

Theorem 3.2. *Assume that* (H1)–(H4) *are satisfied, then for* $u \in L^2(J, \mathbb{U})$, $\Xi(u)$ *is an* R_δ*-set.*

Proof. In view of assumptions (H2)(i), (H3)(i) and the Lasota-Yorke approximation Theorem [17], we can find two local Lipschitz sequences $\{F_n\}_{n=1}^{\infty}, \{G_n\}_{n=1}^{\infty} : J \times C_0 \to \mathbb{X}$ to ensure that

$$\|F_n(t,x) - F(t,x)\| < \varepsilon_n, \quad \|G_n(t,x) - G(t,x)\| < \varepsilon_n, \quad t \in J, \quad x \in C_0, \tag{3.7}$$

where $\varepsilon_n \to 0$, as $n \to \infty$. Hence, according to (H2)(ii) and (H3)(ii), we obtain

$$\begin{aligned} &\|F_n(s,x_s)\| \le 1 + m_F(s) + m_1\|x_s\|_{C_0}, \quad n = 1,2,\dots, \\ &\|G_n(s,x_s)\| \le 1 + m_G(s) + m_2\|x_s\|_{C_0}, \quad n = 1,2,\dots. \end{aligned} \tag{3.8}$$

Next, we define an approximation operator as

$$\begin{aligned} (\Gamma_n x)(t) &= C(t)\varphi(0) + S(t)(x^0 - G(0,\varphi)) + \int_0^t C(t-s)G_n(s,x_s)\mathrm{d}s \\ &\quad + \int_0^t S(t-s)(\mathbb{B}u(s) + F_n(s,x_s))\mathrm{d}s, \quad t \in J, \quad n = 1,2,\dots. \end{aligned}$$

According to (3.7) and (3.8), we can easily see that the operator Γ_n is well defined. For $x \in \Xi(u)$, $t \in J$, one has

$$\begin{aligned} &\|(I-\Gamma_n)(x)(t) - (I-\Gamma)(x)(t)\| \\ &\le M_1 \int_0^t \|G_n(s,x_s) - G(s,x_s)\|\mathrm{d}s + M_2 \int_0^t \|F_n(s,x_s) - F(s,x_s)\|\mathrm{d}s \\ &\le 2(M_1+M_2)T\varepsilon_n \to 0, \quad n \to \infty, \end{aligned} \tag{3.9}$$

where I represents identical operator. It is easy to see from (3.9) that $I - \Gamma_n$ converges to $I - \Gamma$ uniformly on $C(J, \mathbb{X})$.

Moreover, similar to the proof of Theorem 3.1, for any $y \in C(J, \mathbb{X})$, one may deduce that

$$(I - \Gamma_n)(x) = y \tag{3.10}$$

yields at least a mild solution. Further, similar to the argument used in [38, Theorem 3.6], by applying the local Lipschitz continuity of $\{F_n\}_{n=1}^{\infty}$ and $\{G_n\}_{n=1}^{\infty}$, we can prove that the mild solution of equation (3.10) is unique.

Next, we show that the mappings $I - \Gamma_n$ and $I - \Gamma$ are proper for given $n = 1,2,\dots$. According to the Step 2 of the proof in Theorem 3.1, $I - \Gamma_n$ is continuous. Now, for every compact set D in $C(J, \mathbb{X})$, we need only confirm that $(I - \Gamma_n)^{-1}(D)$ is a compact set. Set $(I - \Gamma_n)(K) = D$. Thus, for sequence $\{k_n\}_{n=1}^{\infty} \subset K$, there is a sequence $\{y_n\}_{n=1}^{\infty} \subset D$ satisfies $(I - \Gamma_n)(k_n) = y_n$, that is,

$$\begin{aligned} k_n(t) &= y_n(t) + C(t)\varphi(0) + S(t)(x^0 - G(0,\varphi)) + \int_0^t C(t-s)G_n(s,(k_n)_s)\mathrm{d}s \\ &\quad + \int_0^t S(t-s)(\mathbb{B}u(s) + F_n(s,(k_n)_s))\mathrm{d}s, \quad t \in J, \quad n = 1,2,\dots. \end{aligned} \tag{3.11}$$

