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NON-STEADY MAGNETO-HYDRODYNAMICS-HEAT SYSTEM WITH JOULE AND BUOYANCY EFFECTS UNDER MIXED BOUNDARY CONDITIONS

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Abstract. This article concerns the non-steady magneto-hydro-dynamics (MHD) heat system with Joule and buoyancy effects under mixed boundary conditions. The boundary conditions for fluids can include the stick, total pressure, vorticity, total stress conditions, and for a magnetic field with non-homogeneous mixed boundary conditions. The boundary condition for temperature may include Dirichlet, Neumann and Robin conditions. We prove that if the parameter for buoyancy effect is small in accordance with the data of problem, then there exists a solution with "defect measure" for temperature.

1. Introduction

This article concerns the non-steady magneto-hydro-dynamic (MHD) heat system with Joule and buoyancy effects

$$
\begin{gathered}
\frac{\partial v}{\partial t}-2\nabla\cdot\big(\eta(\theta)\mathcal{E}(v)\big)+(v\cdot\nabla)v+\frac{1}{\rho}\nabla P-\frac{\mu}{\rho}\operatorname{curl}H\times H=\mathbf{f}_0-\alpha_0(\theta-\bar\theta_0)\mathbf{g},\\
\operatorname{div}v=0,\\
\frac{\partial H}{\partial t}+\frac{1}{\mu}\operatorname{curl}\Big(\frac{1}{\sigma(\theta)}\operatorname{curl}H\Big)+\operatorname{curl}(H\times v)=0,\\
\operatorname{div}H=0,\\
\frac{\partial \theta}{\partial t}-\nabla\cdot\Big(\frac{k(\theta)}{\rho C_h}\nabla\theta\Big)+v\cdot\nabla\theta-\frac{1}{\rho C_h\sigma(\theta)}|\operatorname{curl}H|^2-\frac{\eta(\theta)}{\rho C_h}|\mathcal{E}(v)|^2=g_0,\\
v(x,0)=v_0,\quad H(x,0)=H_0,\quad \theta(x,0)=\theta_0,
\end{gathered}\tag{1.1}
$$

under mixed boundary conditions. Here v,P,H,θ are, respectively, velocity, pressure, magnetic field, and temperature. The strain tensor $\mathcal{E}(v)$ is the one with the components $\varepsilon_{ij}(v)=\frac12(\partial_{x_i}v_j+\partial_{x_j}v_i)$, $\mathbf{f}_0$ - body force, α_0 - parameter for buoyancy effect, $\bar\theta_0$ - constant, $\mathbf{g}$ - gravitational acceleration, g_0 - heat source. The density ρ, magnetic permeability μ and the specific heat capacity C_h of fluid are positive constants, and the kinematic viscosity $\eta(\theta)$, electrical conductivity $\sigma(\theta)$ and thermal conductivity $k(\theta)$ depend on the temperature. In what follows $\frac{P}{\rho}$ and $\frac{k(\theta)}{\rho C_h}$ are, respectively, denoted by p and $\kappa(\theta)$. For two matrices $A=\{a_{ij}\}$ and $B=\{b_{ij}\}$, $A:B=\sum_{ij}a_{ij}b_{ij}$, and $|A|^2=\sum_{ij}|a_{ij}|^2$. The terms $\frac{1}{\sigma(\theta)}|\operatorname{curl}H|^2$ and $\eta(\theta)|\mathcal{E}(v)|^2$ in the fifth equation mean the Joule heats, respectively, by the electric current and viscous friction in fluid. The boundary conditions for fluid may include stick on the wall, total pressure, vorticity, total stress conditions (Navier-slip with friction, the normal total stress, the total stress, outflow condition) together, and the boundary conditions for temperature may include Dirichlet, Neumann

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and Robin conditions together. For the magnetic field the non-homogeneous mixed boundary conditions are given.

In [27] (2011) the following density dependent MHD-heat system with Joule effects on a bounded domain Ω of $\mathbb{R}^3$ was studied,

$$\begin{gathered}
\partial_t\rho + \operatorname{div}(\rho v) = 0,\\
\partial_t H + \operatorname{curl}(H \times v) + \operatorname{curl}(\nu \operatorname{curl} H) = 0,\\
\operatorname{div} H = 0,\\
\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) - \operatorname{div}(2\eta\mathcal{E}(v)) + \nabla P = \operatorname{curl} H \times H,\\
\operatorname{div} v = 0,\\
C_v(\partial_t(\rho\theta) + div(\rho v\theta)) - \operatorname{div}(\kappa\nabla\theta) = 2\eta|\mathcal{E}(v)|^2 + \nu|\operatorname{curl} H|^2,
\end{gathered} \tag{1.2}$$

where $\nu = (\mu\sigma)^{-1}$ (magnetic diffusivity), η - dynamic viscosity, C_v - specific heat at constant volume and κ - heat conductivity and these depend on ρ and θ, was studied under the boundary conditions and the initial conditions

$$\begin{gathered}
(H\cdot n, \operatorname{curl} H \times n, v, \theta) = (0,0,0,0) \quad \text{on } \partial\Omega \times [0,T],\\
(\rho, H, \rho v, \rho\theta) = (\rho_0, H_0, \rho_0 v_0, \rho_0\theta_0) \quad \text{in } \Omega.
\end{gathered}$$

Under the regular conditions and the compatibility conditions at the initial instance for the initial data, the existence of a local-in-time unique strong solution to (1.2) was proved. For the other papers dealt with (1.2), we refer to Introduction of [27] and references therein.

In [28](2015) a density dependent non-Newtonian MHD-heat system with Joule effects on a bounded domain Ω of $\mathbb{R}^3$ was studied,

$$\begin{gathered}
\partial_t\rho + \operatorname{div}(\rho v) = 0,\\
\operatorname{div} v = 0,\\
\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla P = \operatorname{curl} H \times H + \operatorname{div} S(\rho, \eta, \mathbf{D}(v)),\\
\partial_t H + \operatorname{curl}(H \times v) + \operatorname{curl}(\nu \operatorname{curl} H) = 0,\\
\operatorname{div} H = 0,\\
\partial_t(\rho Q(\theta)) - \operatorname{div}(\mathbf{q}(\rho,\theta,\nabla\theta)) + \operatorname{div}(\rho Q(\theta)v) - S(\rho,\theta,\mathbf{D}(v)) : \nabla v - \nu|\operatorname{curl} H|^2 = 0,
\end{gathered} \tag{1.3}$$

where Q is a function of θ, $\mathbf{q}$ - thermal flux vector and $\mathbf{D}(v) = \nabla v + \nabla v^T$, under the boundary conditions

$$v = 0, \quad H = 0, \quad \mathbf{q}\cdot n = 0 \quad \text{on } \partial\Omega \times [0,T].$$

However, the assumptions

$$\begin{gathered}
S(\rho,\eta,\mathbf{D}(v)) \sim \mu(\rho,\theta)(\epsilon + |\mathbf{D}|^2)^{\frac{r-2}{2}}\mathbf{D},\\
S(\rho,\eta,\mathbf{D}(v))\cdot\mathbf{D} \ge \underline{\mu}(\epsilon + |\mathbf{D}|^2)^{\frac{r-2}{2}}|\mathbf{D}|^2,
\end{gathered}$$

where $0 < \underline{\mu} \le \mu(\rho,\theta)$, ((6), (7) of [28]) and the condition

$$r \ge \frac{11}{5}$$

(the condition in [28, Theorem 2.2]) problem (1.3) excludes problem (1.1) in the case that $\rho = const$. For similar results for (1.3), we refer the reader to Introduction of [28]. To the best of our knowledge, for the density dependent fluid systems on the bounded domains always the boundary condition for fluid $v|_{\partial\Omega} = 0$ or $v\cdot n|_{\partial\Omega} = 0$ was used, and so are both [27] and [28].

On the other hand, there are several papers concerned with the Navier-Stokes-heat system with Joule heating by the viscose friction (the case that $H \equiv 0$ in (1.1)) ([9]-[19]).

In [24] when $\alpha_0 = 0$ (i.e. without buoyancy effect), the existence of solution to the problem was studied under homogeneous Dirichlet boundary conditions for velocity and temperature. Also, in [25] when $\alpha_0 = 0$, the existence of a solution was studied under homogeneous Dirichlet boundary conditions for velocity and homogeneous Neumann condition for temperature. In [26] the existence of a solution was studied when $|\alpha_0|$ is small enough in accordance with data of the problem, under homogeneous Dirichlet boundary conditions for velocity and mixture of homogeneous Dirichlet condition and homogeneous Neumann condition for temperature. All papers above concerned the homogeneous Dirichlet boundary condition for velocity. In [19] the existence of a solution was studied when $|\alpha_0|$ is small enough in accordance with data of the problem, under mixed boundary conditions for fluid and mixture of homogeneous Dirichlet condition and non-homogeneous Neumann and Robin conditions for temperature. The definitions of solutions in [19, 24, 26] include the "defect measures" in the equation for temperature θ due to bad smoothness of the term for dissipation of energy. In [10] for Newtonian fluid with $\alpha_0 = 0$ the existence of solution to the problem was studied under Navier-slip boundary conditions for velocity and homogeneous Dirichlet condition for temperature. But, the definition of solution for temperature is inequality instead equality (cf. [10, (11)]), and so in essence the solution holds an equality with a "defect measure" as [19], [24]-[26].

Dealing with Newtonian fluid with $\alpha_0 = 0$ and $g = 0$, in [9, 11] instead of the equation for temperature of (1.1) the equation of total energy was used. Under Navier-slip boundary conditions for velocity and homogeneous Neumann condition for temperature in [9] and under periodic boundary conditions for velocity and temperature in [11] the existence of solutions to the problems is proved, but unlike [19], [24]-[26] the equation (1.1) is satisfied without the defect measures. For the reason of success in [9], we refer to Introduction of [19].

In this paper we study the existence of a solution to (1.1) under mixed boundary conditions for fluid, magnetic field and temperature. In our problem the mixed boundary conditions for the magnetic field, which may include the mixture of the normal components of magnetic field on a portions of boundary and electric current on the other portions of boundary (see Theorem 1 of [16]), is somewhat more general than previous ones except [16], [17] (see Remark 3.2). Under the such non-homogeneous boundary conditions for magnetic field, in the estimation of approximate solutions of problem, the terms according to Lorentz force in the momentum equation and the nonlinear term in the magnetic equation are not canceled each other, and so the case of non-homogeneous magnetic boundary conditions is much more difficult than others (see Introduction of [16]). In the MHD-Boussinesq system the temperature is estimated by the data of problem and so the buoyancy term is estimated (see [17]), but in (1.1) we can not do so as. Also the heat source in the fifth equation of (1.1) includes the terms quadratic with respect to gradients of fluid velocity and electric current. Thus, dealing (1.1) is difficult than MHD-Boussinesq system and the Navier-Stoks-heat system with friction Joule effect. Also, as already mentioned, in our problem the viscosity, magnetic permeability and electrical conductivity of the fluid depend on the temperature.

This paper consists of 4 sections. In Section 2, the boundary conditions and assumptions are stated. The boundary conditions for fluid may include the stick, the total pressure, vorticity, total stress conditions together. We exclude the friction boundary conditions, for the reason of which we refer to Remark 5.1 of [17].

In Section 3, we obtain the variational formulation for the problem. To this end, first assuming the existence of a smooth function satisfying non-homogeneous mixed boundary conditions for magnetic fields, we deform the given problem to problem with the homogeneous mixed boundary conditions for magnetic fields. Then, for the problem we obtain a variational formulation which consists of three variational equations (see Problem VE). In the end of Section 3, the main results of this paper are stated (Theorem 3.4). Theorem 3.4 asserts that if the parameter for buoyancy effect is small enough in accordance with the data of problem, then there exists a solution with "defect measure" for temperature.

Following the idea in [19] and correcting some mistakes therein, in Section 4 we prove Theorem 3.4. First in Subsection 4.1 we prove the existence of a solution to an auxiliary problem with the parameter ε for approximation of the viscosity term and buoyancy term in the momentum equation and Joule effects in the heat equation (Theorem 4.2). Next, in Subsection 4.2 under condition $|\alpha_0|$ is small in accordance with the data of the problem we obtain estimation of the solutions independent from ε. Therefore, in Subsection 4.3 passing to the limits as ε goes to zero, we obtain the existence and estimates of a solution to the problem.

Throughout this paper we will use the following notation. Let Ω be a connected bounded open subset of $\mathbb{R}^3$ and

$$\partial\Omega = \cup_{i=1}^{7}\overline{\Gamma}_i = \overline{\Sigma}_\nu \cup \overline{\Sigma}_\tau = \overline{\Gamma}_D \cup \overline{\Gamma}_R,$$

$\Sigma_\tau, \Sigma_\nu, \Gamma_D, \Gamma_R$ be open subsets of $\partial\Omega$ such that $\Sigma_\tau \cap \Sigma_\nu = \emptyset$, $\Gamma_D \cap \Gamma_R = \emptyset$. And $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$, $\Gamma_i = \bigcup_j \Gamma_{ij}$, where Γ_{ij} are connected open subsets of $\partial\Omega$ and $\Gamma_{ij} \in C^{2,1}$ for $i = 2, 3, 7$ and $\Gamma_{ij} \in C^2$ for others. When X is a Banach space, $\mathbf{X} = X^3$. Let $W^{k,p}(\Omega)$ be Sobolev spaces, $H^1(\Omega) = W^{1,2}(\Omega)$, and so $\mathbf{H}^1(\Omega) = \{H^1(\Omega)\}^3$.

An inner product and norm in the space $\mathbf{L}^2(\Omega)$ or $L^2(\Omega)$ are, respectively, denoted by $(\cdot\,,\cdot)$ and $\|\cdot\|$; and $\langle\cdot\,,\cdot\rangle$ means the duality pairing between a Banach space X and its dual one. Especially, $(\cdot,\cdot)_{\Gamma_i}$ is an inner product in $\mathbf{L}^2(\Gamma_i)$ or $L^2(\Gamma_i)$; and $\langle\cdot,\cdot\rangle_{\Gamma_i}$ means the duality pairing between $\mathbf{H}^{\frac12}(\Gamma_i)$ and $\mathbf{H}^{-\frac12}(\Gamma_i)$ or between $H^{\frac12}(\Gamma_i)$ and $H^{-\frac12}(\Gamma_i)$. Sometimes $\langle\cdot\,,\cdot\rangle_{\Gamma_i}$ means the duality pairing between $L^2(0,T;H^{-\frac12}(\Gamma_i))$ and $L^2(0,T;H^{\frac12}(\Gamma_i))$. The inner product and norms in $\mathbb{R}^3$ are, respectively, denoted by $(\cdot\,,\cdot)_{\mathbb{R}^3}$ and $|\cdot|$. Let $n(x)$ and $\tau(x)$ be, respectively, outward normal and tangent unit vectors at x in $\partial\Omega$. For convergence in spaces, $\to$ and $\rightharpoonup$ mean, respectively, strong and weak convergence. $Q = \Omega \times (0,T)$, where $0 < T < \infty$. For the Banach space X its dual space is denoted by X^*.

2. Boundary conditions and assumptions

The total stress tensors $S^t(\theta, v, p)$ is the ones with components $s^t_{ij} = -(p + 1/2|v|^2)\delta_{ij} + 2\eta(\theta)\varepsilon_{ij}(v)$. The total stress vectors on the boundary surface is $\sigma^t(\theta, v, p) = S^t \cdot n$ and the values of normal total stress vectors on the boundary surface is $\sigma^t_n(\theta, v, p) = \sigma^t \cdot n$.

The boundary conditions for fluid are as follows:

$$\begin{gathered}
v|_{\Gamma_1} = 0,\\
v_\tau|_{\Gamma_2} = 0,\ -\big(p + 1/2|v|^2\big)|_{\Gamma_2} = \phi_2,\\
v_n|_{\Gamma_3} = 0,\ \operatorname{curl} v \times n|_{\Gamma_3} = \vec{\phi}_3/\eta(\theta),\\
v_\tau|_{\Gamma_4} = 0,\ \big(-p - 1/2|v|^2 + 2\eta(\theta)\varepsilon_{nn}(v)\big)|_{\Gamma_4} = \phi_4,\\
v_n|_{\Gamma_5} = 0,\ 2(\eta(\theta)\varepsilon_{n\tau}(v) + \alpha v_\tau)|_{\Gamma_5} = \vec{\phi}_5,\ \ \alpha: \text{a matrix},\\
\big(-pn - 1/2|v|^2 n + 2\eta(\theta)\varepsilon_n(v)\big)|_{\Gamma_6} = \vec{\phi}_6,\\
v_\tau|_{\Gamma_7} = 0,\ \big(-p - 1/2|v|^2 + \eta(\theta)\partial v/\partial n \cdot n\big)|_{\Gamma_7} = \phi_7,
\end{gathered}\tag{2.1}$$

where $\varepsilon_n(v) = \mathcal{E}(v)n$, $\varepsilon_{nn}(v) = (\mathcal{E}(v)n, n)_{\mathbb{R}^3}$, $\varepsilon_{n\tau}(v) = \mathcal{E}(v)n - \varepsilon_{nn}(v)n$, $v_n = v \cdot n$, $v_\tau = v - v_n n$, and $\phi_i, \vec{\phi}_i, \alpha_{ij}$ (components of matrix α) are given functions or vectors of functions.

For the magnetic field, we consider the mixed boundary conditions

$$H \cdot n|_{\Sigma_\tau} = q, \quad \Big(\frac{1}{\mu\sigma(\theta)}\operatorname{curl} H + H \times v\Big) \times n\Big|_{\Sigma_\tau} = \vec{\varphi}_\tau, \quad H \times n|_{\Sigma_\nu} = \vec{\phi}_\nu, \tag{2.2}$$

where the compatibility condition

$$\vec{\varphi}_\tau \cdot n = 0, \quad \vec{\phi}_\nu \cdot n = 0 \tag{2.3}$$

must be satisfied since n is normal on the boundary.

Remark 2.1. (1) For the physical background of the boundary conditions in (2.1), we refer to [20, Section 2.2], the Introduction in [21], and references therein.

(2) The boundary condition $\big(\frac{1}{\mu\sigma}\operatorname{curl} H + H \times v\big) \times n\big|_{\Sigma_\tau} = \varphi_\tau$ of (2.2) corresponds to the boundary condition $\frac{1}{\mu}E \times n|_{\Sigma_\tau} = \varphi_\tau$ in [16, (2.3)] for the steady problem since $\operatorname{curl} H = \sigma(E - \mu H \times v)$. The boundary condition

$$H \cdot n|_{\Sigma_\tau} = q, \quad H \times n|_{\Sigma_\nu} = \vec{\phi}_\nu$$

implies

$$H \cdot n|_{\Sigma_\tau} = q, \quad J \cdot n|_{\Sigma_\nu} = a(x),$$

where $J = \operatorname{curl} H$ is the electric current density and $a(x)$ is the surface divergence on Σ_ν of $\vec{\phi}_\nu$, that is, $\operatorname{div}_\tau \vec{\phi}_\nu$ (see [16, Theorem 1]).

(3) In (2.3) the boundary condition $H\cdot n|_{\Sigma_\tau} = 0$, $E\times n|_{\Sigma_\tau} = 0$ describes the boundary conditions on perfectly conducting boundary Σ_τ, and the boundary condition $H \times n|_{\Sigma_\nu} = 0$ corresponds to a perfectly insulating boundary (cf. Introduction of [3] and references therein). Let Ω is a domain in $\mathbb{R}^3$ and $\Omega' = \mathbb{R}^3\backslash\Omega$. The solution of the Maxwell equations over the hole space satisfies the following transmission relations across surface $\partial\Omega$,

$$
\begin{gathered}
(H_0 - H) \times n = -J_{\partial\Omega},\\
(B_0 - B) \cdot n = 0,\\
(E_0 - E) \times n = 0,
\end{gathered}
$$

where $H = \frac{B}{\mu}$, $J_{\partial\Omega}$ the surface density of current and (B, E, H) and (B_0, E_0, H_0) denote the solutions on Ω and Ω', respectively (see [12, (4.25), Ch. 1]). Knowing solutions on Ω', from the above we obtain non-homogeneous boundary conditions (2.3) (See [14, Section 3.8]).

For temperature we consider the boundary conditions

$$
\begin{gathered}
\theta|_{\Gamma_D} = 0,\\
\big(\kappa(\theta)\frac{\partial\theta}{\partial n} + \beta(x)\theta\big)\big|_{\Gamma_R} = g_R(x), \quad \beta(x),\, g_R(x)\text{- given functions on } \Gamma_R.
\end{gathered}
\tag{2.4}
$$

Let

$$
\begin{gathered}
\mathbf{V} = \{u \in \mathbf{H}^1(\Omega) : \operatorname{div} u = 0,\, u|_{\Gamma_1} = 0,\, u_\tau|_{(\Gamma_2\cup\Gamma_4\cup\Gamma_7)} = 0,\, u_n|_{(\Gamma_3\cup\Gamma_5)} = 0\},\\
H_{\mathbf{V}} :\ \text{the closure of } \mathbf{V} \text{ in } \mathbf{L}^2(\Omega),\\
W^{1,r}_{\Gamma_D} = \{y \in W^{1,r}(\Omega) : y|_{\Gamma_D} = 0\}, \quad 1 < r \le \infty.
\end{gathered}
$$

Since $\Gamma_1 \neq \emptyset$ and $\Gamma_D \neq \emptyset$, by Korn's inequality and Friedrichs' inequality we use

$$
(v, u)_{\mathbf{V}} = (\mathcal{E}(v), \mathcal{E}(u)), \quad (y, z)_{W^{1,2}_{\Gamma_D}(\Omega)} = (\nabla y, \nabla z). \tag{2.5}
$$

Assumption 2.2. (1) $\Gamma_1 \neq \emptyset$, $\Gamma_D \neq \emptyset$ and and

$$
\Gamma_R \subset (\cup_{i=1,3,5}\Gamma_i)\,. \tag{2.6}
$$

(2) The boundary $\partial\Omega$ is the union of a finite number of disjoint closed C^2 surfaces; each surface having finite surface area. $\Sigma_\tau \neq \emptyset$, $\Sigma_\nu \neq \emptyset$ and each has a finite number of disjoint nonempty open components. The Euclidean distance between disjoint open components of Σ_τ, and of Σ_ν, is bounded below by a positive number. The boundary of each open component is either empty or a $C^{1,1}$ curve.

(3) For the functions of (1.1) assume that

$$
\begin{gathered}
\mathbf{f}_0 \in L^2(0, T; \mathbf{V}^*);\\
g_0 \in L^2\big(0, T; W^{1,2}_{\Gamma_D}(\Omega)^*\big);\\
\eta \in C(\mathbb{R}), \quad 0 < \eta_0 \le \eta(\xi) \le \eta_1 < \infty \quad \forall \xi \in \mathbb{R};\\
\sigma \in C(\mathbb{R}), \quad 0 < \sigma_0 \le \sigma(\xi) \le \sigma_1 < \infty \quad \forall \xi \in \mathbb{R};\\
\kappa \in C(\mathbb{R}), \quad 0 < \kappa_0 \le \kappa(\xi) \le \kappa_1 < \infty \quad \forall \xi \in \mathbb{R}.
\end{gathered}
\tag{2.7}
$$

(4) For the functions of (2.1), (2.2) and (2.4) assume that

$$\begin{gathered}
\phi_i \in L^2(0,T;H^{-\frac{1}{2}}(\Gamma_i)), i=2,4,7;\\
\vec{\phi}_i \in L^2(0,T;\mathbf{H}^{-\frac{1}{2}}(\Gamma_i)), i=3,5,6;\\
\alpha_{ij} \in L^\infty(\Gamma_5);\\
\vec{\varphi}_\tau \in L^2\big(0,T;\mathbf{H}^{-1/2}(\Sigma_\tau)\big),\ (\vec{\varphi}_\tau, n)_{\Sigma_\tau}=0;\\
\vec{\phi}_\nu \in L^\infty\big(0,T;\mathbf{H}^{1/2}(\Sigma_\nu)\big),\ (\vec{\phi}_\nu, n)_{\Sigma_\nu}=0;\\
q \in L^\infty(0,T;H^{1/2}(\Sigma_\tau);\\
g_R \in L^2(0,T;L^{4/3}(\Gamma_R));\\
\beta_1 \geq \beta(x) \geq 0, \quad \beta_1 \text{ - a constant}, \quad \beta(x) - textmeasurable.
\end{gathered} \tag{2.8}$$

Remark 2.3. For (2) in Assumption 2.2, which is necessary for (3.2), we refer to [7, Conditions B_1, B_2, B_3], or [6, B_1, B_2].

3. Variational formulation and main result

3.1. **Analysis of the boundary conditions for the magnetic fields.** For magnetic fields we use the following spaces.

$$\begin{gathered}
\mathbf{C}_{\Sigma 0}(\bar\Omega)=\{v\in \mathbf{C}(\bar\Omega); v\times n|_{\Sigma_\nu}=0,\ v\cdot n|_{\Sigma_\tau}=0\},\\
\mathcal{H}_\Sigma(\Omega)=\{v\in \mathbf{L}^2(\Omega); \operatorname{curl} v=0,\ \operatorname{div} v=0,\ v\times n|_{\Sigma_\nu}=0,\ v\cdot n|_{\Sigma_\tau}=0\},\\
H_{DC}(\Omega)=\big\{v\in \mathbf{L}^2(\Omega);\ \operatorname{curl} v\in \mathbf{L}^2(\Omega),\ \operatorname{div} v\in L^2(\Omega)\big\},\\
(v,u)_{H_{DC}(\Omega)}=(v,u)_{\mathbf{L}^2(\Omega)}+(\operatorname{div} v,\operatorname{div} u)_{L^2(\Omega)}+(\operatorname{curl} v,\operatorname{curl} u)_{\mathbf{L}^2(\Omega)},\\
H_{DC\Sigma}(\Omega): \text{the clousue of } \mathbf{C}_{\Sigma 0}(\bar\Omega)\cap \mathbf{H}^1(\Omega) \text{ in } H_{DC}(\Omega),\\
\mathbf{V}_\Sigma=\{v\in H_{DC\Sigma}(\Omega): \operatorname{div} v=0\}\cap \mathcal{H}_\Sigma(\Omega)^\perp,\\
H_\Sigma: \text{the closue of } \mathbf{V}_\Sigma \text{ in } \mathbf{L}^2(\Omega),
\end{gathered}$$

here and in what follows $\mathcal{H}_\Sigma(\Omega)^\perp$ is the orthogonal complement of $\mathcal{H}_\Sigma(\Omega)$ in $\mathbf{L}^2(\Omega)$. A field v such that $\operatorname{curl} v=0$, $\operatorname{div} v=0$ is called harmonic, and elements of $\mathcal{H}_\Sigma(\Omega)$ are harmonic.

Note that if $v\in H_{DC\Sigma}(\Omega)$, then $v\times n|_{\Sigma_\nu}=0$ in $\mathbf{H}^{-1/2}(\Sigma_\nu)$ and $v\cdot n|_{\Sigma_\tau}=0$ in $H^{-1/2}(\Sigma_\nu)$ since $v\times n|_{\partial\Omega}\in \mathbf{H}^{-1/2}(\partial\Omega)$ and $v\cdot n|_{\partial\Omega}\in H^{-1/2}(\partial\Omega)$ for $v\in H_{DC}(\Omega)$.

By [6, Theorems 10.2] we have

$$\|\operatorname{curl} v\|^2+\|\operatorname{div} v\|^2\geq c\|v\|^2 \quad \exists c>0,\ \forall v\in H_{DC\Sigma}(\Omega)\cap \mathcal{H}_\Sigma(\Omega)^\perp,$$

which implies

$$\|v\|^2_{H_{DC\Sigma}(\Omega)}\geq \|\operatorname{curl} v\|^2+\|\operatorname{div} v\|^2\geq \tilde c\|v\|^2_{H_{DC\Sigma}(\Omega)} \quad \exists \tilde c>0,\ \forall v\in H_{DC\Sigma}(\Omega)\cap \mathcal{H}_\Sigma(\Omega)^\perp. \tag{3.1}$$

On the other hand, by [6, Theorem 3.3], the norm in $H_{DC\Sigma}(\Omega)$ is equivalent to the one in $\mathbf{H}^1(\Omega)$, and for convenience we denote the $H_{DC\Sigma}(\Omega)$-norm of $\mathbf{V}_\Sigma$ by $\|\cdot\|_{\mathbf{V}_\Sigma}$. Therefore, by (3.1), the norm of $\mathbf{V}_\Sigma$ is equivalent to $\|\operatorname{curl} v\|$, and in what follows we use

$$\|\operatorname{curl} v\|^2=\|v\|^2_{\mathbf{V}_\Sigma(\Omega)} \quad \forall v\in \mathbf{V}_\Sigma. \tag{3.2}$$

Assumption 3.1. (1) For any $t\in[0,T]$ there exists a function $\mathring{H}(t)\in \mathbf{L}^3(\Omega)\cap \mathcal{H}_\Sigma(\Omega)^\perp$ such that

$$\operatorname{curl}\mathring{H}(t)\in \mathbf{L}^2(\Omega), \quad \operatorname{div}\mathring{H}(t)=0, \quad \mathring{H}(t)\cdot n|_{\Sigma_\tau}=q(t), \quad \mathring{H}(t)\times n|_{\Sigma_\nu}=\vec{\phi}_\nu(t).$$

(2) $\mathring{H}\in L^\infty(0,T;\mathbf{H}^1(\Omega))$, $\mathring{H}\in W^{1,2}(0,T;\mathbf{L}^3(\Omega))$.

Remark 3.2. For one case satisfying (1) of Assumption 3.1, we refer to [16, Remark 2].

If we assume the existence of a function $\mathring{H}\in \mathbf{L}^3(\Omega))\cap \mathcal{H}_\Sigma(\Omega)^\perp$ such that

$$\operatorname{curl}\mathring{H}=0, \quad \operatorname{div}\mathring{H}=0, \quad \mathring{H}\cdot n|_{\Sigma_\tau}=q, \quad \mathring{H}\times n|_{\Sigma_\nu}=\vec{\phi}_\nu$$

as in [1, 2, 4] and use (3.3) below, then in the estimation of approximate solutions the terms corresponding to Lorentz force in the momentum equation and the nonlinear term in the magnetic equation are canceled each other and the problem is simplified. However, the assumption $\operatorname{curl} \mathring{H} = 0$ is not usual for the magnetic field and excludes the boundary conditions such that$H \cdot n|_{\Sigma_\tau} = q$ $J \cdot n|_{\Sigma_\nu} \neq 0$ (where J is electric current density) very important in practice (see 1) of Remark 4 of [16]). Thus, as in [16, 17] we assumed in this paper $\operatorname{curl} \mathring{H} \in \mathbf{L}^2(\Omega)$, and so the following calculation is complicate.

We put

$$H(t) = \mathring{H}(t) + \bar{H}(t), \tag{3.3}$$

where $\mathring{H}(t)$ is the function in Assumption 3.1, and $\bar{H}(t) \in \mathbf{V}_\Sigma$. Then, for H of (3.3), $u, v \in \mathbf{H}^1(\Omega)$ and $C \in \mathbf{V}_\Sigma$ we have

$$\begin{aligned} &- \langle \mu \operatorname{curl} H \times H, u \rangle \\ &= -\langle \mu \operatorname{curl} \bar{H} \times \bar{H}, u \rangle - \langle \mu \operatorname{curl} \bar{H} \times \mathring{H}, u \rangle - \langle \mu \operatorname{curl} \mathring{H} \times \bar{H}, u \rangle - \langle \mu \operatorname{curl} \mathring{H} \times \mathring{H}, u \rangle, \\ &\qquad \langle \mu H \times v, \operatorname{curl} C \rangle = \langle \mu \operatorname{curl} C \times \bar{H}, v \rangle + \langle \mu \operatorname{curl} C \times \mathring{H}, v \rangle. \end{aligned} \tag{3.4}$$

Also taking into account (2.2), (3.3) and (3.4), for $H \in \mathbf{H}^2(\Omega), v \in H^1(\Omega)$ and $C \in \mathbf{V}_\Sigma$ we have

$$\begin{aligned} &\Big(\frac{1}{\mu} \operatorname{curl} \Big(\frac{1}{\sigma(\theta)} \operatorname{curl} H\Big) + \operatorname{curl}(H \times v), C\Big) \\ &= \Big(\frac{1}{\mu\sigma(\theta)} \operatorname{curl} H + (H \times v), \operatorname{curl} C\Big) - \Big(\Big(\frac{1}{\mu\sigma(\theta)} \operatorname{curl} H + H \times v\Big) \times n, C\Big)_{\Sigma_\tau} \\ &= \Big(\frac{1}{\mu\sigma(\theta)} \operatorname{curl} \bar{H}, \operatorname{curl} C\Big) + \Big(\frac{1}{\mu\sigma(\theta)} \operatorname{curl} \mathring{H}, \operatorname{curl} C\Big) \\ &\quad + \big(\operatorname{curl} C \times \bar{H}, v\big) + \big(\operatorname{curl} C \times \mathring{H}, v\big) - (\vec{\varphi}_\tau, C)_{\Sigma_\tau}. \end{aligned} \tag{3.5}$$

3.2. **Variational formulation.** For $v \in \mathbf{H}^2(\Omega) \cap \mathbf{V}$, $\theta \in W^{1,2}(\Omega)$ and $u \in \mathbf{V}$, we have

$$\begin{aligned} -2\langle \nabla \cdot (\eta(\theta)\mathcal{E}(v)), u\rangle &= 2(\eta(\theta)\mathcal{E}(v), \mathcal{E}(u)) - 2(\eta(\theta)\mathcal{E}(v)n, u)_{\cup_{i=2}^7 \Gamma_i} \\ &= 2(\eta(\theta)\mathcal{E}(v), \mathcal{E}(u)) + 2(\eta(\theta)k(x)v, u)_{\Gamma_2} - (\eta(\theta) \operatorname{curl} v \times n, u)_{\Gamma_3} \\ &\quad + 2(\eta(\theta)S\tilde{v}, \tilde{u})_{\Gamma_3} - 2(\eta(\theta)\varepsilon_{nn}(v), u_n)_{\Gamma_4} - 2(\eta(\theta)\varepsilon_{n\tau}(v), u)_{\Gamma_5} \\ &\quad - 2(\eta(\theta)\varepsilon_n(v), u)_{\Gamma_6} - \Big(\eta(\theta)\frac{\partial v}{\partial n}, u\Big)_{\Gamma_7} + (\eta(\theta)k(x)v, u)_{\Gamma_7}, \end{aligned} \tag{3.6}$$

where S is the shape operator of boundary surface, $\tilde{v}$ and $\tilde{u}$ are expressions of the vectors v and u in a local orthogonal curvilinear coordinates on Γ_3 and and $k(x) = \operatorname{div} n(x)$ (see [18, Theorems 2.1 and 2.2]).

