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NODAL SETS AND CONTINUITY OF EIGENFUNCTIONS OF KREĬN-FELLER OPERATORS

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Abstract. Let μ be a compactly supported positive finite Borel measure on $\mathbb{R}^d$. Let $0 < \lambda_1 \le \lambda_2 \le \cdots$ be eigenvalues of the Kreĭn-Feller operator Δ_μ. We prove that, on a bounded domain, the nodal set of a continuous λ_n-eigenfunction of a Kreĭn-Feller operator divides the domain into at least 2 and at most $n + r_n - 1$ subdomains, where r_n is the multiplicity of λ_n. This work generalizes the nodal set theorem of the classical Laplace operator to Kreĭn-Feller operators on bounded domains. We also prove that on bounded domains on which the classical Green function exists, the eigenfunctions of a Kreĭn-Feller operator are continuous.

1. Introduction

For a bounded domain (i.e., an open and connected set) $\Omega \subseteq \mathbb{R}^d$, consider the Dirichlet problem

$$\begin{aligned} -\Delta u &= \lambda u, \quad \text{in } \Omega, \\ u &= 0, \quad \text{in } \partial\Omega, \end{aligned} \tag{1.1}$$

with eigenvalue λ and eigenfunction u. The *nodal set* of u is defined as

$$\mathcal{Z}(u) := \{\mathbf{x} \in \Omega : u(\mathbf{x}) = 0\}.$$

It is known that the eigenvalues can be ordered as

$$0 < \lambda_1 \le \lambda_2 \le \cdots$$

with $\lim_{n\to\infty} \lambda_n = \infty$. Properties of nodal set of the eigenfunctions have been studied extensively (see [2, 3, 10, 22, 31, 32, 36, 44, 45, 48, 50] and references therein). Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain. By a λ-*eigenfunction* we mean an eigenfunction corresponding to the eigenvalue λ. The Courant nodal domain theorem says that the nodal set of a λ_n-eigenfunction of (1.1) divides Ω into at most n subdomains (see e.g. [10]). Gladwell and Zhu [22] studied the nodal sets of eigenfunctions of the following Helmholtz equation on a bounded domain $\Omega \subseteq \mathbb{R}^d$:

$$\Delta u + \lambda \rho u = 0,$$

where $\rho(\mathbf{x})$ is positive and bounded. They proved that the nodal set of a λ_n-eigenfunction divides Ω into at most $n + r_n - 1$ subdomains, where r_n is the

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multiplicity of λ_n. One goal of this paper is to generalize the above theorem to Laplace operators defined by measures (see definition in Section 2) on a domain $\Omega \subseteq \mathbb{R}^d$. Such operators are also called *Kreĭn-Feller operators* and are introduced in [17, 27, 30]. These operators are used to describe physical phenomena, such as wave propagation or heat conduction, in media with an inhomogeneous mass distribution modeled by a measure μ, such as a fractal measure.

Kreĭn-Feller operators have been studied extensively. McKean and Ray [33] studied spectral asymptotics of Kreĭn-Feller operator defined by the Cantor measure. Freiberg [18] studied analytic properties of the operators defined on the line. Hu *et al* [25] studied spectral properties of Kreĭn-Feller operators defined on a bounded domain of $\mathbb{R}^d$. Deng and Ngai [13], Pinasco and Scarola [46] studied the eigenvalue estimates of such operators. Kesseböhmer and Niemann [28, 29] studied the relation between the L^q-spectrum and spectral dimension of such an operator. For additional work associated with these operators, including eigenvalues, eigenfunctions, spectral asymptotics, spectral gaps, spectral dimension, wave equation, heat equation and heat kernel estimates, the reader is referred to [7, 8, 9, 13, 14, 18, 19, 20, 24, 25, 28, 29, 37, 38, 39, 40, 41, 42, 46, 47, 52] and references therein.

We will summarize the definition of a Kreĭn-Feller operator in Section 2. We denote by Δ_μ the Kreĭn-Feller operator defined by a measure μ (see Section 2). In this article, we consider eigenvalues and eigenfunctions associated with the Dirichlet problem

$$\begin{aligned} -\Delta_\mu u &= \lambda u, \quad \text{in } \Omega, \\ u &= 0, \quad \text{in } \partial\Omega. \end{aligned} \tag{1.2}$$

Let $\underline{\dim}_\infty(\mu)$ be defined as in (2.1). It is shown in [25, Theorem 1.2] that, under the assumption $\underline{\dim}_\infty(\mu) > d-2$, there exists an orthonormal basis $\{\phi_n\}_{n=1}^\infty$ of $L^2(\Omega,\mu)$ consisting of (Dirichlet) eigenfunctions of Δ_μ. The eigenvalues $\{\lambda_n\}_{n=1}^\infty$ satisfy $0 < \lambda_1 \le \lambda_2 \le \cdots$ and $\lim_{n\to\infty} \lambda_n = \infty$. We let

$$\mathcal{Z}_\mu(u) := \{\mathbf{x} \in \Omega : u(\mathbf{x}) = 0\} \tag{1.3}$$

be the nodal set of an eigenfunction u of Δ_μ. Under the assumption of the continuity of eigenfunctions, we have the following theorem.

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^d$ $(d \ge 1)$ be a bounded domain and μ be a positive finite Borel measure on $\mathbb{R}^d$ with $\operatorname{supp}(\mu) \subseteq \overline{\Omega}$ and $\mu(\Omega) > 0$. Assume $\underline{\dim}_\infty(\mu) > d-2$. Let the eigenvalues $\{\lambda_n\}_{n=1}^\infty$ of (1.2) be arranged in an increasing order and let u_n be a λ_n-eigenfunction. Suppose $u_n \in C(\overline{\Omega})$. Then*

(a) *u_1 is nonzero on Ω.*
(b) *For $n \ge 2$, if λ_n has multiplicity $r_n \ge 1$, then $\mathcal{Z}_\mu(u_n)$ divides Ω into at least 2 and at most $n + r_n - 1$ subdomains.*

To prove this result, we need the maximum principle of continuous μ-subharmonic functions (Definition 3.1) which we will prove in Section 3.

Note that the definition of a nodal set $\mathcal{Z}_\mu(u)$ in (1.3) makes sense only if u is defined everywhere and not just almost everywhere. As the domain of Δ_μ consists of Sobolev functions, we need to study the continuity of the eigenfunctions of Δ_μ. It is known that for the classical Laplacian, the eigenfunctions are continuous and differentiable. If the measure μ is absolutely continuous with respect to Lebesgue measure, then the eigenfunctions of Δ_μ are continuous (see e.g. [16]). On $\mathbb{R}$, the eigenfunctions of Δ_μ are continuous, since Sobolev functions are continuous. To the best of our knowledge, the continuity of eigenfunctions in higher dimensions is unknown in general. Motivated by the requirement of the continuity of eigenfunctions in the definition of $\mathcal{Z}_\mu(u)$ and in Theorem 1.1, we prove the following main theorem. The definition of the regularity of the boundary of a bounded domain is given in Definition 5.2.

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^d$ on which the classical Green function $G(\mathbf{x},\mathbf{y})$ exists and let μ be a finite positive Borel measure with $\operatorname{supp}(\mu)\subseteq\overline{\Omega}$ and $\mu(\Omega)>0$. Assume $\underline{\dim}_\infty(\mu)>d-2$. Then the eigenfunctions of Δ_μ are continuous on Ω. Moreover, if the boundary of Ω is regular, then the eigenfunctions of Δ_μ are continuous on $\overline{\Omega}$.*

To prove Theorem 1.2, we apply the inverse operator of $-\Delta_\mu$, called the Green operator (see Section 5), which is defined by the classical Green function. By expressing the eigenfunctions of Δ_μ in terms of the Green operator and using the continuity of the Green function, we prove Theorem 1.2; details are given in Section 5. This theorem shows that the Dirichlet problem (1.2) has continuous solutions on a bounded domain $\Omega\subseteq\mathbb{R}^d$ on which the Green function exists.

This article is organized as follows. In Section 2, we summarize the definition of Δ_μ in [25]. In Section 3, we study the properties of Δ_μ and prove the maximum principle of continuous μ-subharmonic functions. Sections 4 and 5 are devoted to the proofs of Theorems 1.1 and 1.2, respectively. In Section 6, we construct examples of continuous eigenfunctions corresponding to singular measures on $\mathbb{R}^2$.

2. Preliminaries

In this section, we summarize the definition of Kreĭn-Feller operators; details can be found in [25, 14]. Let $\Omega\subseteq\mathbb{R}^d$ be a bounded domain and μ be a finite positive Borel measure with $\operatorname{supp}(\mu)\subseteq\overline{\Omega}$ and $\mu(\Omega)>0$, where $\overline{\Omega}$ is the closure of Ω. Let $\partial\Omega:=\overline{\Omega}\backslash\Omega$ be the boundary of Ω. Let $d\mathbf{x}$ be the Lebesgue measure on $\mathbb{R}^d$, and let $H^1(\Omega)$ be the Sobolev space equipped with the inner product

$$\langle u,v\rangle_{H^1(\Omega)}:=\int_\Omega uv\,d\mathbf{x}+\int_\Omega\nabla u\cdot\nabla v\,d\mathbf{x}.$$

Let $H_0^1(\Omega)$ be the completion of $C_c^\infty(\Omega)$ under the above inner product, where $C_c^\infty(\Omega)$ is the space of all smooth functions with compact support in Ω. Let $L^2(\Omega,\mu)$ be the space of all square integrable functions with respect to μ. The norm in $L^2(\Omega,\mu)$ is defined by

$$\|u\|_{L^2(\Omega,\mu)}:=\Big(\int_\Omega|u|^2\,d\mu\Big)^{1/2}.$$

Throughout this paper, we write $L^2(\Omega):=L^2(\Omega,d\mathbf{x})$. It follows from the Poincaré inequality (see e.g., [25]), that is, there exists a constant $C>0$ such that

$$\|u\|_{L^2(\Omega)}\le C\|\nabla u\|_{L^2(\Omega)}\,.$$

The space $H_0^1(\Omega)$ admits the equivalent inner product defined by

$$\langle u,v\rangle_{H_0^1(\Omega)}:=\int_\Omega\nabla u\cdot\nabla v\,d\mathbf{x}.$$

The *lower L^∞-dimension* of μ is defined as

$$\underline{\dim}_\infty(\mu) := \liminf_{\delta\to 0^+} \frac{\ln(\sup_{\mathbf{x}} \mu(B_\delta(\mathbf{x})))}{\ln \delta}, \tag{2.1}$$

where $B_\delta(\mathbf{x})$ is the ball with center $\mathbf{x}$ and radius δ, and the supremum is taken over all $\mathbf{x} \in \operatorname{supp}(\mu)$ (see [51] for details). To define Kreĭn-Feller operators, we need the following assumption, which is called the *Poincaré inequality for measures (MPI)*: there exists a constant $C > 0$ such that

$$\int_\Omega |u|^2\, d\mu \le C \int_\Omega |\nabla u|^2\, d\mathbf{x}, \quad \text{for all } u \in C_c^\infty(\Omega).$$

We know that if $\underline{\dim}_\infty(\mu) > d-2$, then μ satisfies (MPI) (see [25, Theorem 1.1]). (MPI) implies that each equivalence class $u \in H_0^1(\Omega)$ contains a unique (in $L^2(\Omega,\mu)$ sense) member $\hat{u} \in L^2(\Omega,\ \mu)$ that satisfies the following two conditions:

(a) there exists a sequence $\{u_n\}$ in $C_c^\infty(\Omega)$ such that $u_n \to \hat{u}$ in $H_0^1(\Omega)$ and $u_n \to \hat{u}$ in $L^2(\Omega,\mu)$;
(b) $\hat{u}$ satisfies the (MPI).