Hence, from (3.8), (3.11), the boundedness of $\{y_n\}_{n=1}^{\infty}$ and Step 1 of Theorem 3.1, we can deduce that $\{k_n\}_{n=1}^{\infty}$ is bounded in $C(J, \mathbb{X})$. In addition, from the compactness of D and the proof of Step 3 of Theorem 3.1, we can obtain that $\{k_n\}_{n=1}^{\infty}$ is relatively compact. Additionally, the continuity of $I - \Gamma_n$ and the closedness of D imply that the set K is closed, it means that K is compact. Thus, $I - \Gamma_n$ is proper. Following analogous arguments, we can acquire that $I - \Gamma$ is also proper. Consequently, it can be obtained by Lemma 2.4 that $\Xi(u)$ is an R_δ-set. The proof is completed. □

4. T-controllability

In this section, we investigate the T-controllability of equation (1.1). To establish our results, we introduce the hypothesis

(H5) The control operator $\mathbb{B}$ is invertible.

Theorem 4.1. *If hypotheses* (H1)–(H5) *hold, then second-order neutral evolution equation* (1.1) *is T-controllable on J.*

Proof. Let $z(t)$ be the given trajectory on Φ. We are looking for a control function $u(t)$ that satisfies

$$z(t) - C(t)\varphi(0) - S(t)(x^0 - G(0,\varphi)) - \int_0^t C(t-s)G(s,z_s)\mathrm{d}s - \int_0^t S(t-s)F(s,z_s)\mathrm{d}s \\ = \int_0^t S(t-s)\mathbb{B}u(s)\mathrm{d}s.$$

Taking the second time derivative of both sides with respect to t, we can obtain

$$\begin{aligned} &z''(t) - AC(t)\varphi(0) - AS(t)x^0 + (S(t)G(0,\varphi))'' - \int_0^t AC(t-s)G(s,z_s)\mathrm{d}s \\ &- G'(t,z_t) - \int_0^t AS(t-s)F(s,z_s)\mathrm{d}s - F(t,z_t) \\ &= \int_0^t AS(t-s)\mathbb{B}u(s)\mathrm{d}s + \mathbb{B}u(t). \end{aligned} \tag{4.1}$$

Equation (4.1) can be rewritten in the form

$$\varpi(t) = \int_0^t \kappa(t,s)\varpi(s)\mathrm{d}s + \varpi_0(t), \tag{4.2}$$

where $\varpi(t) = \mathbb{B}u(t)$, $\kappa(t,s) = -AS(t-s)$, and $\varpi_0(t)$ is the left hand side of equation (4.1). Next, define an operator $\Upsilon : L^2(J,\mathbb{X}) \to L^2(J,\mathbb{X})$ by

$$(\Upsilon\varpi)(t) = \int_0^t \kappa(t,s)\varpi(s)\mathrm{d}s. \tag{4.3}$$

It is easy to verify that Υ is a bounded linear operator [32]. In addition, one can demonstrate that Υ^n is a contraction mapping for large n. Thus, by generalized Banach contraction principle, there is a unique solution $\varpi(t)$ of (4.2) for given $\varpi_0 \in L^2(J,\mathbb{X})$. Hence, we can extract $u(t)$ from the relation

$$\mathbb{B}u(t) = \varpi(t),$$

since the control operator $\mathbb{B}$ is invertible, it can be accomplished by taking the left inverse of $\mathbb{B}$. Hence, equation (1.1) is T-controllable on J. □

5. Application

To illustrate the applicability of the conclusion, we consider the initial boundary value problem for second-order partial differential equation,

$$\begin{gathered} \frac{\partial^2}{\partial t^2}u(t,x) - \frac{\partial}{\partial t}g(t,u_t(\cdot,x)) = \frac{\partial^2}{\partial x^2}u(t,x) + f(t,u_t(\cdot,x)) + \iota v(t,x), \quad t \in [0,1], \quad x \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, \quad t \in [0,1], \\ u(t,x) = \varphi(t,x), \quad t \in [0,1], \quad x \in [0,\pi], \\ \frac{\partial}{\partial t}u(0,x) = x^0, \end{gathered} \tag{5.1}$$

where ι is a constant, $u_t(\cdot,x) = u(\cdot + t, x)$. Let $\mathbb{X} = \mathbb{U} := L^2(0,\pi)$. Define the operator $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ as $Au = \frac{\partial^2 u(t,x)}{\partial x^2}$, $D(A) = H_0^1(0,\pi) \cap H^2(0,\pi)$, then A generates a uniformly bounded strongly continuous cosine family $C(t)$ for $t \geq 0$.