For $p \in H^1(\Omega)$ and $u \in \mathbf{V}$ we have

$$(\nabla p, u) = (p, u_n)_{\cup_{i=2}^7 \Gamma_i} = (p, u_n)_{\Gamma_2} + (p, u_n)_{\Gamma_4 \cup \Gamma_7} + (pn, u)_{\Gamma_6}, \tag{3.7}$$

where $u_n \,|_{\Gamma_1 \cup \Gamma_3 \cup \Gamma_5} = 0$ (see (2.1)) was used.

For $\theta \in W^{1,2}(\Omega)$ and $\varphi \in W^{1,2}_{\Gamma_D}$, by (2.4) we have

$$\langle -\nabla \cdot (\kappa(\theta)\nabla\theta), \varphi\rangle = \big(\kappa(\theta)\nabla\theta, \nabla\varphi\big) - \Big(\kappa(\theta)\frac{\partial\theta}{\partial n}, \varphi\Big)_{\Gamma_R} = (\kappa(\theta)\nabla\theta, \nabla\varphi) + (\beta\theta - g_R, \varphi)_{\Gamma_R}. \tag{3.8}$$

By (2.6) we see that $v_n = 0$ on Γ_R, and so for $v \in \mathbf{V}$, $\theta \in W^{1,2}(\Omega)$ and $\varphi \in W^{1,2}_{\Gamma_D}$ we have

$$\langle v \cdot \nabla\theta, \varphi\rangle = (v_n\theta, \varphi)_{\Gamma_R} - (\theta v, \nabla\varphi) = -(\theta v, \nabla\varphi). \tag{3.9}$$

Taking $(v\cdot\nabla)v = \operatorname{curl} v\times v + \frac{1}{2}\operatorname{grad}|v|^2$ into account, by (3.3)-(3.9) we can see that smooth solutions (v,p,H,θ) of problem (1.1), (2.1), (2.2), (2.4) satisfy the following:

$$
\begin{aligned}
&(v'(t),u) + 2(\eta(\theta)\mathcal{E}(v),\mathcal{E}(u)) + \langle \operatorname{curl} v\times v, u\rangle + 2(\eta(\theta)k(x)v,u)_{\Gamma_2}\\
&+ 2(\eta(\theta)S\tilde v,\tilde u)_{\Gamma_3} + 2(\alpha(x)v,u)_{\Gamma_5} + (\eta(\theta)k(x)v,u)_{\Gamma_7} - \frac{1}{\rho}(\mu\operatorname{curl}\bar H\times\bar H,u)\\
&- \frac{1}{\rho}(\mu\operatorname{curl}\bar H\times \mathring{H},u) - \frac{1}{\rho}(\mu\operatorname{curl}\mathring{H}\times\bar H,u)\\
&= \langle(\mathbf{f}_0+\alpha_0\bar\theta_0\mathbf{g}) - \alpha_0\theta\mathbf{g},u\rangle + \frac{1}{\rho}(\mu\operatorname{curl}\mathring{H}\times\mathring{H},u)\\
&\quad + \sum_{i=2,4,7}\langle\phi_i,u_n\rangle_{\Gamma_i} + \sum_{i=3,5,6}\langle\vec\phi_i,u\rangle_{\Gamma_i} \quad \forall u\in\mathbf{V},\\
&\quad (\bar H'(t),C) + \Big\langle\frac{1}{\mu\sigma(\theta)}\operatorname{curl}\bar H(t),\operatorname{curl} C\Big\rangle + \Big\langle\frac{1}{\mu\sigma(\theta)}\operatorname{curl}\mathring{H}(t),\operatorname{curl} C\Big\rangle\\
&\quad + \big\langle\operatorname{curl} C\times\bar H(t),v(t)\big\rangle + \big\langle\operatorname{curl} C\times\mathring{H}(t),v(t)\big\rangle + (\mathring{H}'(t),C)\\
&\quad = \langle\vec\varphi_\tau,C\rangle_{\Sigma_\tau} \quad \forall C\in\mathbf{V}_\Sigma,\\
&\qquad (\theta'(t),\varphi) + (\kappa(\theta)\nabla\theta(t),\nabla\varphi) - (\theta(t)v(t),\nabla\varphi) + (\beta\theta(t),\varphi)_{\Gamma_R}\\
&\qquad - \Big\langle\frac{1}{\rho C_h\sigma(\theta)}|\operatorname{curl} H|^2,\varphi\Big\rangle - \Big\langle\frac{\eta(\theta)}{\rho C_h}|\mathcal{E}(v)|^2,\varphi\Big\rangle\\
&\qquad = (g_R,\varphi)_{\Gamma_R} + \langle g_0(t),\varphi\rangle \quad \forall\varphi\in W^{1,\infty}_{\Gamma_D}(\Omega).
\end{aligned}
$$

We define $a_0\langle\tilde\theta;\cdot,\cdot\rangle, a_1(\cdot,\cdot,\cdot)$ and $\mathbf{f}_1(t)\in\mathbf{V}^*$ by

$$
\begin{gathered}
a_0\langle\tilde\theta;w,u\rangle = 2\langle\eta(\tilde\theta)\mathcal{E}(w),\mathcal{E}(u)\rangle + 2\langle\eta(\tilde\theta)k(x)w,u\rangle_{\Gamma_2} + 2\langle\eta(\tilde\theta)S\tilde w,\tilde u\rangle_{\Gamma_3}\\
+ 2\langle\alpha(x)w,u\rangle_{\Gamma_5} + \langle\eta(\tilde\theta)k(x)w,u\rangle_{\Gamma_7} \quad \forall w,u\in\mathbf{V},\ \tilde\theta\in W^{1,r}_{\Gamma_D},\\
\mathbf{f}(t) = \mathbf{f}_0(t) + \alpha_0\bar\theta_0\mathbf{g} + \frac{\mu}{\rho}\operatorname{curl}\mathring{H}(t)\times\mathring{H}(t),\\
\langle\mathbf{f}_1(t),u\rangle = \langle\mathbf{f}(t),u\rangle + \sum_{i=2,4,7}\langle\phi_i(t),u_n\rangle_{\Gamma_i} + \sum_{i=3,5,6}\langle\vec\phi_i(t),u\rangle_{\Gamma_i} \quad \forall u\in\mathbf{V}.
\end{gathered}\tag{3.10}
$$

We define $b_0(\tilde\theta;\cdot,\cdot)$ and $\mathbf{r}_1(t)\in\ \mathbf{V}^*_\Sigma$ by

$$
\begin{gathered}
b_0\langle\tilde\theta;\bar H,C\rangle = \Big\langle\frac{1}{\mu\sigma(\tilde\theta)}\operatorname{curl}\bar H,\operatorname{curl} C\Big\rangle \quad \forall\bar H,C\in\mathbf{V}_\Sigma,\tilde\theta\in W^{1,r}_{\Gamma_D},\\
\langle\mathbf{r}_1(t),C\rangle = \langle\vec\varphi_\tau,C\rangle_{\Sigma_\tau} - \langle\mathring{H}'(t),C\rangle \quad \forall C\in\mathbf{V}_\Sigma.
\end{gathered}\tag{3.11}
$$

We define $c_0\langle\tilde\theta;\cdot,\cdot\rangle$ and $g_1(t)\in(W^{1,2}_{\Gamma_D}(\Omega))^*$ by

$$
\begin{gathered}
c_0\langle\tilde\theta;\theta,\varphi\rangle = \langle\kappa(\tilde\theta)\nabla\theta,\nabla\varphi\rangle + \langle\beta(x)\theta,\varphi\rangle_{\Gamma_R} \quad \forall\tilde\theta,\theta\in W^{1,r}_{\Gamma_D}(\Omega),\varphi\in W^{1,r'}_{\Gamma_D}(\Omega),\\
\langle g_1(t),\varphi\rangle = \langle g_R,\varphi\rangle_{\Gamma_R} + \langle g_0(t),\varphi\rangle \quad \forall\varphi\in W^{1,2}_{\Gamma_D}(\Omega).
\end{gathered}\tag{3.12}
$$

By (2.7), (2.8) and (2) of Assumption 3.1,

$$
\mathbf{f}_1\in L^2(0,T;\mathbf{V}^*),\quad \mathbf{r}_1\in L^2(0,T;\mathbf{V}^*_\Sigma),\quad g_1\in L^2(0,T;(W^{1,2}_{\Gamma_D})^*).\tag{3.13}
$$

Then, we introduce the following variational formulation for problem (1.1), (2.1), (2.2), (2.4).

Problem VE. Find $(v,\bar H,\theta)\in\big(L^\infty(0,T;H_{\mathbf V})\cap L^2(0,T;\mathbf V)\big)\times\big(L^\infty(0,T;H_\Sigma)\cap L^2(0,T;\mathbf V_\Sigma)\big)\times\big(L^\infty(0,T;L^2(\Omega))\cap L^r(0,T;W^{1,r}_{\Gamma_D})\big)$, $1<r<\frac54$, such that

$$
\begin{aligned}
&\int_0^T\Big[\langle -v(t),u'(t)\rangle + a_0\langle\theta;v(t),u(t)\rangle+\langle\operatorname{curl}v(t)\times v(t),u(t)\rangle\\
&+\Big\langle -\frac{\mu}{\rho}\operatorname{curl}\bar H(t)\times\bar H(t)-\frac{\mu}{\rho}\operatorname{curl}\bar H(t)\times\mathring H(t)-\frac{\mu}{\rho}\operatorname{curl}\mathring H(t)\times\bar H(t)\\
&-\frac{\mu}{\rho}\operatorname{curl}\mathring H(t)\times\mathring H(t)+\alpha_0\theta(t)\mathbf g-\mathbf f_1(t),u(t)\Big\rangle\Big]\,dt\\
&=\langle v_0,u(x,0)\rangle\quad\forall u\in C^1([0,T];\mathbf V)\text{ with }u(\cdot,T)=0,\\
&\int_0^T\Big[\langle-\bar H(t),C'(t)\rangle+b_0\langle\theta(t);\bar H(t),C(t)\rangle+\Big\langle\frac{1}{\mu\sigma(\theta)}\operatorname{curl}\mathring H(t),\operatorname{curl}C\Big\rangle\\
&+\Big\langle\operatorname{curl}C(t)\times\bar H(t),v(t)\Big\rangle+\Big\langle\operatorname{curl}C(t)\times\mathring H(t),v(t)\Big\rangle-\langle\mathbf r_1(t),C(t)\rangle\Big]\,dt\\
&=\langle\bar H_0,C(x,0)\rangle\quad\forall C\in C^1([0,T];\mathbf V_\Sigma)\text{ with }C(\cdot,T)=0,\\
&\int_0^T\Big[\langle-\theta(t),\varphi'(t)\rangle+c_0\langle\theta(t);\theta(t),\varphi(t)\rangle-\langle\theta(t)v(t),\nabla\varphi(t)\rangle\\
&-\Big\langle\frac{1}{\rho C_h\sigma(\theta)}|\operatorname{curl}H|^2,\varphi\Big\rangle-\Big\langle\frac{\eta(\theta)}{\rho C_h}|\mathcal E(v)|^2,\varphi\Big\rangle-\langle g_1(t),\varphi(t)\rangle\Big]\,dt\\
&=\langle\theta_0(x),\varphi(x,0)\rangle\quad\forall\varphi\in C^1([0,T];W^{1,\infty}_{\Gamma_D}(\Omega))\text{with }\varphi(\cdot,T)=0,
\end{aligned}
\tag{3.14}
$$

where $\bar H_0=H_0-\mathring H(0,x)$.

Remark 3.3. For the case of the variational problem above equivalent to the original PDE problem (1.1), (2.1), (2.2), (2.4), we refer to [17, Theorem 3.1].

3.3. **Main result.** We want to obtain a solution satisfying (3.14), but we prove existence of $(v,\bar H,\theta)$ with a "defect measure" in the third equation of (3.14).

Theorem 3.4. *Let Assumptions 2.2 and 3.1 be satisfied, and let $v_0\in H_{\mathbf V}$, $\bar H_0\in H_\Sigma$, $\theta_0\in L^1(\Omega)$. If $|\alpha_0|$ is small enough in accordance with date of the problem (4.101), then there exists $(v,\bar H,\theta)\in\big(L^\infty(0,T;H_{\mathbf V})\cap L^2(0,T;\mathbf V)\big)\times\big(L^\infty(0,T;H_\Sigma)\cap L^2(0,T;\mathbf V_\Sigma)\big)\times\big(L^\infty(0,T;L^1(\Omega))\cap L^r(0,T;W^{1,r}_{\Gamma_D})\big)$ for all $r\in(1,5/4)$ and a Radon measure $\bar\mu$ such that*

$$
\begin{aligned}
&\int_0^T\big[-\langle v(t),u'(t)\rangle + a_0\langle\theta;v(t),u(t)\rangle+\langle\operatorname{curl}v(t)\times v(t),u(t)\rangle\\
&+\Big\langle -\frac{\mu}{\rho}\operatorname{curl}\bar H(t)\times\bar H(t)-\frac{\mu}{\rho}\operatorname{curl}\bar H(t)\times\mathring H(t)-\frac{\mu}{\rho}\operatorname{curl}\mathring H(t)\times\bar H(t)\\
&-\frac{\mu}{\rho}\operatorname{curl}\mathring H(t)\times\mathring H(t)+\alpha_0\theta(t)\mathbf g-\mathbf f_1(t),u(t)\Big\rangle\Big]\,dt\\
&=\langle v_0,u(x,0)\rangle\quad\forall u\in C^1([0,T];\mathbf V)\ \textit{with}\ u(\cdot,T)=0,\\
&\int_0^T\Big[\langle-\bar H(t),C'(t)\rangle+b_0\langle\theta(t);\bar H(t),C(t)\rangle+\Big\langle\frac{1}{\mu\sigma(\theta)}\operatorname{curl}\mathring H(t),\operatorname{curl}C\Big\rangle\\
&+\Big\langle\operatorname{curl}C(t)\times\bar H(t),v(t)\Big\rangle+\Big\langle\operatorname{curl}C(t)\times\mathring H(t),v(t)\Big\rangle-\langle\mathbf r_1(t),C(t)\rangle\Big]\,dt\\
&=\langle\bar H_0,C(x,0)\rangle\qquad\forall C\in C^1([0,T];\mathbf V_\Sigma)\ \textit{with}\ C(\cdot,T)=0,\\
&\int_0^T\Big[\langle-\theta,\frac{\partial\varphi}{\partial t}\rangle+c_0\langle\theta;\theta,\varphi\rangle-\langle\theta v,\nabla\varphi\rangle-\Big\langle\frac{1}{\rho C_h\sigma(\theta)}|\operatorname{curl}H|^2,\varphi\Big\rangle\\
&-\Big\langle\frac{\eta(\theta)}{\rho C_h}|\mathcal E(u)|^2,\varphi\Big\rangle-\langle g_1,\varphi\rangle\Big]\,dt\\
&=\langle\theta_0(x),\varphi(x,0)\rangle+\int_0^T\varphi\,d\bar\mu\quad\forall\varphi\in C^1([0,T];C^1_{\Gamma_D}(\Omega))\ \textit{with}\ \varphi(\cdot,T)=0,
\end{aligned}
\tag{3.15}
$$

where $C^1_{\Gamma_D}(\Omega) = \{\phi \in C^1(\bar\Omega) : \phi|_{\Gamma_D} = 0\}$. *And* $(v, \bar H, \theta)$ *satisfies the estimate*

$$\begin{aligned}
&\sup_{t\in[0,T]} \|v(t)\|^2 + \|v\|^2_{L^2(0,T;\mathbf{V})} + \sup_{t\in[0,T]} \|\bar H(t)\|^2 + \|\bar H\|^2_{L^2(0,T;\mathbf{V}_\Sigma)}\\
&+ \operatorname*{ess\,sup}_{t\in[0,T]} \|\theta(t)\|_{L^1(\Omega)} + \int_Q |\nabla\theta|^r\,dx\,dt + \int_Q \frac{|\nabla\theta|^2}{(1+|\theta|)^{1+\delta}}\,dx\,dt\\
&\le C_1\Big(\|v_0\|, \|\mathbf{f}_1\|_{L^2(0,T;\mathbf{V}^*)}, \sum_{i=2,4,7} \|\phi_i\|_{L^2(0,T;H^{-\frac12}(\Gamma_i))},\\
&\quad \sum_{i=3,5,6} \|\vec\phi_i\|_{L^2(0,T;\mathbf{H}^{-\frac12}(\Gamma_i))}, \|\bar H_0\|, c_*, \|\mathring{H}(t)\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))}, \frac{2^{(1-\delta)/\delta}}{1-\delta},\\
&\quad |\alpha_0|, \|\theta_0\|_{L^1(\Omega)}, \|g_R\|_{L^2(0,T;L^{4/3}(\Gamma_R))}, \|g_0\|_{L^2(0,T;(W^{1,2}_{\Gamma_D})^*)}\Big)\\
&\quad \forall r\big(1<r<\frac54\big), \quad \forall\delta\big(0<\delta<\frac{5-4r}{3}\big),
\end{aligned} \tag{3.16}$$

where c_* *is the one in* (4.77).

4. Proof of Theorem 3.4

4.1. **Existence of a solution to an approximate problem.** Let $v_0 \in \mathbf{V}$, $\bar H_0(\equiv H_0 - \mathring{H}(0,x)) \in \mathbf{V}_\Sigma$, $\theta_0 \in W^{1,2}_{\Gamma_D}(\Omega)$, $0 < \varepsilon \le 1$.

Problem VEA. Find $(v, \bar H, \theta) \in \big(C([0,T]; H_{\mathbf{V}}) \cap L^6(0,T;\mathbf{V})\big) \times \big(C([0,T]; H_\Sigma) \cap L^6(0,T;\mathbf{V}_\Sigma)\big) \times \big(C([0,T]; L^2(\Omega)) \cap L^2(0,T;W^{1,2}_{\Gamma_D})\big)$ such that

$$\begin{aligned}
&\big\langle \frac{\partial v}{\partial t}, u\big\rangle + 2\Big\langle [\eta(\theta) + \varepsilon\|v\|^4_{\mathbf{V}}]\mathcal{E}(v), \mathcal{E}(u)\Big\rangle + \langle \operatorname{curl} v \times v, u\rangle\\
&+ 2(\eta(\theta)k(x)v, u)_{\Gamma_2} + 2(\eta(\theta)S\tilde v, \tilde u)_{\Gamma_3} + 2(\alpha(x)v, u)_{\Gamma_5}\\
&+ (\eta(\theta)k(x)v, u)_{\Gamma_7} - \langle \frac{\mu}{\rho}\operatorname{curl}\bar H(t) \times \bar H(t), u\rangle - \langle \frac{\mu}{\rho}\operatorname{curl}\bar H(t) \times \mathring{H}(t), u\rangle\\
&- \langle \frac{\mu}{\rho}\operatorname{curl}\mathring{H}(t) \times \bar H(t), u\rangle + \langle \frac{\alpha_0\theta}{1+\varepsilon\theta^2}\mathbf{g}, u\rangle\\
&= \langle \mathbf{f}_1(t), u(t)\rangle \quad \forall u \in L^6(0,T;\mathbf{V}),\\
&\qquad \langle \bar H'(t), C\rangle + \Big\langle [\frac{1}{\mu\sigma(\theta)} + \varepsilon\|\operatorname{curl}\bar H(t)\|^4]\operatorname{curl}\bar H(t), \operatorname{curl} C\Big\rangle\\
&\qquad + \Big\langle \frac{1}{\mu\sigma(\theta)}\operatorname{curl}\mathring{H}(t), \operatorname{curl} C\Big\rangle + \big\langle \operatorname{curl} C \times \bar H(t), v(t)\big\rangle\\
&\qquad + \big\langle \operatorname{curl} C \times \mathring{H}(t), v(t)\big\rangle\\
&\qquad = \langle \mathbf{r}_1(t), C\rangle \quad \forall C \in L^6(0,T;\mathbf{V}_\Sigma),\\
&\qquad \big\langle \frac{\partial\theta}{\partial t}, \varphi\big\rangle + (\kappa(\theta)\nabla\theta, \nabla\varphi) + (\beta(x)\theta, \varphi)_{\Gamma_R} - \langle v\theta, \nabla\varphi\rangle\\
&\qquad - \Big\langle \frac{1}{\rho C_h\sigma(\theta)}\frac{|\operatorname{curl} H|^2}{1+\varepsilon|\operatorname{curl} H|^2}, \varphi\Big\rangle - \Big\langle \frac{\eta(\theta)}{\rho C_h}\frac{|\mathcal{E}(v)|^2}{1+\varepsilon|\mathcal{E}(v)|^2}, \varphi\Big\rangle\\
&\qquad = \langle g_1, \varphi\rangle \quad \forall\varphi \in L^2(0,T;W^{1,2}_{\Gamma_D}(\Omega)),\\
&\qquad\qquad\qquad v(0) = v_0, \quad \bar H(0) = \bar H_0, \quad \theta(0) = \theta_0.
\end{aligned} \tag{4.1}$$

To prove the main result, we use the following result, see [19, Proposition 4.1 and Remark 4.1].

Proposition 4.1. *Let* X *be a real reflexive Banach space and norms of* X *and* X^* *be strictly convex. Let a linear operator* L*, where* $D(L)$ *is a dense linear subspace of* X*, be maximal monotone. The operator* $A : D(L) \to X^*$ *satisfies the following:*

(i) *For every sequence such that* $x_n \rightharpoonup x$ *in* X, $x_n, x \in D(L)$, $Lx_n \rightharpoonup Lx$ *in* X^* *and* $\limsup_{n\to\infty} \langle Ax_n, x_n - x\rangle \le 0$, *there exists a subsequence such that*
$$\liminf_{k\to\infty} \langle Ax_k, x_k - v\rangle \ge \langle Ax, x - v\rangle \quad \forall v \in X; \tag{4.2}$$

(ii) *There exists a function* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$, *which is bounded on any compact interval, and a number* $\theta, 0 \le \theta < 1$, *such that*
$$\|A(x)\|_{X^*} \le \psi(\|x\|_X) + \theta\|Lx\|_{X^*} \quad \forall x \in D(L); \tag{4.3}$$

(iii)
$$\frac{\langle A(x), x\rangle}{\|x\|_X} \to \infty, \quad \text{as } \|x\|_X \to \infty. \tag{4.4}$$

Then, for any $f \in X^*$ *there exists a solution to* $Lx + Ax = f$.

Let
$$\mathscr{V} := L^6(0,T;\mathbf{V}) \times L^6(0,T;\mathbf{V}_\Sigma) \times L^2(0,T;W^{1,2}_{\Gamma_D}),$$
$$\|(\cdot,\cdot)\|_{\mathscr{V}} = \big(\|\cdot\|^2_{L^6(0,T;\mathbf{V})} + \|\cdot\|^2_{L^6(0,T;\mathbf{V}_\Sigma)} + \|\cdot\|^2_{L^2(0,T;W^{1,2}_{\Gamma_D})}\big)^{1/2}.$$

Then
$$\mathscr{V}^* := L^{6/5}(0,T;\mathbf{V}^*) \times L^{6/5}\big(0,T;\mathbf{V}^*_\Sigma) \times L^2\big(0,T;(W^{1,2}_{\Gamma_D})^*\big),$$
$$\|(\cdot,\cdot)\|_{\mathscr{V}^*} = \big(\|\cdot\|^2_{L^{6/5}(0,T;\mathbf{V}^*)} + \|\cdot\|^2_{L^{6/5}\big(0,T;\mathbf{V}^*_\Sigma\big)} + \|\cdot\|^2_{L^2\big(0,T;(W^{1,2}_{\Gamma_D})^*\big)}\big)^{1/2}.$$

Theorem 4.2. *There exists a solution* $(v_\varepsilon, \bar{H}_\varepsilon, \theta_\varepsilon) \in \mathscr{V}$ *to Problem VEA.*

Proof. In (4.1) let us make changes of the unknown functions
$$(v, \bar{H}, \theta) \longrightarrow (w, z, \theta)$$
by $v = e^{k_1 t} w$, $\bar{H} = e^{k_1 t} z$ where k_1 is a constant to be determined later((4.34)). Owing to the changes, differently from [16], where the steady problems were studied, we need not assume that $\Gamma_{2j}, \Gamma_{3j}, \Gamma_{7j}$ are convex and the matrix α is positive, and the datum corresponding to the non-homogeneous magnetic boundary condition is small. Then putting
$$\hat{w} = w - v_0, \quad \hat{z} = z - \bar{H}_0, \quad \hat{\theta} = \theta - \theta_0$$
and multiplying the first and second equations of (4.1), respectively, by $e^{-k_1 t}$ and $\frac{\mu}{\rho} e^{-k_1 t}$, we have a modified evolution problem with zero initial conditions equivalent to Problem VEA. □

Problem VEA' Find $(\hat{w}, \hat{H}, \hat{\theta}) \in \big(C([0,T]; H_{\mathbf{V}}) \cap L^6(0,T;\mathbf{V})\big) \times \big(C([0,T]; H_\Sigma) \cap L^6(0,T;\mathbf{V}_\Sigma)\big) \times \big(C([0,T]; L^2(\Omega)) \cap L^2(0,T; W^{1,2}_{\Gamma_D})\big)$ such that
$$\begin{aligned}
&\Big\langle \frac{\partial \hat{w}}{\partial t}, u\Big\rangle + 2\Big\langle [\eta(\theta) + e^{4k_1 t}\varepsilon\|\hat{w} + v_0\|^4_{\mathbf{V}}]\mathcal{E}(\hat{w} + v_0), \mathcal{E}(u)\Big\rangle \\
&+ e^{k_1 t}\langle \operatorname{curl}(\hat{w} + v_0) \times (\hat{w} + v_0), u\rangle + k_1\langle \hat{w} + v_0, u\rangle + 2\langle \eta(\theta)k(x)(\hat{w} + v_0), u\rangle_{\Gamma_2} \\
&+ 2\langle \eta(\theta)S(\tilde{\hat{w}} + v_0), \tilde{u}\rangle_{\Gamma_3} + 2\langle \alpha(x)(\hat{w} + v_0), u\rangle_{\Gamma_5} + \langle \eta(\theta)k(x)(\hat{w} + v_0), u\rangle_{\Gamma_7} \\
&- \frac{\mu}{\rho}\Big\langle e^{k_1 t} \operatorname{curl}(\hat{z}(t) + \bar{H}_0) \times (\hat{z}(t) + \bar{H}_0) + \operatorname{curl}(\hat{z}(t) + \bar{H}_0) \times \mathring{H}(t) \\
&+ \operatorname{curl} \mathring{H}(t) \times (\hat{z}(t) + \bar{H}_0), u\Big\rangle + e^{-k_1 t}\Big\langle \frac{\alpha_0(\hat{\theta} + \theta_0)}{1 + \varepsilon|\hat{\theta} + \theta_0|^2}\mathbf{g}, u\Big\rangle \\
&= e^{-k_1 t}\langle \mathbf{f}_1(t), u\rangle \quad \forall u \in L^6(0,T;\mathbf{V}),
\end{aligned}$$

$$
\begin{aligned}
&\frac{\mu}{\rho}\Big[\langle\frac{\partial\hat z(t)}{\partial t},C\rangle+\Big\langle[\frac{1}{\mu\sigma(\theta)}+\varepsilon e^{4k_1t}\|\operatorname{curl}(\hat z(t)+\bar H_0)\|^4]\operatorname{curl}(\hat z(t)+\bar H_0),\operatorname{curl}C\Big\rangle\\
&+\Big\langle e^{-k_1t}\frac{1}{\mu\sigma(\theta)}\operatorname{curl}\mathring{H}(t),\operatorname{curl}C\Big\rangle+k_1\langle(\hat z(t)+\bar H_0),C\rangle\\
&+\big\langle\operatorname{curl}C\times(\hat z(t)+\bar H_0),e^{k_1t}w(t)\big\rangle+\big\langle\operatorname{curl}C\times\mathring{H}(t),w(t)\big\rangle\Big]\\
&=\frac{\mu}{\rho}e^{-k_1t}\langle\mathbf{r}_1(t),C\rangle\quad\forall C\in L^6(0,T;\mathbf{V}_\Sigma),\\
&\big\langle\frac{\partial\hat\theta}{\partial t},\varphi\big\rangle+\langle\kappa(\theta)(\nabla\hat\theta+\nabla\theta_0),\nabla\phi\rangle+\langle\beta(x)(\hat\theta+\theta_0),\phi\rangle_{\Gamma_R}\\
&-e^{k_1t}\langle(\hat w+v_0)(\hat\theta+\theta_0),\nabla\phi\rangle-\Big\langle\frac{1}{\rho C_h\sigma(\theta)}\frac{|\operatorname{curl}(\bar H+\mathring{H})|^2}{1+\varepsilon|\operatorname{curl}(\bar H+\mathring{H})|^2},\varphi\Big\rangle\\
&-\Big\langle\frac{\eta(\theta)}{\rho C_h}\frac{|\mathcal{E}(\hat w+v_0)|^2}{e^{-2k_1t}+\varepsilon|\mathcal{E}(\hat w+v_0)|^2},\varphi\Big\rangle\\
&=\langle g_1(t),\varphi\rangle\quad\forall\varphi\in L^2(0,T;W^{1,2}_{\Gamma_D}(\Omega)),\\
&\qquad\qquad\hat w(0)=0,\quad\hat H(0)=0,\quad\hat\theta(0)=0.
\end{aligned}\tag{4.5}
$$

We define an operator $\mathscr{L}:D(\mathscr{L})\to\mathscr{V}^*$ by

$$
\begin{gathered}
D(\mathscr{L}):=\big\{(\hat w,\hat z,\hat\theta)\in\mathscr{V}:(\hat w',\hat z',\hat\theta')\in\mathscr{V}^*,\hat w(0)=0,\hat z(0)=0,\hat\theta(0)=0\big\},\\
\|(\hat w,\hat z,\hat\theta)\|_{D(\mathscr{L})}:=\|(\hat w,\hat z,\hat\theta)\|_{\mathscr{V}}+\|(\hat w',\frac{\mu}{\rho}\hat z',\hat\theta')\|_{\mathscr{V}^*},\\
\mathscr{L}:=\big(\frac{d}{dt},\,\frac{\mu}{\rho}\frac{d}{dt},\,\frac{d}{dt}\big).
\end{gathered}
$$

Then, $\mathscr{L}$ is a linear, maximal monotone operator; see [23, Section 2.1, Ch. 3].