We call $\hat{u}$ the *$L^2(\Omega,\mu)$-representative* of u. Under the assumption (MPI), we define a map $\mathcal{I}$: $H_0^1(\Omega) \to L^2(\Omega,\mu)$ by

$$\mathcal{I}(u) = \hat{u}. \tag{2.2}$$

The mapping $\mathcal{I}$ is in general not injective. We define the following closed subset of $H_0^1(\Omega)$:

$$\mathcal{N} := \{u \in H_0^1(\Omega) : \|\mathcal{I}(u)\|_{L^2(\Omega,\mu)} = 0\}.$$

Let $\mathcal{N}^\perp$ be the orthogonal complement of $\mathcal{N}$ in $H_0^1(\Omega)$. Then $\mathcal{I} : \mathcal{N}^\perp \to L^2(\Omega,\mu)$ is injective. We denote $\hat{u}$ simply by u if there is no confusion possible.

Consider a nonnegative bilinear form $\mathcal{E}(\cdot,\cdot)$ in $L^2(\Omega,\mu)$ defined as

$$\mathcal{E}(u,v) := \int_\Omega \nabla u \cdot \nabla v\, d\mathbf{x}, \tag{2.3}$$

with $\operatorname{dom}(\mathcal{E}) = \mathcal{N}^\perp$. (MPI) implies that $(\mathcal{E}, \operatorname{dom}(\mathcal{E}))$ is a closed quadratic form on $L^2(\Omega,\mu)$ (see [25, Proposition 2.1]). Hence, there exists a nonnegative self-adjoint operator $-\Delta_\mu$ such that

$$\operatorname{dom}(\mathcal{E}) = \operatorname{dom}((-\Delta_\mu)^{1/2}),$$
$$\mathcal{E}(u,v) = \langle (-\Delta_\mu)^{1/2}u, (-\Delta_\mu)^{1/2}v\rangle_{L^2(\Omega,\mu)}, \quad \text{for all } u,v \in \operatorname{dom}(\mathcal{E})$$

(see [12]), where the $\langle\cdot,\cdot\rangle_{L^2(\Omega,\mu)}$ is the inner product on $L^2(\Omega,\mu)$. We call the above Δ_μ the *(Dirichlet) Laplacian* with respect to μ or the *Kreĭn-Feller operator* defined by μ. It follows from [25, Proposition 2.2] that $u \in \operatorname{dom}(\Delta_\mu)$ and $-\Delta_\mu u = f$ if and only if $-\Delta u = f\, d\mu$ in the sense of distribution, i.e.,

$$\int_\Omega \nabla u \cdot \nabla \varphi\, d\mathbf{x} = \int_\Omega f\varphi\, d\mu, \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

3. Maximum principle

In this section, we prove the maximum principle for a continuous μ-subharmonic function. We first define μ-subharmonic functions.

Definition 3.1. We call $u \in \operatorname{dom}(\Delta_\mu)$ a *μ-subharmonic function* if $\Delta_\mu u \ge 0$ (μ-a.e.). Call $u \in \operatorname{dom}(\Delta_\mu)$ a *μ-superharmonic function* if $\Delta_\mu u \le 0$ (μ-a.e.).

Note that the class of μ-subharmonic (resp. μ-superharmonic) functions and the class of classical subharmonic (resp. superharmonic) functions are in general not equal. In fact, the function u in Example 6.4 is μ-superharmonic but not superharmonic. Nevertheless, we will prove that the maximum principle still holds for continuous μ-subharmonic (resp. μ-superharmonic) functions. To prove this, we use mollifiers.

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain and let $u \in H_0^1(\Omega)$. Let $\widetilde{u}$ be the zero-extension of u, i.e.,

$$\widetilde{u}(\mathbf{x}) = \begin{cases} u(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega, \\ 0 & \text{if } \mathbf{x} \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

It is known that $\widetilde{u} \in H^1(\mathbb{R}^d)$ (see [1, Lemma 3.27]). Let $\epsilon > 0$. Define

$$\widetilde{u}^\epsilon := \eta_\epsilon * \widetilde{u}, \tag{3.1}$$

where $\eta_\epsilon \geq 0$ are mollifiers. It is known that for each $\epsilon > 0$, η_ϵ is smooth and satisfies $\int_{\mathbb{R}^d} \eta_\epsilon(\mathbf{x})\, d\mathbf{x} = 1$. Moreover, $\operatorname{supp}(\eta_\epsilon(\mathbf{x})) \subseteq B_\epsilon(\mathbf{0})$ and

$$\operatorname{supp}(\eta_\epsilon(\mathbf{x} - \cdot)) \subseteq B_\epsilon(\mathbf{x}) \tag{3.2}$$

(see [16, 1] for details). The following proposition follows from [1, Theorem 2.29].

Proposition 3.2. *Let $\widetilde{u}^\epsilon$ be defined in* (3.1)*. Then*

(a) $\widetilde{u}^\epsilon \in C^\infty(\mathbb{R}^d)$.
(b) *If $u \in C(\overline{\Omega})$, then $\widetilde{u}^\epsilon \to u$ uniformly on Ω.*
(c) $\widetilde{u}^\epsilon \to u$ *in* $H^1(\Omega)$ *as* $\epsilon \to 0$.

Proposition 3.3. *Let $\Omega \subseteq \mathbb{R}^d$ $(d \geq 1)$ be a bounded domain and μ be a positive finite Borel measure on $\mathbb{R}^d$ with $\operatorname{supp}(\mu) \subseteq \overline{\Omega}$ and $\mu(\Omega) > 0$. Assume that (MPI) holds. Let $u \in \operatorname{dom}(\Delta_\mu)$ be a μ-subharmonic function and $\widetilde{u}^\epsilon$ be defined as in* (3.1)*. Let $\mathbf{z} \in \Omega$. Then for any $0 < r < \operatorname{dist}(\mathbf{z}, \partial\Omega)$,*

$$\lim_{\epsilon \to 0} \int_{B_r(\mathbf{z})} \Delta(\widetilde{u}^\epsilon|_\Omega)\, d\mathbf{x} = \int_{B_r(\mathbf{z})} \Delta_\mu u\, d\mu.$$

Proof. Let $\mathbf{z} \in \Omega$ and $\epsilon > 0$ be sufficiently small so that $\epsilon < \operatorname{dist}(\mathbf{z}, \partial\Omega)/4$. Let $r \in (\epsilon, \operatorname{dist}(\mathbf{z}, \partial\Omega) - 3\epsilon)$. Then by Proposition 3.2(a), we have

$$\begin{aligned} \int_{B_r(\mathbf{z})} \Delta(\widetilde{u}^\epsilon|_\Omega)\, d\mathbf{x} &= \int_{B_r(\mathbf{z})} \Delta\big((\eta_\epsilon * \widetilde{u})|_\Omega\big)\, d\mathbf{x} \\ &= \int_{B_r(\mathbf{z})} (\Delta\eta_\epsilon * \widetilde{u})|_\Omega\, d\mathbf{x} \\ &= \int_{B_r(\mathbf{z})} \int_{B_\epsilon(\mathbf{x})} \Delta\eta_\epsilon(\mathbf{x} - \mathbf{y})\widetilde{u}(\mathbf{y})\, d\mathbf{y}d\mathbf{x}. \quad \text{(by (3.2))} \end{aligned}$$

Using (3.2) and the fact that $\eta_\epsilon(\mathbf{x} - \mathbf{y}) = 0$ on $\partial B_\epsilon(\mathbf{x})$ (see [1, 16]), we have

$$\begin{aligned} \int_{B_r(\mathbf{z})} \Delta(\widetilde{u}^\epsilon|_\Omega)\, d\mathbf{x} &= -\int_{B_r(\mathbf{z})} \int_{B_\epsilon(\mathbf{x})} \nabla\eta_\epsilon(\mathbf{x} - \mathbf{y}) \cdot \nabla\widetilde{u}(\mathbf{y})\, d\mathbf{y}d\mathbf{x} \\ &= -\int_{B_r(\mathbf{z})} \int_{B_\epsilon(\mathbf{x})} \nabla\eta_\epsilon(\mathbf{x} - \mathbf{y}) \cdot \nabla u(\mathbf{y})\, d\mathbf{y}d\mathbf{x} \\ &= \int_{B_r(\mathbf{z})} \int_{B_\epsilon(\mathbf{x})} \eta_\epsilon(\mathbf{x} - \mathbf{y})\Delta_\mu u(\mathbf{y})\, d\mu(\mathbf{y})d\mathbf{x}, \end{aligned}$$

where the second inequality holds as $\widetilde{u} = u$ in $B_\epsilon(\mathbf{x})$, and the last equality follows from [25, Proposition 2.2] and (3.2). Therefore, if we let $\chi_{B_\epsilon(\mathbf{x})}$ be the characteristic function on $B_\epsilon(\mathbf{x})$, then