We set $\ell_n = n^2\pi^2$ and $\xi_n(x) = \sqrt{\frac{2}{\pi}} sin(n\pi x)$, $n \in \mathbb{N}^+$. Obviously, $\{-\ell_n, \xi_n\}_{n=1}^{\infty}$ is the characteristic system of A, then $0 < \ell_1 \le \ell_2 \le \dots, \ell_n \to \infty$, as $n \to \infty$, and $\{\xi_n\}_{n=1}^{\infty}$ is a standard orthogonal basis in $\mathbb{X}$. Thus, the following properties are valid.

(i) If $u \in D(A)$, then
$$Au = -\sum_{n=1}^{\infty} \ell_n \langle u, \xi_n \rangle \xi_n, \quad n \in \mathbb{N}^+.$$

(ii) For every $u \in \mathbb{X}$, the strong continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ generated by A be defined as
$$C(t)u = \sum_{n=1}^{\infty} \cos(\sqrt{\ell_n} t) \langle u, \xi_n \rangle \xi_n, \quad n \in \mathbb{N}^+,$$
and the associated sine family $\{S(t) : t \in \mathbb{R}\}$ is given by
$$S(t)u = \sum_{n=1}^{\infty} \frac{sin(\sqrt{\ell_n} t)}{\sqrt{\ell_n}} \langle u, \xi_n \rangle \xi_n, \quad n \in \mathbb{N}^+.$$

Obviously, $\{C(t) : t \in \mathbb{R}\}$ and $\{S(t) : t \in \mathbb{R}\}$ are periodic functions, then for any $t \in \mathbb{R}$, we have $\|C(t)\| \le 1 = M_1$, $\|S(t)\| \le 1 = M_2$.

We define $u(t,x) = u(t)(x)$, $f(t, u_t(\cdot, x)) = F(t, u_t)(x)$, $g(t, u_t(\cdot, x)) = G(t, u_t)(x)$, $v(t,x) = v(t)(x)$ and $\varphi(t,x) = \varphi(t)(x)$.

If function $f(t,u)$ satisfies the following conditions:

(i) $f(t,\cdot) : C_0 \to \mathbb{X}$ is continuous for $t \in [0,1]$ and $f(\cdot, u) : J \to \mathbb{X}$ is measurable for each $u \in C_0$.

(ii) There exist a function $m_F \in L^2(J, \mathbb{R}^+)$ and $\frac{1}{3}$ such that
$$\|f(t,u)\| \le m_F(t) + \frac{1}{3}\|u\|_{C_0},$$
for all $u \in [0,\pi]$, a.e. $t \in [0,1]$.

(iii) There exists a constant $L_1 > 0$ such that for any bounded subset $D \subset C_0$ and a.e. $t \in [0,1]$ have
$$\alpha(f(t,D)) \le L_1 \sup_{\theta \in [-\tau,0]} \alpha(D(\theta)).$$

Based on (i)-(iii), it can be concluded that $f(t, u_t(\cdot, x))$ satisfies (H2). Similarly, we can verify that $g(t, u_t(\cdot, x))$ satisfies (H3).

In addition, we define the control operator $\mathbb{B} : L^2([0,1]; \mathbb{U}) \to L^2([0,1]; \mathbb{X})$ by $(\mathbb{B}v)(t) = \iota(v)(t)$ for $v \in L^2([0,1]; \mathbb{U})$. Then hypotheses (H1) and (H5) are met.

Accordingly, hypotheses (H1)–(H5) are all fulfilled. Hence, by Theorems 3.1, 3.2, and 4.1, we obtain:

(i) For control $u \in L^2([0,1]; \mathbb{U})$, the mild solution set of equation (5.1) is nonempty and compact.

(ii) For control $u \in L^2([0,1]; \mathbb{U})$, the mild solution set of equation (5.1) is an R_δ-set.

(iii) Since ι is a constant, then equation (5.1) is T-controllable on [0,1].

Acknowledgments. This work was supported by the Foundation for Innovative Fundamental Research Group Project of Gansu Province (Grant No. 25JRRA805). The authors are grateful to Professor Hongxia Fan for her guidance and revision of this article.

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