We define an operator $\mathscr{A}_\varepsilon:D(\mathscr{L})\to\mathscr{V}^*$ by

$$
\begin{aligned}
&\big\langle\mathscr{A}_\varepsilon(\hat w,\hat z,\hat\theta),(u,C,\phi)\big\rangle=\int_0^T\Big[2\big\langle[\eta(\theta)+e^{4k_1t}\varepsilon\|\hat w+v_0\|^4_{\mathbf{V}}]\mathcal{E}(\hat w+v_0),\mathcal{E}(u)\big\rangle\\
&+e^{k_1t}\langle\operatorname{curl}(\hat w+v_0)\times(\hat w+v_0),u\rangle+k_1(\hat w+v_0,u)+2\langle\eta(\theta)k(x)(\hat w+v_0),u\rangle_{\Gamma_2}\\
&+2\langle\eta(\theta)S(\tilde{\hat w}+\tilde v_0),\tilde u\rangle_{\Gamma_3}+2\langle\alpha(x)(\hat w+v_0),u\rangle_{\Gamma_5}+\langle\eta(\theta)k(x)(\hat w+v_0),u\rangle_{\Gamma_7}\\
&-\frac{\mu}{\rho}\Big\langle e^{k_1t}\operatorname{curl}(\hat z(t)+\bar H_0)\times(\hat z(t)+\bar H_0)+\operatorname{curl}(\hat z(t)+\bar H_0)\times\mathring{H}(t)\\
&+\operatorname{curl}\mathring{H}(t)\times(\hat z(t)+\bar H_0),u\Big\rangle+e^{-k_1t}\Big\langle\frac{\alpha_0(\hat\theta+\theta_0)}{1+\varepsilon|\hat\theta+\theta_0|^2}\mathbf{g},u\Big\rangle\\
&+\Big\langle[\frac{1}{\rho\sigma(\theta)}+\frac{\mu\varepsilon}{\rho}e^{4k_1t}\|\operatorname{curl}(\hat z(t)+\bar H_0)\|^4]\operatorname{curl}(\hat z(t)+\bar H_0),\operatorname{curl}C\Big\rangle\\
&+\Big\langle e^{-k_1t}\frac{1}{\rho\sigma(\theta)}\operatorname{curl}\mathring{H}(t),\operatorname{curl}C\Big\rangle+k_1\langle(\hat z+\bar H_0),C\rangle\\
&+\Big\langle\frac{\mu}{\rho}e^{k_1t}\operatorname{curl}C\times(\hat z(t)+\bar H_0),(\hat w(t)+v_0)\Big\rangle+\Big\langle\frac{\mu}{\rho}\operatorname{curl}C\times\mathring{H}(t),(\hat w(t)+v_0)\Big\rangle\\
&+\langle\kappa(\theta)(\nabla\hat\theta+\nabla\theta_0),\nabla\phi\rangle+\langle\beta(x)(\hat\theta+\theta_0),\phi\rangle_{\Gamma_R}-e^{k_1t}\langle(\hat w+v_0)(\hat\theta+\theta_0),\nabla\phi\rangle\\
&-\Big\langle\frac{1}{\rho C_h\sigma(\theta)}\frac{|\operatorname{curl}\big(\bar H(t)+\mathring{H}(t)\big)|^2}{1+\varepsilon|\operatorname{curl}\big(\bar H(t)+\mathring{H}(t)\big)|^2},\phi\Big\rangle-\Big\langle\frac{\eta(\theta)}{\rho C_h}\frac{|\mathcal{E}(\hat w+v_0)|^2}{e^{-2k_1t}+\varepsilon|\mathcal{E}(\hat w+v_0)|^2},\phi\Big\rangle\Big]\,dt
\end{aligned}\tag{4.6}
$$

for $(\hat w,\hat z,\hat\theta)\in D(\mathscr{L})$ and all $(u,C,\phi)\in\mathscr{V}$, where and in what follows $\bar H(t)=e^{k_1t}z(t)\equiv e^{k_1t}(\hat z+\bar H_0)$ and $\theta=\hat\theta+\theta_0$. Estimate (4.43) below shows that this operator is well-defined.

Then, the existence of a solution to Problem VEA' is equivalent to the existence of a solution to

$$\mathscr{L}(\hat{w},\hat{z},\hat{\theta})+\mathscr{A}_\varepsilon(\hat{w},\hat{z},\hat{\theta})=\mathscr{F},\quad \mathscr{F}(t)=\begin{pmatrix} e^{-k_1t}\mathbf{f}_1(t)\\ \frac{\mu}{\rho}e^{-k_1t}\mathbf{r}_1(t)\\ g_1(t)\end{pmatrix}.$$

Remark 4.3. By definition of $D(\mathscr{L})$, from the fact that $(\hat{w},\hat{z},\hat{\theta})\in D(\mathscr{L})$ it follows that $(\hat{w},\hat{z},\hat{\theta})\in \big(C([0,T];H_{\mathbf{V}})\times C([0,T];H_\Sigma)\times C([0,T];L^2(\Omega))\big)$. (See [20, Theorem 1.37].)

Relying on Proposition 4.1, we will prove the existence of a solution to the equation above.

(i) Let us prove property (iii) of Proposition 4.1 for $\mathscr{A}_\varepsilon(\hat{w},\hat{z},\hat{\theta})$:

$$\frac{1}{\|(\hat{w},\hat{z},\hat{\theta})\|_{\mathscr{V}}}\big\langle \mathscr{A}_\varepsilon(\hat{w},\hat{z},\hat{\theta}),(\hat{w},\hat{z},\hat{\theta})\big\rangle\to\infty\quad\text{as } \|(\hat{w},\hat{z},\hat{\theta})\|_{\mathscr{V}}\to\infty. \tag{4.7}$$

$$\begin{aligned}
&\big\langle \mathscr{A}_\varepsilon(\hat{w},\hat{z},\hat{\theta}),(\hat{w},\hat{z},\hat{\theta})\big\rangle=\int_0^T\Big[2\Big\langle[\eta(\theta)+e^{4k_1t}\varepsilon\|\hat{w}+v_0\|^4_{\mathbf{V}}]\mathcal{E}(\hat{w}+v_0),\mathcal{E}(\hat{w})\Big\rangle\\
&+e^{k_1t}\langle \operatorname{curl}(\hat{w}+v_0)\times(\hat{w}+v_0),\hat{w}\rangle+k_1(\hat{w}+v_0,\hat{w})+2\langle\eta(\theta)k(x)(\hat{w}+v_0),\hat{w}\rangle_{\Gamma_2}\\
&+2\langle\eta(\theta)S(\tilde{\hat{w}}+\tilde{v}_0),\tilde{\hat{w}}\rangle_{\Gamma_3}+2\langle\alpha(x)(\hat{w}+v_0),\hat{w}\rangle_{\Gamma_5}+\langle\eta(\theta)k(x)(\hat{w}+v_0),\hat{w}\rangle_{\Gamma_7}\\
&-\frac{\mu}{\rho}\Big\langle e^{k_1t}\operatorname{curl}(\hat{z}(t)+\bar{H}_0)\times(\hat{z}(t)+\bar{H}_0)+\operatorname{curl}(\hat{z}(t)+\bar{H}_0)\times\mathring{H}(t)\\
&+\operatorname{curl}\mathring{H}(t)\times(\hat{z}(t)+\bar{H}_0),\hat{w}\Big\rangle+e^{-k_1t}\langle\frac{\alpha_0\theta}{1+\varepsilon\theta^2}\mathbf{g},\hat{w}\rangle\\
&+\Big\langle[\frac{1}{\rho\sigma(\theta)}+\frac{\mu\varepsilon}{\rho}e^{4k_1t}\|\operatorname{curl}(\hat{z}(t)+\bar{H}_0)\|^4]\operatorname{curl}(\hat{z}(t)+\bar{H}_0),\operatorname{curl}\hat{z}\Big\rangle\\
&+\Big\langle e^{-k_1t}\frac{1}{\rho\sigma(\theta)}\operatorname{curl}\mathring{H}(t),\operatorname{curl}\hat{z}\Big\rangle+k_1\langle(\hat{z}(t)+\bar{H}_0),\hat{z}(t)\rangle\\
&+\Big\langle\frac{\mu}{\rho}e^{k_1t}\operatorname{curl}\hat{z}(t)\times\mathring{H}(t),(\hat{w}(t)+v_0)\Big\rangle+\Big\langle\frac{\mu}{\rho}\operatorname{curl}\hat{z}(t)\times\bar{H}_0,(\hat{w}(t)+v_0)\Big\rangle\\
&+\langle\kappa(\theta)(\nabla\hat{\theta}+\nabla\theta_0),\nabla\hat{\theta}\rangle+\langle\beta(x)(\hat{\theta}+\theta_0),\hat{\theta}\rangle_{\Gamma_R}-e^{k_1t}\langle(\hat{w}+v_0)(\hat{\theta}+\theta_0),\nabla\hat{\theta}\rangle\\
&-\Big\langle\frac{1}{\rho C_h\sigma(\theta)}\frac{|\operatorname{curl}\big(\bar{H}(t)+\mathring{H}(t)\big)|^2}{1+\varepsilon|\operatorname{curl}\big(\bar{H}(t)+\mathring{H}(t)\big)|^2},\hat{\theta}\Big\rangle-\Big\langle\frac{\eta(\theta)}{\rho C_h}\frac{|\mathcal{E}(\hat{w}+v_0)|^2}{e^{-2k_1t}+\varepsilon|\mathcal{E}(\hat{w}+v_0)|^2},\hat{\theta}\Big\rangle\Big]\,dt
\end{aligned}\tag{4.8}$$

for all $(\hat{w},\hat{z},\hat{\theta})\in D(\mathscr{L})$.

Using $\langle\operatorname{rot}w\times v,v\rangle=0$ and $\langle w\hat{\theta},\nabla\hat{\theta}\rangle=0$ (cf. (3.9)) and arranging, we obtain

$$\begin{aligned}
&\big\langle \mathscr{A}_\varepsilon(\hat{w},\hat{z},\hat{\theta}),(\hat{w},\hat{z},\hat{\theta})\big\rangle\\
&=\int_0^T\Big[2\Big\langle[\eta(\theta)+e^{4k_1t}\varepsilon\|\hat{w}+v_0\|^4_{\mathbf{V}}]\mathcal{E}(\hat{w}+v_0),\mathcal{E}(\hat{w})\Big\rangle\\
&\quad+e^{k_1t}\langle\operatorname{curl}\hat{w}\times v_0,\hat{w}\rangle+e^{k_1t}\langle\operatorname{curl}v_0\times v_0,\hat{w}\rangle+k_1(\hat{w}+v_0,\hat{w})\\
&\quad+2\langle\eta(\theta)k(x)(\hat{w}+v_0),\hat{w}\rangle_{\Gamma_2}+2\langle\eta(\theta)S(\tilde{\hat{w}}+\tilde{v}_0),\tilde{\hat{w}}\rangle_{\Gamma_3}\\
&\quad+2\langle\alpha(x)(\hat{w}+v_0),\hat{w}\rangle_{\Gamma_5}+\langle\eta(\theta)k(x)(\hat{w}+v_0),\hat{w}\rangle_{\Gamma_7}\\
&\quad-\frac{\mu}{\rho}\langle e^{k_1t}\operatorname{curl}\bar{H}_0\times(\hat{z}(t)+\bar{H}_0)+\operatorname{curl}\bar{H}_0\times\mathring{H}(t)+\operatorname{curl}\mathring{H}(t)\times(\hat{z}(t)+\bar{H}_0),\hat{w}\rangle\\
&\quad+e^{-k_1t}\langle\frac{\alpha_0\theta}{1+\varepsilon\theta^2}\mathbf{g},\hat{w}\rangle\\
&\quad+\Big\langle[\frac{1}{\rho\sigma(\theta)}+\frac{\mu\varepsilon}{\rho}e^{4k_1t}\|\operatorname{curl}(\hat{z}(t)+\bar{H}_0)\|^4]\operatorname{curl}(\hat{z}(t)+\bar{H}_0),\operatorname{curl}\hat{z}\Big\rangle\\
&\quad+\Big\langle e^{-k_1t}\frac{1}{\rho\sigma(\theta)}\operatorname{curl}\mathring{H}(t),\operatorname{curl}\hat{z}\Big\rangle+k_1\langle(\hat{z}(t)+\bar{H}_0),\hat{z}(t)\rangle
\end{aligned}$$

$$
\begin{aligned}
&+ \Big\langle \frac{\mu}{\rho} e^{k_1 t} \operatorname{curl} \hat{z} \times (\hat{z}(t) + \bar{H}_0), v_0 \Big\rangle + \Big\langle \frac{\mu}{\rho} \operatorname{curl} \hat{z} \times \mathring{H}(t), v_0 \Big\rangle \\
&+ \langle \kappa(\theta)(\nabla\hat{\theta} + \nabla\theta_0), \nabla\hat{\theta}\rangle + \langle \beta(x)(\hat{\theta} + \theta_0), \hat{\theta}\rangle_{\Gamma_R} - e^{k_1 t}\langle (\hat{w} + v_0)\theta_0, \nabla\hat{\theta}\rangle \\
&- \Big\langle \frac{1}{\rho C_h \sigma(\theta)} \frac{|\operatorname{curl}\big(\bar{H}(t) + \mathring{H}(t)\big)|^2}{1 + \varepsilon|\operatorname{curl}\big(\bar{H}(t) + \mathring{H}(t)\big)|^2}, \hat{\theta}\Big\rangle - \Big\langle \frac{\eta(\theta)}{\rho C_h} \frac{|\mathcal{E}(\hat{w} + v_0)|^2}{e^{-2k_1 t} + \varepsilon|\mathcal{E}(\hat{w} + v_0)|^2}, \hat{\theta}\Big\rangle\Big]\, dt
\end{aligned} \tag{4.9}
$$

for all $(\hat{w}, \hat{z}, \hat{\theta}) \in D(\mathscr{L})$.

Let us estimate the terms on the right-hand side the above. Taking into account (2.5), by Hölder's and Young's inequalities, we have

$$
\begin{aligned}
&\int_0^T \Big[2\Big\langle [\eta(\theta) + e^{4k_1 t}\varepsilon\|\hat{w} + v_0\|_{\mathbf{V}}^4]\mathcal{E}(\hat{w} + v_0), \mathcal{E}(\hat{w})\Big\rangle \\
&\geq 2\eta_0\|\hat{w}\|_{L^2(0,T;\mathbf{V})}^2 + \int_0^T 2\big(\eta(\theta)\mathcal{E}(v_0), \mathcal{E}(\hat{w})\big)\, dt \\
&\quad + \varepsilon\int_0^T 2\Big\langle e^{4k_1 t}\|\hat{w} + v_0\|_{\mathbf{V}}^4 \mathcal{E}(\hat{w} + v_0), \mathcal{E}(\hat{w} + v_0)\Big\rangle\, dt \\
&\quad - \varepsilon\int_0^T 2\Big\langle e^{4k_1 t}\|\hat{w} + v_0\|_{\mathbf{V}}^4 \mathcal{E}(\hat{w} + v_0), \mathcal{E}(v_0)\Big\rangle\, dt \\
&\geq \eta_0\|\hat{w}\|_{L^2(0,T;\mathbf{V})}^2 + 2\varepsilon\|\hat{w} + v_0\|_{L^6(0,T;\mathbf{V})}^6 \\
&\quad - 2c\varepsilon\|\hat{w} + v_0\|_{L^6(0,T;\mathbf{V})}^5\|v_0\|_{L^6(0,T;\mathbf{V})} - kT\|v_0\|_{\mathbf{V}}^2 \\
&\geq \eta_0\|\hat{w}\|_{L^2(0,T;\mathbf{V})}^2 + \varepsilon\|\hat{w}\|_{L^6(0,T;\mathbf{V})}^6 - K_1,
\end{aligned} \tag{4.10}
$$

where K_1 depends on $\|v_0\|_{\mathbf{V}}, \varepsilon, T$. By Hölder's and Young's inequalities, an interpolation inequality $\|\hat{w}\|_{\mathbf{L}^3(\Omega)} \leq \|\hat{w}\|_{\mathbf{W}^{1,2}(\Omega)}^{1/2}\|\hat{w}\|^{1/2}$, we have

$$
\begin{aligned}
e^{k_1 T}\langle \operatorname{curl}\hat{w} \times v_0, \hat{w}\rangle &\leq e^{k_1 T}\|\hat{w}\|_{\mathbf{V}}^2\|v_0\|_{\mathbf{L}^3} \leq \frac{\varepsilon}{3}\|\hat{w}\|_{\mathbf{V}}^6 + \frac{2}{3\varepsilon}\big(e^{k_1 T}\|v_0\|_{\mathbf{L}^3}\big)^{3/2}, \\
e^{k_1 T}\langle \operatorname{curl} v_0 \times v_0, \hat{w}\rangle &\leq e^{k_1 T}\|v_0\|_{\mathbf{V}}^2 + c\|v_0\|_{\mathbf{V}}^2\|\hat{w}\|_{\mathbf{L}^3}^2 \\
&\leq e^{k_1 T}\|v_0\|_{\mathbf{V}}^2 + c\|v_0\|_{\mathbf{V}}^2\|\hat{w}\|_{\mathbf{V}}\|\hat{w}\| \\
&\leq \frac{\eta_0}{14}\|\hat{w}\|_{\mathbf{V}}^2 + e^{k_1 T}\|v_0\|_{\mathbf{V}}^2 + c_1\|v_0\|_{\mathbf{V}}^4\|\hat{w}\|^2,
\end{aligned}
$$

where c_1 depends on η_0, but is independent from k_1. Then

$$
\begin{aligned}
&\Big|\int_0^T e^{k_1 t}\big[\langle \operatorname{curl}\hat{w} \times v_0, \hat{w}\rangle + \langle \operatorname{curl} v_0 \times v_0, \hat{w}\rangle\big]\, dt\Big| \\
&\leq \frac{\eta_0}{14}\|\hat{w}\|_{L^2(0,T;\mathbf{V})}^2 + \frac{\varepsilon}{3}\|\hat{w}\|_{L^6(0,T;\mathbf{V})}^6 + K_2 + k_{11}\|\hat{w}\|_{L^2(0,T;\mathbf{L}^2)}^2,
\end{aligned} \tag{4.11}
$$

where $K_2 = \frac{2T}{3\varepsilon}\big(e^{k_1 T}\|v_0\|_{\mathbf{L}^3}\big)^{3/2} + Te^{k_1 T}\|v_0\|_{\mathbf{V}}^2$ and $k_{11} = c_1\|v_0\|_{\mathbf{V}}^4$. Also,

$$
\int_0^T k_1(\hat{w} + v_0, \hat{w}) \geq k_1\|\hat{w}\|_{L^2(0,T;\mathbf{L}^2)}^2 - L_1\|\hat{w}\|_{L^2(0,T;\mathbf{L}^2)}, \tag{4.12}
$$

where L_1 depends on $\|v_0\|, k_1$.

Because $\Gamma_{2j}, \Gamma_{3j}, \Gamma_{7j}$ are in $C^{2.1}(\Gamma_{ij})$, there exists a constant M such that

$$
\|S(x)\|_\infty,\ \|k(x)\|_\infty, \|\alpha\|_{L^\infty(\Gamma_5)} \leq M.
$$

Thus,

$$
\begin{aligned}
&\Big|\int_0^T \big(2\langle \eta(\theta)k(x)\hat{w}, \hat{w}\rangle_{\Gamma_2} + 2\langle \eta(\theta)S\tilde{\hat{w}}, \tilde{\hat{w}}\rangle_{\Gamma_3} + 2\langle \alpha(x)\hat{w}, \hat{w}\rangle_{\Gamma_5} + \langle \eta(\theta)k(x)\hat{w}, \hat{w}\rangle_{\Gamma_7}\big)\, dt\Big| \\
&\leq \frac{\eta_0}{14}\|\hat{w}\|_{L^2(0,T;\mathbf{V})}^2 + k_{12}\|\hat{w}\|_{L^2(0,T;\mathbf{L}^2)}^2
\end{aligned} \tag{4.13}
$$

(cf. [15, Theorem 1.5.1.10]), and

$$
\begin{aligned}
&\Big| \int_0^T \big(2\langle \eta(\theta)k(x)v_0, \hat{w}\rangle_{\Gamma_2} + 2\langle \eta(\theta)S\tilde{v}_0, \tilde{\hat{w}}\rangle_{\Gamma_3} + 2\langle \alpha(x)v_0, \hat{w}\rangle_{\Gamma_5} + \langle \eta(\theta)k(x)v_0, \hat{w}\rangle_{\Gamma_7}\big)\, dt\Big| \\
&\leq \frac{\eta_0}{14}\|\hat{w}\|^2_{L^2(0,T;\mathbf{V})} + K_3,
\end{aligned} \tag{4.14}
$$

where K_3 depends on $\|v_0\|_{\mathbf{V}}, \eta_0, T$.

Let us estimate

$$
\begin{aligned}
&\int_0^T \frac{\mu}{\rho} e^{k_1 t}\big\langle \operatorname{curl} \bar{H}_0 \times (\hat{z}(t) + \bar{H}_0), \hat{w}\big\rangle\, dt \\
&= \int_0^T \frac{\mu}{\rho} e^{k_1 t}\big\langle \operatorname{curl} \bar{H}_0 \times \hat{z}(t), \hat{w}\big\rangle\, dt + \int_0^T \frac{\mu}{\rho} e^{k_1 t}\big\langle \operatorname{curl} \bar{H}_0 \times \bar{H}_0, \hat{w}\big\rangle\, dt
\end{aligned}
$$

By Hölder's and Young's inequalities and the interpolation inequality

$$
\|u\|_{\mathbf{L}^3(\Omega)} \leq \|u\|^{1/2}_{\mathbf{W}^{1,2}(\Omega)}\|u\|^{1/2}_{\mathbf{L}^2(\Omega)},
$$

we obtain

$$
\begin{aligned}
&\int_0^T \frac{\mu}{\rho} e^{k_1 t}|\big\langle \operatorname{curl} \bar{H}_0 \times \hat{z}(t), \hat{w}\big\rangle|\, dt \\
&\leq \int_0^T \frac{\mu}{\rho} e^{k_1 t}\|\operatorname{curl} \bar{H}_0\|\|\hat{z}(t)\|_{\mathbf{V}_\Sigma}\|\hat{w}\|_{\mathbf{L}^3}\, dt \\
&\leq T\frac{\mu^2}{\rho^2} e^{2k_1 T}\|\bar{H}_0\|^2_{\mathbf{H}^1(\Omega)} + \int_0^T \|\hat{z}\|^2_{\mathbf{V}_\Sigma}\|\hat{w}\|_{\mathbf{V}}\|\hat{w}\|\, dt \\
&\leq T\frac{\mu^2}{\rho^2} e^{2k_1 T}\|\bar{H}_0\|^2_{\mathbf{H}^1(\Omega)} + \frac{\mu\varepsilon}{4\rho}\|\hat{z}\|^6_{L^6(0,T;\mathbf{V}_\Sigma)} + \frac{2}{3}\sqrt{\frac{4\rho}{3\mu\varepsilon}}\int_0^T \|\hat{w}\|^{3/2}_{\mathbf{V}}\|\hat{w}\|^{3/2}\, dt \\
&\leq \frac{\mu\varepsilon}{4\rho}\|\hat{z}\|^6_{L^6(0,T;\mathbf{V}_\Sigma)} + \frac{\varepsilon}{6}\|\hat{w}\|^6_{L^6(0,T;\mathbf{V})} + k_{13}\|\hat{w}\|^2_{L^2(0,T;\mathbf{L}^2)} + K_{41},
\end{aligned} \tag{4.15}
$$

where k_{13} depends on $\frac{1}{\varepsilon^2}, \rho, \mu$, and $K_{41} = T\frac{\mu^2}{\rho^2} e^{k_1 T}\|\bar{H}_0\|^2_{\mathbf{H}^1(\Omega)}$.

$$
\int_0^T \frac{\mu}{\rho} e^{k_1 t}|\operatorname{curl} \bar{H}_0 \times \bar{H}_0, \hat{w}|\, dt \leq \int_0^T \frac{\mu}{\rho} e^{k_1 t}\|\bar{H}_0\|^2_{\mathbf{H}^1}\|\hat{w}\|_{\mathbf{V}}\, dt \leq \frac{\eta_0}{14}\|\hat{w}\|^2_{L^2(0,T;\mathbf{V})} + K_{42}, \tag{4.16}
$$

where K_{42} depends on $\|H_0\|^4_{\mathbf{H}^1(\Omega)}$, η_0, ρ, μ and $e^{k_1 T}$. By (4.15) and (4.16), we have

$$
\begin{aligned}
&\Big|\int_0^T \frac{\mu}{\rho} e^{k_1 t}\big\langle \operatorname{curl} \bar{H}_0 \times (\hat{z}(t) + \bar{H}_0), \hat{w}\big\rangle\, dt\Big| \\
&\leq \frac{\mu\varepsilon}{4\rho}\|\hat{z}\|^6_{L^6(0,T;\mathbf{V}_\Sigma)} + \frac{\eta_0}{14}\|\hat{w}\|^2_{L^2(0,T;\mathbf{V})} + \frac{\varepsilon}{3}\|\hat{w}\|^6_{L^6(0,T;\mathbf{V})} + k_{13}\|\hat{w}\|^2_{L^2(0,T;\mathbf{L}^2)} + K_4,
\end{aligned} \tag{4.17}
$$

where $K_4 = K_{41} + K_{42}$.

Also, by Hölder's and Young's inequalities we have

$$
\Big|\int_0^T \frac{\mu}{\rho}\big\langle \operatorname{curl} \bar{H}_0 \times \mathring{H}(t), \hat{w}\big\rangle\, dt\Big| \leq \frac{\eta_0}{14}\|\hat{w}\|^2_{L^2(0,T;\mathbf{V})} + K_5, \tag{4.18}
$$

where K_5 depends on η_0 and $(\|\bar{H}_0\|^2_{\mathbf{H}^1(\Omega)}\|\mathring{H}\|^2_{L^\infty(0,T;\mathbf{H}^1(\Omega))})$.

Let us estimate

$$
\begin{aligned}
&\int_0^T \frac{\mu}{\rho}\big\langle \operatorname{curl} \mathring{H}(t) \times (\hat{z}(t) + \bar{H}_0), \hat{w}\big\rangle\, dt \\
&= \int_0^T \frac{\mu}{\rho}\big\langle \operatorname{curl} \mathring{H}(t) \times \hat{z}(t), \hat{w}\big\rangle\, dt + \int_0^T \frac{\mu}{\rho}\big\langle \operatorname{curl} \mathring{H}(t) \times \bar{H}_0, \hat{w}\big\rangle\, dt.
\end{aligned}
$$

Similarly to (4.15) we have

$$\begin{aligned}
&\Big|\int_0^T \frac{\mu}{\rho}\big\langle \operatorname{curl}\mathring{H}(t)\times\hat{z}(t),\hat{w}\big\rangle\,dt\Big| \\
&\le \Big|\frac{\mu}{\rho}\int_0^T \|\operatorname{curl}\mathring{H}(t)\|\|\hat{z}(t)\|_{\mathbf{V}_\Sigma}\|\hat{w}\|_{\mathbf{L}^3}\,dt\Big| \\
&\le \frac{1}{6\rho\sigma_1}\|\hat{z}\|^2_{L^2(0,T;\mathbf{V}_\Sigma)} + c\|\mathring{H}\|^2_{L^\infty(0,T;\mathbf{H}^1(\Omega))}\int_0^T \|\hat{w}\|_{\mathbf{V}}\|\hat{w}\|\,dt \\
&\le \frac{1}{6\rho\sigma_1}\|\hat{z}\|^2_{L^2(0,T;\mathbf{V}_\Sigma)} + \frac{\eta_0}{28}\|\hat{w}\|^2_{L^2(0,T;\mathbf{V})} + k_{14}\|\hat{w}\|^2_{L^2(0,T;\mathbf{L}^2)},
\end{aligned} \tag{4.19}$$

where k_{14} depends on $\|\mathring{H}\|^2_{L^\infty(0,T;\mathbf{H}^1(\Omega))}$, ρ,σ_1,η_0.

$$\begin{aligned}
\Big|\int_0^T \frac{\mu}{\rho}\big\langle \operatorname{curl}\mathring{H}(t)\times\bar{H}_0,\hat{w}\big\rangle\,dt\Big| &\le \Big|\frac{\mu}{\rho}\int_0^T \|\operatorname{curl}\mathring{H}(t)\|\|\bar{H}_0\|_{\mathbf{V}_\Sigma}\|\hat{w}\|_{\mathbf{L}^3}\,dt\Big| \\
&\le c\|\mathring{H}\|^2_{L^\infty(0,T;\mathbf{H}^1(\Omega))}\|\bar{H}_0\|^2_{\mathbf{V}_\Sigma} + \int_0^T \|\hat{w}\|_{\mathbf{V}}\|\hat{w}\|\,dt \\
&\le \frac{\eta_0}{28}\|\hat{w}\|^2_{L^2(0,T;\mathbf{V})} + k_{15}\|\hat{w}\|^2_{L^2(0,T;\mathbf{L}^2)} + K_6,
\end{aligned} \tag{4.20}$$

where k_{15} depends on η_0 and K_6 depends on $\|\mathring{H}\|^2_{L^\infty(0,T;\mathbf{H}^1(\Omega))}, \|\bar{H}_0\|^2_{\mathbf{V}_\Sigma},\rho,\mu$.

From (4.19) and (4.20), we obtain

$$\begin{aligned}
&\Big|\int_0^T \frac{\mu}{\rho}\big\langle \operatorname{curl}\mathring{H}(t)\times(\hat{z}(t)+\bar{H}_0),\hat{w}\big\rangle\,dt\Big| \\
&\le \frac{1}{6\rho\sigma_1}\|\hat{z}\|^2_{L^2(0,T;\mathbf{V}_\Sigma)} + \frac{\eta_0}{14}\|\hat{w}\|^2_{L^2(0,T;\mathbf{V})} + (k_{14}+k_{15})\|\hat{w}\|^2_{L^2(0,T;\mathbf{L}^2)} + K_6.
\end{aligned} \tag{4.21}$$

Since

$$\frac{\xi}{1+\varepsilon\xi^2} \le \frac{1}{2\sqrt{\varepsilon}} \quad \forall \xi\in[0,\infty), \tag{4.22}$$

we have

$$\Big|\int_0^T e^{-k_1 t}\langle \frac{\alpha_0\theta}{1+\varepsilon\theta^2}\mathbf{g},\hat{w}\rangle\,dt\Big| \le \frac{c}{\sqrt{\varepsilon}}\|\hat{w}\|_{L^2(0,T;\mathbf{L}^2)}. \tag{4.23}$$

Taking into account (3.2), in the same way as (4.10), we have

$$\begin{aligned}
&\int_0^T \Big\langle [\frac{1}{\rho\sigma(\theta)} + \frac{\mu\varepsilon}{\rho}e^{4k_1 t}\|\operatorname{curl}(\hat{z}(t)+\bar{H}_0)\|^4]\operatorname{curl}(\hat{z}(t)+\bar{H}_0),\operatorname{curl}\hat{z}\Big\rangle\,dt \\
&\ge \frac{1}{\rho\sigma_1}\|\hat{z}\|^2_{L^2(0,T;\mathbf{V}_\Sigma)} + \frac{\mu\varepsilon}{\rho}\|\hat{z}\|^6_{L^6(0,T;\mathbf{V}_\Sigma)} - K_7,
\end{aligned} \tag{4.24}$$

where K_7 depends on $\|\bar{H}_0\|_{\mathbf{V}_\Sigma},T,\varepsilon,\rho,\mu$. Also we obtain

$$\int_0^T \Big\langle e^{-k_1 t}\frac{1}{\rho\sigma(\theta)}\operatorname{curl}\mathring{H}(t),\operatorname{curl}\hat{z}\Big\rangle\,dt \le L_2\|\hat{z}(t)\|_{L^2(0,T;\mathbf{V}_\Sigma)}, \tag{4.25}$$

where L_2 depends on $\|\bar{H}_0\|_{L^\infty(0,T;\mathbf{H}^1)},\rho,\sigma_0$.

$$\int_0^T k_1\langle(\hat{z}(t)+\bar{H}_0),\hat{z}(t)\rangle\,dt \ge k_1\|\hat{z}\|^2_{L^2(0,T;\mathbf{L}^2)} - L_3\|\hat{z}\|_{L^2(0,T;\mathbf{L}^2)}, \tag{4.26}$$

where L_3 depends on $k_1,\|\bar{H}_0\|$.

Let us estimate

$$\begin{aligned}
&\int_0^T \Big\langle \frac{\mu}{\rho}e^{k_1 t}\operatorname{curl}\hat{z}\times(\hat{z}(t)+\bar{H}_0),v_0\Big\rangle\,dt \\
&= \int_0^T \Big\langle \frac{\mu}{\rho}e^{k_1 t}\operatorname{curl}\hat{z}\times\hat{z}(t),v_0\Big\rangle\,dt + \int_0^T \Big\langle \frac{\mu}{\rho}e^{k_1 t}\operatorname{curl}\hat{z}\times\bar{H}_0,v_0\Big\rangle\,dt.
\end{aligned}$$

Taking into account $\|\hat z\|_{\mathbf{L}^3(\Omega)} \le \|\hat z\|^{1/2}_{\mathbf{V}_\Sigma}\|\hat z\|^{1/2}_{\mathbf{L}^2(\Omega)}$, we obtain

$$
\begin{aligned}
&\Big|\int_0^T \Big\langle \frac{\mu}{\rho} e^{k_1 t} \operatorname{curl} \hat z \times \hat z(t), v_0\Big\rangle \, dt\Big| \\
&\le c\int_0^T \frac{\mu}{\rho} e^{k_1 t} \|\hat z(t)\|_{\mathbf{V}_\Sigma} \|\hat z(t)\|_{\mathbf{L}^3} \|v_0\|_{\mathbf{L}^6}\, dt \\
&\le c' \frac{T\mu^2}{\rho^2} e^{2k_1 T}\|v_0\|^2_{\mathbf{V}} + \int_0^T \|\hat z(t)\|^3_{\mathbf{V}_\Sigma}\|\hat z(t)\|\, dt \\
&\le c' \frac{T\mu^2}{\rho^2} e^{2k_1 T}\|v_0\|^2_{\mathbf{V}} + \frac{\mu\varepsilon}{4\rho}\|\hat z\|^6_{L^6(0,T;\mathbf{V}_\Sigma)} + k_{16}\|\hat z\|^2_{L^2(0,T;\mathbf{L}^2)},
\end{aligned}
\tag{4.27}
$$

where k_{16} depends on ρ, μ, ε.

$$
\begin{aligned}
&\Big|\int_0^T \Big\langle \frac{\mu}{\rho} e^{k_1 t} \operatorname{curl} \hat z \times \bar H_0, v_0\Big\rangle \, dt\Big| \\
&\le c\int_0^T \frac{\mu}{\rho} e^{k_1 t} \|\hat z(t)\|_{\mathbf{V}_\Sigma} \|\bar H_0\|_{\mathbf{L}^3} \|v_0\|_{\mathbf{L}^6}\, dt \\
&\le \frac{1}{6\rho\sigma_1}\|\hat z\|^2_{L^2(0,T;\mathbf{V}_\Sigma)} + c'' \frac{T\mu^2}{\rho^2} e^{2k_1 T}\|\bar H_0\|^2_{\mathbf{L}^3}\|v_0\|^2_{\mathbf{L}^6},
\end{aligned}
\tag{4.28}
$$

where c'' depends on ρ, σ_1.