$$\begin{aligned}
&\int_{B_r(\mathbf{z})} \Delta(\widetilde{u}^\epsilon|_\Omega)\, d\mathbf{x} \\
&= \int_{B_r(\mathbf{z})} \int_{B_{r+2\epsilon}(\mathbf{z})} \chi_{B_\epsilon(\mathbf{x})}(\mathbf{y}) \eta_\epsilon(\mathbf{x}-\mathbf{y}) \Delta_\mu u(\mathbf{y})\, d\mu(\mathbf{y}) d\mathbf{x} \\
&= \int_{B_{r+2\epsilon}(\mathbf{z})} \int_{B_r(\mathbf{z})} \chi_{B_\epsilon(\mathbf{x})}(\mathbf{y}) \eta_\epsilon(\mathbf{x}-\mathbf{y}) \Delta_\mu u(\mathbf{y})\, d\mathbf{x} d\mu(\mathbf{y}) \qquad \text{(Fubini)} \\
&= \int_{B_{r+2\epsilon}(\mathbf{z})} \int_{B_r(\mathbf{z}) \cap B_\epsilon(\mathbf{y})} \chi_{B_\epsilon(\mathbf{y})}(\mathbf{x}) \eta_\epsilon(\mathbf{x}-\mathbf{y}) \Delta_\mu u(\mathbf{y})\, d\mathbf{x} d\mu(\mathbf{y}) \\
&\le \int_{B_{r+2\epsilon}(\mathbf{z})} \Big(\int_{B_\epsilon(\mathbf{y})} \eta_\epsilon(\mathbf{x}-\mathbf{y})\, d\mathbf{x} \Big) \Delta_\mu u(\mathbf{y})\, d\mu(\mathbf{y}) \\
&= \int_{B_{r+2\epsilon}(\mathbf{z})} \Delta_\mu u\, d\mu
\end{aligned} \tag{3.3}$$

(see Figure 2(a)). On the other hand, since $\eta_\epsilon \ge 0$ and $\Delta_\mu u \ge 0$ μ-a.e., by (3.3) we have

$$\begin{aligned}
&\int_{B_r(\mathbf{z})} \Delta(\widetilde{u}^\epsilon|_\Omega)\, d\mathbf{x} \\
&= \int_{B_{r+2\epsilon}(\mathbf{z})} \int_{B_r(\mathbf{z}) \cap B_\epsilon(\mathbf{y})} \chi_{B_\epsilon(\mathbf{y})}(\mathbf{x}) \eta_\epsilon(\mathbf{x}-\mathbf{y}) \Delta_\mu u(\mathbf{y})\, d\mathbf{x} d\mu(\mathbf{y}) \\
&\ge \int_{B_{r-\epsilon}(\mathbf{z})} \int_{B_r(\mathbf{z}) \cap B_\epsilon(\mathbf{y})} \chi_{B_\epsilon(\mathbf{y})}(\mathbf{x}) \eta_\epsilon(\mathbf{x}-\mathbf{y}) \Delta_\mu u(\mathbf{y})\, d\mathbf{x} d\mu(\mathbf{y}) \\
&= \int_{B_{r-\epsilon}(\mathbf{z})} \Big(\int_{B_\epsilon(\mathbf{y})} \eta_\epsilon(\mathbf{x}-\mathbf{y})\, d\mathbf{x} \Big) \Delta_\mu u(\mathbf{y})\, d\mu(\mathbf{y}) \\
&= \int_{B_{r-\epsilon}(\mathbf{z})} \Delta_\mu u\, d\mu
\end{aligned} \tag{3.4}$$

(see Figure 2(b)).

Combining (3.3) and (3.4), we have

$$\int_{B_{r-\epsilon}(\mathbf{z})} \Delta_\mu u\, d\mu \le \int_{B_r(\mathbf{z})} \Delta(\widetilde{u}^\epsilon|_\Omega)\, d\mathbf{x} \le \int_{B_{r+2\epsilon}(\mathbf{z})} \Delta_\mu u\, d\mu.$$

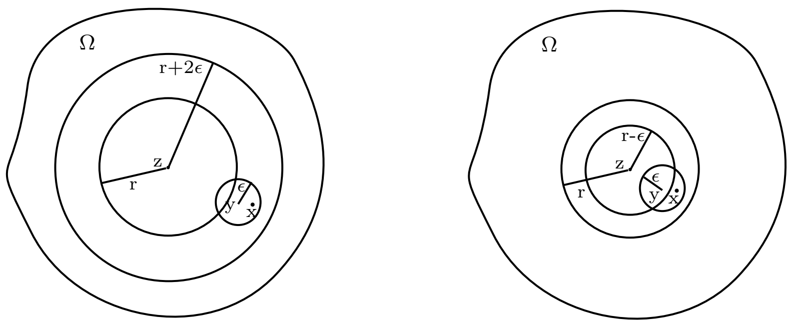
Letting $\epsilon \to 0$ completes the proof. □

Remark 3.4. We know that $\Delta_\mu u \in L^2(\Omega, \mu) \subseteq L^1(\Omega, \mu)$ for $u \in \operatorname{dom}(\Delta_\mu)$ (see [25, Proposition 2.2]). Therefore, if u is a μ-subharmonic function, then for any $B_r(\mathbf{z}) \subseteq \Omega$, the limit

$$\lim_{\epsilon \to 0} \int_{B_r(\mathbf{z})} \Delta(\widetilde{u}^\epsilon|_\Omega)\, d\mathbf{x} = \int_{B_r(\mathbf{z})} \Delta_\mu u\, d\mu$$

is nonnegative and finite.

In [52, Theorem 3.3], it is shown that if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a μ-subharmonic function, then the maximum principle holds. The following theorem generalizes this theorem to $u \in C(\overline{\Omega})$.



(a) (b)

FIGURE 1. Sets $B_r(\mathbf{z})$, $B_{r+2\epsilon}(\mathbf{z})$, $B_{r-\epsilon}(\mathbf{z})$, $B_\epsilon(\mathbf{y})$ and the positions of the points $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$.

Theorem 3.5. *Let $\Omega \subseteq \mathbb{R}^d$ $(d \geq 1)$ be a bounded domain and μ be a positive finite Borel measure on $\mathbb{R}^d$ with $\operatorname{supp}(\mu) \subseteq \overline{\Omega}$ and $\mu(\Omega) > 0$. Assume that (MPI) holds. If $u \in C(\overline{\Omega})$ is a nonconstant μ-subharmonic function, then u cannot attain its maximum value in Ω.*

Proof. Some basic derivations are similar to the proof of the mean-value formula (see e.g., [16, §2.2.2 Theorem 2]); we will omit some details. Let u be a nonconstant continuous μ-subharmonic function. For any fixed $\mathbf{x} \in \Omega$, let $\widetilde{u}^\epsilon$ be defined as in (3.1) and let $r > 0$ be sufficiently small such that $B_r(\mathbf{x}) \subseteq \Omega$. Define

$$\varphi_\epsilon(r) := \fint_{\partial B_r(\mathbf{x})} \widetilde{u}^\epsilon|_\Omega(\mathbf{y})\, dS(\mathbf{y}) = \fint_{\partial B_1(\mathbf{0})} \widetilde{u}^\epsilon|_\Omega(\mathbf{x} + r\mathbf{z})\, dS(\mathbf{z}),$$

where

$$\fint_{\partial B_r(\mathbf{x})} f\, dS := \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B_r(\mathbf{x})} f\, dS \tag{3.5}$$

is the average of f over the sphere $\partial B_r(\mathbf{x})$ and $\alpha(n) = \pi^{n/2}/\Gamma(n/2+1)$ is the volume of the unit ball $B_1(\mathbf{0})$ in $\mathbb{R}^n$. Let

$$\fint_{B_r(\mathbf{x})} f\, d\mathbf{y} := \frac{1}{\alpha(n) r^n} \int_{B_r(\mathbf{x})} f\, d\mathbf{y} \tag{3.6}$$

be the average of f over $B_r(\mathbf{x})$ (see [16, Appendix A]). By the calculation as in the proof of [16, §2.2.2 Theorem 2], we have

$$\begin{aligned} \varphi_\epsilon'(r) &= \fint_{\partial B_1(\mathbf{0})} D\widetilde{u}^\epsilon|_\Omega(\mathbf{x} + r\mathbf{z}) \cdot \mathbf{z}\, dS(\mathbf{z}) \\ &= \frac{1}{n\alpha(n) r^{n-1}} \int_{B_r(\mathbf{x})} \Delta(\widetilde{u}^\epsilon|_\Omega)(\mathbf{y})\, d\mathbf{y}. \end{aligned}$$

By Remark 3.4, $\lim_{\epsilon\to 0} \varphi_\epsilon'(r) \geq 0$. It follows that for each $t > 0$, there exists $\delta_t > 0$ such that for all $\epsilon \in (0, \delta_t)$, $\varphi_\epsilon'(r) + t \geq 0$. This implies that for each $\epsilon \in (0, \delta_t)$, $\varphi_\epsilon(r) + tr$ is an increasing function of r. For each $t > 0$, we choose $\epsilon_t > 0$ so that the function ϵ_t is decreasing and tends to 0 as $t \to 0^+$. Moreover, $\varphi_{\epsilon_t}(r) + tr$ is an increasing function of r. Letting $t \to 0^+$ and using Proposition 3.2(b), we have

$$\lim_{t\to 0^+} (\varphi_{\epsilon_t}(r) + tr) = \lim_{\epsilon_t \to 0} \fint_{\partial B_r(\mathbf{x})} \widetilde{u}^{\epsilon_t}|_\Omega(\mathbf{y})\, dS(\mathbf{y}) = \fint_{\partial B_r(\mathbf{x})} u(\mathbf{y})\, dS(\mathbf{y}) =: \varphi(r).$$

Observe that $\varphi(r)$ is an increasing function of r. Hence, by using this and the continuity of u, we have that for all $s > 0$,

$$\varphi(s) \ge \lim_{\xi \to 0} \varphi(\xi) = \lim_{\xi\to 0} \fint_{\partial B_\xi(\mathbf{x})} u(\mathbf{y})\, dS(\mathbf{y}) = u(\mathbf{x}). \tag{3.7}$$

Using (3.6) and the equation

$$\int_{B_r(\mathbf{x})} u(\mathbf{y})\, d\mathbf{y} = \int_0^r \Big(\int_{\partial B_s(\mathbf{x})} u(\mathbf{y})\, dS(\mathbf{y})\Big)\, ds,$$

which can be derived by using [16, §C3]. Then by (3.5) and (3.7) we have

$$\begin{aligned}
\fint_{B_r(\mathbf{x})} u(\mathbf{y})\, d\mathbf{y} &= \frac{1}{\alpha(n) r^n} \int_0^r \Big(\int_{\partial B_s(\mathbf{x})} u(\mathbf{y})\, dS(\mathbf{y})\Big)\, ds \\
&= \frac{1}{\alpha(n) r^n} \int_0^r n\alpha(n) s^{n-1} \Big(\fint_{\partial B_s(\mathbf{x})} u(\mathbf{y})\, dS(\mathbf{y})\Big)\, ds \\
&\ge \frac{1}{\alpha(n) r^n} \int_0^r u(\mathbf{x}) n\alpha(n) s^{n-1} \\
&= \frac{u(\mathbf{x})}{r^n} \int_0^r n s^{n-1}\, ds \\
&= u(\mathbf{x}).
\end{aligned} \tag{3.8}$$

Suppose there exists a point $\mathbf{x}_0 \in \Omega$ such that $u(\mathbf{x}_0) = M := \max_\Omega u(\mathbf{x})$. Then for $0 < r < \operatorname{dist}(\mathbf{x}_0, \partial\Omega)$, it follows from (3.8) that

$$M = u(\mathbf{x}_0) \le \fint_{B_r(\mathbf{x}_0)} u\, d\mathbf{y} \le M.$$

Hence,

$$\fint_{B_r(\mathbf{x}_0)} u\, d\mathbf{y} = M.$$

Thus $u(\mathbf{y}) = M$ for all $y \in B_r(\mathbf{x}_0)$. Since Ω is a domain, and hence is connected, it follows that $u(\mathbf{x}) = M$ for all $\mathbf{x} \in \Omega$. □

Remark 3.6. (a) By replacing u in the proof of Theorem 3.5 with $-u$, one can prove that a nonconstant continuous μ-superharmonic function attains its minimum only on $\partial\Omega$.

(b) From the proof of Theorem 3.5, one can see that the reason for introducing the function $\widetilde{u}^\epsilon|_\Omega$ is that Δu may exist as a distribution but need not exist as a function (see Example 6.4).

Let Ω be a bounded domain. For a continuous *μ-harmonic function* u, i.e., $\Delta_\mu u = 0$, we have the following proposition.

Proposition 3.7. *Let Ω be a bounded domain. If $u \in C(\overline{\Omega})$ is a μ-harmonic function on Ω and vanishes on $\partial\Omega$, then $u \equiv 0$.*

To prove Proposition 3.7, we need the following Weyl's Lemma (see [26, Corollary 2.2.1] or [53, 6, 21] and references therein).

Lemma 3.8 (Weyl's Lemma)**.** *Let $u : \Omega \to \mathbb{R}$ be measurable and locally integrable on Ω. Suppose that for any $v \in C_c^\infty(\Omega)$,*

$$\int_\Omega u(\mathbf{x})\Delta v(\mathbf{x})\, d\mathbf{x} = 0.$$

Then u is harmonic and, in particular, smooth.

Proof of Proposition 3.7. Let u be μ-harmonic, i.e., $\Delta_\mu u = 0$. Then for all $v \in C_c^\infty(\Omega)$,

$$0 = \int_\Omega v\Delta_\mu u\, d\mu = -\int_\Omega \nabla v \cdot \nabla u\, d\mathbf{x} = \int_\Omega u\Delta v\, d\mathbf{x}.$$

Moreover, u is locally integrable as $u \in \operatorname{dom}(\Delta_\mu) \subset H_0^1(\Omega)$. It follows from Lemma 3.8 that u is harmonic and smooth on Ω. Therefore, by the classical maximum principle (see [16, §6.4]) and the fact that u vanish on $\partial\Omega$, we have $u \equiv 0$. □

4. Courant nodal domain theorem

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain and μ be a positive finite Borel measure on $\mathbb{R}^d$ with $\operatorname{supp}(\mu) \subseteq \overline{\Omega}$ and $\mu(\Omega) > 0$. Let $\mathcal{E}(u,u)$ be defined as in (2.3). We define the Rayleigh quotient of μ, an important and useful quantity in studying eigenvalues. For related applications, see [3, 22, 50, 14, 9].

Definition 4.1. Use the above assumption and notation. For any $u \in \operatorname{dom}(\mathcal{E})$, the *Rayleigh quotient* associated with μ is defined as

$$R_\mu(u) := \frac{\mathcal{E}(u,u)}{(u,u)_{L^2(\Omega,\mu)}} = \frac{\int_\Omega |\nabla u|^2\, d\mathbf{x}}{\int_\Omega |u|^2\, d\mu}.$$

The following lemma is inspired by [14, 43]. It can be proved by using similar methods as in the proofs of [14, Theorem 1.3] and [43, Corollary 4.3]; we omit the proof.

Lemma 4.2. *Let $\lambda_1, \dots, \lambda_n$ be eigenvalues of* (1.2) *and let $u_1(\mathbf{x}), \dots, u_n(\mathbf{x})$ be corresponding eigenfunctions. Then*

- (a) $\lambda_1 = \min\{R_\mu(u) : u \in \operatorname{dom}(\mathcal{E})\}$ *and* $R_\mu(u_1) = \lambda_1$. *Moreover, if there exists some $u \in \operatorname{dom}(\mathcal{E})$ such that $R_\mu(u) = \lambda_1$, then u is a λ_1-eigenfunction.*
- (b) *For $n \geq 2$, $\lambda_n = \min\{R_\mu(u) : u \in \operatorname{dom}(\mathcal{E}), (u,u_i)_{L^2(\Omega,\mu)} = 0, i = 1, \dots, n-1\}$ and $R_\mu(u_n) = \lambda_n$. Moreover, if there exists some $u \in \operatorname{dom}(\mathcal{E})$ such that $(u,u_i)_{L^2(\Omega,\mu)} = 0$ for $i = 1, \dots, n-1$, and $R_\mu(u) = \lambda_n$, then u is a λ_n-eigenfunction.*

For each $f \in H_0^1(\Omega)$, under the assumption $\underline{\dim}_\infty(\mu) > d-2$, it is known that there exists a unique $L^2(\Omega,\mu)$-representative $\widehat{f}$ of f. Let

$$\mathcal{H}_\mu(\Omega) := \{f \in H_0^1(\Omega) : \|\widehat{f}\|_{L^2(\Omega,\mu)} > 0\}.$$

For each $f \in \mathcal{H}_\mu(\Omega)$, we define

$$\widetilde{R}_\mu(f) := \frac{\int_\Omega |\nabla f|^2\, d\mathbf{x}}{\int_\Omega |\widehat{f}|^2\, d\mu}.$$

Lemma 4.3. *Assume the hypotheses of Lemma 4.2. For $n \geq 1$, let*

$$\mathcal{H}_\mu^n(\Omega) := \begin{cases} \mathcal{H}_\mu(\Omega) & \text{if } n = 1, \\ \{f \in \mathcal{H}_\mu(\Omega) : (\widehat{f}, u_i)_{L^2(\Omega,\mu)} = 0, i = 1, \dots, n-1\} & \text{if } n \geq 2. \end{cases}$$

Then $\lambda_n = \min\{\widetilde{R}_\mu(f) : f \in \mathcal{H}_\mu^n(\Omega)\}$. Moreover, for each $n \geq 1$, if $\widetilde{R}_\mu(f) = \lambda_n$ holds for some $f \in \mathcal{H}_\mu^n(\Omega)$, then f is a λ_n-eigenfunction.

Proof. For any fixed $n \geq 1$, we first prove that if $\widetilde{R}_\mu(f)$ attains minimum at f in $\mathcal{H}_\mu^n(\Omega)$, then $f \in \operatorname{dom}(\mathcal{E})$. In fact, for any $f \in \mathcal{H}_\mu^n(\Omega)$, let $f = f_\mathcal{E} + f_\mathcal{N}$ be the direct sum of f, where $f_\mathcal{E} \in \operatorname{dom}(\mathcal{E}) = \mathcal{N}^\perp$ and $f_\mathcal{N} \in \mathcal{N}$. Then

$$\langle f_\mathcal{E}, f_\mathcal{N} \rangle_{H_0^1(\Omega)} = \int_\Omega \nabla f_\mathcal{E} \cdot \nabla f_\mathcal{N} \, d\mathbf{x} = 0. \tag{4.1}$$

Moreover, $\|f_\mathcal{E}\|_{L^2(\Omega,\mu)} = \|\widehat{f}\|_{L^2(\Omega,\mu)}$ as $\|f_\mathcal{N}\|_{L^2(\Omega,\mu)} = 0$. Thus,

$$\begin{aligned} \widetilde{R}_\mu(f) &= \frac{\int_\Omega |\nabla f|^2 \, d\mathbf{x}}{\int_\Omega |\widehat{f}|^2 \, d\mu} = \frac{\int_\Omega |\nabla f_\mathcal{E} + \nabla f_\mathcal{N}|^2 \, d\mathbf{x}}{\int_\Omega |f_\mathcal{E}|^2 \, d\mu} \\ &= \frac{\int_\Omega |\nabla f_\mathcal{E}|^2 + |\nabla f_\mathcal{N}|^2 \, d\mathbf{x}}{\int_\Omega |f_\mathcal{E}|^2 \, d\mu}. \quad \text{(by (4.1))} \end{aligned} \tag{4.2}$$

Since $f_\mathcal{E} \in \mathcal{H}_\mu^n(\Omega)$, it follows by (4.2) that $\widetilde{R}_\mu(f)$ attains minimum in $\mathcal{H}_\mu(\Omega)$ if and only if $\|f_\mathcal{N}\|_{H_0^1(\Omega)} = 0$, i.e., $f = f_\mathcal{E} \in \operatorname{dom}(\mathcal{E})$. Moreover, we can conclude that

$$\min\{\widetilde{R}_\mu(f) | f \in \mathcal{H}_\mu\} = \min\{R_\mu(u) | u \in \operatorname{dom}(\mathcal{E})\} = \lambda_1 \tag{4.3}$$

and for $n \geq 2$,

$$\begin{aligned} &\min\{\widetilde{R}_\mu(f) | f \in \mathcal{H}_\mu^n\} \\ &= \min\{R_\mu(u) | u \in \operatorname{dom}(\mathcal{E}), (u, u_i)_{L^2(\Omega,\mu)} = 0, i = 1, \dots, n-1\} \\ &= \lambda_n. \end{aligned} \tag{4.4}$$

Therefore, the last assertion of the lemma follows from Lemma 4.2. □

Proof of theorem 1.1. We follow [50] for the proof of (a). We use some methods and techniques in [22] to prove (b).