By (4.27) and (4.28), we have

$$
\begin{aligned}
&\int_0^T \Big\langle \frac{\mu}{\rho} e^{k_1 t} \operatorname{curl} \hat z \times (\hat z(t) + \bar H_0), v_0\Big\rangle \, dt \\
&\le \frac{\mu\varepsilon}{4\rho}\|\hat z\|^6_{L^6(0,T;\mathbf{V})} + \frac{1}{6\rho\sigma_1}\|\hat z\|^2_{L^2(0,T;\mathbf{V}_\Sigma)} + k_{16}\|\hat z\|^2_{L^2(0,T;\mathbf{L}^2)} + K_8,
\end{aligned}
\tag{4.29}
$$

where $K_8 = c'\frac{T\mu^2}{\rho^2} e^{2k_1 T}\|v_0\|^2_V + c''\frac{T\mu^2}{\rho^2} e^{2k_1 T}\|\bar H_0\|^2_{\mathbf{L}^3}\|v_0\|^2_{\mathbf{L}^6}$. Also we obtain

$$
\Big|\int_0^T \Big\langle \frac{\mu}{\rho} \operatorname{curl} \hat z \times \mathring{H}(t), v_0\Big\rangle \, dt\Big| \le \frac{1}{6\rho\sigma_1}\|\hat z\|^2_{L^2(0,T;\mathbf{V}_\Sigma)} + K_9,
\tag{4.30}
$$

where K_9 depends on $\|\mathring{H}\|^2_{L^\infty(0,T;\mathbf{L}^3)}, \|v_0\|^2_{\mathbf{L}^6}, \rho, \mu, \sigma_1$. Also, we have

$$
\begin{gathered}
\int_0^T \langle \kappa(\theta)(\nabla\hat\theta + \nabla\theta_0), \nabla\hat\theta\rangle\, dt \ge \kappa_0\|\hat\theta\|^2_{L^2(0,T;W^{1,2}_{\Gamma_D})} - \frac{\kappa_0}{6}\|\hat\theta\|^2_{L^2(0,T;W^{1,2}_{\Gamma_D})} - K_{10}, \\
\int_0^T (\beta(x)\hat\theta, \hat\theta)_{\Gamma_R}\, dt \ge 0, \\
\Big|\int_0^T (\beta(x)\theta_0, \hat\theta)_{\Gamma_R}\, dt\Big| \le \frac{\kappa_0}{6}\|\hat\theta\|^2_{L^2(0,T;W^{1,2}_{\Gamma_D})} + K_{11},
\end{gathered}
\tag{4.31}
$$

where $K_{10} = T\|\theta_0\|^2_{W^{1,2}_{\Gamma_D}}$ and K_{11} depends on $\|\theta_0\|_{W^{1,2}(\Omega)}$ and T.

Let us estimate

$$
\Big|\int_0^T e^{k_1 t}\langle(\hat w + v_0)\theta_0, \nabla\hat\theta\rangle\, dt\Big| = \Big|\int_0^T \big[e^{k_1 t}\langle\hat w\theta_0, \nabla\hat\theta\rangle + e^{k_1 t}\langle v_0\theta_0, \nabla\hat\theta\rangle\big]\, dt\Big|.
$$

First, we have

$$
\begin{aligned}
\Big|\int_0^T e^{k_1 t}\langle\hat w\theta_0, \nabla\hat\theta\rangle\, dt\Big| &\le \int_0^T e^{k_1 t}\|\hat w\|_{\mathbf{L}^3}\|\theta_0\|_{L^6}\|\nabla\hat\theta\|\, dt \\
&\le \frac{\kappa_0}{12}\|\hat\theta\|^2_{L^2(0,T;W^{1,2}_{\Gamma_D})} + \frac{c' T e^{4k_1 T}}{\varepsilon}\|\theta_0\|^4_{W^{1,2}} + \frac{\varepsilon}{6}\|\hat w\|^6_{L^6(0,T;\mathbf{V})}.
\end{aligned}
$$

Also

$$
\Big|\int_0^T e^{k_1 t}\langle v_0\theta_0, \nabla\hat\theta\rangle\, dt\Big| \le \frac{\kappa_0}{12}\|\hat\theta\|^2_{L^2(0,T;W^{1,2}_{\Gamma_D})} + c'' T e^{2k_1 T}\|v_0\|^2_{\mathbf{V}}\|\theta_0\|^2_{W^{1,2}},
$$

where c'' depends on κ_0. Combining above two inequalities yields

$$\Big|\int_0^T e^{k_1 t}\langle(\hat w+v_0)\theta_0,\nabla\hat\theta\rangle\,dt\Big|\le\frac{\kappa_0}{6}\|\hat\theta\|^2_{L^2(0,T;W^{1,2}_{\Gamma_D})}+\frac{\varepsilon}{6}\|\hat w\|^6_{L^6(0,T;\mathbf{V})}+K_{12},\tag{4.32}$$

where $K_{12}=\frac{c'Te^{4k_1T}}{\varepsilon}\|\theta_0\|^4_{W^{1,2}}+c''Te^{2k_1T}\|v_0\|^2_{\mathbf{V}}\|\theta_0\|^2_{W^{1,2}}$.

In the same way as for (4.22), we obtain

$$\begin{gathered}\Big|\int_0^T\Big\langle\frac{1}{\rho C_h\sigma(\theta)}\frac{|\operatorname{curl}\big(\bar H(t)+\mathring{H}(t)\big)|^2}{1+\varepsilon|\operatorname{curl}\big(\bar H(t)+\mathring{H}(t)\big)|^2},\hat\theta\Big\rangle\,dt\Big|\le\frac{c}{\varepsilon}\|\hat\theta\|_{L^2(0,T;L^2(\Omega))},\\ \Big|\int_0^T\Big\langle\frac{\eta(\theta)}{\rho C_h}\frac{|\mathcal{E}(w)|^2}{e^{-2k_1t}+\varepsilon|\mathcal{E}(w)|^2},\hat\theta\Big\rangle\,dt\Big|\le\frac{c}{\varepsilon}\|\hat\theta\|_{L^2(0,T;L^2(\Omega))}.\end{gathered}\tag{4.33}$$

Taking

$$k_1=\max\Big\{\sum_{i=1}^5 k_{1i},\ k_{16}\Big\},\tag{4.34}$$

by (4.10)-(4.14), (4.17)-(4.18), (4.21)-(4.26), (4.29)-(4.33), we have from (4.9) that

$$\begin{aligned}&\big\langle\mathscr{A}_\varepsilon(\hat w,\hat z,\hat\theta),(\hat w,\hat z,\hat\theta)\big\rangle\\ &\ge\min\Big\{\frac{\eta_0}{2},\frac{\varepsilon}{3},\frac{1}{2\rho\sigma_1},\frac{\mu\varepsilon}{2\rho},\frac{\kappa_0}{2}\Big\}\Big(\|\hat w\|^2_{L^2(0,T;\mathbf{V})}+\|\hat w\|^6_{L^6(0,T;\mathbf{V})}+\|\hat z\|^2_{L^2(0,T;\mathbf{V}_\Sigma)}\\ &\quad+\|\hat z\|^6_{L^6(0,T;\mathbf{V}_\Sigma)}+\|\hat\theta\|^2_{L^2(0,T;W^{1,2}_{\Gamma_D})}\Big)-\sum_{i=1}^{12}K_i-(L_1+\frac{c}{\sqrt{\varepsilon}})\|\hat w\|_{L^2(0,T;\mathbf{L}^2)}\\ &\quad-L_2\|\hat z\|_{L^2(0,T;\mathbf{V}_\Sigma)}-L_3\|\hat z\|_{L^2(0,T;\mathbf{L}^2)}-\frac{2c}{\varepsilon}\|\hat\theta\|_{L^2(0,T;L^2)}\quad\forall(\hat w,\hat z,\hat\theta)\in D(\mathscr{L}),\end{aligned}\tag{4.35}$$

which implies (4.7).

(ii) Let us prove (ii) of Proposition 4.1 for $\mathscr{A}_\varepsilon(\hat w,\hat z,\hat\theta)$. Let us estimate

$$\begin{aligned}I_1\equiv\frac{1}{\|u\|_{L^6(0,T;\mathbf{V})}}\Big|\int_0^T&\Big[2\Big\langle[\eta(\theta)+e^{4k_1t}\varepsilon\|\hat w+v_0\|^4_{\mathbf{V}}]\mathcal{E}(\hat w+v_0),\mathcal{E}(u)\Big\rangle\\ &+e^{k_1t}\langle\operatorname{curl}(\hat w+v_0)\times(\hat w+v_0),u\rangle+k_1(\hat w+v_0,u)+2\langle\eta(\theta)k(x)(\hat w+v_0),u\rangle_{\Gamma_2}\\ &+2\langle\eta(\theta)S(\tilde{\hat w}+\tilde v_0),\tilde u\rangle_{\Gamma_3}+2\langle\alpha(x)(\hat w+v_0),u\rangle_{\Gamma_5}+\langle\eta(\theta)k(x)(\hat w+v_0),u\rangle_{\Gamma_7}\Big]\,dt\Big|.\end{aligned}$$

Applying Hölder's and Young's inequalities and $|a+b|^p\le 2^p(|a|^p+|b|^p)$, $p\in(1,\infty)$, we have

$$I_1\le c\big(\|\hat w\|^5_{L^6(0,T;\mathbf{V})}+K\big),\tag{4.36}$$

where K depends on v_0 and T.

We see that for $(\hat w,\hat z,\hat\theta)\in D(\mathscr{L})$

$$\begin{gathered}(\hat w,\hat z,\hat\theta))\in C([0,T];H_{\mathbf{V}})\times C([0,T];\mathbf{L}^2(\Omega))\times C([0,T];L^2(\Omega)),\\ \|\hat w\|_{C([0,T];H_{\mathbf{V}})}\le c\|\hat w'\|^{1/2}_{L^{6/5}(0,T;\mathbf{V}^*)}\|\hat w\|^{1/2}_{L^6(0,T;\mathbf{V})},\\ \|\hat z\|_{C([0,T];\mathbf{L}^2)}\le c\|\hat z'\|^{1/2}_{L^{6/5}(0,T;(\mathbf{V}_\Sigma)^*)}\|\hat z\|^{1/2}_{L^6(0,T;\mathbf{V}_\Sigma)},\\ \|\hat\theta\|_{C([0,T];L^2)}\le c\|\hat\theta'\|^{1/2}_{L^2\big(0,T;(W^{1,2}_{\Gamma_D})^*\big)}\|\hat\theta\|^{1/2}_{L^2(0,T;W^{1,2}_{\Gamma_D})}\end{gathered}\tag{4.37}$$

(cf. [20, Theorem 1.37 and (1.27)].) and

$$\begin{gathered}\|w\|_{\mathbf{L}^3(\Omega)}\le c\|w\|^{1/2}_{\mathbf{L}^2(\Omega)}\|w\|^{1/2}_{\mathbf{V}}\quad\forall w\in\mathbf{V},\\ \|z\|_{\mathbf{L}^3(\Omega)}\le c\|z\|^{1/2}_{\mathbf{L}^2(\Omega)}\|z\|^{1/2}_{\mathbf{V}_\Sigma}\quad\forall z\in\mathbf{V}_\Sigma,\\ \|\theta\|_{L^3(\Omega)}\le c\|\theta\|^{1/2}_{L^2(\Omega)}\|\theta\|^{1/2}_{W^{1,2}_{\Gamma_D}(\Omega)}\quad\forall\theta\in W^{1,2}_{\Gamma_D}(\Omega).\end{gathered}\tag{4.38}$$

Taking into account (4.37), (4.38) and applying the inequality $|a+b|^p \le 2^p(|a|^p+|b|^p), p \in (1,\infty)$, we have

$$
\begin{aligned}
I_2 &\equiv \frac{1}{\|u\|_{L^6(0,T;\mathbf{V})}}\Big|\int_0^T \Big[e^{k_1 t}\langle \operatorname{curl}(\hat w+v_0)\times(\hat w+v_0),u\rangle\Big]\,dt\Big| \\
&\le \frac{c}{\|u\|_{L^6(0,T;\mathbf{V})}}\int_0^T \|\operatorname{curl}(\hat w+v_0)\|_{\mathbf{L}^2}\|(\hat w+v_0)\|_{\mathbf{L}^2}^{1/2}\|(\hat w+v_0)\|_{\mathbf{V}}^{1/2}\|u\|_{\mathbf{L}^6}\,dt \\
&\le \|\hat w+v_0\|_{C(0,T;\mathbf{L}^2)}^{1/2}\Big(\frac{c}{\|u\|_{L^6(0,T;\mathbf{V})}}\|\hat w+v_0\|_{L^{9/5}(0,T;\mathbf{V})}^{3/2}\|u\|_{L^6(0,T;\mathbf{V})}\Big) \\
&\le \|\hat w+v_0\|_{C([0,T];H_{\mathbf{V}})}+c\|(\hat w+v_0)\|_{\mathbf{L}^{9/5}(0.T;\mathbf{V})}^3 \\
&\le c\|\hat w'\|_{L^{6/5}(0,T;\mathbf{V}^*)}^{1/2}\|\hat w\|_{L^6(0,T;\mathbf{V})}^{1/2}+\|v_0\|+c\|\hat w+v_0\|_{L^6(0,T;\mathbf{V})}^3 \\
&\le \frac12\|\hat w'\|_{L^{6/5}(0,T;\mathbf{V}^*)}+c\big(\|\hat w\|_{L^6(0,T;\mathbf{V})}^3+K\big).
\end{aligned} \tag{4.39}
$$

where K depends on v_0 and T.

Let us estimate

$$
\begin{aligned}
I_3 \equiv \frac{1}{\|u\|_{L^6(0,T;\mathbf{V})}}\Big|\int_0^T \Big[&-\frac{\mu}{\rho}\Big\langle e^{k_1 t}\operatorname{curl}(\hat z(t)+\bar H_0)\times(\hat z(t)+\bar H_0) \\
&+\operatorname{curl}(\hat z(t)+\bar H_0)\times \mathring{H}(t)+\operatorname{curl}\mathring{H}(t)\times(\hat z(t)+\bar H_0),u\Big\rangle \\
&+e^{-k_1 t}\Big\langle \frac{\alpha_0(\hat\theta+\theta_0)}{1+\varepsilon|\hat\theta+\theta_0|^2}\mathbf{g},u\Big\rangle\Big]\,dt.
\end{aligned}
$$

Taking into account (4.37), (4.38), we obtain

$$
\begin{aligned}
&\frac{1}{\|u\|_{L^6(0,T;\mathbf{V})}}\Big|\int_0^T \frac{\mu}{\rho}\Big\langle e^{k_1 t}\operatorname{curl}(\hat z(t)+\bar H_0)\times(\hat z(t)+\bar H_0),u\Big\rangle\,dt\Big| \\
&\le \frac{c}{\|u\|_{L^6(0,T;\mathbf{V})}}\int_0^T \|\hat z(t)+\bar H_0\|_{\mathbf{H}^1}\|\hat z(t)+\bar H_0\|_{\mathbf{L}^2}^{1/2}\|\hat z(t)+\bar H_0\|_{\mathbf{H}^1}^{1/2}\|u(t)\|_{\mathbf{V}}\,dt \\
&\le \|\hat z+\bar H_0\|_{C(0,T;\mathbf{L}^2)}^{1/2}\Big(\frac{c}{\|u\|_{L^6(0,T;\mathbf{V})}}\|\hat z(t)+\bar H_0\|_{L^{9/5}(0,T;\mathbf{V})}^{3/2}\|u\|_{L^6(0,T;\mathbf{V})}\Big) \\
&\le \|\hat z+\bar H_0\|_{C(0,T;\mathbf{L}^2)}+c\|\hat z(t)+\bar H_0\|_{L^2(0,T;\mathbf{V})}^3 \\
&\le \|\hat z\|_{C(0,T;\mathbf{L}^2)}+\|\bar H_0\|+c\big(\|\hat z\|_{L^2(0,T;\mathbf{V})}^3+\|\hat z\|_{L^2(0,T;\mathbf{V})}^2+\|\hat z\|_{L^2(0,T;\mathbf{V})}+1\big) \\
&\le \frac14\|\hat z'\|_{L^{6/5}(0,T;(\mathbf{V}_\Sigma)^*)}+c\|\hat z\|_{L^6(0,T;\mathbf{V}_\Sigma)}+c\big(\|\hat z\|_{L^2(0,T;\mathbf{V}_\Sigma)}^3+K\big).
\end{aligned}
$$

Taking into account Assumption 3.1, we obtain

$$
\begin{aligned}
&\frac{1}{\|u\|_{L^6(0,T;\mathbf{V})}}\Big|\int_0^T \frac{\mu}{\rho}\Big\langle e^{k_1 t}\operatorname{curl}(\hat z(t)+\bar H_0)\times\mathring{H}(t),u\Big\rangle\,dt\Big| \\
&\le \frac{c}{\|u\|_{L^6(0,T;\mathbf{V})}}\int_0^T \|\hat z(t)+\bar H_0\|_{\mathbf{H}^1}\|\mathring{H}(t)\|_{\mathbf{L}^3}\|u(t)\|_{\mathbf{V}}\,dt \\
&\le \|\mathring{H}\|_{L^\infty(0,T;\mathbf{L}^3)}\Big(\frac{c}{\|u\|_{L^6(0,T;\mathbf{V})}}\|\hat z(t)+\bar H_0\|_{L^{6/5}(0,T;\mathbf{V}_\Sigma)}\|u\|_{L^6(0,T;\mathbf{V})}\Big) \\
&\le c\big(\|\hat z(t)\|_{L^6(0,T;\mathbf{V}_\Sigma)}+K\big),
\end{aligned}
$$

where c depends on $\|\mathring{H}\|_{L^\infty(0,T;\mathbf{L}^3)}$, and K depends on $\bar H_0$. Taking into account (4.37), (4.38), we obtain

$$
\frac{1}{\|u\|_{L^6(0,T;\mathbf{V})}}\Big|\int_0^T \frac{\mu}{\rho}\Big\langle e^{k_1 t}\operatorname{curl}\mathring{H}(t)\times(\hat z(t)+\bar H_0),u\Big\rangle\,dt\Big|
$$

$$
\begin{aligned}
&\le \frac{c}{\|u\|_{L^6(0,T;\mathbf{V})}} \int_0^T \|\mathring{H}\|_{\mathbf{H}^1} \|\hat{z}(t)+\bar{H}_0\|_{\mathbf{L}^2}^{1/2} \|\hat{z}(t)+\bar{H}_0\|_{\mathbf{H}^1}^{1/2} \|u(t)\|_{\mathbf{V}}\,dt \\
&\le \frac{c}{\|u\|_{L^6(0,T;\mathbf{V})}} \int_0^T \|\bar{H}_0\|_{\mathbf{H}^1} \|\hat{z}(t)+\bar{H}_0\|_{\mathbf{L}^2}^{1/2} \|\hat{z}(t)+\bar{H}_0\|_{\mathbf{H}^1}^{1/2} \|u(t)\|_{\mathbf{V}}\,dt \\
&\le \|\hat{z}+\bar{H}_0\|_{C(0,T;\mathbf{L}^2)}^{1/2} \Big(\frac{c\|\mathring{H}\|_{L^\infty(0,T;\mathbf{H}^1)}}{\|u\|_{L^6(0,T;\mathbf{V})}} \|\hat{z}(t)+\bar{H}_0\|_{L^1(0,T;\mathbf{V}_\Sigma)}^{1/2} \|u\|_{L^2(0,T;\mathbf{V})} \Big) \\
&\le \frac{1}{4}\|\hat{z}'\|_{L^{6/5}(0,T;(\mathbf{V}_\Sigma)^*)} + c\big(\|\hat{z}\|_{L^6(0,T;\mathbf{V}_\Sigma)} + \|\hat{z}\|_{L^2(0,T;\mathbf{V}_\Sigma)} + K\big),
\end{aligned}
$$

where c depends on $\|\mathring{H}\|_{L^\infty(0,T;\mathbf{H}^1)}$ and K depends on $\bar{H}_0$. Also taking into account (4.22), we obtain

$$
\frac{1}{\|u\|_{L^6(0,T;\mathbf{V})}} \Big| \int_0^T e^{-k_1 t} \Big\langle \frac{\alpha_0(\hat{\theta}+\theta_0)}{1+\varepsilon|\hat{\theta}+\theta_0|^2}\mathbf{g}, u \Big\rangle\,dt \le \frac{c}{2\sqrt{\varepsilon}}.
$$

The four estimations above and Young's inequality yield

$$
I_3 \le \frac{1}{4}\|\hat{z}'\|_{L^{6/5}(0,T;(\mathbf{V}_\Sigma)^*)} + c\big(\|\hat{z}\|_{L^6(0,T;\mathbf{V}_\Sigma)} + \|\hat{z}\|_{L^2(0,T;\mathbf{V}_\Sigma)}^3 + K + \frac{1}{\sqrt{\varepsilon}}\big). \tag{4.40}
$$

Let us estimate

$$
\begin{aligned}
I_4 \equiv \frac{1}{\|C\|_{L^6(0,T;\mathbf{V}_\Sigma)}} \Big| \int_0^T &\Big[\Big\langle [\frac{1}{\rho\sigma(\theta)} + \frac{\mu\varepsilon}{\rho} e^{4k_1 t} \|\operatorname{curl}(\hat{z}(t)+\bar{H}_0)\|^4] \operatorname{curl}(\hat{z}(t)+\bar{H}_0), \operatorname{curl} C \Big\rangle \\
&+ \Big\langle e^{-k_1 t} \frac{1}{\rho\sigma(\theta)} \operatorname{curl} \mathring{H}(t), \operatorname{curl} C \Big\rangle + k_1\langle(\bar{z}+\bar{H}_0), C\rangle \\
&+ \Big\langle \frac{\mu}{\rho} e^{k_1 t} \operatorname{curl} C \times (\hat{z}(t)+\bar{H}_0), (\hat{w}(t)+v_0) \Big\rangle + \Big\langle \frac{\mu}{\rho} \operatorname{curl} C \times \mathring{H}(t), (\hat{w}(t)+v_0) \Big\rangle \Big]\,dt \Big|.
\end{aligned}
$$

First, we obtain

$$
\begin{aligned}
&\frac{1}{\|C\|_{L^6(0,T;\mathbf{V}_\Sigma)}} \Big| \int_0^T \Big[\Big\langle [\frac{1}{\rho\sigma(\theta)} + \frac{\mu\varepsilon}{\rho} e^{4k_1 t} \|\operatorname{curl}(\hat{z}(t)+\bar{H}_0)\|^4] \operatorname{curl}(\hat{z}(t)+\bar{H}_0), \operatorname{curl} C \Big\rangle \\
&+ \Big\langle e^{-k_1 t} \frac{1}{\rho\sigma(\theta)} \operatorname{curl} \mathring{H}(t), \operatorname{curl} C \Big\rangle + k_1\langle(\bar{z}+\bar{H}_0), C\rangle \Big]\,dt \\
&\le c\big(\|\hat{z}\|_{L^6(0,T;\mathbf{V}_\Sigma)}^5 + \|\mathring{H}\|_{L^2(0,T;\mathbf{V}_\Sigma)} + K\big),
\end{aligned}
$$

where K depends on $\bar{H}_0, \rho, \sigma_1, \mu, k_1$. In much the same way as (4.39), we obtain

$$
\begin{aligned}
&\frac{1}{\|C\|_{L^6(0,T;\mathbf{V}_\Sigma)}} \Big| \int_0^T \Big\langle \frac{\mu}{\rho} e^{k_1 t} \operatorname{curl} C \times (\hat{z}(t)+\bar{H}_0), (\hat{w}(t)+v_0) \Big\rangle\,dt \Big| \\
&\le \frac{c}{\|C\|_{L^6(0,T;\mathbf{V}_\Sigma)}} \int_0^T \|C(t)\|_{\mathbf{V}_\Sigma} \|\hat{z}(t)+\bar{H}_0\|_{\mathbf{L}^2}^{1/2} \|\hat{z}(t)+\bar{H}_0\|_{\mathbf{H}^1}^{1/2} \|\hat{w}(t)+v_0\|_{\mathbf{V}}\,dt \\
&\le \|\hat{z}(t)+\bar{H}_0\|_{C(0,T;\mathbf{L}^2)}^{1/2} \Big(\|\hat{z}(t)+\bar{H}_0\|_{L^{3/2}(0,T;\mathbf{H}^1)}^{1/2} \|\hat{w}+v_0\|_{L^2(0,T;\mathbf{V})} \Big) \\
&\le \frac{1}{2}\|\hat{z}'\|_{L^{6/5}(0,T;(\mathbf{V}_\Sigma)^*)} + c\Big(\|\hat{z}\|_{L^6(0,T;\mathbf{V}_\Sigma)} + \|\hat{z}(t)\|_{L^2(0,T;\mathbf{V}_\Sigma)}^2 + \|\hat{w}\|_{L^6(0,T;\mathbf{V})}^4 + K \Big),
\end{aligned}
$$

where K depends on $\bar{H}_0, v_0$. Also, we obtain

$$
\begin{aligned}
&\frac{1}{\|C\|_{L^6(0,T;\mathbf{V}_\Sigma)}} \Big| \int_0^T \Big\langle \frac{\mu}{\rho} \operatorname{curl} C \times \mathring{H}(t), (\hat{w}(t)+v_0) \Big\rangle\,dt \\
&\le \frac{c}{\|C\|_{L^6(0,T;\mathbf{V}_\Sigma)}} \int_0^T \|C(t)\|_{\mathbf{V}_\Sigma} \|\mathring{H}(t)\|_{\mathbf{L}^3} \|\hat{w}(t)+v_0\|_{\mathbf{V}}\,dt \le c(\|\hat{w}\|_{L^2(0,T;\mathbf{V})}^2 + K),
\end{aligned}
$$

where K depends on $\|\mathring{H}\|_{L^\infty(0,T;\mathbf{L}^3)}$ and v_0.

The three estimations above yield

$$I_4 \le \frac{1}{4}\|\hat z'\|_{L^{6/5}(0,T;(\mathbf{V}_\Sigma)^*)} + c\Big(\|\hat z\|^5_{L^6(0,T;\mathbf{V}_\Sigma)} + \|\hat w\|^4_{L^6(0,T;\mathbf{V})} + K_1\Big), \tag{4.41}$$

where K_1 depends on $\bar H_0, v_0$ and $\|\mathring{H}\|^2_{L^\infty(0,T;\mathbf{L}^3)}$. Let us estimate

$$\begin{aligned}
I_5 \equiv \frac{1}{\|\phi\|_{L^2(0,T;W^{1,2}_{\Gamma_D})}}\Big|\int_0^T \Big[&\langle \kappa(\theta)(\nabla\hat\theta + \nabla\theta_0), \nabla\phi\rangle \\
&+ \langle \beta(x)(\hat\theta+\theta_0),\phi\rangle_{\Gamma_R} - e^{k_1 t}\langle(\hat w + v_0)(\hat\theta+\theta_0),\nabla\phi\rangle \\
&- \Big\langle \frac{1}{\rho C_h\sigma(\theta)}\frac{|\operatorname{curl}\big(\bar H(t)+\mathring{H}(t)\big)|^2}{1+\varepsilon|\operatorname{curl}\big(\bar H(t)+\mathring{H}(t)\big)|^2},\phi\Big\rangle - \Big\langle\frac{\eta(\theta)}{\rho C_h}\frac{|\mathcal{E}(\hat w+v_0)|^2}{e^{-2k_1 t}+\varepsilon|\mathcal{E}(\hat w+v_0)|^2},\phi\Big\rangle\Big]\,dt\Big|.
\end{aligned}$$

It is easy to obtain

$$\begin{aligned}
&\frac{1}{\|\phi\|_{L^2(0,T;W^{1,2}_{\Gamma_D})}}\Big|\int_0^T\Big[\langle\kappa(\theta)(\nabla\hat\theta+\nabla\theta_0),\nabla\phi\rangle + \langle\beta(x)(\hat\theta+\theta_0),\phi\rangle_{\Gamma_R}\Big]\,dt\Big| \\
&\le c\big(\|\hat\theta\|_{L^2(0,T;W^{1,2}_{\Gamma_D})} + K\big),
\end{aligned}$$

where K depends on θ_0 and κ_1. In much the same way as (4.39), we obtain

$$\begin{aligned}
&\frac{c}{\|\phi\|_{L^2(0,T;W^{1,2}_{\Gamma_D})}}\int_0^T \|\hat w+v_0\|_{\mathbf{L}^6}\|\hat\theta+\theta_0\|^{1/2}_{L^2}\|\hat\theta+\theta_0\|^{1/2}_{W^{1,2}_{\Gamma_D}}\|\nabla\phi\|_{\mathbf{L}^2}\,dt \\
&\le \|\hat\theta+\theta_0\|^{1/2}_{C([0,T];L^2)}\frac{1}{\|\phi\|_{L^2(0,T;W^{1,2}_{\Gamma_D})}}\int_0^T\|\hat w+v_0\|_{\mathbf{L}^6}\|\hat\theta+\theta_0\|^{1/2}_{W^{1,2}_{\Gamma_D}}\|\nabla\phi\|_{\mathbf{L}^2}\,dt \\
&\le c\|\hat\theta+\theta_0\|_{C([0,T];L^2)} + \|\hat w+v_0\|^2_{L^6(0,T;\mathbf{V})}\|\hat\theta+\theta_0\|_{L^{3/2}(0,T;W^{1,2}_{\Gamma_D})} \\
&\le c\|\hat\theta'\|^{1/2}_{L^2(0,T;(W^{1,2}_{\Gamma_D})^*)}\|\hat\theta\|^{1/2}_{L^2(0,T;W^{1,2}_{\Gamma_D})} + \|\theta_0\| \\
&\quad + \|\hat w+v_0\|^2_{L^6(0,T;\mathbf{V})}\|\hat\theta+\theta_0\|_{L^{3/2}(0,T;W^{1,2}_{\Gamma_D})} \\
&\le \frac{1}{2}\|\hat\theta'\|_{L^2(0,T;(W^{1,2}_{\Gamma_D})^*)} + c\|\hat\theta\|_{L^2(0,T;W^{1,2}_{\Gamma_D})} + \|\theta_0\| \\
&\quad + \|\hat w+v_0\|^4_{L^6(0,T;\mathbf{V})} + \|\hat\theta+\theta_0\|^2_{L^{3/2}(0,T;W^{1,2}_{\Gamma_D})} \\
&\le \frac{1}{2}\|\hat\theta'\|_{L^2(0,T;(W^{1,2}_{\Gamma_D})^*)} + c\Big(\|\hat\theta\|^2_{L^2(0,T;W^{1,2}_{\Gamma_D})} + \|\hat w\|^4_{L^6(0,T;\mathbf{V})} + K\Big),
\end{aligned}$$

where K depends on v_0, θ_0. It is easy to obtain

$$\begin{aligned}
&\frac{1}{\|\phi\|_{L^2(0,T;W^{1,2}_{\Gamma_D})}}\int_0^T\Big|\Big\langle\frac{1}{\rho C_h\sigma(\theta)}\frac{|\operatorname{curl}\big(\bar H(t)+\mathring{H}(t)\big)|^2}{1+\varepsilon|\operatorname{curl}\big(\bar H(t)+\mathring{H}(t)\big)|^2} \\
&+ \frac{\eta(\theta)}{\rho C_h}\frac{|\mathcal{E}(\hat w+v_0)|^2}{e^{-2k_1 t}+\varepsilon|\mathcal{E}(\hat w+v_0)|^2},\phi\Big\rangle\Big|\,dt \le \frac{c}{\varepsilon}.
\end{aligned}$$

The three estimations above yield

$$I_5 \le \frac{1}{2}\|\hat\theta'\|_{L^2(0,T;(W^{1,2}_{\Gamma_D})^*)} + c\Big(\|\hat\theta\|^2_{L^2(0,T;W^{1,2}_{\Gamma_D})} + \|\hat w\|^4_{L^6(0,T;\mathbf{V})} + K + \frac{c_1}{\varepsilon}\Big). \tag{4.42}$$

Owing to (4.36), (4.39)-(4.42), from (4.9) we have

$$
\begin{aligned}
&\|\mathscr{A}_\varepsilon(\hat w,\hat z,\hat\theta)\|_{\mathscr{V}^*}\\
&= \sup_{\|(u,C,\phi)\|_{\mathscr{V}}=1} \big\langle \mathscr{A}_\varepsilon(\hat w,\hat z,\hat\theta),(u,C,\phi)\big\rangle\\
&\le c\Big(\|\hat w\|^5_{L^6(0,T;\mathbf{V})}+\|\hat z\|^5_{L^6(0,T;\mathbf{V}_\Sigma)}+\|\hat\theta\|^2_{L^2(0,T;W^{1,2}_{\Gamma_D})}+K+\frac{1}{\varepsilon}+\frac{1}{\sqrt{\varepsilon}}\Big)\\
&\quad+\frac12\|\mathscr{L}(\hat w,\hat z,\hat\theta)\|_{\mathscr{V}^*},
\end{aligned}
\tag{4.43}
$$

which shows the property (4.3) for $\mathscr{A}_\varepsilon(\hat w,\hat z,\hat\theta)$.