(a) We divide the proof into two steps as follows: *Step 1.* Suppose on the contrary that the λ_1-eigenfunction u_1 has a node. i.e., there exists $\mathbf{x}_0 \in \Omega$ such that

$$u_1(\mathbf{x}_0) = 0. \tag{4.5}$$

Let

$$\Omega^+ := \{\mathbf{x} \in \Omega \,|\, u_1(\mathbf{x}) > 0\} \quad \text{and} \quad \Omega^- := \{\mathbf{x} \in \Omega \,|\, u_1(\mathbf{x}) < 0\}.$$

We claim that Ω^+ and Ω^- are nonempty. In fact, if $\Omega^+ = \emptyset$, then for any $\mathbf{x} \in \Omega$,

$$u_1(\mathbf{x}) \leq 0. \tag{4.6}$$

Since $-\Delta_\mu u_1 = \lambda_1 u_1$ and $\lambda_1 > 0$, we have $-\Delta_\mu u_1(\mathbf{x}) = \lambda_1 u_1(\mathbf{x}) \leq 0$. Thus

$$\Delta_\mu u_1(\mathbf{x}) \geq 0 \quad \text{for all } \mathbf{x} \in \Omega.$$

Hence u_1 is a μ-subharmonic function. Combining this with Theorem 3.5, we see that u_1 cannot attain 0 in Ω. Thus, by (4.6),

$$u_1(\mathbf{x}) < 0 \quad \text{for all } \mathbf{x} \in \Omega,$$

which contradicts (4.5). Hence $\Omega^+ \neq \emptyset$. By the same argument, $\Omega^- \neq \emptyset$.

Step 2. Let

$$u^+(\mathbf{x}) = \begin{cases} u_1(\mathbf{x}), & \mathbf{x} \in \Omega^+, \\ 0, & \mathbf{x} \in \Omega \setminus \Omega^+, \end{cases}$$

and let $u^-(\mathbf{x}) = u_1(\mathbf{x}) - u^+(\mathbf{x})$. Note that $|u_1(\mathbf{x})| = u^+(\mathbf{x}) - u^-(\mathbf{x})$ and

$$\nabla u^+(\mathbf{x}) = \begin{cases} \nabla u_1(\mathbf{x}), & \mathbf{x} \in \Omega^+, \\ 0, & \mathbf{x} \in \Omega\setminus\Omega^+, \end{cases} \quad \nabla u^-(\mathbf{x}) = \begin{cases} 0, & \mathbf{x} \in \Omega\setminus\Omega^-, \\ \nabla u_1(\mathbf{x}) & \mathbf{x} \in \Omega^-. \end{cases}$$

Obviously, u^+ and u^- belong to $H_0^1(\Omega)$. It follows by the linearity of $H_0^1(\Omega)$ that $|u_1| \in H_0^1(\Omega)$.

Step 3. Since $|u_1| \in H_0^1(\Omega)$ and $\||u_1|\|_{L^2(\Omega,\mu)} > 0$, we have $|u_1| \in \mathcal{H}_\mu$ and thus

$$\begin{aligned} \widetilde{R}_\mu(|u_1|) &= \frac{\int_\Omega \big|\nabla|u_1|\big|^2 \, d\mathbf{x}}{\int_\Omega |u_1|^2 \, d\mu} = \frac{\int_{\Omega^+} \big|\nabla u_1\big|^2 \, d\mathbf{x}}{\int_\Omega |u_1|^2 \, d\mu} + \frac{\int_{\Omega^-} \big|\nabla u_1\big|^2 \, d\mathbf{x}}{\int_\Omega |u_1|^2 \, d\mu} \\ &= \frac{\int_\Omega |\nabla u_1|^2 \, d\mathbf{x}}{\int_\Omega |u_1|^2 \, d\mu} = \frac{\int_\Omega (-\Delta_\mu u_1) \cdot u_1 \, d\mu}{\int_\Omega |u_1|^2 \, d\mu} \\ &= \frac{\lambda_1 \int_\Omega u_1^2 \, d\mu}{\int_\Omega |u_1|^2 \, d\mu} = \lambda_1. \end{aligned}$$

It follows from Lemma 4.2(a) that $\lambda_1 = \min\{R_\mu(u) : u \in \operatorname{dom}(\mathcal{E})\}$. Combining this and (4.3), we have

$$\widetilde{R}_\mu(|u_1|) = \lambda_1.$$

Hence, by Lemma 4.3, $|u_1|$ is a λ_1-eigenfunction, i.e., $-\Delta_\mu |u_1| = \lambda_1 |u_1|$. Combining this with $\lambda_1 > 0$, we have

$$\Delta_\mu |u_1(\mathbf{x})| \le 0 \quad \text{for all } \mathbf{x} \in \Omega.$$

Hence $|u_1|$ is a μ-superharmonic function. By Remark 3.6, u_1 cannot attain 0 in Ω. Thus,

$$|u_1(\mathbf{x})| > 0 \quad \text{for all } \mathbf{x} \in \Omega,$$

i.e., $|u_1|$ does not have nodes in Ω. This contradicts Step 1 and completes the proof of the first part of the theorem.

(b) By the proof of (a), $u_1(\mathbf{x}) \neq 0$, for any $\mathbf{x} \in \Omega$. Without loss of generality, we assume $u_1(\mathbf{x}) > 0$. Since u_n is orthogonal to u_1, i.e.,

$$\int_\Omega u_n u_1 \, d\mu = 0,$$

u_n must change sign in Ω. Thus u_n must be positive on some subdomians of Ω and negative on some other subdomains of Ω. By the continuity of u_n, these subdomains must be separated by the nodal set of u_n. Hence, for $n \ge 2$, the nodal set of u_n divides Ω into at least two subdomains.

For the second part of (b), let $\mathcal{Z}_\mu$ be defined as in (1.3). Then $\Omega \setminus \mathcal{Z}_\mu(u_n) = \{\mathbf{x} \in \Omega \mid u_n(\mathbf{x}) \neq 0\}$. Assume $\mathcal{Z}_\mu(u_n)$ divides Ω into m ($m \ge 2$) subdomains: $\Omega_1, \dots, \Omega_m$, where the Ω_i are pairwise disjoint and separated by a subset of $\mathcal{Z}_\mu(u_n)$. Moreover,

$$\Omega \setminus \mathcal{Z}_\mu(u_n) = \cup_{j=1}^m \Omega_j.$$

Let

$$w_j(\mathbf{x}) = \begin{cases} u_n(\mathbf{x}), & \mathbf{x} \in \Omega_j, \\ 0, & \mathbf{x} \in \Omega\setminus\Omega_j. \end{cases}$$

Then

$$\nabla w_j(\mathbf{x}) = \begin{cases} \nabla u_n(\mathbf{x}), & \mathbf{x} \in \Omega_j, \\ 0, & \mathbf{x} \in \Omega \backslash \Omega_j. \end{cases}$$

Let

$$w = \sum_{j=1}^{m} c_j w_j, \tag{4.7}$$

where $c_1, \dots, c_m$ are arbitrary constants. Note that $w \in \mathcal{H}_\mu$ and

$$\begin{aligned} \widetilde{R}_\mu(w) &= \frac{\int_\Omega |\nabla w|^2 \, d\mathbf{x}}{\int_\Omega |w|^2 \, d\mu} = \frac{\sum_{j=1}^m c_j^2 \int_{\Omega_j} |\nabla u_n|^2 \, d\mathbf{x}}{\sum_{j=1}^m c_j^2 \int_{\Omega_j} |u_n|^2 \, d\mu} \\ &= \frac{\sum_{j=1}^m c_j^2 \int_{\Omega_j} (-\Delta_\mu u_n) \cdot u_n \, d\mu}{\sum_{j=1}^m c_j^2 \int_{\Omega_j} |u_n|^2 \, d\mu} = \frac{\lambda_n \sum_{j=1}^m c_j^2 \int_{\Omega_j} u_n^2 \, d\mu}{\sum_{j=1}^m c_j^2 \int_{\Omega_j} |u_n|^2 \, d\mu} = \lambda_n. \end{aligned} \tag{4.8}$$

Since the system (4.9) below has $m-1$ equations in m unknowns c_j, it has a nonzero solution $\{c_1, \dots, c_m\}$. Hence, we can choose the coefficients $\{c_j\}_{j=1}^m$ of w in (4.7) so that

$$\big(w, u_i\big)_{L^2(\Omega,\mu)} = 0, \quad i = 1, \dots, m-1, \tag{4.9}$$

where $\{u_i\}_{i=1}^{m-1}$ are the first $m-1$ eigenfunctions. For this choice of $\{c_j\}_{j=1}^m$, $w \in \mathcal{H}_\mu^m$. By (4.4), we have

$$\widetilde{R}_\mu(w) \geq \lambda_m.$$

Combining this and (4.8), we have $\lambda_m \leq \lambda_n$. Since $\lambda_n < \lambda_{n+r_n}$, we have $\lambda_m < \lambda_{n+r_n}$. Thus $m \leq n + r_n - 1$. Therefore, the nodal set of u_n divides Ω into at most $n + r_n - 1$ subdomains. □

From Theorem 1.1(a), we can immediately derive the following corollary.

Corollary 4.4. *The multiplicity of the first eigenvalue λ_1 is 1.*

Proof. Suppose, on the contrary, that the multiplicity of λ_1 is not 1. Then there exists another λ_1-eigenfunction v so that u_1 and v are linearly independent in $L^2(\Omega, \mu)$. Write $v = c_1 u_1 + w$ in $L^2(\Omega, \mu)$, where c_1 is a constant and $w \in (\mathrm{span}(u_1))^\perp$. Thus

$$\int_\Omega u_1 w \, d\mu = 0. \tag{4.10}$$

As in the proof of Theorem 1.1, w does not change sign in Ω. Combining this and (4.10), we have $w(\mathbf{x}) = 0$ (μ-a.e.) in Ω. Thus $v = c_1 u_1$ in $L^2(\Omega, \mu)$, contradicting the fact that u_1 and v are linearly independent in $L^2(\Omega, \mu)$. This completes the proof. □

5. Continuity of eigenfunctions

In this section, we prove Theorem 1.2. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain. We recall the Green function of the classical Laplacian Δ. For $u \in C^2(\Omega)$,

$$\Delta u = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}.$$

Let

$$g(\mathbf{x}, \mathbf{y}) = \begin{cases} -\frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{y}| & \text{if } d = 2, \\ -|\mathbf{x} - \mathbf{y}|^{2-d} & \text{if } d \geq 3. \end{cases} \tag{5.1}$$

The following definition of Green's function follows from [15].

Definition 5.1. Let $d \geq 2$, $\Omega \subseteq \mathbb{R}^d$ be a bounded domain. If there exists a real valued function $h_\Omega(\cdot,\cdot)$ on $\Omega \times \Omega$ such that for each $\mathbf{x} \in \Omega$, $h_\Omega(\mathbf{x},\cdot)$ is the harmonic minorant of $g(\mathbf{x},\cdot)$ on Ω, i.e., $h_\Omega(\mathbf{x},\cdot)$ is the greatest harmonic function satisfying

$$h_\Omega(\mathbf{x},\cdot) \leq g(\mathbf{x},\cdot) \tag{5.2}$$

on Ω, then the function

$$G(\mathbf{x},\mathbf{y}) := g(\mathbf{x},\mathbf{y}) - h_\Omega(\mathbf{x},\mathbf{y}) \tag{5.3}$$

is called the *Green function* of Ω.

It is known that $h_\Omega(\mathbf{x},\mathbf{y})$, if it exists, is a symmetric continuous function on $\overline{\Omega} \times \overline{\Omega}$ [25]. Hence the Green function $G(\mathbf{x},\mathbf{y})$ is symmetric on $\Omega \times \Omega$ [15]. Some basic properties of the Green function are summarized below. Fix any point $\mathbf{x} \in \Omega$.

(a) $G(\mathbf{x},\cdot)$ is defined on $\Omega \times \Omega$ and $G(\mathbf{x},\mathbf{x}) = +\infty$ [15, Chapter VII.4].
(b) $G(\mathbf{x},\cdot)$ is continuous and harmonic on $\Omega - \{\mathbf{x}\}$ [15, Chapter VII.4].
(c) $G(\mathbf{x},\mathbf{y}) \geq 0$ for any $\mathbf{y} \in \Omega$ (by (5.2)).

For a bounded domain on which the Green function exists, we provide an equivalent definition of a regular boundary (see [15, VIII 14] or [35]).

Definition 5.2. A bounded domain $\Omega \subseteq \mathbb{R}^d$ ($d \geq 2$) on which the Green function exists is said to have a *regular boundary* if for any $\mathbf{z} \in \partial\Omega$ and $\mathbf{y} \in \Omega$,

$$\lim_{\mathbf{x}\to\mathbf{z}} G(\mathbf{x},\mathbf{y}) = 0.$$

Remark 5.3. For any open set $\Omega \subseteq \mathbb{R}^d$, the Green function always exists when $d \geq 3$. When $d = 2$, the Green function exists if $\mathbb{R}^2 \setminus \partial\Omega$ is not connected (see [4, Theorm 4.1.2] and [35, 5]). Examples of domains with regular boundary include those with smooth or Lipschitz boundaries (see [35, Section 4]).

For $f \in C^1(\Omega)$, the unique solution of the equation in $C^2(\Omega)$:

$$\begin{gathered} -\Delta u = f \\ u|_{\partial\Omega} = 0 \end{gathered}$$

can be represented through the Green function $G(\mathbf{x},\mathbf{y})$ by

$$u(\mathbf{x}) = \int_\Omega G(\mathbf{x},\mathbf{y}) f(\mathbf{y})\, d\mathbf{y}.$$

More details about the Green function can be found in, for example, [4, 16, 35, 10, 11, 15].

According to [25], the Green function $G(\mathbf{x},\mathbf{y})$ for Δ, if exists, is also the Green function for Δ_μ. It means that for the equation $-\Delta_\mu u = f$, there exists a Green operator defined on $L^p(\Omega,\mu)$ ($p \geq 1$) by

$$(G_\mu f)(\mathbf{x}) := \int_\Omega G(\mathbf{x},\mathbf{y}) f(\mathbf{y})\, d\mu(\mathbf{y}) \tag{5.4}$$

such that $u = G_\mu f$. The operator G_μ is the inverse of $-\Delta_\mu$ [25, Theorem 1.3]. To ensure that G_μ has good properties, we need the following assumption in [25]:

$$\sup_{\mathbf{x}\in\Omega} \int_\Omega G(\mathbf{x},\mathbf{y})\, d\mu(\mathbf{y}) \leq C < +\infty \quad \text{for some constant } C > 0. \tag{5.5}$$

It is proved in [25] that the condition $\underline{\dim}_\infty(\mu) > d - 2$ implies (5.5).

Proposition 5.4. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain on which the classical Green function $G(\mathbf{x},\mathbf{y})$ exists and let $f \in \operatorname{dom}(\Delta_\mu)$. Let μ be a finite positive Borel measure with $\operatorname{supp}(\mu) \subseteq \overline{\Omega}$. Assume $\underline{\dim}_\infty(\mu) > d-2$. Then $G_\mu f$ is bounded, i.e., there exists a constant $\widetilde{C} > 0$ such that $|G_\mu f(\mathbf{x})| \le \widetilde{C}$ for all $\mathbf{x} \in \Omega$.*

Proof. Since the case $d = 1$ is clear, we divide the proof into two cases: $d = 2$ and $d \ge 3$.

Case 1. $d = 2$. We claim that for each $\mathbf{x} \in \Omega$, $G(\mathbf{x},\mathbf{y}) \in L^2(\Omega,\mu)$. By (5.3), it suffices to prove that there exists some constant $\widetilde{C} > 0$ such that

$$\int_\Omega (\ln|\mathbf{x}-\mathbf{y}|)^2 \, d\mu(\mathbf{y}) \le \widetilde{C} \tag{5.6}$$

for all $\mathbf{x} \in \Omega$. Using the same method in the proof of [25, Proposition 4.1], one can prove that (5.6) holds. Hence,

$$\big|G_\mu f(\mathbf{x})\big| = \Big|\int_\Omega G(\mathbf{x},\mathbf{y}) f(\mathbf{y}) \, d\mu(\mathbf{y})\Big| \le \|G(\mathbf{x},\cdot)\|_{L^2(\Omega,\mu)} \|f\|_{L^2(\Omega,\mu)}. \tag{5.7}$$

Case 2. $d \ge 3$. Since $f \in \operatorname{dom}(\Delta_\mu) \subseteq H_0^1(\Omega)$ and $\underline{\dim}_\infty(\mu) > d-2$, there exists a sequence $f_m \in C_c^\infty(\Omega)$ such that $f_m \to f$ in $L^2(\Omega,\mu)$. We claim that there exists some constant $C_1 > 0$ such that

$$\lim_{m\to\infty} \big|G_\mu(f^2 - f_m^2)\big| \le C_1. \tag{5.8}$$

To see this, by (5.3), it suffices to prove that

$$\lim_{m\to\infty} \int_\Omega |g(\mathbf{x},\mathbf{y})| \cdot |f^2(\mathbf{y}) - f_m^2(\mathbf{y})| \, d\mu(\mathbf{y}) \le C_1.$$

Note that, by Hölder's inequality,

$$\begin{aligned} \int_\Omega |f^2 - f_m^2| \, d\mu &= \int_\Omega |f - f_m||f + f_m| \, d\mu \\ &\le \|f - f_m\|_{L^2(\Omega,\mu)} \|f + f_m\|_{L^2(\Omega,\mu)} \to 0 \end{aligned} \tag{5.9}$$

as $m \to \infty$. Let $\operatorname{diam}(\Omega) = r_0$. We have

$$\begin{aligned} &\int_\Omega |g(\mathbf{x},\mathbf{y})||f^2(\mathbf{y}) - f_m^2(\mathbf{y})| \, d\mu(\mathbf{y}) \\ &= \int_{|\mathbf{x}-\mathbf{y}|\le 1} |\mathbf{x}-\mathbf{y}|^{-(d-2)} |f^2(\mathbf{y}) - f_m^2(\mathbf{y})| \, d\mu(\mathbf{y}) \\ &\quad + \int_{1\le|\mathbf{x}-\mathbf{y}|\le r_0} |\mathbf{x}-\mathbf{y}|^{-(d-2)} |f^2(\mathbf{y}) - f_m^2(\mathbf{y})| \, d\mu(\mathbf{y}). \end{aligned}$$

By (5.9), the second integral on the right-hand side tends to 0, since

$$\begin{aligned} &\int_{1\le|\mathbf{x}-\mathbf{y}|\le r_0} |\mathbf{x}-\mathbf{y}|^{-(d-2)} |f^2(\mathbf{y}) - f_m^2(\mathbf{y})| \, d\mu(\mathbf{y}) \\ &\le \int_{1\le|\mathbf{x}-\mathbf{y}|\le r_0} |f^2(\mathbf{y}) - f_m^2(\mathbf{y})| \, d\mu(\mathbf{y}). \end{aligned} \tag{5.10}$$

Furthermore, we let $V_k(\mathbf{x}) := \{\mathbf{y} : 2^{-k} \le |\mathbf{x}-\mathbf{y}| \le 2^{-(k-1)}\}$. Then

$$\begin{aligned}
&\int_{|\mathbf{x}-\mathbf{y}|\le 1} |\mathbf{x}-\mathbf{y}|^{-(d-2)} |f^2(\mathbf{y}) - f_m^2(\mathbf{y})| \, d\mu(\mathbf{y}) \\
&= \sum_{k=1}^{\infty} \int_{V_k(\mathbf{x})} |\mathbf{x}-\mathbf{y}|^{-(d-2)} |f^2(\mathbf{y}) - f_m^2(\mathbf{y})| \, d\mu(\mathbf{y}) \\
&\le \lim_{N\to\infty} \sum_{k=1}^{N} 2^{k(d-2)} \int_{V_k(\mathbf{x})} |f^2(\mathbf{y}) - f_m^2(\mathbf{y})| \, d\mu(\mathbf{y}).
\end{aligned} \tag{5.11}$$