(iii) Let us prove property (i) of Proposition 4.1 for $\mathscr{A}_\varepsilon(\hat w,\hat z,\hat\theta)$. Let $\{(\hat w_k,\hat z_k,\hat\theta_k)\}\subset D(\mathscr{L})$ be a sequence such that

$$
\begin{gathered}
(\hat w_k,\hat z_k,\hat\theta_k)\rightharpoonup(\hat w,\hat z,\hat\theta)\in D(\mathscr{L})\quad\text{in } \mathscr{V},\\
\mathscr{L}(\hat w_k,\hat z_k,\hat\theta_k)\rightharpoonup\mathscr{L}(\hat w,\hat z,\hat\theta)\quad\text{in } \mathscr{V}^*,\\
\limsup_{k\to\infty}\Big\langle \mathscr{A}_\varepsilon(\hat w_k,\hat z_k,\hat\theta_k),(\hat w_k,\hat z_k,\hat\theta_k)-(\hat w,\hat z,\hat\theta)\Big\rangle\le 0.
\end{gathered}
\tag{4.44}
$$

In what follows $w_k=\hat w_k+w_0$, $z_k=\hat z_k+\bar H_0$ and $\theta_k=\hat\theta_k+\theta_0$. Then, by (4.9),

$$
\begin{aligned}
&c\min\{\eta_0,\frac{1}{\rho\sigma_1},\kappa_0\}\int_0^T\big(\|\hat w_k(t)-\hat w(t)\|^2_{\mathbf{V}}+\|\hat z_k(t)-\hat z(t)\|^2_{\mathbf{V}_\Sigma}+\|\hat\theta_k(t)-\hat\theta(t)\|^2_{W^{1,2}_{\Gamma_D}}\big)\,dt\\
&=c\min\{\eta_0,\frac{1}{\rho\sigma_1},\kappa_0\}\int_0^T\big(\|w_k(t)-w(t)\|^2_{\mathbf{V}}+\|z_k(t)-z(t)\|^2_{\mathbf{V}_\Sigma}+\|\theta_k(t)-\theta(t)\|^2_{W^{1,2}_{\Gamma_D}}\big)\,dt\\
&\le\int_0^T 2\Big([\eta(\theta_k)+e^{4k_1t}\varepsilon\|w_k\|^4_{\mathbf{V}}]\mathcal{E}(w_k),\mathcal{E}(w_k-w)\Big)\,dt-\int_0^T 2\Big(\eta(\theta_k)\mathcal{E}(w),\mathcal{E}(w_k-w)\Big)\,dt\\
&\quad-\int_0^T 2e^{4k_1t}\varepsilon\|w_k\|^4_{\mathbf{V}}\langle\mathcal{E}(w_k),\mathcal{E}(w_k-w)\rangle\,dt\\
&\quad+\int_0^T\Big\langle[\frac{1}{\rho\sigma(\theta_k)}+\frac{\mu\varepsilon}{\rho}e^{4k_1t}\|\operatorname{curl} z_k\|^4]\operatorname{curl} z_k,\operatorname{curl}(z_k-z)\Big\rangle\,dt\\
&\quad+\int_0^T\Big\langle\frac{\mu\varepsilon}{\rho}e^{4k_1t}\|\operatorname{curl} z_k\|^4\operatorname{curl} z_k,\operatorname{curl}(z-z_k)\Big\rangle\,dt-\int_0^T\Big\langle\frac{1}{\rho\sigma(\theta_k)}\operatorname{curl} z,\operatorname{curl}(z_k-z)\Big\rangle\,dt\\
&\quad+\int_0^T\big(\kappa(\theta_k)\nabla\theta_k,\nabla\theta_k-\nabla\theta\big)\,dt-\int_0^T\big(\kappa(\theta_k)\nabla\theta,\nabla\theta_k-\nabla\theta\big)\,dt\\
&\le\langle\mathscr{A}_\varepsilon(\hat w_k,\hat z_k,\hat\theta_k),(w_k-w,z_k-z,\theta_k-\theta)\rangle-\int_0^T 2\Big(\eta(\theta_k)\mathcal{E}(w),\mathcal{E}(w_k-w)\Big)\,dt\\
&\quad+\int_0^T 2e^{4k_1t}\varepsilon\Big\langle\|w_k\|^4_{\mathbf{V}}\mathcal{E}(w_k),\mathcal{E}(w-w_k)\Big\rangle\,dt-\int_0^T e^{k_1t}\langle\operatorname{curl} w_k\times w_k,w_k-w\rangle\,dt\\
&\quad-\int_0^T\Big[\langle k_1(w_k,w_k-w\rangle+2\langle\eta(\theta_k)k(x)w_k,w_k-w\rangle_{\Gamma_2}+2\langle(\eta(\theta_k)S(\tilde w_k,\tilde w_k-\tilde w\rangle_{\Gamma_3}\\
&\quad+2\langle\alpha(x)w_k,w_k-w\rangle_{\Gamma_5}+\langle\eta(\theta_k)k(x)w_k,w_k-w\rangle_{\Gamma_7}\Big]\,dt\\
&\quad+\int_0^T\Big[\frac{\mu}{\rho}\Big\langle e^{k_1t}\operatorname{curl} z_k(t)\times z_k(t)+\operatorname{curl} z_k(t)\times\mathring{H}(t)+\operatorname{curl}\mathring{H}(t)\times z_k(t)\\
&\quad-\frac{e^{-k_1t}\alpha_0\theta_k}{1+\varepsilon{\theta_k}^2}\mathbf{g},w_k-w\Big\rangle\Big]\,dt\\
&\quad+\int_0^T\Big\langle\frac{\mu\varepsilon}{\rho}e^{4k_1t}\|\operatorname{curl} z_k\|^4\operatorname{curl} z_k,\operatorname{curl}(z-z_k)\Big\rangle\,dt
\end{aligned}
$$

$$
\begin{aligned}
&- \int_0^T \Big\langle \frac{1}{\rho\sigma(\theta_k)} \operatorname{curl} z_k, \operatorname{curl}(z_k - z) \Big\rangle \, dt \\
&- \int_0^T \Big[\Big\langle \frac{e^{-k_1 t}}{\rho\sigma(\theta_k)} \operatorname{curl} \mathring{H}(t), \operatorname{curl}(z_k - z) \Big\rangle + k_1 \langle z_k, z_k - z \rangle \Big] \, dt \\
&- \int_0^T \Big[\Big\langle \frac{\mu}{\rho} e^{k_1 t} \operatorname{curl}(z_k - z) \times z_k(t), w_k(t) \Big\rangle + \Big\langle \frac{\mu}{\rho} \operatorname{curl}(z_k - z) \times \mathring{H}(t), w_k(t) \Big\rangle \Big] \, dt \\
&- \int_0^T \Big[\big\langle \kappa(\theta_k) \nabla\theta, \nabla\theta_k - \nabla\theta \big\rangle + \big\langle \beta(x)\theta_k, \theta_k - \theta \big\rangle_{\Gamma_R} - e^{k_1 t} \langle w_k \theta_k, \nabla(\theta_k - \theta) \rangle \\
&- \Big\langle \frac{1}{\rho C_h \sigma(\theta_k)} \frac{|\operatorname{curl}\big(\bar{H}_k(t) + \mathring{H}(t)\big)|^2}{1 + \varepsilon |\operatorname{curl}\big(\bar{H}_k(t) + \mathring{H}(t)\big)|^2} + \frac{\eta(\theta_k)}{\rho C_h} \frac{|\mathcal{E}(w_k)|^2}{e^{-2k_1 t} + \varepsilon |\mathcal{E}(w_k)|^2}, \theta_k - \theta \Big\rangle \Big] \, dt.
\end{aligned} \tag{4.45}
$$

Let us estimate the right-hand side of the above expression By (4.44) (see [20, Corollary 1.1]), we see that

$$
\int_0^T 2\Big(\eta(\theta_k) \mathcal{E}(w), \mathcal{E}(w_k - w) \Big) \, dt \to 0. \tag{4.46}
$$

We have

$$
\begin{aligned}
&\lim_{k\to\infty} \sup \int_0^T 2\Big(e^{4k_1 t} \varepsilon \|w_k\|_{\mathbf{V}}^4 \mathcal{E}(w_k), \mathcal{E}(w - w_k) \Big) \, dt \\
&\le \int_0^T 2e^{4k_1 t} \varepsilon \lim_{k\to\infty} \sup \|w_k\|_{\mathbf{V}}^4 \cdot \lim_{k\to\infty} \sup \Big(\mathcal{E}(w_k), \mathcal{E}(w - w_k) \Big) \, dt \\
&\le \int_0^T 2e^{4k_1 t} \varepsilon \lim_{k\to\infty} \sup \|w_k\|_{\mathbf{V}}^4 \cdot \Big(\|\mathcal{E}(w)\|^2 - \lim_{k\to\infty} \inf \|\mathcal{E}(w_k)\|^2 \Big) \, dt \\
&\le \int_0^T 2e^{4k_1 t} \varepsilon \lim_{k\to\infty} \sup \|w_k\|_{\mathbf{V}}^4 \cdot \Big(\|\mathcal{E}(w)\|^2 - \|\mathcal{E}(w)\|^2 \Big) \, dt = 0.
\end{aligned} \tag{4.47}
$$

By taking a subsequence and denoting with same subindex if necessary, from (4.44) we obtain

$$
\begin{gathered}
w_k \to w \quad \text{in } L^6(0,T; \mathbf{W}^{\alpha,2}(\Omega)) \quad (9/10 < \alpha < 1), \\
z_k \to z \quad \text{in } L^6(0,T; \mathbf{W}^{\alpha,2}(\Omega)) \quad (9/10 < \alpha < 1), \\
\theta_k \to \theta \quad \text{in } L^2(0,T; W^{\alpha,2}(\Omega)) \quad (9/10 < \alpha < 1).
\end{gathered} \tag{4.48}
$$

Then, we have

$$
\begin{aligned}
&\int_0^T e^{k_1 t} \langle \operatorname{curl} w_k \times w_k, w_k - w \rangle \, dt \\
&\le c \|w_k\|_{\mathbf{L}^4(Q)} \|\nabla w_k\|_{\mathbf{L}^2(Q)} \|w_k - w\|_{\mathbf{L}^4(Q)} \to 0 \quad \text{as } k \to \infty.
\end{aligned} \tag{4.49}
$$

Also, by trace theorem and Remark 3.4 of [20] we have

$$
\begin{aligned}
&- \int_0^T \Big[(k_1(\hat{w}_k, w_k - w) + 2(\mu(\hat{\theta}_k) k(x) \hat{w}_k, w_k - w)_{\Gamma_2} + 2(\mu(\hat{\theta}_k) S \tilde{\hat{w}}_k, \tilde{w}_k - \tilde{w})_{\Gamma_3} \\
&+ 2(\alpha(x)\hat{w}_k, w_k - w)_{\Gamma_5} + (\mu(\hat{\theta}_k) k(x) \hat{w}_k, w_k - w)_{\Gamma_7} + (\beta(x)\hat{\theta}_k, \theta_k - \theta)_{\Gamma_R} \Big] \, dt \to 0
\end{aligned} \tag{4.50}
$$

as $k \to \infty$. Let us prove that

$$
\begin{aligned}
I_6 \equiv \int_0^T \frac{\mu}{\rho} \Big\langle & e^{k_1 t} \operatorname{curl} z_k(t) \times z_k(t) + \operatorname{curl} z_k(t) \times \mathring{H}(t) + \operatorname{curl} \mathring{H}(t) \times z_k(t) \\
&+ e^{-k_1 t} \frac{\alpha_0 \theta_k}{1 + \varepsilon {\theta_k}^2} \mathbf{g}, w_k - w \Big\rangle \, dt \to 0 \quad \text{as } k \to \infty.
\end{aligned} \tag{4.51}
$$

Since $\{z_k\}$ is bounded in $L^6(0,T; \mathbf{L}^6)$,

$$
\begin{aligned}
I_6 \le \Big[&\|\operatorname{curl} z_k\|_{\mathbf{L}^2(Q)} \|z_k\|_{\mathbf{L}^6(Q)} + \|\operatorname{curl} z_k\|_{\mathbf{L}^2(Q)} \|\mathring{H}\|_{\mathbf{L}^6(Q)} \\
&+ \|\operatorname{curl} \mathring{H}\|_{\mathbf{L}^2(Q)} \|z_k\|_{\mathbf{L}^6(Q)} + \frac{g}{\sqrt{\varepsilon}} \Big] \|w_k - w\|_{L^3(Q)} \to 0,
\end{aligned}
$$

which implies (4.51). In much the same way as for (4.47), we obtain

$$\lim_{k\to\infty}\sup\int_0^T \Big\langle \frac{\mu\varepsilon}{\rho}e^{4k_1 t}\|\operatorname{curl} z_k\|^4 \operatorname{curl} z_k, \operatorname{curl}(z_k - z)\Big\rangle\, dt \le 0. \tag{4.52}$$

Let us estimate

$$\begin{aligned} I_7 \equiv -\int_0^T \Big[&\Big\langle \frac{1}{\rho\sigma(\theta_k)}\operatorname{curl}(z_k(t)), \operatorname{curl}(z_k - z)\Big\rangle \\ &+ \Big\langle e^{-k_1 t}\frac{1}{\rho\sigma(\theta_k)}\operatorname{curl}\mathring{H}(t), \operatorname{curl}(z_k - z)\Big\rangle + k_1\langle z_k, z_k - z\rangle\Big]\, dt. \end{aligned}\tag{4.53}$$

By [20, Corollaries 1.1 and 1.2], we obtain

$$\begin{aligned} &-\liminf \Big\langle \frac{1}{\rho\sigma(\theta_k)}\operatorname{curl}(z_k(t)), \operatorname{curl}(z_k(t))\Big\rangle \\ &\le -\Big\langle \frac{1}{\rho\sigma(\theta)}\operatorname{curl}(z(t)), \operatorname{curl}(z(t))\Big\rangle, \Big\langle \frac{1}{\rho\sigma(\theta_k)}\operatorname{curl}(z_k(t)), \operatorname{curl}(z(t))\Big\rangle \\ &\to \Big\langle \frac{1}{\rho\sigma(\theta)}\operatorname{curl}(z(t)), \operatorname{curl}(z(t))\Big\rangle. \end{aligned}\tag{4.54}$$

Taking into account (4.54), from (4.53) we have

$$\liminf_{k\to\infty} I_7 \le 0. \tag{4.55}$$

It is easy to prove

$$-\int_0^T \Big[\Big\langle e^{-k_1 t}\frac{1}{\rho\sigma(\theta_k)}\operatorname{curl}\mathring{H}(t), \operatorname{curl}(z_k - z)\Big\rangle + k_1\langle z_k, z_k - z\rangle\Big]\, dt \to 0 \quad \text{as } k\to\infty. \tag{4.56}$$

Let us prove

$$I_8 \equiv -\int_0^T \Big[\Big\langle \frac{\mu}{\rho}e^{k_1 t}\operatorname{curl}(z_k - z)\times z_k(t), w_k(t)\Big\rangle + \Big\langle \frac{\mu}{\rho}\operatorname{curl}(z_k - z)\times \mathring{H}(t), w_k(t)\Big\rangle\Big]\, dt \to 0 \tag{4.57}$$

as $k\to\infty$. Taking into account (4.48), the fact that $z_i w_j \in L^3(Q)$ and $\operatorname{curl}(z_k - z) \rightharpoonup 0$ in $\mathbf{L}^2(0,T;\mathbf{L}^2)$, we have

$$\begin{aligned} &\int_0^T \Big\langle \frac{\mu}{\rho}e^{k_1 t}\operatorname{curl}(z_k - z)\times z_k(t), w_k(t)\Big\rangle\, dt \\ &\le c\|\operatorname{curl}(z_k - z)\|_{\mathbf{L}^2(Q)}\|z_k\|_{\mathbf{L}^6(Q)}\|w_k(t) - w(t)\|_{\mathbf{L}^3(Q)} \\ &\quad + c\|\operatorname{curl}(z_k - z)\|_{\mathbf{L}^2(Q)}\|(z_k - z)\|_{\mathbf{L}^3(Q)}\|w(t)\|_{\mathbf{L}^6(Q)} \\ &\quad + c\int_0^T \Big|\Big\langle \operatorname{curl}(z_k - z)\times z(t), w(t)\Big\rangle\Big|\, dt \rightharpoonup 0. \end{aligned}$$

In a similar way we have

$$\int_0^T \Big|\Big\langle \frac{\mu}{\rho}\operatorname{curl}(z_k - z)\times \mathring{H}(t), w_k(t)\Big\rangle\Big|\, dt \rightharpoonup 0.$$

The 2 formulas above give (4.57).

Let us prove

$$\begin{aligned} I_9 \equiv -\int_0^T \Big[&\Big\langle \kappa(\theta_k)\nabla\theta, \nabla\theta_k - \nabla\theta\Big\rangle + \Big\langle \beta(x)\theta_k, \theta_k - \theta\Big\rangle_{\Gamma_R} - e^{k_1 t}\langle w_k\theta_k, \nabla(\theta_k - \theta)\rangle \\ &- \Big\langle \frac{1}{\rho C_h\sigma(\theta)}\frac{|\operatorname{curl}\big(\bar{H}_k(t) + \mathring{H}(t)\big)|^2}{1 + \varepsilon|\operatorname{curl}\big(\bar{H}_k(t) + \mathring{H}(t)\big)|^2}, \theta_k - \theta\Big\rangle \\ &- \Big\langle \frac{\eta(\theta_k)}{\rho C_h}\frac{|\mathcal{E}(w_k)|^2}{e^{-2k_1 t} + \varepsilon|\mathcal{E}(w_k)|^2}, \theta_k - \theta\Big\rangle\Big]\, dt \quad \to 0. \end{aligned}\tag{4.58}$$

It is easy to verify that

$$\int_0^T \Big[\big\langle \kappa(\theta_k)\nabla\theta, \nabla\theta_k - \nabla\theta\big\rangle + \big\langle \beta(x)\theta_k, \theta_k - \theta\big\rangle_{\Gamma_R}\Big]\,dt \to 0 \quad \text{as } k\to\infty. \tag{4.59}$$

(See [20, Corollary 1.1] and (4.48).) Let us verify that

$$\int_0^T e^{k_1 t}\langle w_k\theta_k, \nabla(\theta_k-\theta)\rangle\,dt \to 0. \tag{4.60}$$

Since $\{\theta_k\}$ is bounded in $L^2(0,T;W^{1,2}(\Omega))\cap L^\infty(0,T;L^2(\Omega))$, by [20, Theorem 1.33 and (1.22)] it is bounded in $L^3(0,T;W^{2/3,2}(\Omega))$, and $\{\theta_k\}$ is relatively compact in $L^3(Q)$. Then, by taking a subsequence and denoting with same subindex if necessary, we have that $\theta_k\to\theta$ in $L^3(Q)$. By taking into account (3.9), we have

$$\int_0^T \Big| e^{k_1 t}\langle w_k\theta_k, \nabla(\theta_k-\theta)\rangle\Big|\,dt \le c\|w_k\|_{\mathbf{L}^6(Q)}\|\theta_k-\theta\|_{L^3(Q)}\|\nabla\theta_k\|_{\mathbf{L}^2(Q)} \to 0,$$

which yields (4.60). Taking into account $\theta_k\to\theta$ in $L^3(Q)$ for the subsequence, we can see easily the convergence of the last two terms of I_9 to zero. Thus, (4.59)-(4.60) implies (4.58).

Taking into account (4.46)-(4.47), (4.49)-(4.52), (4.55)-(4.58) and the last formula of (4.44), from (4.45) we have

$$\begin{aligned}
&c\min\{\eta_0, \frac{1}{\rho\sigma_0}, \kappa_0\}\lim_{k\to\infty}\sup\int_0^T \big(\|\hat w_k(t)-\hat w(t)\|^2_{\mathbf{V}} + \|\hat z_k(t)-\hat z(t)\|^2_{\mathbf{V}_\Sigma} + \|\hat\theta_k(t)-\hat\theta(t)\|^2_{W^{1,2}_{\Gamma_D}}\big)\,dt\\
&\le \lim_{k\to\infty}\sup\big\langle \mathscr{A}_\varepsilon(\hat w_k,\hat z_k,\hat\theta_k), (w_k-w, z_k-z, \theta_k-\theta)\big\rangle \le 0,
\end{aligned}$$

which implies that

$$\begin{gathered}
\hat w_k \to \hat w \ \text{ in } L^2(0,T;\mathbf{V}),\\
\hat z_k \to \hat z \ \text{ in } L^2(0,T;\mathbf{V}_\Sigma),\\
\hat\theta_k \to \hat\theta \ \text{ in } L^2(0,T;W^{1,2}_{\Gamma_D}),\\
\nabla\hat w_k\to\nabla\hat w, \quad \nabla\hat z_k\to\nabla\hat z, \quad \nabla\hat\theta_k\to\nabla\hat\theta \quad \text{a.e. in } Q.
\end{gathered} \tag{4.61}$$

When $(u,C,\phi)\in\mathscr{V}$, by definition of $\mathscr{A}_\varepsilon$, we have

$$\begin{aligned}
&\big\langle \mathscr{A}_\varepsilon(\hat w_k,\hat z_k,\hat\theta_k), (\hat w_k-u, \hat z_k-C, \hat\theta_k-\phi)\big\rangle\\
&= \int_0^T \Big[2\big\langle [\eta(\theta_k) + e^{4k_1 t}\varepsilon\|\hat w_k+v_0\|^4_{\mathbf{V}}]\mathcal{E}(\hat w_k+v_0), \mathcal{E}(\hat w_k-u)\big\rangle\\
&\quad + e^{k_1 t}\langle \operatorname{curl}(\hat w_k+v_0)\times(\hat w_k+v_0), \hat w_k-u\rangle + k_1(\hat w_k+v_0, \hat w_k-u)\\
&\quad + 2\langle \eta(\theta_k)k(x)(\hat w_k+v_0), \hat w_k-u\rangle_{\Gamma_2} + 2\langle \eta(\theta_k)S(\tilde{\hat w}_k+\tilde v_0), \tilde{\hat w}_k-\tilde u\rangle_{\Gamma_3}\\
&\quad + 2\langle \alpha(x)(\hat w_k+v_0), \hat w_k-u\rangle_{\Gamma_5} + \langle \eta(\theta_k)k(x)(\hat w_k+v_0), \hat w_k-u\rangle_{\Gamma_7}\\
&\quad - \frac{\mu}{\rho}\Big\langle e^{k_1 t}\operatorname{curl}(\hat z_k(t)+\bar H_0)\times(\hat z_k(t)+\bar H_0) + \operatorname{curl}(\hat z_k(t)+\bar H_0)\times\mathring{H}(t)\\
&\quad + \operatorname{curl}\mathring{H}(t)\times(\hat z_k(t)+\bar H_0), \hat w_k-u\Big\rangle + e^{-k_1 t}\Big\langle \frac{\alpha_0(\hat\theta_k+\theta_0)}{1+\varepsilon|\hat\theta_k+\theta_0|^2}\mathbf{g}, \hat w_k-u\Big\rangle\\
&\quad + \Big\langle [\frac{1}{\rho\sigma(\theta_k)} + \frac{\mu\varepsilon}{\rho}e^{4k_1 t}\|\operatorname{curl}(\hat z_k(t)+\bar H_0)\|^4]\operatorname{curl}(\hat z_k(t)+\bar H_0), \operatorname{curl}(\hat z_k-C)\Big\rangle\\
&\quad + \Big\langle e^{-k_1 t}\frac{1}{\rho\sigma(\theta_k)}\operatorname{curl}\mathring{H}(t), \operatorname{curl}(\hat z_k-C)\Big\rangle + k_1\langle(\bar z_k+\bar H_0), (\hat z_k-C)\rangle\\
&\quad + \Big\langle \frac{\mu}{\rho}e^{k_1 t}\operatorname{curl}(\hat z_k-C)\times(\hat z_k(t)+\bar H_0), (\hat w_k(t)+v_0)\Big\rangle\\
&\quad + \Big\langle \frac{\mu}{\rho}\operatorname{curl}(\hat z_k-C)\times\mathring{H}(t), (\hat w_k(t)+v_0)\Big\rangle + \langle \kappa(\theta_k)(\nabla\hat\theta_k+\nabla\theta_0), \nabla(\hat\theta_k-\phi)\rangle\\
&\quad + \langle \beta(x)(\hat\theta_k+\theta_0), (\hat\theta_k-\phi)\rangle_{\Gamma_R} - e^{k_1 t}\langle(\hat w_k+v_0)(\hat\theta_k+\theta_0), \nabla(\hat\theta_k-\phi)\rangle
\end{aligned}$$

$$
\begin{aligned}
&- \Big\langle \frac{1}{\rho C_h \sigma(\theta_k)} \frac{|\operatorname{curl}\big(\bar H_k(t) + \mathring{H}(t)\big)|^2}{1+\varepsilon|\operatorname{curl}\big(\bar H_k(t) + \mathring{H}(t)\big)|^2}, \hat\theta_k - \phi \Big\rangle \\
&- \Big\langle \frac{\eta(\theta_k)}{\rho C_h} \frac{|\mathcal{E}(\hat w_k + v_0)|^2}{e^{-2k_1 t} + \varepsilon |\mathcal{E}(\hat w_k + v_0)|^2}, \hat\theta_k - \phi \Big\rangle \Big]\, dt.
\end{aligned} \tag{4.62}
$$

By [20, Corollary 1.1] and (4.61),

$$
\lim_{k\to\infty} \int_0^T 2\big(\eta(\theta_k)\mathcal{E}(\hat w_k + v_0), \mathcal{E}(\hat w_k - u)\, dt = \int_0^T 2\big(\eta(\theta)\mathcal{E}(\hat w + v_0), \mathcal{E}(\hat w - u)\, dt. \tag{4.63}
$$

The sequence $\{e^{4k_1 t}\|\hat w_k + v_0\|_{\mathbf{V}}^4 \varepsilon_{ij}(\hat w_k + v_0)\}$ is bounded in $L^{6/5}(0,T;\mathbf{L}^2(\Omega))$ and $\varepsilon_{ij}(u) \in L^6(0,T;\mathbf{L}^2(\Omega))$, and by taking a subsequence and denoting with same subindex, we know that

$$
\begin{aligned}
&\lim_{k\to\infty} \int_0^T 2e^{4k_1 t} \Big\langle \|\hat w_k + v_0\|_{\mathbf{V}}^4 \mathcal{E}(\hat w_k + v_0), \mathcal{E}(u) \Big\rangle\, dt \\
&\to \int_0^T 2e^{4k_1 t} \Big\langle \|\hat w + v_0\|_{\mathbf{V}}^4 \mathcal{E}(\hat w + v_0), \mathcal{E}(u) \Big\rangle\, dt.
\end{aligned} \tag{4.64}
$$

Since $w_k \rightharpoonup w$ in $L^6(0,T;\mathbf{V})$ and (4.61), there exist subsequences, respectively, (which is expressed as before with sub-indexes k) such that $w_k(t) \rightharpoonup w(t)$ and $\langle \mathcal{E}(\hat w_k + v_0)(t), \mathcal{E}(\hat w_k)(t)\rangle \to \langle \mathcal{E}(\hat w + v_0)(t), \mathcal{E}(\hat w)(t)\rangle$ for a.e. $t \in [0,T]$ (cf. [22, Sec. 2, Ch. VII]). Thus

$$
\begin{aligned}
&\liminf_{k\to\infty} \int_0^T \|\hat w_k + v_0\|_{\mathbf{V}}^4 \big(\mathcal{E}(\hat w_k + v_0)(t), \mathcal{E}(\hat w_k)(t)\big)\, dt \\
&\geq \int_0^T \|\hat w + v_0\|_{\mathbf{V}}^4 \big(\mathcal{E}(\hat w + v_0), \mathcal{E}(\hat w)\big)\, dt.
\end{aligned} \tag{4.65}
$$

(See [5, Exercises 2, (c) in pp. 173].)

By (4.63)-(4.65), we have

$$
\begin{aligned}
&\liminf_{k\to\infty} \int_0^T 2\Big\langle [\eta(\theta_k) + e^{4k_1 t}\varepsilon \|\hat w_k + v_0\|_{\mathbf{V}}^4 \mathcal{E}(\hat w_k + v_0), \mathcal{E}(\hat w_k - u) \Big\rangle\, dt \\
&\geq \int_0^T 2\Big\langle [\eta(\theta) + e^{4k_1 t}\varepsilon \|\hat w + v_0\|_{\mathbf{V}}^4] \mathcal{E}(\hat w + v_0), \mathcal{E}(\hat w - u) \Big\rangle\, dt.
\end{aligned} \tag{4.66}
$$

Since $e^{k_1 t}\langle \operatorname{curl}(\hat w_k + v_0) \times (\hat w_k + v_0), \hat w_k - u\rangle = e^{k_1 t}\langle \operatorname{curl}(\hat w_k + v_0) \times (\hat w_k + v_0), -v_0 - u\rangle$, taking into account the fact that $\operatorname{curl}(\hat w_k + v_0) \to \operatorname{curl}(\hat w + v_0)$ in $\mathbf{L}^2(Q)$ ((4.61)), $\hat w_k + v_0 \to \hat w + v_0$ in $\mathbf{L}^3(Q)$ ((4.48)) and $-v_0 - u \in \mathbf{L}^6(Q)$, we can see that

$$
\begin{aligned}
&\int_0^T e^{k_1 t}\langle \operatorname{curl}(\hat w_k + v_0) \times (\hat w_k + v_0), \hat w_k - u\rangle\, dt \\
&\to \int_0^T e^{k_1 t}\langle \operatorname{curl}(\hat w + v_0) \times (\hat w + v_0), \hat w - u\rangle\, dt.
\end{aligned} \tag{4.67}
$$

Let us consider

$$
\begin{aligned}
&\int_0^T -\frac{\mu}{\rho}\Big\langle e^{k_1 t} \operatorname{curl}(\hat z_k(t) + \bar H_0) \times (\hat z_k(t) + \bar H_0), \hat w_k - u \Big\rangle\, dt \\
&= \int_0^T -\frac{\mu}{\rho}\Big\langle e^{k_1 t} \operatorname{curl}(\hat z_k(t) + \bar H_0) \times (\hat z_k(t) + \bar H_0), \hat w_k - \hat w \Big\rangle\, dt \\
&\quad + \int_0^T -\frac{\mu}{\rho}\Big\langle e^{k_1 t} \operatorname{curl}(\hat z_k(t) + \bar H_0) \times (\hat z_k(t) + \bar H_0), \hat w - u \Big\rangle\, dt.
\end{aligned}
$$

Since $\operatorname{curl}(\hat z_k + \bar H_0)$ and $(\hat z_k(t) + \bar H_0)$ are bounded in $\mathbf{L}^2(Q)$ and $\mathbf{L}^6(Q)$, respectively, and $\hat w_k \to \hat w$ in $\mathbf{L}^3(Q)$ ((4.48)), the first term of the right hand side of the above equality converges to zero.