By (5.9), for each $k \in \{1, 2, \dots, N\}$, there exists m_N sufficiently large such that

$$\int_{V_k(\mathbf{x})} |f^2(\mathbf{y}) - f_{m_N}^2(\mathbf{y})| \, d\mu(\mathbf{y}) \le 2^{-2k(d-2)}.$$

Hence

$$\sum_{k=1}^{N} 2^{k(d-2)} \int_{V_k(\mathbf{x})} |f^2(\mathbf{y}) - f_{m_N}^2(\mathbf{y})| \, d\mu(\mathbf{y}) \le \sum_{k=1}^{N} 2^{-k(d-2)}. \tag{5.12}$$

Letting $N \to \infty$, we have, by (5.11) and (5.12),

$$\int_{|\mathbf{x}-\mathbf{y}|\le 1} |\mathbf{x}-\mathbf{y}|^{-(d-2)} |f^2(\mathbf{y}) - f_m^2(\mathbf{y})| \, d\mu(\mathbf{y}) \le \sum_{k=1}^{\infty} 2^{-k(d-2)} < +\infty.$$

Combining this and (5.10) completes the proof of the claim in (5.8). It follows from (5.8) that there exists some sufficiently large integer N_0 such that for all $m > N_0$ and all $\mathbf{x} \in \Omega$,

$$\big|G_\mu(f^2(\mathbf{x}) - f_m^2(\mathbf{x}))\big| \le C_1 + 1.$$

It follows from Property (c) that for any $\mathbf{x}, \mathbf{y} \in \Omega$, $G(\mathbf{x}, \mathbf{y}) \ge 0$. Hence,

$$\begin{aligned}
\big|G_\mu f^2\big| &\le \big|G_\mu f_{N_0+1}^2\big| + C_2 \\
&\le \int_\Omega \big|G(\mathbf{x}, \mathbf{y}) f_{N_0+1}^2\big| \, d\mu + C_2 \quad \text{(by (5.4))} \\
&\le \|f_{N_0+1}^2\|_{L^\infty(\Omega)} \int_\Omega G(\mathbf{x}, \mathbf{y}) \, d\mu(y) + C_2 \\
&\le C\|f_{N_0+1}^2\|_{L^\infty(\Omega)} + C_2 \quad \text{(by (5.5))} \\
&=: C_3,
\end{aligned}$$

where $C_2 = C_1 + 1$. This proves that

$$\big|G_\mu f^2(\mathbf{x})\big| = \int_\Omega G(\mathbf{x}, \mathbf{y}) f^2(\mathbf{y}) \, d\mu(\mathbf{y}) \le C_3. \tag{5.13}$$

Now, for $d \ge 3$, by Hölder's inequality, (5.5) and (5.13), we have

$$\begin{aligned}
\big|G_\mu f(\mathbf{x})\big|^2 &= \Big| \int_\Omega G(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) \, d\mu(\mathbf{y}) \Big|^2 \\
&\le \Big(\int_\Omega \big|G(\mathbf{x}, \mathbf{y})^{1/2} f(\mathbf{y}) \cdot G(\mathbf{x}, \mathbf{y})^{1/2}\big| \, d\mu(\mathbf{y}) \Big)^2 \\
&\le \int_\Omega G(\mathbf{x}, \mathbf{y}) f^2(\mathbf{y}) \, d\mu(\mathbf{y}) \cdot \int_\Omega G(\mathbf{x}, \mathbf{y}) \, d\mu(\mathbf{y}) \\
&\le C \cdot C_3. \quad \text{(by (5.13))}
\end{aligned}$$

Therefore $\big|G_\mu f\big|$ is bounded by $\sqrt{C \cdot C_3}$. Combining this and (5.7) completes the proof. $\square$

We give an outline of the proof of Theorem 1.2. First, we prove any λ-eigenfunction u can be expressed as $\lambda G_\mu u$. Second, we use the properties of Green's function, Lebesgue's dominated convergence theorem and ideas from [35, Proposition 1.26.7] to study the continuity of $G_\mu u$ on Ω.

Proof of Theorem 1.2. Let $f \in \operatorname{dom}(\Delta_\mu)$, we claim that f is a λ-eigenfunction of Δ_μ if and only if $f = \lambda G_\mu f$. In fact, on the one hand, if $f \in \operatorname{dom}(\Delta_\mu)$ satisfies $f = \lambda G_\mu f$, then by [25, Theorem 1.3], we have

$$-\Delta_\mu f = -\lambda \Delta_\mu G_\mu f = \lambda f,$$

which implies that f is a λ-eigenfunction of Δ_μ. On the other hand, if $f \in \operatorname{dom}(\Delta_\mu)$ is a λ-eigenfunction, i.e., $-\Delta_\mu f = \lambda f$, in view of the fact that $G_\mu = \big(-\Delta_\mu\big)^{-1}$ (see [25, Theorem 1.3]), we have

$$G_\mu(\lambda f) = G_\mu(-\Delta_\mu f) = f.$$

The linearity of G_μ implies that $f = \lambda G_\mu f$. Therefore, to prove Theorem 1.2, it suffices to prove that for any $f \in \operatorname{dom}(\Delta_\mu)$, $G_\mu f$ is continuous. We divide the proof into three steps.

Step 1. We claim that for any $\epsilon > 0$, there exists some $f_1 \in L^2(\Omega, \mu)$ such that $|G_\mu f_1| < \epsilon$. In the case $d = 2$, by (5.6), we see that for each $\mathbf{y} \in \Omega$, $G(\cdot, \mathbf{y})$ belongs to $L^2(\Omega, \mu)$, and $\|G(\cdot, \mathbf{y})\|_{L^2(\Omega,\mu)}$ has a uniform bound independent of $\mathbf{y}$. Hence, there exists a constant $\widehat{C}_1$ such that for all $\mathbf{y} \in \Omega$, $\|G(\cdot, \mathbf{y})\|_{L^2(\Omega,\mu)} \le \widehat{C}_1$. Since $\underline{\dim}_\infty(\mu) > d - 2 = 0$, μ does not have point masses, i.e., μ is a continuous measure [43, Proposition 6.5]. Therefore, for any $\mathbf{z} \in \Omega$,

$$\lim_{r \to 0^+} \int_{B_r(\mathbf{z})} |G(\mathbf{x}, \mathbf{y})|^2 \, d\mu(\mathbf{y}) = 0 \quad \text{for all } \mathbf{x} \in \Omega.$$

Hence, for any $\epsilon > 0$, there exists $\widetilde{r}_1 > 0$ sufficiently small such that for all $\mathbf{x} \in \Omega$,

$$\int_{B_{\widetilde{r}_1}(\mathbf{z})} |G(\mathbf{x}, \mathbf{y})|^2 \, d\mu(\mathbf{y}) \le \epsilon^2. \tag{5.14}$$

Now consider the case $d \ge 3$. Let $\epsilon > 0$. Then by (5.5) and the continuity of μ again, there exists $\widetilde{r}_2 > 0$ sufficiently small such that for all $\mathbf{x} \in \Omega$,

$$\int_{B_{\widetilde{r}_2}(\mathbf{z})} G(\mathbf{x}, \mathbf{y}) \, d\mu(\mathbf{y}) < \epsilon^2. \tag{5.15}$$

Let $\widetilde{r} := \min\{\widetilde{r}_1, \widetilde{r}_2\}$ and $f_1 := f \chi_{B_{\widetilde{r}}(\mathbf{z})}$. For the case $d = 2$, by Hölder's inequality and (5.14), we have

$$\begin{aligned}
|G_\mu f_1| &= \Big| \int_\Omega G(\mathbf{x}, \mathbf{y}) f_1(\mathbf{y}) \, d\mu(\mathbf{y}) \Big| \\
&\le \|f_1\|_{L^2(\Omega,\mu)} \Big(\int_{B_{\widetilde{r}}(z)} G^2(\mathbf{x}, \mathbf{y}) \, d\mu(\mathbf{y}) \Big)^{1/2} \\
&\le \epsilon \|f\|_{L^2(\Omega,\mu)}.
\end{aligned} \tag{5.16}$$

For the case $d \ge 3$, by Hölder's inequality and (5.15), we have

$$\big|G_\mu f_1\big|^2 = \Big| \int_\Omega G(\mathbf{x}, \mathbf{y}) f_1(\mathbf{y}) \, d\mu(\mathbf{y}) \Big|^2$$

$$\begin{aligned}&\le \int_{B_{\widetilde{r}}(\mathbf{z})} G(\mathbf{x},\mathbf{y}) f_1^2(\mathbf{y})\,d\mu(\mathbf{y}) \cdot \int_{B_{\widetilde{r}}(\mathbf{z})} G(\mathbf{x},\mathbf{y})\,d\mu(\mathbf{x})\\ &\le \epsilon^2 \int_\Omega G(\mathbf{x},\mathbf{y}) f^2(\mathbf{y})\,d\mu(\mathbf{y}).\end{aligned}$$

Combining this and (5.13), we have

$$\big|G_\mu f_1\big| \le C_3 \cdot \epsilon. \tag{5.17}$$

Therefore, combining (5.16) and (5.17) proves the claim.