Also, since $\operatorname{curl}(\hat z_k+\bar H_0)\to\operatorname{curl}(\hat z+\bar H_0)$ in $\mathbf{L}^2(Q)$ ((4.61)), $(\hat z_k+\bar H_0)\to(\hat z+\bar H_0)$ in $\mathbf{L}^3(Q)$ ((4.48)) and $\hat w-u\in\mathbf{L}^6(Q)$, we have for a subsequence (which is denoted with the same subindex k)

$$\begin{aligned}&\int_0^T -\frac{\mu}{\rho}\Big\langle e^{k_1t}\operatorname{curl}(\hat z_k(t)+\bar H_0)\times(\hat z_k(t)+\bar H_0),\hat w_k-u\Big\rangle\,dt\\&\to\int_0^T -\frac{\mu}{\rho}\Big\langle e^{k_1t}\operatorname{curl}(\hat z(t)+\bar H_0)\times(\hat z(t)+\bar H_0),\hat w-u\Big\rangle\,dt.\end{aligned}\tag{4.68}$$

In much the same way as above, we obtain

$$\begin{aligned}&\int_0^T\frac{\mu}{\rho}\Big\langle\operatorname{curl}(\hat z_k(t)+\bar H_0)\times\mathring{H}(t)+\operatorname{curl}\mathring{H}(t)\times(\hat z_k(t)+\bar H_0),\hat w_k-u\Big\rangle\\&\to\int_0^T\frac{\mu}{\rho}\Big\langle\operatorname{curl}(\hat z(t)+\bar H_0)\times\mathring{H}(t)+\operatorname{curl}\mathring{H}(t)\times(\hat z(t)+\bar H_0),\hat w-u\Big\rangle.\end{aligned}\tag{4.69}$$

Since

$$\frac{\alpha_0(\hat\theta_k+\theta_0)}{1+\varepsilon|\hat\theta_k+\theta_0|^2}\to\frac{\alpha_0(\hat\theta+\theta_0)}{1+\varepsilon|\hat\theta+\theta_0|^2}\quad\text{a.e. in }Q$$

and

$$\frac{\alpha_0(\hat\theta_k+\theta_0)}{1+\varepsilon|\hat\theta_k+\theta_0|^2}\quad\text{is bounded in }L^2(0,T;L^2),$$

we have

$$\frac{\alpha_0(\hat\theta_k+\theta_0)}{1+\varepsilon|\hat\theta_k+\theta_0|^2}\rightharpoonup\frac{\alpha_0(\hat\theta+\theta_0)}{1+\varepsilon|\hat\theta+\theta_0|^2}\quad\text{in }L^2(0,T;L^2)$$

(cf. [23, Lemma 1.3, Ch. 1]). Owing to (4.61) and the above, we have

$$\begin{aligned}&\int_0^T e^{-k_1t}\langle\frac{\alpha_0(\hat\theta_k+\theta_0)}{1+\varepsilon|\hat\theta_k+\theta_0|^2}\mathbf{g},w_k-u\rangle\,dt\\&=\int_0^T e^{-k_1t}\langle\frac{\alpha_0(\hat\theta_k+\theta_0)}{1+\varepsilon|\hat\theta_k+\theta_0|^2}\mathbf{g},w_k-w\rangle\,dt+\int_0^T e^{-k_1t}\langle\frac{\alpha_0(\hat\theta_k+\theta_0)}{1+\varepsilon|\hat\theta_k+\theta_0|^2}\mathbf{g},w-u\rangle\,dt\\&\to\int_0^T e^{-k_1t}\langle\frac{\alpha_0(\hat\theta+\theta_0)}{1+\varepsilon|\hat\theta+\theta_0|^2}\mathbf{g},w-u\rangle\,dt.\end{aligned}\tag{4.70}$$

In much the same way as (4.66), we obtain

$$\begin{aligned}&\liminf_{k\to\infty}\int_0^T\Big\langle[\frac{1}{\rho\sigma(\theta_k)}+\frac{\mu\varepsilon}{\rho}e^{4k_1t}\|\operatorname{curl}(\hat z_k(t)+\bar H_0)\|^4]\operatorname{curl}(\hat z_k(t)+\bar H_0),\operatorname{curl}(\hat z_k-C)\Big\rangle\,dt\\&\ge\int_0^T\Big\langle[\frac{1}{\rho\sigma(\theta)}+\frac{\mu\varepsilon}{\rho}e^{4k_1t}\|\operatorname{curl}(\hat z(t)+\bar H_0)\|^4]\operatorname{curl}(\hat z(t)+\bar H_0),\operatorname{curl}(\hat z-C)\Big\rangle\,dt.\end{aligned}\tag{4.71}$$

This is rewritten as

$$\begin{aligned}&\int_0^T\Big\langle\frac{\mu}{\rho}e^{k_1t}\operatorname{curl}(\hat z_k-C)\times(\hat z_k(t)+\bar H_0),(\hat w_k(t)+v_0)\Big\rangle\,dt\\&=\int_0^T\Big\langle\frac{\mu}{\rho}e^{k_1t}\operatorname{curl}(\hat z_k-\hat z)\times(\hat z_k(t)+\bar H_0),(\hat w_k(t)+v_0)\Big\rangle\,dt\\&\quad+\int_0^T\Big\langle\frac{\mu}{\rho}e^{k_1t}\operatorname{curl}(\hat z-C)\times(\hat z_k(t)+\bar H_0),(\hat w_k(t)+v_0)\Big\rangle\,dt.\end{aligned}$$

By the same argument as (4.68), for a subsequence (which is denoted with the same subindex k), we have

$$\begin{aligned}&\int_0^T\Big\langle\frac{\mu}{\rho}e^{k_1t}\operatorname{curl}(\hat z_k-C)\times(\hat z_k(t)+\bar H_0),(\hat w_k(t)+v_0)\Big\rangle\,dt\\&\to\int_0^T\Big\langle\frac{\mu}{\rho}e^{k_1t}\operatorname{curl}(\hat z-C)\times(\hat z(t)+\bar H_0),(\hat w(t)+v_0)\Big\rangle\,dt.\end{aligned}\tag{4.72}$$

Similarly, taking a subsequence, we have

$$\begin{gathered}\int_0^T \Big\langle \frac{\mu}{\rho}\operatorname{curl}(\hat z_k - C)\times \mathring{H}(t), (\hat w_k(t)+v_0)\Big\rangle\,dt \to \int_0^T \Big\langle \frac{\mu}{\rho}\operatorname{curl}(\hat z - C)\times \mathring{H}(t), (\hat w(t)+v_0)\Big\rangle\,dt,\\ \int_0^T e^{k_1 t}\langle(\hat w_k+v_0)(\hat\theta_k+\theta_0), \nabla(\hat\theta_k-\phi)\rangle\,dt \to \int_0^T e^{k_1 t}\langle(\hat w+v_0)(\hat\theta+\theta_0), \nabla(\hat\theta-\phi)\rangle\,dt.\end{gathered} \tag{4.73}$$

By [20, Corollary 1.1] and (4.61), we obtain

$$\begin{aligned}\int_0^T \langle \kappa(\theta_k)\nabla\theta_k, \nabla(\hat\theta_k-\phi)\rangle\,dt &= \int_0^T \langle \kappa(\theta_k)\nabla\theta_k, \nabla(\hat\theta_k-\hat\theta)\rangle\,dt - \int_0^T \langle \kappa(\theta_k)\nabla\theta_k, \nabla(\hat\theta-\phi)\rangle\,dt\\ &\to \int_0^T (\kappa(\theta)\nabla\theta, \nabla(\hat\theta-\phi))\,dt.\end{aligned} \tag{4.74}$$

In much the same arguments as for (4.70) and (4.71), we have

$$\begin{aligned}&\int_0^T \Big\langle \frac{1}{\rho C_h\sigma(\theta_k)}\frac{|\operatorname{curl}\big(\bar H_k(t)+\mathring{H}(t)\big)|^2}{1+\varepsilon|\operatorname{curl}\big(\bar H_k(t)+\mathring{H}(t)\big)|^2}, \hat\theta_k-\phi\Big\rangle\,dt\\ &\to \int_0^T \Big\langle \frac{1}{\rho C_h\sigma(\theta)}\frac{|\operatorname{curl}\big(\bar H(t)+\mathring{H}(t)\big)|^2}{1+\varepsilon|\operatorname{curl}\big(\bar H(t)+\mathring{H}(t)\big)|^2}, \hat\theta-\phi\Big\rangle\,dt,\\ &\int_0^T \Big\langle \frac{\eta(\theta_k)}{\rho C_h}\frac{|\mathcal{E}(\hat w_k+v_0)|^2}{e^{-2k_1t}+\varepsilon|\mathcal{E}(\hat w_k+v_0)|^2}, \hat\theta_k-\phi\Big\rangle\,dt\\ &\to \int_0^T \Big\langle \frac{\eta(\theta)}{\rho C_h}\frac{|\mathcal{E}(\hat w+v_0)|^2}{e^{-2k_1t}+\varepsilon|\mathcal{E}(\hat w+v_0)|^2}, \hat\theta-\phi\Big\rangle\,dt.\end{aligned} \tag{4.75}$$

It is easy to prove convergence of other terms in the right hand side of (4.62). Thus, by (4.62) and (4.66)-(4.75), we have the existence of a subsequence $\{(\hat w_k, \hat z_k, \hat\eta_k)\}$ such that

$$\begin{aligned}&\liminf_{k\to\infty} \big\langle \mathscr{A}_\varepsilon(\hat w_k, \hat z_k, \hat\theta_k), (\hat w_k-u, \hat z_k-C, \hat\theta_k-\phi)\big\rangle\\ &\ge \big\langle \mathscr{A}_\varepsilon(\hat w, \hat z, \hat\theta), (\hat w-u, \hat z-C, \hat\theta-\phi)\big\rangle,\end{aligned}$$

by which property (i) of Proposition 4.1 for $\mathscr{A}_\varepsilon(w,\theta)$ is proved.

Therefore, by Proposition 4.1 there exists a solution to Problem VEA', which gives us the conclusion of Theorem 4.2. □

4.2. **Estimates of solutions to the approximate problem.** Let $0<\varepsilon<1$. For $v_0\in H_{\mathbf V}$, $\bar H_0\in H_\Sigma$, $\theta_0\in L^1(\Omega)$, there exist $v_{0\varepsilon}\in\mathbf V$, $\bar H_{0\varepsilon}\in\mathbf V_\Sigma$, $\theta_{0\varepsilon}\in W^{1,2}_{\Gamma_D}$, such that

$$\|v_0-v_{0\varepsilon}\|_{H_{\mathbf V}}\le\varepsilon,\quad \|\bar H_0-\bar H_{0\varepsilon}\|_{H_\Sigma}\le\varepsilon,\quad \|\theta_0-\theta_{0\varepsilon}\|_{L^1(\Omega)}\le\varepsilon. \tag{4.76}$$

Let $(v_\varepsilon(t), \bar H_\varepsilon(t), \theta_\varepsilon(t))$ be a solution to Problem VEA with the initial function $(v_{0\varepsilon}, \bar H_{0\varepsilon}, \theta_{0\varepsilon})$ by Theorem 4.2.

We define $A_\varepsilon(\theta):\mathbf V\to\mathbf V^*$ by

$$\langle A_\varepsilon(\theta)v, w\rangle = a_0(\theta; v, w) + 2\varepsilon\|v\|^4_{\mathbf V}\big\langle\mathcal{E}(v), \mathcal{E}(w)\big\rangle + \langle\operatorname{curl} v\times v, w\rangle\quad \forall v, w\in\mathbf V,$$

where $a_0(\theta; v, w)$ is as in (3.10). Them there exists $c_*\ge 0$ such that

$$|\nu(k(x)z,z)_{\Gamma_2} + 2\nu(S\tilde z,\tilde z)_{\Gamma_3} + (\alpha(x)z,z)_{\Gamma_5} + \nu(k(x)z,z)_{\Gamma_7}| \le \eta_0\|z\|^2_{\mathbf V} + c_*\|z\|^2\quad \forall z\in\mathbf V \tag{4.77}$$

(cf. [15, Theorem 1.5.1.10]).

Remark 4.4. Note that c_* depends on shape of $\Gamma_i, i=2,3,7$, and the norm of matrix α on Γ_5, and so for the fixed domain Ω it depends only on $|\alpha|$.

By (2.5), (3.10), (4.77), we have

$$\begin{aligned}&\langle A_\varepsilon(\theta_\varepsilon)z, z\rangle \ge \eta_0\|z\|^2_{\mathbf V} + 2\varepsilon\|z\|^6_{\mathbf V} - c_*\|z\|^2,\quad \forall z\in\mathbf V,\\ &|\langle A_\varepsilon(\theta_\varepsilon)z, w\rangle| \le c_2\big(\|z\|_{\mathbf V} + \varepsilon\|z\|^5_{\mathbf V}\big)\|w\|_{\mathbf V}\quad \forall z, w\in\mathbf V.\end{aligned} \tag{4.78}$$

Putting $u = v_\varepsilon$ in the first equation of (4.1) and taking (4.78), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v_\varepsilon(t)\|^2 + \eta_0\|v_\varepsilon(t)\|^2_{\mathbf{V}} + 2\varepsilon\|v_\varepsilon(t)\|^6_{\mathbf{V}} - c_*\|v_\varepsilon(t)\|^2 - \Big[\langle\frac{\mu}{\rho}\operatorname{curl}\bar{H}_\varepsilon(t)\times\bar{H}_\varepsilon(t), v_\varepsilon(t)\rangle \\
&+ \langle\frac{\mu}{\rho}\operatorname{curl}\bar{H}_\varepsilon(t)\times\mathring{H}(t), v_\varepsilon(t)\rangle + \langle\frac{\mu}{\rho}\operatorname{curl}\mathring{H}(t)\times\bar{H}_\varepsilon(t), v_\varepsilon(t)\rangle\Big] + \langle\frac{\alpha_0\theta_\varepsilon}{1+\varepsilon\theta_\varepsilon^2}\mathbf{g}, v_\varepsilon(t)\rangle \\
&\le \langle\mathbf{f}_1(t), v_\varepsilon(t)\rangle.
\end{aligned} \tag{4.79}$$

Putting $C = \bar{H}_\varepsilon$ and multiplying $\frac{\mu}{\rho}$, from the second equation of (4.1), we obtain

$$\begin{aligned}
&\frac{\mu}{2\rho}\frac{d}{dt}\|\bar{H}_\varepsilon(t)\|^2 + \Big\langle\big[\frac{1}{\rho\sigma(\theta_\varepsilon)} + \varepsilon\|\operatorname{curl}\bar{H}_\varepsilon(t)\|^4\big]\operatorname{curl}\bar{H}_\varepsilon(t), \operatorname{curl}\bar{H}_\varepsilon(t)\Big\rangle \\
&+ \frac{1}{\mu\sigma(\theta_\varepsilon)}\big\langle\operatorname{curl}\mathring{H}(t), \operatorname{curl}\bar{H}_\varepsilon\big\rangle + \big\langle\frac{\mu}{\rho}\operatorname{curl}\bar{H}_\varepsilon(t)\times\bar{H}_\varepsilon(t), v_\varepsilon(t)\big\rangle \\
&+ \big\langle\frac{\mu}{\rho}\operatorname{curl}\bar{H}_\varepsilon(t)\times\mathring{H}(t), v_\varepsilon(t)\big\rangle \\
&= \langle\frac{\mu}{\rho}\mathbf{r}_1(t), \bar{H}_\varepsilon(t)\rangle.
\end{aligned} \tag{4.80}$$

Adding (4.79) and (4.80), and integrating on $[0,t]$ yield

$$\begin{aligned}
&\|v_\varepsilon(t)\|^2 + \frac{\mu}{\rho}\|\bar{H}_\varepsilon(t)\|^2 + 2\int_0^t \big(\eta_0\|v_\varepsilon\|^2_{\mathbf{V}} + 2\varepsilon\|v_\varepsilon\|^6_{\mathbf{V}}\big)\,ds \\
&+ 2\int_0^t \big(\frac{1}{\rho\sigma(\theta_\varepsilon)}\|\bar{H}_\varepsilon(t)\|^2_{\mathbf{V}_\Sigma} + \varepsilon\|\bar{H}_\varepsilon(t)\|^6_{\mathbf{V}_\Sigma}\big)\,ds - 2\int_0^t\langle\frac{\mu}{\rho}\operatorname{curl}\mathring{H}(t)\times\bar{H}_\varepsilon(t), v_\varepsilon(t)\rangle\,ds \\
&+ 2\int_0^t\langle\frac{\alpha_0\theta_\varepsilon}{1+\varepsilon\theta_\varepsilon^2}\mathbf{g}, v_\varepsilon(t)\rangle\,ds + 2\int_0^t \frac{1}{\mu\sigma(\theta_\varepsilon)}\big\langle\operatorname{curl}\mathring{H}(t), \operatorname{curl}\bar{H}_\varepsilon(t)\big\rangle\,ds \\
&\le \|v_{0\varepsilon}\|^2 + \frac{\mu}{\rho}\|\bar{H}_{0\varepsilon}\|^2 + 2\int_0^t c_*\|v_\varepsilon(s)\|^2\,ds + 2\int_0^t\big(\langle\mathbf{f}_1(s), v_\varepsilon(t)\rangle + \langle\frac{\mu}{\rho}\mathbf{r}_1(s), \bar{H}_\varepsilon(t)\rangle\big)\,ds.
\end{aligned} \tag{4.81}$$

Taking into account the first one of (4.38), by Hölder's inequality, we have

$$\begin{gathered}
\begin{aligned}
2\Big|\frac{\mu}{\rho}\big\langle\operatorname{curl}\mathring{H}\times\bar{H}_\varepsilon, v_\varepsilon\big\rangle\Big| &\le c\|\bar{H}_\varepsilon\|_{\mathbf{V}_\Sigma}\|\operatorname{curl}\mathring{H}\|\|v_\varepsilon\|^{1/2}_{\mathbf{V}}\|v_\varepsilon\|^{1/2} \\
&\le \frac{1}{4\rho\sigma_1}\|\bar{H}_\varepsilon\|^2_{\mathbf{V}_\Sigma} + \frac{\eta_0}{4}\|v_\varepsilon\|^2_{\mathbf{V}} + c_{\rho\sigma\eta 1}\|\operatorname{curl}\mathring{H}\|^4\|v_\varepsilon\|^2,
\end{aligned} \\
2|\langle\frac{\alpha_0\theta_\varepsilon}{1+\varepsilon\theta_\varepsilon^2}\mathbf{g}, v_\varepsilon(t)\rangle| \le c|\alpha_0\mathbf{g}|\|\theta_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)}\|v_\varepsilon\|_{\mathbf{V}}, \\
\begin{aligned}
2\Big|\frac{1}{\rho\sigma(\theta_\varepsilon)}\big\langle\operatorname{curl}\mathring{H}, \operatorname{curl}\bar{H}_\varepsilon\big\rangle\Big| &\le c\|\operatorname{curl}\mathring{H}\|\|\bar{H}_\varepsilon\|_{\mathbf{V}_\Sigma} \\
&\le \frac{1}{4\rho\sigma_1}\|\bar{H}_\varepsilon\|^2_{\mathbf{V}_\Sigma} + c_{\rho\sigma 1}\|\operatorname{curl}\mathring{H}\|^2,
\end{aligned} \\
2|\langle\mathbf{f}_1(t), v_\varepsilon\rangle| \le \frac{\eta_0}{4}\|v_\varepsilon\|^2_{\mathbf{V}} + c_\eta\|\mathbf{f}_1(t)\|^2_{\mathbf{V}^*} \quad \text{(see (3.13))}, \\
2\langle\frac{\mu}{\rho}\mathbf{r}_1(t), \bar{H}_\varepsilon(t)\rangle \le \frac{1}{4\rho\sigma_1}\|\bar{H}_\varepsilon\|^2_{\mathbf{V}_\Sigma} + c_{\rho\sigma 2}\|\mathbf{r}_1(t)\|^2_{\mathbf{V}^*_\Sigma}.
\end{gathered} \tag{4.82}$$

Taking into account (4.78) and (4.82), from (4.81), we have

$$
\begin{aligned}
&\|v_\varepsilon(t)\|^2+\frac{\mu}{\rho}\|\bar H_\varepsilon(t)\|^2+\int_0^t\Big(\frac{3}{2}\eta_0\|v_\varepsilon\|_{\mathbf V}^2+4\varepsilon\|v_\varepsilon\|_{\mathbf V}^6\Big)\,ds\\
&+\int_0^t\big(\frac{5}{4\rho\sigma_1}\|\bar H_\varepsilon(t)\|_{\mathbf V_\Sigma}^2+2\varepsilon\|\bar H_\varepsilon(t)\|_{\mathbf V_\Sigma}^6\big)\,ds\\
&\le\|v_{0\varepsilon}\|^2+\frac{\mu}{\rho}\|\bar H_{0\varepsilon}\|^2+\int_0^t\big(2c_*+c_{\rho\sigma\eta1}\|\operatorname{curl}\mathring H\|^4\big)\|v_\varepsilon\|^2\,ds\\
&\quad+c|\alpha_0\mathbf g|\int_0^t\|\theta_\varepsilon\|_{L^{\frac65}(\Omega)}\|v_\varepsilon\|_{\mathbf V}\,ds+c_{\rho\sigma1}\int_0^t\|\operatorname{curl}\mathring H\|^2\,ds\\
&\quad+c_\eta\int_0^t\|\mathbf f_1(t)\|_{\mathbf V^*}^2\,ds+c_{\rho\sigma2}\int_0^t\|\mathbf r_1(t)\|_{\mathbf V_\Sigma^*}^2\,ds.
\end{aligned}\tag{4.83}
$$

For a fixed $0<\delta<1$, we define

$$
\begin{gathered}
\Phi(\xi):=\Big(1-\frac{1}{(1+|\xi|)^\delta}\Big)\operatorname{sign}\xi,\quad \xi\in\mathbb R,\\
\Psi(\xi):=\begin{cases}\int_0^\xi\Phi(\tau)\,d\tau & \text{for } \xi\ge0\\ -\int_\xi^0\Phi(\tau)\,d\tau & \text{for } \xi<0.\end{cases}
\end{gathered}
$$

(cf. [24]). Then

$$
\begin{gathered}
\Phi'(\xi)=\frac{\delta}{(1+|\xi|)^{1+\delta}}\quad\forall\xi\in\mathbb R,\\
|\Phi(\xi)|\le1,\quad 0\le\Phi'(\xi)\le\delta\quad\forall\xi\in\mathbb R,\\
\Psi'(\xi)=\Phi(\xi),\\
\Psi(\xi)=|\xi|+\frac{1}{1-\delta}\big(1-(1+|\xi|)^{1-\delta}\big)\quad\forall\xi\in\mathbb R,\\
\frac{|\xi|}{2}-\frac{2^{(1-\delta)/\delta}}{1-\delta}\le\Psi(\xi)\le|\xi|\quad\forall\xi\in\mathbb R.
\end{gathered}\tag{4.84}
$$

Since $\theta_\varepsilon\in L^2(0,T;W^{1,2}_{\Gamma_D}(\Omega))\cap C([0,T];L^2(\Omega))$, we see that

$$
\begin{gathered}
\Phi(\theta_\varepsilon(t))\in W^{1,2}_{\Gamma_D}(\Omega),\quad \Psi(\theta_\varepsilon(t))\in W^{1,2}_{\Gamma_D}(\Omega)\quad\text{for a.a. } t\in[0,T],\\
\langle v_\varepsilon\theta_\varepsilon,\nabla\Phi(\theta_\varepsilon(t))\rangle=-\langle(v_\varepsilon\cdot\nabla\theta_\varepsilon),\Phi(\theta_\varepsilon(t))\rangle=-\langle v_\varepsilon,\nabla\Psi(\theta_\varepsilon(t))\rangle=\langle\operatorname{div}v_\varepsilon,\Psi(\theta_\varepsilon(t))\rangle=0,
\end{gathered}\tag{4.85}
$$

where $v\cdot n|_{\Gamma_R}=0$(cf. (2.3)) and $\Phi(\theta_\varepsilon)=0$ on Γ_D were used. Also

$$
\begin{gathered}
\int_0^t\big\langle\frac{\partial\theta_\varepsilon}{\partial t},\Phi(\theta_\varepsilon)\big\rangle\,ds=\int_\Omega\Psi(\theta_\varepsilon(t))\,dx-\int_\Omega\Psi(\theta_{0\varepsilon})\,dx\quad\forall t\in[0,T],\\
(\kappa(\theta_\varepsilon)\nabla\theta_\varepsilon,\nabla\Phi(\theta_\varepsilon))=\int_\Omega\kappa(\theta_\varepsilon)|\nabla\theta_\varepsilon|^2\Phi'(\theta_\varepsilon)\,dx,\\
(\beta(x)\theta_\varepsilon,\Phi(\theta_\varepsilon))_{\Gamma_R}\ge0.
\end{gathered}\tag{4.86}
$$

Putting $\varphi(t)=\Phi(\theta_\varepsilon(t))$ in the third equation of (4.1) and using (4.85), (4.86), we have

$$
\begin{aligned}
&\int_\Omega\Psi(\theta_\varepsilon(t))\,dx+\int_0^t\int_\Omega\kappa(\theta_\varepsilon)|\nabla\theta_\varepsilon|^2\Phi'(\theta_\varepsilon)\,dx+\int_0^t(\beta(x)\theta_\varepsilon,\Phi(\theta_\varepsilon))_{\Gamma_R}\,ds\\
&=\int_\Omega\Psi(\theta_{0\varepsilon})\,dx+\int_0^t\Big\langle\frac{1}{\rho C_h\sigma(\theta_\varepsilon)}\frac{|\operatorname{curl}(\bar H_\varepsilon+\mathring H)|^2}{1+\varepsilon|\operatorname{curl}(\bar H_\varepsilon+\mathring H)|^2},\Phi(\theta_\varepsilon)\Big\rangle\,dt\\
&\quad+\int_0^t\Big\langle\frac{\eta(\theta_\varepsilon)}{\rho C_h}\frac{|\mathcal E(v_\varepsilon)|^2}{1+\varepsilon|\mathcal E(v_\varepsilon)|^2},\Phi(\theta_\varepsilon)\Big\rangle\,ds+\int_0^t\big\langle g_1(t),\Phi(\theta_\varepsilon)\big\rangle\,ds.
\end{aligned}\tag{4.87}
$$

By (4.84), (4.86), from (4.87), we have

$$\begin{aligned}&\frac{1}{2}\|\theta_\varepsilon(t)\|_{L^1(\Omega)}-\frac{2^{(1-\delta)/\delta}}{1-\delta}\operatorname{meas}\Omega+\delta\int_0^t\int_\Omega\kappa_0\frac{|\nabla\theta_\varepsilon|^2}{(1+|\theta_\varepsilon|)^{1+\delta}}\,dxds\\&\le\|\theta_{0\varepsilon}\|_{L^1(\Omega)}+\frac{2}{\rho C_h\sigma_0}\int_0^t\|\bar H\|^2_{\mathbf{V}_\Sigma}\,dt+\frac{2}{\rho C_h\sigma_0}\int_0^t\|\mathring{H}\|^2_{\mathbf{H}^1}\,dt\\&\quad+\frac{\eta_1}{\rho C_h}\int_0^t\|v_\varepsilon\|^2_{\mathbf{V}}\,ds+\int_0^t\|g_1(t)\|_{(W^{1,2}_{\Gamma_D})^*}\,ds.\end{aligned}\tag{4.88}$$

Putting

$$\beta=\min\Big\{\frac{\rho C_h}{\eta_1}\cdot\frac{\eta_0}{2},\quad\frac{\rho C_h\sigma_0}{2}\cdot\frac{1}{4\rho\sigma_1},\quad 1\Big\}\tag{4.89}$$

and summing (4.83) and (4.88) multiplied by β, we have

$$\begin{aligned}&\|v_\varepsilon(t)\|^2+\frac{\mu}{\rho}\|\bar H_\varepsilon(t)\|^2+\frac{\beta}{2}\|\theta_\varepsilon(t)\|_{L^1(\Omega)}+\int_0^t\Big(\eta_0\|v_\varepsilon\|^2_{\mathbf{V}}+4\varepsilon\|v_\varepsilon\|^6_{\mathbf{V}}\Big)\,ds\\&+\int_0^t\big(\frac{1}{\rho\sigma_1}\|\bar H_\varepsilon(t)\|^2_{\mathbf{V}_\Sigma}+2\varepsilon\|\bar H_\varepsilon(t)\|^6_{\mathbf{V}_\Sigma}\big)\,ds+\delta\beta\int_0^t\int_\Omega\kappa_0\frac{|\nabla\theta_\varepsilon|^2}{(1+|\theta_\varepsilon|)^{1+\delta}}\,dxds\\&\le\Lambda_{\delta\varepsilon}+\int_0^t\big(2c_*+c_{\rho\sigma\eta1}\|\operatorname{curl}\mathring{H}\|^4\big)\|v_\varepsilon\|^2\,ds+c|\alpha_0\mathbf{g}|\int_0^t\|\theta_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)}\|v_\varepsilon\|_{\mathbf{V}}\,ds,\end{aligned}\tag{4.90}$$

where

$$\begin{aligned}\Lambda_{\delta\varepsilon}=c\Big(&\frac{2^{(1-\delta)/\delta}}{1-\delta}\operatorname{meas}\Omega+\|v_{0\varepsilon}\|^2+\|\bar H_{0\varepsilon}\|^2+\|\theta_{0\varepsilon}\|_{L^1(\Omega)}+\int_0^T\|\mathring{H}\|^2_{\mathbf{H}^1}\,dt\\&+\int_0^T\|\operatorname{curl}\mathring{H}\|^2\,dt+\int_0^T\|\mathbf{f}_1(t)\|^2_{\mathbf{V}^*}\,dt+\int_0^T\|\mathbf{r}_1(t)\|^2_{\mathbf{V}^*_\Sigma}\,dt+\int_0^T\|g_1(t)\|_{(W^{1,2}_{\Gamma_D})^*}\,dt\Big).\end{aligned}$$

Taking into account (4.76) and putting

$$\begin{aligned}\Lambda_\delta=c\Big(&\frac{2^{(1-\delta)/\delta}}{1-\delta}\operatorname{meas}\Omega+1+\|v_0\|^2+\|\bar H_0\|^2+\|\theta_0\|_{L^1(\Omega)}+\int_0^T\|\mathring{H}\|^2_{\mathbf{H}^1}\,dt\\&+\int_0^T\|\operatorname{curl}\mathring{H}\|^2\,dt+\int_0^T\|\mathbf{f}_1(t)\|^2_{\mathbf{V}^*}\,dt+\int_0^T\|\mathbf{r}_1(t)\|^2_{\mathbf{V}^*_\Sigma}\,dt+\int_0^T\|g_1(t)\|_{(W^{1,2}_{\Gamma_D})^*}\,dt\Big),\end{aligned}\tag{4.91}$$

we obtain $\Lambda_{\delta\varepsilon}<\Lambda_\delta$. By Hölder's inequality and Young's inequality,

$$\begin{aligned}c|\alpha_0\mathbf{g}|\int_0^t\|\theta_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)}\|v_\varepsilon\|_{\mathbf{V}}\,ds&\le c|\alpha_0\mathbf{g}|\Big(\int_0^t\|\theta_\varepsilon\|^2_{L^{6/5}}\,ds\Big)^{1/2}\Big(\int_0^t\|v_\varepsilon\|^2_{\mathbf{V}}\,ds\Big)^{1/2}\\&\le|\alpha_0|c_{\eta_0}\int_0^t\|\theta_\varepsilon\|^2_{L^{6/5}}\,ds+\frac{\eta_0}{2}\int_0^t\|v_\varepsilon\|^2_{\mathbf{V}}\,ds.\end{aligned}\tag{4.92}$$

By complex interpolation with exponent $\frac{13}{18}$ between L^1 and $L^{5/2}$(cf. [20, Theorem 1.12]),

$$\|\theta_\varepsilon\|_{L^{6/5}(\Omega)}\le\|\theta_\varepsilon\|^{13/18}_{L^1(\Omega)}\|\theta_\varepsilon\|^{5/18}_{L^{5/2}(\Omega)}.\tag{4.93}$$

Taking into account (4.91)-(4.93), from (4.90), we have

$$\begin{aligned}&\|v_\varepsilon(t)\|^2+\frac{\mu}{\rho}\|\bar H_\varepsilon(t)\|^2+\frac{\beta}{2}\|\theta_\varepsilon(t)\|_{L^1(\Omega)}+\int_0^t\Big(\frac{\eta_0}{2}\|v_\varepsilon\|^2_{\mathbf{V}}+4\varepsilon\|v_\varepsilon\|^6_{\mathbf{V}}\Big)\,ds\\&+\int_0^t\Big(\frac{1}{\rho\sigma_1}\|\bar H_\varepsilon(t)\|^2_{\mathbf{V}_\Sigma}+2\varepsilon\|\bar H_\varepsilon(t)\|^6_{\mathbf{V}_\Sigma}\Big)\,ds+\delta\beta\kappa_0\int_0^t\int_\Omega\frac{|\nabla\theta_\varepsilon|^2}{(1+|\theta_\varepsilon|)^{1+\delta}}\,dxds\\&\le\Lambda_\delta+c^*\int_0^t\|v_\varepsilon\|^2\,ds+|\alpha_0|c_{\eta_0}\int_0^t\|\theta_\varepsilon\|^{13/9}_{L^1}\|\theta_\varepsilon\|^{5/9}_{L^{5/2}}\,ds,\end{aligned}\tag{4.94}$$

where $c^*=2c_*+c_{\rho\sigma\eta1}\|\mathring{H}\|^4_{L^\infty(0,T;\mathbf{H}^1(\Omega))}$. For $0<\zeta<1$ we define

$$\eta_\varepsilon:=\frac{|\theta_\varepsilon|}{(1+|\theta_\varepsilon|)^{(1+\zeta)/2}}\quad\text{a.e. in }Q.$$

Then, by elementary calculation we see that

$$(1+|\theta_\varepsilon|)^{(1-\zeta)/2} \le 1+\eta_\varepsilon, \quad |\nabla\eta_\varepsilon| \le \frac{|\nabla\theta_\varepsilon|}{(1+|\theta_\varepsilon|)^{(1+\zeta)/2}} \quad \text{a.e. in } Q. \tag{4.95}$$

In fact, adding $|\theta_\varepsilon|$ and dividing by $(1+|\theta_\varepsilon|)^{(1+\zeta)/2}$, from

$$1 \le (1+|\theta_\varepsilon|)^{(1+\zeta)/2}$$

we obtain the first inequality of (4.95). From

$$\nabla\eta_\varepsilon = \frac{\text{sign}\theta_\varepsilon(1+|\theta_\varepsilon|)^{(1+\zeta)/2} - |\theta_\varepsilon|\cdot\frac{1+\zeta}{2}(1+|\theta_\varepsilon|)^{(-1+\zeta)/2}\text{sign}\theta_\varepsilon}{(1+|\theta_\varepsilon|)^{(1+\zeta)}}\nabla\theta_\varepsilon,$$

we obtain

$$|\nabla\eta_\varepsilon| \le |\nabla\theta_\varepsilon|\frac{1-|\theta_\varepsilon|\cdot\frac{1+\zeta}{2}(1+|\theta_\varepsilon|)^{-1}}{(1+|\theta_\varepsilon|)^{(1+\zeta)/2}} \le \frac{|\nabla\theta_\varepsilon|}{(1+|\theta_\varepsilon|)^{(1+\zeta)/2}},$$

which is the second part of (4.95).

Taking $\zeta=\frac{1}{6}$, from the first inequality of (4.95) we obtain $|\theta_\varepsilon|^{\frac{5}{2}\cdot\frac{1}{6}} \le 1+\eta_\varepsilon$, which yields

$$\|\theta_\varepsilon\|^{5/2}_{L^{5/2}(\Omega)} \le 2^6\Big(\text{meas}\,\Omega + \|\eta_\varepsilon\|^6_{L^6(\Omega)}\Big).$$

Taking $\sqrt[3]{\ }$ in two sides of the above and using $|a|+|b| \le \big(|a|^{\frac{1}{3}}+|b|^{\frac{1}{3}}\big)^3$, we obtain

$$\|\theta_\varepsilon\|^{5/6}_{L^{5/2}(\Omega)} \le 2^2\Big((\text{meas}\,\Omega)^{\frac{1}{3}} + \|\eta_\varepsilon\|^2_{L^6(\Omega)}\Big),$$

which yields

$$\|\theta_\varepsilon(s)\|^{5/6}_{L^{5/2}(\Omega)} \le c_6(1+\|\eta_\varepsilon(s)\|^2_{L^6(\Omega)}).$$

Thus, from (4.94), we have

$$\begin{aligned}
&\|v_\varepsilon(t)\|^2 + \frac{\mu}{\rho}\|\bar H_\varepsilon(t)\|^2 + \frac{\beta}{2}\|\theta_\varepsilon(t)\|_{L^1(\Omega)} + \int_0^t \Big(\frac{\eta_0}{2}\|v_\varepsilon\|^2_{\mathbf V} + 4\varepsilon\|v_\varepsilon\|^6_{\mathbf V}\Big)\,ds \\
&+ \int_0^t \Big(\frac{1}{\rho\sigma_1}\|\bar H_\varepsilon(t)\|^2_{\mathbf V_\Sigma} + 2\varepsilon\|\bar H_\varepsilon(t)\|^6_{\mathbf V_\Sigma}\Big)\,ds + \delta\beta\kappa_0\int_0^t\int_\Omega \frac{|\nabla\theta_\varepsilon|^2}{(1+|\theta_\varepsilon|)^{1+\delta}}\,dxds \\
&\le \Lambda_\delta + c^*\int_0^t \|v_\varepsilon(s)\|^2\,ds + |\alpha_0|c_{\eta_0}\int_0^t \|\theta_\varepsilon(s)\|^{13/9}_{L^1(\Omega)}(1+\|\eta\|^2_{L^6(\Omega)})^{2/3}\,ds.
\end{aligned} \tag{4.96}$$

By Gronwall's inequality, from (4.96), we have

$$\|v_\varepsilon(t)\|^2 \le \Big(\Lambda_\delta + |\alpha_0|c_{\eta_0}\int_0^t \|\theta_\varepsilon(s)\|^{13/9}_{L^1(\Omega)}\big(1+\|\eta\|^2_{L^6(\Omega)}\big)^{2/3}\,ds\Big)e^{c^*t},$$

which yields

$$c^*\int_0^t \|v_\varepsilon(s)\|^2\,ds \le c^*e^{c^*T}T\Big(\Lambda_\delta + |\alpha_0|c_{\eta_0}\int_0^t \|\theta_\varepsilon(s)\|^{13/9}_{L^1(\Omega)}\big(1+\|\eta\|^2_{L^6(\Omega)}\big)^{2/3}\,ds\Big).$$

Taking into account the above, from (4.96), we have

$$\begin{aligned}
&\|v_\varepsilon(t)\|^2 + \frac{\mu}{\rho}\|\bar H_\varepsilon(t)\|^2 + \frac{\beta}{2}\|\theta_\varepsilon(t)\|_{L^1(\Omega)} + \int_0^t \Big(\frac{\eta_0}{2}\|v_\varepsilon\|^2_{\mathbf V} + 4\varepsilon\|v_\varepsilon\|^6_{\mathbf V}\Big)\,ds \\
&+ \int_0^t \Big(\frac{1}{\rho\sigma_1}\|\bar H_\varepsilon(t)\|^2_{\mathbf V_\Sigma} + 2\varepsilon\|\bar H_\varepsilon(t)\|^6_{\mathbf V_\Sigma}\Big)\,ds + \delta\beta\kappa_0\int_0^t\int_\Omega \frac{|\nabla\theta_\varepsilon|^2}{(1+|\theta_\varepsilon|)^{1+\delta}}\,dxds \\
&\le \Lambda_\delta(1+c^*e^{c^*T}T) + (1+c^*e^{c^*T}T)|\alpha_0|c_{\eta_0}\int_0^t \|\theta_\varepsilon(s)\|^{13/9}_{L^1(\Omega)}(1+\|\eta(s)\|^2_{L^6(\Omega)})^{2/3}\,ds.
\end{aligned} \tag{4.97}$$

We put

$$\phi(t) := \|\theta_\varepsilon\|_{C([0,t];L^1(\Omega))} + \|\eta\|^2_{L^2(0,t;L^6(\Omega))}.$$

Taking into account the second inequality of (4.95) and

$$\|\eta_\varepsilon(s)\|_{L^6(\Omega)} \le \bar c\|\nabla\eta_\varepsilon(s)\|_{\mathbf L^2(\Omega)},$$

from (4.97), we have

$$\begin{aligned}
\phi(t) &\le \|\theta_\varepsilon\|_{C([0,t];L^1(\Omega))} + \bar{c}\int_0^t \int_\Omega \frac{|\nabla\theta_\varepsilon|^2}{(1+|\theta_\varepsilon|)^{1+\delta}}\,dxds \\
&\le c_3\Lambda_\delta(1+c^*e^{c^*T}T) + (1+c^*e^{c^*T}T)|\alpha_0|c_4\int_0^t \|\theta_\varepsilon(s)\|_{L^1(\Omega)}^{13/9}(1+\|\eta(s)\|_{L^6(\Omega)}^2)^{\frac{2}{3}}\,ds,
\end{aligned} \tag{4.98}$$

where $c_3 = \max\{\frac{2}{\beta}, \frac{\bar{c}}{\delta\beta\kappa_0}\}$ and $c_4 = c_3 c_{\eta_0}$. Let us estimate the second term of the right hand side of (4.98). Using Hölder's inequality with exponents $3/2, 3$, we have

$$\begin{aligned}
&(1+c^*e^{c^*T}T)|\alpha_0|c_4\int_0^t \|\theta_\varepsilon(s)\|_{L^1(\Omega)}^{13/9}(1+\|\eta(s)\|_{L^6(\Omega)}^2)^{2/3}\,ds \\
&\le (1+c^*e^{c^*T}T)|\alpha_0|c_4\|\theta_\varepsilon(s)\|_{C([0,t];L^1(\Omega))}^{13/9}\Big(\int_0^t(1+\|\eta(s)\|_{L^6(\Omega)}^2)\,ds\Big)^{2/3} t^{1/3} \\
&\le (1+c^*e^{c^*T}T)|\alpha_0|c_4T^{1/3}\|\theta_\varepsilon(s)\|_{C([0,t];L^1(\Omega))}^{13/9}\big(t+\|\eta(s)\|_{L^2(0,t;L^6(\Omega))}^2\big)^{2/3} \\
&\le (1+c^*e^{c^*T}T)|\alpha_0|c_4T^{1/3}\big(t+\|\phi(t)\|\big)^{19/9}.
\end{aligned} \tag{4.99}$$

Therefore, from (4.98), we have

$$t+\phi(t) \le K_0 + (1+c^*e^{c^*T}T)|\alpha_0|c_4T^{1/3}\big(t+\phi(t)\big)^{19/9}, \tag{4.100}$$

where

$$K_0 = c_3\Lambda_\delta(1+c^*e^{c^*T}T) + T.$$

Now we use the following lema.

Lemma 4.5 ([26, Lemma 1 in Appendix])**.** *Let $\Psi : [0,T] \to [0,+\infty)$ be a continuous non-decreasing function such that*

$$\begin{gathered}
\Psi(0) \le K_0, \\
\Psi(t) \le K_0 + \beta\Psi(t)^\gamma \quad \forall t \in [0,T],
\end{gathered}$$

where $K_0 = const > 0$, $\gamma = const > 1$ and $\beta = \frac{1}{3}(2K_0)^{1-\gamma}$. Then

$$\Psi(t) \le 2K_0 \quad \forall t \in [0,T].$$

By (4.98) we see that $\phi(0) \le K_0$. According to assumption of the main theorem that the buoyancy effect ($|\alpha_0|$) is small enough than the data of problem and K_0 depends to the data of problem, we can assume

$$(1+c^*e^{c^*T}T)\alpha_0c_4T^{1/3} \le \frac{1}{3}(2K_0)^{(1-19/9)}. \tag{4.101}$$

Then, putting $\Psi(t) = t+\phi(t)$ and applying Lemma 4.5 to (4.100), we have

$$t+\phi(t) \le 2K_0 \quad \forall t \in [0,T]. \tag{4.102}$$

Taking into account (4.99), (4.101) and (4.102) in the second term of the right hand side of (4.97), we have

$$\begin{aligned}
&\|v_\varepsilon(t)\|^2 + \frac{\mu}{\rho}\|\bar{H}_\varepsilon(t)\|^2 + \frac{\beta}{2}\|\theta_\varepsilon(t)\|_{L^1(\Omega)} + \int_0^t\Big(\frac{\eta_0}{2}\|v_\varepsilon\|_{\mathbf{V}}^2 + 4\varepsilon\|v_\varepsilon\|_{\mathbf{V}}^6\Big)\,ds \\
&+ \int_0^t\Big(\frac{1}{\rho\sigma_1}\|\bar{H}_\varepsilon(t)\|_{\mathbf{V}_\Sigma}^2 + 2\varepsilon\|\bar{H}_\varepsilon(t)\|_{\mathbf{V}_\Sigma}^6\Big)\,ds + \delta\beta\kappa_0\int_0^t\int_\Omega \frac{|\nabla\theta_\varepsilon|^2}{(1+|\theta_\varepsilon|)^{1+\delta}}\,dxds \\
&\le C(\Lambda_\delta, c^*),
\end{aligned} \tag{4.103}$$

where $C(\Lambda_\delta, c^*)$ is independent from ε and depends on $\delta \in (0,1)$ and c^*.

We will obtain an estimate for θ_ε in $L^r(0,T;W^{1,r})$ $(1 < r < 5/4)$. To this end, we need to obtain an estimate on $\nabla\theta_\varepsilon$. By (4.103), Hölder's inequality with exponents $\frac{2}{r}$, $\frac{2}{2-r}$, inequalities $|a+b|^p \le 2^p(|a|^p + |b|^p)$, $|a|+|b| \le (|a|^{\frac{1}{p}} + |b|^{\frac{1}{p}})^p$, $p \in (1,\infty)$, we have

$$\begin{aligned}\int_Q |\nabla\theta_\varepsilon|^r\,dx\,dt &\le \Big(\int_Q \frac{|\nabla\theta_\varepsilon|^2}{(1+|\theta_\varepsilon|)^{1+\delta}}\,dx\,dt\Big)^{r/2}\Big(\int_Q \big(1+|\theta_\varepsilon|\big)^{r(1+\delta)/(2-r)}\,dx\,dt\Big)^{(2-r)/2}\\ &\le c2^{r(1+\delta)/2}C(\Lambda_\delta,c^*)^{r/2}\Big[(\operatorname{meas} Q)^{(2-r)/2} + \Big(\int_Q |\theta_\varepsilon|^{r(1+\delta)/(2-r)}\,dx\,dt\Big)^{(2-r)/2}\Big],\end{aligned} \tag{4.104}$$

where $C(\Lambda_\delta,c^*)$ is the one in (4.103).

To use the property $W^{1,r}(\Omega) \subset L^q(\Omega)$, let us take q such that $\frac{1}{r} - \frac{1}{3} = \frac{1}{q}$, i.e. $q = \frac{3r}{3-r}$. Let us take $1 > \delta_0 > 0$ such that

$$1 < r(1+\delta_0)/(2-r) < q,$$

which holds if

$$0 < \delta_0 < (3-2r)/(3-r). \tag{4.105}$$

Set $s = r(1+\delta_0)/(2-r)$ and take λ such that

$$\frac{1}{s} = \frac{\lambda}{1} + \frac{1-\lambda}{q}.$$

Then, $\lambda = \frac{q-s}{s(q-1)}$, $0 < \lambda < 1$ and

$$\|\theta_\varepsilon\|_{L^s(\Omega)} \le c\|\theta_\varepsilon\|_{L^1(\Omega)}^{\lambda}\|\theta_\varepsilon\|_{L^q(\Omega)}^{1-\lambda} \le c\|\theta_\varepsilon\|_{L^1(\Omega)}^{\lambda}\|\nabla\theta_\varepsilon\|_{\mathbf{L}^r(\Omega)}^{1-\lambda} \quad \forall\theta_\varepsilon \in W^{1,r}_{\Gamma_D}(\Omega) \tag{4.106}$$

(see [20, Theorem 1.12]). Therefore, by (4.103) and (4.106) we have

$$\int_Q |\theta_\varepsilon|^{r(1+\delta_0)/(2-r)}\,dx\,dt \le cC(\Lambda_{\delta_0},c^*)^{\lambda s}\int_0^T \Big(\int_\Omega |\nabla\theta_\varepsilon|^r\,dx\Big)^{(1-\lambda)s/r}\,dt.$$

We will fix r and δ_0 so that

$$1 < r < \frac{5}{4}, \quad 0 < \delta_0 < \frac{5-4r}{3},$$

which ensures that (4.105). Taking into account $\lambda = \frac{q-s}{s(q-1)}$, $q = \frac{3r}{3-r}$ and $s = \frac{r(1+\delta_0)}{2-r}$, we have

$$(1-\lambda)s/r = \frac{1}{r}\frac{q(s-1)}{q-1} = \frac{3(2r-2+r\delta_0)}{(4r-3)(2-r)} < \frac{3\big(2r-2+r\frac{5-4r}{3}\big)}{(4r-3)(2-r)} = 1.$$

Then, by Hölder's inequality, we have

$$\int_Q |\theta_\varepsilon|^{r(1+\delta_0)/(2-r)}\,dx\,dt \le cC(\Lambda_{\delta_0},c^*)^{\lambda s}\Big(\int_0^T\int_\Omega |\nabla\theta_\varepsilon|^r\,dx\,dt\Big)^{(1-\lambda)s/r},$$

and

$$\Big(\int_Q |\theta_\varepsilon|^{r(1+\delta_0)/(2-r)}\,dx\,dt\Big)^{(2-r)/2} \le cC(\Lambda_{\delta_0},c^*)^{\lambda s(2-r)/2}\Big(\int_Q |\nabla\theta_\varepsilon|^r\,dx\,dt\Big)^{(1-\lambda)s(2-r)/2r}, \tag{4.107}$$

where $\frac{(1-\lambda)s(2-r)}{2r} = \frac{1}{2}(1-\lambda)(1+\delta_0) < 1$. Then, by Young's inequality and (4.107), we obtain

$$2^{r(1+\delta_0)/2}C(\Lambda_{\delta_0},c^*)^{r/2}\Big(\int_Q |\theta_\varepsilon|^{r(1+\delta_0)/(2-r)}\,dx\,dt\Big)^{(2-r)/2} \le \frac{1}{2}\int_Q |\nabla\theta_\varepsilon|^r\,dx\,dt + R(C(\Lambda_{\delta_0},c^*)), \tag{4.108}$$

where $R(\cdot)$ is a nonnegative continuous function on $[0,\infty)$. Substituting (4.108) into (4.104) with $\delta = \delta_0$, we have

$$\int_Q |\nabla\theta_\varepsilon|^r\,dx\,dt \le cC(\Lambda_{\delta_0},c^*)^{r/2} + 2R(C(\Lambda_{\delta_0},c^*)), \quad 1 < r < \frac{5}{4}, \quad 0 < \delta_0 < \frac{5-4r}{3}, \tag{4.109}$$

which yields

$$
\begin{gathered}
\theta_\varepsilon \in L^r(0,T:W^{1,r}_{\Gamma_D}), \quad 1<r<\frac{5}{4},\\
\|\theta_\varepsilon\|_{L^r(0,T:W^{1,r}_{\Gamma_D})} \le \Big(cC(\Lambda_{\delta_0},c^*)^{r/2}+2R(C(\Lambda_{\delta_0},c^*))\Big)^{1/r}, \quad 0<\delta_0<\frac{5-4r}{3}.
\end{gathered}\tag{4.110}
$$

Taking $r=6/5$, from (4.110), we have

$$
\|\theta_\varepsilon\|^{\frac{6}{5}}_{L^{6/5}(0,T;L^2)} \le cC(\Lambda_{\delta_0},c^*)^{3/5}+2R(C(\Lambda_{\delta_0},c^*)).\tag{4.111}
$$

On the other hand, from the first equation of (4.1), we obtain

$$
\begin{aligned}
&\big\langle \frac{\partial v_\varepsilon}{\partial t}(t),u\big\rangle+\big\langle A_\varepsilon(\theta_\varepsilon)v_\varepsilon(t),u\big\rangle+\langle \operatorname{curl} v_\varepsilon(t)\times v_\varepsilon(t),u\rangle-\langle\frac{\mu}{\rho}\operatorname{curl}\bar H_\varepsilon(t)\times\bar H_\varepsilon(t),u\rangle\\
&-\langle\frac{\mu}{\rho}\operatorname{curl}\bar H_\varepsilon(t)\times\mathring{H}(t),u\rangle-\langle\frac{\mu}{\rho}\operatorname{curl}\mathring{H}(t)\times\bar H_\varepsilon(t),u\rangle+\langle\frac{\alpha_0\theta_\varepsilon(t)}{1+\varepsilon\theta_\varepsilon(t)^2}\mathbf{g},u\rangle\\
&=\langle \mathbf{f}_1(t),u(t)\rangle \quad \forall u\in\mathbf{V}.
\end{aligned}
$$

Taking (4.111) implies

$$
\begin{aligned}
\|v'_\varepsilon(t)\|_{\mathbf{V}^*} &\le \|A_\varepsilon(\theta_\varepsilon)v_\varepsilon\|_{\mathbf{V}^*}+\|\operatorname{curl} v_\varepsilon\|_{\mathbf{L}^2}\|v_\varepsilon\|_{\mathbf{L}^3}+\frac{\mu}{\rho}\|\operatorname{curl}\bar H_\varepsilon(t)\|_{\mathbf{L}^2}\|\bar H_\varepsilon(t)\|_{\mathbf{L}^3}\\
&\quad+\frac{\mu}{\rho}\|\operatorname{curl}\bar H_\varepsilon(t)\|_{\mathbf{L}^2}\|\mathring{H}(t)\|_{\mathbf{L}^3}+\frac{\mu}{\rho}\|\operatorname{curl}\mathring{H}(t)\|_{\mathbf{L}^2}\|\bar H_\varepsilon(t)\|_{\mathbf{L}^3}+|\alpha_0\mathbf{g}|\|\theta_\varepsilon\|+\|\mathbf{f}_1(t)\|_{\mathbf{V}^*}.
\end{aligned}
$$

Thus, by (4.103) and (4.111) the right hand side of the next inequality is bounded.

$$
\begin{aligned}
\|v'_\varepsilon\|_{L^1(0,T;\mathbf{V}^*)} &\le c\big(\|v_\varepsilon\|_{L^2(0,T;\mathbf{V})}+\varepsilon\|v_\varepsilon\|^5_{L^5(0,T;\mathbf{V})}+\|v_\varepsilon\|^2_{L^2(0,T;\mathbf{V})}+\|\bar H_\varepsilon\|^2_{L^2(0,T;\mathbf{V}_\Sigma)}\\
&\quad+\|\bar H_\varepsilon\|_{L^2(0,T;\mathbf{V}_\Sigma)}\|\mathring{H}\|_{L^\infty(0,T;\mathbf{L}^3)}+\|\operatorname{curl}\mathring{H}\|_{L^\infty(0,T;\mathbf{L}^2)}\|\bar H_\varepsilon\|_{L^2(0,T;\mathbf{V}_\Sigma)}\\
&\quad+\|\theta_\varepsilon\|_{L^{6/5}(0,T;L^2)}+\|\mathbf{f}_1\|_{L^2(0,T;\mathbf{V}^*)}\big).
\end{aligned}\tag{4.112}
$$

Therefore, by (4.103), there exists v and a subsequence $\{v_{\varepsilon_k}\}$ such that

$$
\begin{gathered}
v_{\varepsilon_k}\overset{*}{\rightharpoonup} v \text{ in } L^\infty(0,T;H),\\
v_{\varepsilon_k}\to v \text{ in } L^2(0,T;\mathbf{W}^{\frac{9}{10},2}(\Omega)),\\
v_{\varepsilon_k}\rightharpoonup v \text{ in } L^2(0,T;\mathbf{V})
\end{gathered}\tag{4.113}
$$

as $\varepsilon_k\to 0$.

From the second equation of (4.1), we obtain

$$
\begin{aligned}
&\langle \bar H'_\varepsilon(t),C\rangle+\Big\langle[\frac{1}{\mu\sigma(\theta)}+\varepsilon\|\operatorname{curl}\bar H(t)\|^4]\operatorname{curl}\bar H(t),\operatorname{curl}C\Big\rangle+\Big\langle\frac{1}{\mu\sigma(\theta_\varepsilon)}\operatorname{curl}\mathring{H}(t),\operatorname{curl}C\Big\rangle\\
&+\big\langle \operatorname{curl}C\times\bar H_\varepsilon(t),v_\varepsilon(t)\big\rangle+\big\langle\operatorname{curl}C\times\mathring{H}(t),v_\varepsilon(t)\big\rangle=\langle\mathbf{r}_1(t),C\rangle\quad\forall C\in\mathbf{V}_\Sigma,
\end{aligned}
$$

which yields

$$
\begin{aligned}
\|\bar H'(t)\|_{\mathbf{V}^*_\Sigma}&\le c\big(\|\bar H(t)\|_{\mathbf{V}_\Sigma}+\varepsilon\|\operatorname{curl}\bar H(t)\|^5_{\mathbf{V}_\Sigma}+\|\operatorname{curl}\mathring{H}(t)\|_{\mathbf{L}^2}\\
&\quad+\|\bar H(t)\|_{\mathbf{L}^6}\|v(t)\|_{\mathbf{L}^3}+\|\mathring{H}(t)\|_{\mathbf{L}^6}\|v(t)\|_{\mathbf{L}^3}+\|\mathbf{r}_1(t)\|_{\mathbf{V}^*_\Sigma}\big).
\end{aligned}
$$

Thus, by (4.103), (2) of Assumption 3.1 and (3.13) we have

$$
\begin{aligned}
\|\bar H'_\varepsilon\|_{L^1(0,T;\mathbf{V}^*_\Sigma)}&\le c\big(\|\bar H_\varepsilon\|_{L^2(0,T;\mathbf{V}_\Sigma)}+\varepsilon\|\operatorname{curl}\bar H(t)\|^5_{L^6(0,T;\mathbf{V}_\Sigma)}+\|\mathring{H}\|_{L^\infty(0,T;\mathbf{L}^2)}\\
&\quad+\|\bar H_\varepsilon\|_{L^2(0,T;\mathbf{V}_\Sigma)}\|v_\varepsilon(t)\|_{L^2(0,T;\mathbf{V})}+\|\mathring{H}\|_{L^\infty(0,T;\mathbf{H}^1)}\|v_\varepsilon\|_{L^2(0,T;\mathbf{V})}\\
&\quad+\|\mathbf{r}_1(t)\|_{L^2(0,T;\mathbf{V}^*_\Sigma)}\big).
\end{aligned}\tag{4.114}
$$

Therefore, by (4.103), there exists $\bar H$ and a subsequence $\{\bar H_{\varepsilon_k}\}$ such that

$$
\begin{gathered}
\bar H_{\varepsilon_k}\overset{*}{\rightharpoonup}\bar H \text{ in } L^\infty(0,T;\mathbf{L}^2(\Omega)),\\
\bar H_{\varepsilon_k}\to\bar H \text{ in } L^2(0,T;\mathbf{W}^{\frac{9}{10},2}(\Omega)),\\
\bar H_{\varepsilon_k}\rightharpoonup\bar H \text{ in } L^2(0,T;\mathbf{V}_\Sigma)
\end{gathered}\tag{4.115}
$$

as $\varepsilon_k \to 0$.

We will obtain an estimate on θ'_ε. As above, let $1 < r < \frac{5}{4}$, and $r' := r/(r-1) > 5$.

$$
\begin{aligned}
|\langle \kappa(\theta_\varepsilon)\nabla\theta_\varepsilon, \nabla\varphi\rangle| &\le \kappa_1 \|\nabla\theta_\varepsilon\|_{\mathbf{L}^r}\|\nabla\varphi\|_{\mathbf{L}^{r'}} \le c\|\nabla\theta_\varepsilon\|_{\mathbf{L}^r}\|\varphi\|_{W^{1,r'}_{\Gamma_D}(\Omega)},\\
|(\beta(x)\theta_\varepsilon, \varphi)_{\Gamma_R}| &\le \beta_1\|\theta_\varepsilon\|_{L^1(\Gamma_R)}\|\varphi\|_{L^\infty(\Gamma_R)} \le c\|\theta_\varepsilon\|_{W^{1,r}_{\Gamma_D}(\Omega)}\|\varphi\|_{W^{1,r'}_{\Gamma_D}(\Omega)}.
\end{aligned} \tag{4.116}
$$

In the last inequality a trace theorem (cf. [20, Theorem 1.25]) was used. As [24] let us estimate $\langle v_\varepsilon\theta_\varepsilon, \nabla\varphi\rangle$ (cf. [24, pp. 1901]). Let r satisfy $\frac{21}{20} < r < \frac{5}{4}$. Then there exists $\frac{4}{3} < \sigma < q$ such that

$$
\frac{q(\sigma-1)}{\sigma(q-1)} = \frac{5r}{8},
$$

where $q = \frac{3r}{3-r}$. We define

$$
\varrho := \frac{4\sigma}{3\sigma-4}, \quad \lambda_1 := \frac{q-\sigma}{\sigma(q-1)}.
$$

Then

$$
\frac{1}{4} + \frac{1}{\sigma} + \frac{1}{\varrho} = 1, \quad \frac{1}{\sigma} = \frac{\lambda_1}{1} + \frac{1-\lambda_1}{q}, \quad 1-\lambda_1 = \frac{5r}{8}.
$$

Taking into account

$$
\begin{gathered}
\|\theta_\varepsilon\|_{L^\sigma} \le c\|\theta_\varepsilon\|^{\lambda_1}_{L^1}\|\theta_\varepsilon\|^{1-\lambda_1}_{L^q} \le c\|\theta_\varepsilon\|^{\lambda_1}_{L^1}\|\nabla\theta_\varepsilon\|^{1-\lambda_1}_{\mathbf{L}^r},\\
\|v_\varepsilon\|_{\mathbf{L}^4(\Omega)} \le c\|v_\varepsilon\|^{1/4}_{\mathbf{L}^2(\Omega)}\|\nabla v_\varepsilon\|^{3/4}_{\mathbf{L}^2(\Omega)},
\end{gathered}
$$

we have

$$
\begin{aligned}
|\langle v_\varepsilon\theta_\varepsilon, \nabla\varphi\rangle| &\le c\|v_\varepsilon\|_{\mathbf{L}^4}\|\theta_\varepsilon\|_{\mathbf{L}^\sigma}\|\nabla\varphi\|_{\mathbf{L}^\varrho}\\
&\le c\|v_\varepsilon\|^{1/4}_{L^\infty(0,T;H)}\|\theta_\varepsilon\|^{\lambda_1}_{L^\infty(0,T;L^1)}\|\nabla v_\varepsilon\|^{3/4}_{\mathbf{L}^2(\Omega)}\|\nabla\theta_\varepsilon\|^{\frac{5r}{8}}_{\mathbf{L}^r}\|\varphi\|_{W^{1,\varrho}_{\Gamma_D}},
\end{aligned} \tag{4.117}
$$

where $1-\lambda_1 = \frac{5r}{8}$ was used. Also,

$$
\begin{aligned}
\Big|\Big\langle \frac{1}{\rho C_h\sigma(\theta_\varepsilon)}\frac{|\operatorname{curl} H_\varepsilon|^2}{1+\varepsilon|\operatorname{curl} H_\varepsilon|^2}, \varphi\Big\rangle\Big| &\le c\frac{2}{\sigma_0}\Big(\|\bar H_\varepsilon\|^2_{\mathbf{V}_\Sigma} + \|\mathring{H}\|^2_{\mathbf{H}^1}\Big)\max_{x\in\Omega}|\varphi|\\
&\le c\Big(\|\bar H_\varepsilon\|^2_{\mathbf{V}_\Sigma} + \|\mathring{H}\|^2_{\mathbf{H}^1}\Big)\|\varphi\|_{W^{1,r'}_{\Gamma_D}} \quad \forall\varphi \in W^{1,r'}_{\Gamma_D}(\Omega),
\end{aligned} \tag{4.118}
$$

$$
\Big|\Big\langle \frac{\eta(\theta_\varepsilon)}{\rho C_h}\frac{|\mathcal{E}(v_\varepsilon)|^2}{1+\varepsilon|\mathcal{E}(v_\varepsilon)|^2}, \varphi\Big\rangle\Big| \le c\eta_1\|v_\varepsilon\|^2_{\mathbf{V}}\max_{x\in\Omega}|\varphi| \le c\|v_\varepsilon\|^2_{\mathbf{V}}\|\varphi\|_{W^{1,r'}_{\Gamma_D}} \quad \forall\varphi\in W^{1,r'}_{\Gamma_D}(\Omega). \tag{4.119}
$$

Let $\tau := \max\{r', \varrho\}$. Then, by (4.116)-(4.119), from the third equation of (4.1), we have

$$
\begin{aligned}
\Big|\Big\langle \frac{\partial\theta_\varepsilon}{\partial t}, \varphi\Big\rangle\Big| \le c\Big(&\|\nabla\theta_\varepsilon\|_{\mathbf{L}^r} + \|v_\varepsilon\|^{1/4}_{L^\infty(0,T;H)}\|\theta_\varepsilon\|^{\lambda}_{L^\infty(0,T;L^1)}\|\nabla v_\varepsilon\|^{3/4}_{\mathbf{L}^2(\Omega)}\|\nabla\theta_\varepsilon\|^{\frac{5r}{8}}_{\mathbf{L}^r}\\
&+ \big(\|\bar H_\varepsilon\|^2_{\mathbf{V}_\Sigma} + \|\mathring{H}\|^2_{\mathbf{H}^1}\big) + \|v_\varepsilon\|^2_{\mathbf{V}} + \|g_1\|_{(W^{1,2}_{\Gamma_D})^*}\Big)\|\varphi\|_{W^{1,\tau}_{\Gamma_D}} \ \ \forall\varphi\in W^{1,\tau}_{\Gamma_D}(\Omega).
\end{aligned}
$$

Hence, taking into account (4.103), the second one of (4.110), 2) of Assumption 3.1 and (3.13), we see that $\theta'_\varepsilon \in L^1\big(0,T;(W^{1,\tau}_{\Gamma_D})^*\big)$ and

$$
\begin{aligned}
&\|\theta'_\varepsilon\|_{L^1\left(0,T;(W^{1,\tau}_{\Gamma_D})^*\right)}\\
&\le c\Big(\|\theta_\varepsilon\|_{L^1\left(0,T;W^{1,r}_{\Gamma_D}\right)} + \|v_\varepsilon\|^{1/4}_{L^\infty(0,T;H)}\|\theta_\varepsilon\|^{1-\frac{5r}{8}}_{L^\infty(0,T;L^1)}\|v_\varepsilon\|^{3/4}_{L^2(0,T;\mathbf{V})}\|\theta_\varepsilon\|^{\frac{5r}{8}}_{L^r(0,T;W^{1,r}_{\Gamma_D})}\\
&\quad + \|\bar H_\varepsilon\|^2_{L^2(0,T;\mathbf{V}_\Sigma)} + \|\mathring{H}\|^2_{L^\infty(0,T;\mathbf{H}^1)} + \|v_\varepsilon\|^2_{L^2(0,T;\mathbf{V})} + \|g_1\|_{L^2(0,T;(W^{1,2}_{\Gamma_D})^*)}\Big),
\end{aligned} \tag{4.120}
$$

where $\frac{21}{20} < r < \frac{5}{4}$. To obtain the second term of the right hand side of the above it was used the fact that by Hölder's inequality with exponents 8/3 and 8/5

$$
\int_0^T \|\nabla v_\varepsilon(t)\|^{3/4}_{\mathbf{L}^2(\Omega)}\|\nabla\theta_\varepsilon(t)\|^{\frac{5r}{8}}_{\mathbf{L}^r(\Omega)}\,dt \le \|v_\varepsilon\|^{3/4}_{L^2(0,T;\mathbf{V})}\|\theta_\varepsilon\|^{\frac{5r}{8}}_{L^r(0,T;W^{1,r}_{\Gamma_D})}.
$$

Thus, by (4.110) and (4.120), there exists θ and a subsequence $\{\theta_{\varepsilon_k}\}$ such that

$$\begin{gathered}\theta_{\varepsilon_k} \overset{*}{\rightharpoonup} \theta \quad \text{in } L^\infty(0,T;L^1(\Omega)),\\ \theta_{\varepsilon_k} \rightharpoonup \theta \quad \text{in } L^r(0,T;W^{1,r}_{\Gamma_D}(\Omega)) \quad \forall r\,(1<r<5/4),\\ \theta_{\varepsilon_k} \to \theta \quad \text{in } L^r(0,T;W^{\alpha,r}_{\Gamma_D}(\Omega)) \quad \forall r\;(1<r<5/4)\;\forall \alpha\;(\frac{1}{r}<\alpha<1) \text{ and a.e. in } Q.\end{gathered}\tag{4.121}$$

The last formula the above is obtained as follows. First, by $W^{1,\tau}_{\Gamma_D}(\Omega) \hookrightarrow L^6(\Omega)$, we obtain $L^{6/5}(\Omega) \hookrightarrow (W^{1,\tau}_{\Gamma_D}(\Omega))^*$ [13, Remark 5.14, Ch. 1], which combing with the fact that $W^{\alpha,r}_{\Gamma_D}(\Omega) \subset L^{3/2}(\Omega)$ yields $W^{\alpha,r}_{\Gamma_D}(\Omega) \subset (W^{1,\tau}_{\Gamma_D})^*$. Then, since $W^{1,r}_{\Gamma_D}(\Omega) \hookrightarrow W^{\alpha,r}_{\Gamma_D}(\Omega)$ is compact, we obtain the fact that $\{\theta_\varepsilon\}$ is relatively compact in $L^r(0,T;W^{\alpha,r}_{\Gamma_D}(\Omega))$ (cf. [20, Theorem 1.39]).

4.3. **Passing to the limit as $\varepsilon \to 0$.** By passing to limit of solutions in Theorem 4.2 we will prove Theorem 3.4. Owing to (4.113), (4.115) and (4.121), we can extract subsequences, which is denoted as before, such that

$$\begin{gathered}v_{\varepsilon_k} \rightharpoonup v \quad \text{in } L^2(0,T;\mathbf{V}),\\ v_{\varepsilon_k} \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty(0,T;H),\\ v_{\varepsilon_k} \to v \quad \text{in } L^2(0,T;\mathbf{W}^{\frac{9}{10},2}(\Omega)),\\ \bar H_{\varepsilon_k} \overset{*}{\rightharpoonup} \bar H \;\; \text{in } L^\infty(0,T;\mathbf{L}^2(\Omega)),\\ \bar H_{\varepsilon_k} \to \bar H \;\; \text{in } L^2(0,T;\mathbf{W}^{\frac{9}{10},2}(\Omega)),\\ \bar H_{\varepsilon_k} \rightharpoonup \bar H \;\; \text{in } L^2(0,T;\mathbf{V}_\Sigma),\\ \theta_{\varepsilon_k} \rightharpoonup \theta \quad \text{in } L^r(0,T;W^{1,r}_{\Gamma_D}(\Omega)) \quad \forall r\,(1<r<5/4),\\ \theta_{\varepsilon_k} \to \theta \;\text{ in } L^r(0,T;W^{\alpha,r}_{\Gamma_D}(\Omega)(\Omega))\;\forall r\,(1<r<5/4),\;\forall \alpha\,\big(\frac{1}{r}<\alpha<1\big) \text{ and a.e. in } Q\end{gathered}\tag{4.122}$$

as $\varepsilon_k \to 0$.

We will obtain the first equation of (3.15). Let $u \in C^1([0,T];\mathbf{V})$ with $u(T)=0$. By (4.76) and (4.122), we have

$$\int_0^T \langle \frac{\partial v_{\varepsilon_k}}{\partial t}, u\rangle\,dt \to \int_0^T \big\langle -v(t), \frac{\partial u}{\partial t}\big\rangle\,dt - \langle v_0, u(x,0)\rangle \tag{4.123}$$

Taking into account the above, by [20, Corollary 1.1] we obtain

$$\int_0^T \langle \mu(\theta_{\varepsilon_k})\mathcal{E}(v_{\varepsilon_k}), \mathcal{E}(u)\rangle\,dt \to \int_0^T \langle \mu(\theta)\mathcal{E}(v), \mathcal{E}(u)\rangle\,dt. \tag{4.124}$$

By (4.103) we have

$$\begin{aligned}&\Big|\int_0^T \varepsilon_k \|v_{\varepsilon_k}\|^4_{\mathbf{V}}\big(\mathcal{E}(v_{\varepsilon_k}), \mathcal{E}(u)\big)\,dt\Big|\\ &\le c{\varepsilon_k}^{1/6}\int_0^T {\varepsilon_k}^{5/6}\|v_{\varepsilon_k}\|^5_{\mathbf{V}}\|u(t)\|_{\mathbf{V}}\,dt\\ &\le c{\varepsilon_k}^{1/6}\Big(\int_0^T \big({\varepsilon_k}^{5/6}\|v_{\varepsilon_k}\|^5_{\mathbf{V}}\big)^{6/5}\,dt\Big)^{5/6}\Big(\int_0^T \|u(t)\|^6_{\mathbf{V}}\,dt\Big)^{1/6}\\ &\le c{\varepsilon_k}^{1/6}C(\Lambda_\delta, c^*)^{5/6}\|u\|_{L^6(0,T;\mathbf{V})} \to 0 \quad \text{as } \varepsilon_k \to 0.\end{aligned}\tag{4.125}$$

In a rather routine way we can prove that

$$\int_0^T \langle \operatorname{curl} v_{\varepsilon_k}(t) \times v_{\varepsilon_k}(t), u(t)\rangle\,dt \to \int_0^T \langle \operatorname{curl} v \times v, u\rangle\,dt \quad \text{as } \varepsilon_k \;\; \to 0. \tag{4.126}$$

(See [20, (6.46)-(6.49)].) Owing to (4.122),

$$v_{\varepsilon_k} \to v \quad \text{in } L^2(0,T;\mathbf{L}^2(\partial\Omega)),$$

and taking a subsequence if necessary, by [20, Lemma 1.3]

$$\begin{aligned} &2(\mu(\theta_{\varepsilon_k})k(x)v_{\varepsilon_k},u)_{\Gamma_2}+2(\mu(\theta_{\varepsilon_k})S\tilde{v}_{\varepsilon_k},\tilde{u})_{\Gamma_3}+2(\alpha(x)v_{\varepsilon_k},u)_{\Gamma_5}+(\mu(\theta_{\varepsilon_k})k(x)v_{\varepsilon_k},u)_{\Gamma_7}\\ &\to 2(\mu(\theta)k(x)v,u)_{\Gamma_2}+2(\mu(\theta)S\tilde{v},\tilde{u})_{\Gamma_3}+2(\alpha(x)v,u)_{\Gamma_5}+(\mu(\theta)k(x)v,u)_{\Gamma_7}. \end{aligned} \tag{4.127}$$

Let us prove

$$\int_0^T\Big\langle\frac{\mu}{\rho}\operatorname{curl}\bar{H}_{\varepsilon_k}(t)\times\bar{H}_{\varepsilon_k}(t),u(t)\Big\rangle\,dt\to\int_0^T\Big\langle\frac{\mu}{\rho}\operatorname{curl}\bar{H}\times\bar{H},u(t)\Big\rangle\,dt. \tag{4.128}$$

Writing as

$$\begin{aligned} &\frac{\mu}{\rho}\int_0^T\Big|\langle\operatorname{curl}\bar{H}_{\varepsilon_k}(t)\times\bar{H}_{\varepsilon_k}(t),u(t)\rangle-\langle\operatorname{curl}\bar{H}\times\bar{H},u(t)\rangle\Big|\,dt\\ &\le\frac{\mu}{\rho}\int_0^T|\langle\operatorname{curl}\bar{H}_{\varepsilon_k}(t)\times(\bar{H}_{\varepsilon_k}(t)-\bar{H}),u(t)\rangle|\,dt+\frac{\mu}{\rho}\int_0^T|\langle\operatorname{curl}\bar{H}_{\varepsilon_k}(t)-\operatorname{curl}\bar{H},\bar{H}\times u(t)\rangle|\,dt\\ &\le\frac{\mu}{\rho}\|\operatorname{curl}\bar{H}_{\varepsilon_k}(t)\|_{L^2(0,T;\mathbf{L}^2)}\|(\bar{H}_{\varepsilon_k}(t)-\bar{H})\|_{L^2(0,T;\mathbf{L}^3)}\|u\|_{L^\infty(0,T;\mathbf{L}^6)}\\ &\quad+\frac{\mu}{\rho}\int_0^T|\langle\operatorname{curl}\bar{H}_{\varepsilon_k}(t)-\operatorname{curl}\bar{H},\bar{H}\times u(t)\rangle|\,dt \end{aligned}$$

and taking into account (4.122) and that $\bar{H}\times u\in L^2(0,T;\mathbf{L}^2)$, we obtain (4.128).

Similarly, taking into account $\mathring{H}(t)\times u(t)\in L^2(0,T;\mathbf{L}^2(\Omega)), u(t)\times\operatorname{curl}\mathring{H}(t)\in L^\infty(0,T;\mathbf{L}^{\frac{6}{5}}(\Omega))$ and (4.122), we obtain

$$\begin{gathered} \frac{\mu}{\rho}\int_0^T\big\langle\operatorname{curl}\bar{H}_{\varepsilon_k}\times\mathring{H},u(t)\big\rangle\,dt=\frac{\mu}{\rho}\int_0^T\big\langle\mathring{H}\times u(t),\operatorname{curl}\bar{H}_{\varepsilon_k}\big\rangle\,dt\to\int_0^T\Big\langle\frac{\mu}{\rho}\operatorname{curl}\bar{H}\times\mathring{H},u(t)\Big\rangle\,dt,\\ \frac{\mu}{\rho}\int_0^T\big\langle\operatorname{curl}\mathring{H}\times\bar{H}_{\varepsilon_k},u(t)\big\rangle\,dt=\frac{\mu}{\rho}\int_0^T\big\langle u(t)\times\operatorname{curl}\mathring{H}(t),\bar{H}_{\varepsilon_k}\big\rangle\,dt\\ \to\int_0^T\Big\langle\frac{\mu}{\rho}\operatorname{curl}\mathring{H}\times\bar{H},u(t)\Big\rangle\,dt \end{gathered} \tag{4.129}$$

as $\varepsilon_k\to 0$. By (4.122) and [20, Lemma 1.5], we have

$$\begin{aligned} &\int_0^T\Big\langle\Big(\frac{\alpha_0\theta_{\varepsilon_k}}{1+\varepsilon_k\theta_{\varepsilon_k}^2}-\alpha_0\theta\Big)\mathbf{g},u(t)\Big\rangle\,dt\\ &=\int_0^T\Big\langle\frac{\alpha_0(\theta_{\varepsilon_k}-\theta)}{1+\varepsilon_k\theta_{\varepsilon_k}^2}\mathbf{g},u(t)\Big\rangle\,dt+\int_0^T\Big\langle\Big(\frac{1}{1+\varepsilon_k\theta_{\varepsilon_k}^2}-1\Big)\alpha_0\theta\mathbf{g},u(t)\Big\rangle\,dt\to 0, \end{aligned}$$

which shows that

$$\int_0^T\Big\langle\frac{\alpha_0\theta_{\varepsilon_k}}{1+\varepsilon_k\theta_{\varepsilon_k}^2}\mathbf{g},u(t)\Big\rangle\,dt\to\int_0^T\langle\alpha_0\theta\mathbf{g},u(t)\rangle\;dt\to 0\quad\text{as }\varepsilon_k\to 0. \tag{4.130}$$

Therefore, by (4.123)-(4.130) and (4.76), from the first equation of (4.1) we have the first equation of (3.16).

We will obtain the second equation of (3.16). Let $C\in C^1([0,T];\mathbf{V}_\Sigma)$ with $C(\cdot,T)=0$. As in (4.123) we obtain

$$\int_0^T\langle\bar{H}'_{\varepsilon_k}(t),C(t)\rangle\,dt\to\int_0^T\langle-\bar{H}'(t),C(t)\rangle\,dt+\langle\bar{H}_0,C(x,0)\rangle\quad\text{as }\varepsilon_k\to 0. \tag{4.131}$$

At the same way as (4.124) and (4.125), we see that

$$
\begin{aligned}
&\int_0^T b_0(\theta_{\varepsilon_k};\bar H_{\varepsilon_k},C(t))\,dt\to\int_0^T b_0(\theta;\bar H,C(t))\,dt,\\
&\int_0^T \Big\langle \varepsilon_k\|\operatorname{curl}\bar H_{\varepsilon_k}(t)\|^4\operatorname{curl}\bar H_{\varepsilon_k}(t),\operatorname{curl}C\Big\rangle\,dt\to 0,\\
&\int_0^T \big\langle \frac{1}{\mu\sigma(\theta_{\varepsilon_k})}\operatorname{curl}\mathring{H},\operatorname{curl}C(t)\big\rangle\,dt\to\int_0^T \big\langle \frac{1}{\mu\sigma(\theta)}\operatorname{curl}\mathring{H},\operatorname{curl}C(t)\big\rangle\,dt
\end{aligned}
\tag{4.132}
$$

as $\varepsilon_k\to 0$. Since

$$
\begin{aligned}
&\int_0^T |\big\langle \operatorname{curl}C(t)\times\bar H_{\varepsilon_k},v_{\varepsilon_k}\big\rangle-\big\langle \operatorname{curl}C(t)\times\bar H,v\big\rangle|\,dt\\
&\le \|\operatorname{curl}C\|_{L^\infty(0,T;\mathbf{L}^2)}\|\bar H_{\varepsilon_k}-\bar H\|_{L^2(0,T;\mathbf{L}^3)}\|v_{\varepsilon_k}\|_{L^2(0,T;\mathbf{L}^6)}\\
&\quad+\|\operatorname{curl}C\|_{L^\infty(0,T;\mathbf{L}^2)}\|\bar H\|_{L^2(0,T;\mathbf{L}^6)}\|v_{\varepsilon_k}-v\|_{L^2(0,T;\mathbf{L}^3)},
\end{aligned}
$$

we have

$$
\int_0^T \big\langle \operatorname{curl}C(t)\times\bar H_{\varepsilon_k},v_{\varepsilon_k}\big\rangle\to\int_0^T \big\langle \operatorname{curl}C(t)\times\bar H,v\big\rangle\,dt. \tag{4.133}
$$

Also, by (2) of Assumption 3.1,

$$
\operatorname{curl}C(t)\times\mathring{H}(t)\in L^2(0,T;\mathbf{L}^{\frac65}(\Omega)),
$$

and taking into account (4.122), we have

$$
\int_0^T \big\langle \operatorname{curl}C(t)\times\mathring{H}(t),v_{\varepsilon_k}\big\rangle\to\int_0^T \big\langle \operatorname{curl}C(t)\times\mathring{H}(t),v\big\rangle\,dt\quad\text{as }\varepsilon_k\to 0. \tag{4.134}
$$

Using (4.131)-(4.134) and (4.76), from the second equation of (4.1) we obtain the second one of (3.16).

We will obtain the third equation of (3.16). Let $\varphi\in C^1([0,T];C^1_{\Gamma_D}(\Omega))$. We have from the third equation of (4.1)

$$
\begin{aligned}
&\int_0^t \big\langle\frac{\partial\theta_{\varepsilon_k}}{\partial t},\varphi\big\rangle\,ds+\int_0^t(\kappa(\theta_{\varepsilon_k})\nabla\theta_{\varepsilon_k},\nabla\varphi)\,ds+\int_0^t(\beta(x)\theta_{\varepsilon_k},\varphi)_{\Gamma_R}\,ds\\
&-\int_0^t\langle v_{\varepsilon_k}\theta_{\varepsilon_k},\nabla\varphi\rangle\,ds-\int_0^t\Big\langle\frac{1}{\rho C_h\sigma(\theta_{\varepsilon_k})}\frac{\big|\operatorname{curl}\big(\bar H_{\varepsilon_k}(s)+\mathring{H}(s)\big)\big|^2}{1+\varepsilon\big|\operatorname{curl}\big(\bar H_{\varepsilon_k}(s)+\mathring{H}(s)\big)\big|^2},\varphi\Big\rangle\,ds\\
&-\int_0^t\Big\langle\frac{\eta(\theta_{\varepsilon_k})}{\rho C_h}\frac{|\mathcal{E}(v_{\varepsilon_k})|^2}{1+\varepsilon|\mathcal{E}(v_{\varepsilon_k})|^2},\varphi\Big\rangle\,ds\\
&=\int_0^t\big\langle g_1,\varphi\big\rangle\,ds\quad\forall t\in[0,T].
\end{aligned}
$$

As (4.123), we obtain

$$
\int_0^T \big\langle\frac{\partial\theta_{\varepsilon_k}}{\partial t},\varphi\big\rangle\,dt\to\int_0^T\big\langle -\theta(t),\frac{\partial\varphi}{\partial t}\big\rangle\,dt-\langle\theta_0,\varphi(x,0)\rangle. \tag{4.135}
$$

By [20, Corollary 1.1] we have

$$
\int_0^t(\kappa(\theta_{\varepsilon_k})\nabla\theta_{\varepsilon_k},\nabla\varphi)\,ds\to\int_0^t(\kappa(\theta)\nabla\theta,\nabla\varphi)\,ds\quad\text{as }\varepsilon_k\to 0. \tag{4.136}
$$

It is easy to prove that

$$
\int_0^t(\beta(x)\theta_{\varepsilon_k},\varphi)_{\Gamma_R}\,ds\to\int_0^t(\beta(x)\theta,\varphi)_{\Gamma_R}\,ds\quad\text{as }\varepsilon_k\to 0. \tag{4.137}
$$

Since $v_{\varepsilon_k}\to v$ in $L^2(0,T;\mathbf{W}^{\frac{9}{10},2}(\Omega))$ and $\mathbf{W}^{\frac{9}{10},2}(\Omega)\subset\mathbf{L}^5(\Omega)$ (cf. [20, Theorem 1.20]), we have

$$
v_{\varepsilon_k}\to v\quad\text{in }L^2(0,T;\mathbf{L}^5(\Omega)). \tag{4.138}
$$

Since $\{\theta_{\varepsilon_k}\} \subset \big(L^{6/5}(0,T;W^{1,6/5}(\Omega)) \cap L^\infty(0,T;L^1(\Omega))\big)$ and $W^{1,6/5}(\Omega) \subset L^2(\Omega)$, by complex interpolation with exponent $\frac{2}{5}$ we have

$$\{\theta_{\varepsilon_k}\} \subset L^2(0,T;L^{\frac{10}{7}}(\Omega)) \subset L^2(0,T;L^{\frac{5}{4}}(\Omega))$$

(cf. [20, Theorems 1.34 and 1.12]). Taking into account (4.138) and the above, we have

$$\int_0^t \langle v_{\varepsilon_k}\theta_{\varepsilon_k}, \nabla\varphi\rangle\, ds \to \int_0^t \langle v\theta, \nabla\varphi\rangle\, ds \quad \text{as } \varepsilon_k \to 0. \tag{4.139}$$

Since when $\varphi \geq 0$,

$$\begin{aligned}
&\int_0^t \Big\langle \frac{1}{\rho C_h \sigma(\theta_{\varepsilon_k})} \frac{\big|\operatorname{curl}\big(\bar H_{\varepsilon_k}(s) + \mathring{H}(s)\big)\big|^2}{1+\varepsilon\big|\operatorname{curl}\big(\bar H_{\varepsilon_k}(s) + \mathring{H}(s)\big)\big|^2}, \varphi\Big\rangle\, ds \\
&= \int_0^t \Big\langle \frac{1}{\rho C_h \sigma(\theta_{\varepsilon_k})} \frac{\big|\operatorname{curl}\big(\bar H_{\varepsilon_k}(s) + \mathring{H}(s)\big)\big|\sqrt{\varphi}}{\sqrt{1+\varepsilon\big|\operatorname{curl}\big(\bar H_{\varepsilon_k}(s) + \mathring{H}(s)\big)\big|^2}}, \frac{\big|\operatorname{curl}\big(\bar H_{\varepsilon_k}(s) + \mathring{H}(s)\big)\big|\sqrt{\varphi}}{\sqrt{1+\varepsilon\big|\operatorname{curl}\big(\bar H_{\varepsilon_k}(s) + \mathring{H}(s)\big)\big|^2}}\Big\rangle\, ds, \\
&\int_0^t \Big\langle \frac{\eta(\theta_{\varepsilon_k})}{\rho C_h} \frac{|\mathcal{E}(v_{\varepsilon_k})|^2}{1+\varepsilon_k|\mathcal{E}(v_{\varepsilon_k})|^2}, \varphi\Big\rangle\, ds = \int_0^t \Big\langle \frac{\eta(\theta_{\varepsilon_k})}{\rho C_h} \frac{|\mathcal{E}(v_{\varepsilon_k})|\sqrt{\varphi}}{\sqrt{1+\varepsilon_k|\mathcal{E}(v_{\varepsilon_k})|^2}}, \frac{|\mathcal{E}(v_{\varepsilon_k})|\sqrt{\varphi}}{\sqrt{1+\varepsilon_k|\mathcal{E}(v_{\varepsilon_k})|^2}}\Big\rangle\, ds,
\end{aligned}$$

by [20, Lemma 1.4 and Corollary 1.2] we have

$$\begin{aligned}
&\liminf_{\varepsilon_k\to 0} \int_0^t \Big\langle \frac{1}{\rho C_h \sigma(\theta_{\varepsilon_k})} \frac{\big|\operatorname{curl}\big(\bar H_{\varepsilon_k}(s) + \mathring{H}(s)\big)\big|^2}{1+\varepsilon\big|\operatorname{curl}\big(\bar H_{\varepsilon_k}(s) + \mathring{H}(s)\big)\big|^2}, \varphi\Big\rangle\, ds \\
&\geq \int_0^t \Big\langle \frac{1}{\rho C_h \sigma(\theta)} \big|\operatorname{curl}\big(\bar H(s) + \mathring{H}(s)\big)\big|^2, \varphi\Big\rangle\, ds, \\
&\liminf_{\varepsilon_k\to 0} \int_\Omega \Big\langle \frac{\eta(\theta_{\varepsilon_k})}{\rho C_h} \frac{|\mathcal{E}(v_{\varepsilon_k})|^2}{1+\varepsilon_k|\mathcal{E}(v_{\varepsilon_k})|^2}, \varphi\Big\rangle\, ds \geq \int_0^t \Big\langle \frac{\eta(\theta)}{\rho C_h} |\mathcal{E}(v)|^2, \varphi\Big\rangle\, ds.
\end{aligned} \tag{4.140}$$

Therefore, taking into account (4.135)-(4.137), (4.139), (4.140) and (4.76), we have

$$\begin{aligned}
&\big\langle \theta(t), \varphi(t)\big\rangle + \int_0^t \big[\langle -\theta, \frac{\partial\varphi}{\partial s}\rangle + \langle \kappa(\theta)\nabla\theta, \nabla\varphi\rangle + \langle \beta(x)\theta, \varphi\rangle_{\Gamma_R} - \langle v\theta, \nabla\varphi\rangle\big]\, ds \\
&\geq \int_0^t \Big[\Big\langle \frac{1}{\rho C_h \sigma(\theta)} \big|\operatorname{curl}\big(\bar H(s) + \mathring{H}(s)\big)\big|^2, \varphi\Big\rangle + \Big\langle \frac{\eta(\theta)}{\rho C_h} |\mathcal{E}(v)|^2, \varphi\Big\rangle + \langle g_1(s), \varphi\rangle\Big]\, ds \\
&\quad + \big\langle \theta_0, \varphi(x,0)\big\rangle \quad \text{for a.a. } t \in [0,T] \quad \forall \varphi \in C^1([0,T]; C^1_{\Gamma_D}(\Omega)),\ \varphi \geq 0.
\end{aligned} \tag{4.141}$$

Now, following [24] we can prove that θ satisfies the third formula of (3.16) with "defect measure". For convenience of readers we give here proof.

By (4.103), $\frac{\eta(\theta_{\varepsilon_k})}{\rho C_h}|\mathcal{E}(v_{\varepsilon_k})|^2 \in L^1(Q)$, and these give continuous linear functionals on $C(\overline{Q})$. Since $\overline{Q}$ is compact, every $\frac{\eta(\theta_{\varepsilon_k})}{\rho C_h}|\mathcal{E}(v_{\varepsilon_k})|^2$ is identified with a Radon measure(cf. [8, Theorem 7.10.4, pp.1]). Since $\big\{\frac{\eta(\theta_{\varepsilon_k})}{\rho C_h}|\mathcal{E}(v_{\varepsilon_k})|^2\big\}$ is bounded in $L^1(Q)$ and $C(\overline{Q})$ is separable, there exists a weak* convergent subsequence (cf. Corollary, 1. Polar Sets, Appendix to Chap. V of [29]) and a positive Radon measure $\bar\mu_0 \in \mathscr{M}(\bar Q)$ such that

$$\int_0^T \Big\langle \frac{\eta(\theta_{\varepsilon_k})}{\rho C_h} |\mathcal{E}(v_{\varepsilon_k})|^2, \varphi\Big\rangle\, dt \to \int_{\bar Q} \varphi\, d\bar\mu_0 \quad \forall \varphi \in C(\overline{Q}). \tag{4.142}$$

By the same argument, there exists a weak* convergent subsequence and a positive Radon measure $\bar\mu_1 \in \mathscr{M}(\bar Q)$ such that

$$\int_0^T \Big[\Big\langle \frac{1}{\rho C_h \sigma(\theta_{\varepsilon_k})} \big|\operatorname{curl}\big(\bar H_{\varepsilon_k}(t) + \mathring{H}(s)\big)\big|^2, \varphi\Big\rangle\, dt \to \int_{\bar Q} \varphi\, d\bar\mu_1 \quad \forall \varphi \in C(\overline{Q}). \tag{4.143}$$

Therefore, taking into account (4.135)-(4.137), (4.139), (4.142) and (4.143), we have

$$\begin{aligned}&\int_0^T \big[\langle -\theta, \frac{\partial \varphi}{\partial t}\rangle + \langle \kappa(\theta)\nabla\theta, \nabla\varphi\rangle + \langle \beta(x)\theta, \varphi\rangle_{\Gamma_R} - \langle v\theta, \nabla\varphi\rangle\big]\, dt \\ &= \int_{\bar{Q}} \varphi\, d(\bar{\mu}_1 + \bar{\mu}_0) + \big\langle \theta_0, \varphi(x,0)\big\rangle + \int_0^T \big\langle g_1, \varphi\big\rangle\, dt\end{aligned} \tag{4.144}$$

for all $\varphi \in C^1([0,T]; C^1_{\Gamma_D}(\Omega))$, $\varphi(\cdot,T) = 0$. From (4.142) and (4.144) we have

$$\int_{\bar{Q}} \varphi\, d(\bar{\mu}_1 + \bar{\mu}_0) \geq \int_0^T \Big[\Big\langle \frac{1}{\rho C_h \sigma(\theta)} \big|\operatorname{curl}\big(\bar{H}(t) + \mathring{H}(t)\big)\big|^2 + \frac{\eta(\theta)}{\rho C_h} |\mathcal{E}(v)|^2, \varphi\Big\rangle\Big]\, dt$$

for all $\varphi \in C^1([0,T]; C^1_{\Gamma_D}(\Omega))$, $\varphi \geq 0$, and $\varphi(\cdot,T) = 0$. We define

$$\bar{\mu} = (\bar{\mu}_1 + \bar{\mu}_0) - \Big(\frac{1}{\rho C_h \sigma(\theta)} \big|\operatorname{curl}\big(\bar{H} + \mathring{H}\big)\big|^2 + \frac{\eta(\theta)}{\rho C_h} |\mathcal{E}(v)|^2\Big)\lambda_4|_{\mathscr{B}(\overline{Q})},$$

where λ_4 is the Lebesgue measure in $\mathbb{R}^4$ and $\mathscr{B}(\overline{Q})$ is the σ-algebra of Borel subsets of $\overline{Q}$. Then, the third formula of (3.16) is valid with a "defect measure" $\bar{\mu}$.

To get estimate (**??**), we use (4.103) and (4.109). Let us prove that

$$\int_Q \frac{|\nabla\theta|^2}{(1+|\theta|)^{1+\delta}}\, dx\, dt \leq \liminf_{\varepsilon_k \to 0} \int_Q \frac{|\nabla\theta_{\varepsilon_k}|^2}{(1+|\theta_{\varepsilon_k}|)^{1+\delta}}\, dx\, dt \quad 0 < \delta < 1. \tag{4.145}$$

We put

$$z_{\varepsilon_k,i} := \frac{\partial\theta_{\varepsilon_k}/\partial x_i}{(1+|\theta_{\varepsilon_k}|)^{(1+\delta)/2}} \quad \text{a.e. in } Q\ (i = 1,2,3).$$

By (4.122), $\{z_{\varepsilon_k}\}$ is bounded in $L^r(Q)$, and by passing to a subsequence if necessary, we have

$$z_{\varepsilon_k,i} \rightharpoonup z_i \quad \text{in } L^r(Q). \tag{4.146}$$

To obtain (4.145), it is sufficient to prove that

$$z_i = \frac{\partial\theta/\partial x_i}{(1+|\theta|)^{(1+\delta)/2}}. \tag{4.147}$$

By (4.122),

$$\partial\theta_{\varepsilon_k}/\partial x_i \rightharpoonup \partial\theta/\partial x_i \quad \text{in } L^r(Q),\ 1 < r < \frac{5}{4},$$

and by Lebesgue Theorem,

$$\frac{1}{(1+|\theta_{\varepsilon_k}|)^{(1+\delta)/2}} \to \frac{1}{(1+|\theta|)^{(1+\delta)/2}} \quad \text{in } L^{r'}(Q).$$

Then

$$\frac{\partial\theta_{\varepsilon_k}/\partial x_i}{(1+|\theta_{\varepsilon_k}|)^{(1+\delta)/2}} \to \frac{\partial\theta/\partial x_i}{(1+|\theta|)^{(1+\delta)/2}} \quad \text{in } L^1(Q),$$

from which we obtain

$$\frac{\partial\theta_{\varepsilon_k}/\partial x_i}{(1+|\theta_{\varepsilon_k}|)^{(1+\delta)/2}} \to \frac{\partial\theta/\partial x_i}{(1+|\theta|)^{(1+\delta)/2}} \quad \text{a.e. in } Q \tag{4.148}$$

for a subsequence if necessary. By (4.146) and (4.148), we obtain (4.147) (cf. [23, Lemma 1.3, Ch.1]), which yields (4.145). By (4.109), we obtain

$$\begin{gathered}\int_Q |\nabla\theta|^r\, dx\, dt \leq \liminf_{\varepsilon_k \to 0} \int_Q |\nabla\theta_{\varepsilon_k}|^r\, dx\, dt \leq cC(\Lambda_{\delta_0}, c^*)^{r/2} + 2R(C(\Lambda_{\delta_0}, c^*)), \\ 1 < r < \frac{5}{4}, \quad 0 < \delta_0 < \frac{5-4r}{3}.\end{gathered} \tag{4.149}$$

Taking into account (4.91), (4.145), and (4.149), from (4.103), we obtain (**??**). □

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