Step 2. Let $f_2 := f - f_1$. We claim that $G_\mu f_2$ is continuous at $\mathbf{z}$. In fact, by the continuity of $G(\mathbf{x},\mathbf{y})$, for any $\mathbf{y} \in \Omega \setminus B_{\widetilde{r}}(\mathbf{z})$, $G(\mathbf{z},\mathbf{y})$ is continuous at $\mathbf{z}$, i.e.,

$$\lim_{\mathbf{x}\to\mathbf{z}} G(\mathbf{x},\mathbf{y}) = G(\mathbf{z},\mathbf{y}). \tag{5.18}$$

Note that by (5.1) and (5.3) there exists $\delta \in (0, \widetilde{r}/2)$ such that for any $\mathbf{x} \in B_\delta(\mathbf{z})$ and all $\mathbf{y} \in \Omega \setminus B_{\widetilde{r}}(\mathbf{z})$,

$$|G(\mathbf{x},\mathbf{y})| \le C_4 := \max\Big\{\Big|\log\frac{\widetilde{r}}{2}\Big| + C_5, \Big(\frac{\widetilde{r}}{2}\Big)^{2-d} + C_5\Big\},$$

where $C_5 > 0$ is a constant. Moreover, $f_2 \in L^1(\Omega,\mu)$ as $\|f_2\|_{L^1(\Omega,\mu)} \le \|f\|_{L^1(\Omega,\mu)}$. Therefore, by (5.18) and Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}\lim_{\mathbf{x}\to z} G_\mu f_2(\mathbf{x}) &= \lim_{\mathbf{x}\to\mathbf{z}} \int_\Omega G(\mathbf{x},\mathbf{y}) f_2(\mathbf{y})\,d\mu(\mathbf{y})\\ &= \lim_{x\to\mathbf{z}} \int_{\Omega\setminus B_{\widetilde{r}}(\mathbf{z})} G(\mathbf{x},\mathbf{y}) f_2(\mathbf{y})\,d\mu(\mathbf{y})\\ &= \int_{\Omega\setminus B_{\widetilde{r}}(\mathbf{z})} G(\mathbf{z},\mathbf{y}) f_2(\mathbf{y})\,d\mu(\mathbf{y})\\ &= G_\mu f_2(\mathbf{z}).\end{aligned}$$

Step 3. By Step 2, for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $\widetilde{\mathbf{z}} \in B_\delta(\mathbf{z})$, $\big|G_\mu f_2(\mathbf{z}) - G_\mu f_2(\widetilde{\mathbf{z}})\big| < \epsilon$. Combining this, Step 1 and the definitions of f_1 and f_2, we have

$$\begin{aligned}|G_\mu f(\mathbf{z}) - G_\mu f(\widetilde{\mathbf{z}})| &= |G_\mu f_1(\mathbf{z}) + G_\mu f_2(\mathbf{z}) - G_\mu f_1(\widetilde{\mathbf{z}}) - G_\mu f_2(\widetilde{\mathbf{z}})|\\ &\le |G_\mu f_1(\mathbf{z})| + |G_\mu f_1(\widetilde{\mathbf{z}})| + |G_\mu f_2(\mathbf{z}) - G_\mu f_2(\widetilde{\mathbf{z}})|\\ &\le 3\epsilon,\end{aligned}$$

which shows that $G_\mu f$ is continuous at $\mathbf{z}$. Since $\mathbf{z}$ is arbitrary, $G_\mu f$ is continuous on Ω.

Finally, if the boundary of Ω is regular, then, applying the argument in Steps 1–3 to $\overline{\Omega}$, we can prove that the eigenfunctions of Δ_μ are continuous on $\overline{\Omega}$. □

6. Examples of continuous eigenfunctions

In this section, we assume that $\Omega \subseteq \mathbb{R}^2$. We will construct some examples of continuous eigenfunctions of Δ_μ. As mentioned in Section 1, we are interested in the case that the measures are singular with respect to Lebesgue measure.

Let $\mathcal{D}(\Omega) := C_c^\infty(\Omega)$ be the vector space of test functions. Recall that a distribution $T : \mathcal{D}(\Omega) \to \mathbb{C}$ is a continuous linear map [23, 49]. For any locally integrable function $f \in L^1_{\rm loc}(\Omega)$, define a distribution T_f as

$$\langle T_f, v\rangle := \int_\Omega fv\, d\mathbf{x}, \quad v \in \mathcal{D}(\Omega).$$

The i-th partial distributional derivative $\partial T/\partial x_i$ of a distribution T is defined by

$$\Big\langle \frac{\partial T}{\partial x_i}, v\Big\rangle = -\Big\langle T, \frac{\partial v}{\partial x_i}\Big\rangle, \quad v \in \mathcal{D}(\Omega).$$

We consider the square domain $\Omega = (-1,1)\times(-1,1)$. To describe the distributional derivative of functions we construct, we need to define the following specific distributions in $\mathcal{D}'(\Omega)$, where $\mathcal{D}'(\Omega)$ is the dual space of $\mathcal{D}(\Omega)$.

Definition 6.1. *Let $\Omega = (-1,1)\times(-1,1)$ and $\beta \in (-1,1)$ be a fixed constant. Define $\delta^{\rm I,\beta}$ and $\delta^{\beta,\rm II}$ as distributions in $\mathcal{D}'(\Omega)$ so that the following equations hold for all $v(x,y) \in \mathcal{D}(\Omega)$:*

$$\big\langle \delta^{\rm I,\beta}, v(x,y)\big\rangle := \int_\Omega v(x,y)\, dx d\delta_\beta(y) = \int_{-1}^1 v(x,\beta)\, dx, \tag{6.1}$$

$$\big\langle \delta^{\beta,\rm II}, v(x,y)\big\rangle := \int_\Omega v(x,y)\, d\delta_\beta(x) dy = \int_{-1}^1 v(\beta,y)\, dy, \tag{6.2}$$

where δ_β is the Dirac *measure at β defined on $\mathbb{R}$.*

Remark 6.2. The superscript β in $\delta^{\rm I,\beta}$ and $\delta^{\beta,\rm II}$ represents the point at which the Dirac measure takes the value 1. The position of β indicates the axis on which the Dirac measure is defined. Hence, δ_β in (6.1) is defined on the y-axis, and δ_β in (6.2) is defined on the x-axis. The Roman superscripts I and II represent dx and dy, respectively.

It can be checked directly that $\delta^{\rm I,\beta}$ and $\delta^{\beta,\rm II}$ are distributions. Moreover, the following property holds.

Proposition 6.3. *Use the above notations. For any $f(x,y) \in C(\Omega)$, $f(x,\beta)\delta^{\rm I,\beta}$ and $f(\beta,y)\delta^{\beta,\rm II}$ are distributions in $\mathcal{D}'(\Omega)$.*

Proof. For each $v(x,y) \in \mathcal{D}(\Omega)$, by (6.2) and (6.1), we have

$$\big\langle f(x,\beta)\delta^{\rm I,\beta}, v(x,y)\big\rangle = \int_\Omega f(x,\beta)v(x,y)\, d\delta_\beta(y)\, dx = \int_{-1}^1 f(x,\beta)v(x,\beta)\, dx,$$

$$\big\langle f(\beta,y)\delta^{\beta,\rm II}, v(x,y)\big\rangle = \int_\Omega f(\beta,y)v(x,y)\, d\delta_\beta(x)\, dy = \int_{-1}^1 f(\beta,y)v(\beta,y)\, dy\,.$$

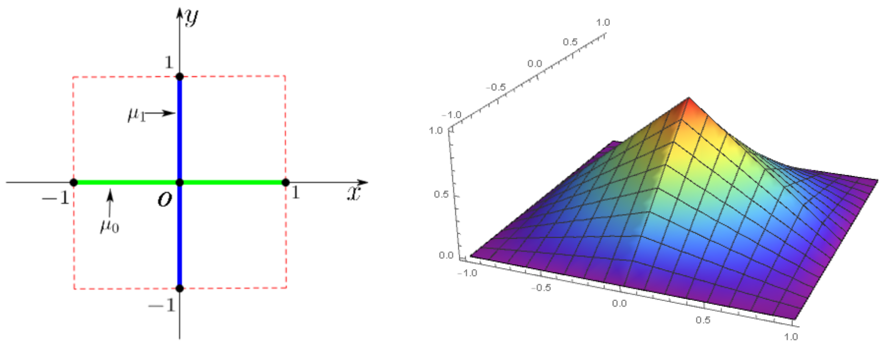
The proof can be completed by checking the linearity and continuity. □

For the above square domain $\Omega = (-1,1)\times(-1,1)$, let μ_0 be the 1-dimensional Lebesgue measure defined on $[-1,1]\times\{0\}$ and μ_1 be the 1-dimensional Lebesgue measure defined on $\{0\}\times[-1,1]$, as shown in Figure 2(a). We will use μ_0 and μ_1 to construct a measure on Ω, which is singular respect to the Lebesgue measure on $\mathbb{R}^2$.

Example 6.4. Use the above notation. Let $\mu := \mu_0 + \mu_1$ be defined on Ω. Then μ is singular respect to the 2-dimensional Lebesgue measure $d\mathbf{x}$. Let Δ_μ be defined as in Section 2. Then

$$u(x,y) = 1 + |xy| - |x| - |y| \tag{6.3}$$

is a 2-eigenfunction of (1.2). $u(x,y)$ (see Figure 2(b)) is continuous and has no nodal points in Ω. Hence, it is a first eigenfunction of $-\Delta_\mu$ in equation (1.2).



(a) (b)

FIGURE 2. Measures and eigenfunctions in Example 6.4.

Proof. According to [25, Proposition 2.2] and (1.2), we have, for any $v(x,y) \in \mathcal{D}(\Omega)$,

$$-\int_\Omega \Delta u(x,y)v(x,y)\,dx\,dy = \lambda \int_\Omega u(x,y)v(x,y)\,d\mu. \tag{6.4}$$

We first calculate the second-order distributional derivative of $u(x,y)$. For any $v(x,y) \in \mathcal{D}(\Omega)$, we have

$$\begin{aligned}
\Big\langle \frac{\partial^2 T_{|x|}}{\partial x^2}, v(x,y)\Big\rangle &= -\Big\langle \frac{\partial T_{|x|}}{\partial x}, v_x(x,y)\Big\rangle = \langle T_{|x|}, v_{xx}(x,y)\rangle \\
&= \int_\Omega |x| v_{xx}(x,y)\,dx\,dy \\
&= \int_{-1}^1 dy \int_0^1 x v_{xx}(x,y)\,dx - \int_{-1}^1 dy \int_{-1}^0 x v_{xx}(x,y)\,dx \\
&= 2\int_{-1}^1 v(0,y)\,dy \\
&= 2\big\langle \delta^{0,\mathrm{II}}, v(x,y)\big\rangle.
\end{aligned}$$

Hence, $\Delta|x| = 2\delta^{0,\mathrm{II}}$ in the sense of distribution. By the same argument, we have

$$\Delta|y| = 2\delta^{\mathrm{I},0}$$

and

$$\Delta|xy| = 2\left(|x|\delta^{\mathrm{I},0} + |y|\delta^{0,\mathrm{II}}\right)$$

in the sense of distribution. In summary,

$$\Delta u(x,y) = \Delta(1 + |xy| - |x| - |y|) = 2(|x|\delta^{\mathrm{I},0} + |y|\delta^{0,\mathrm{II}} - \delta^{\mathrm{I},0} - \delta^{0,\mathrm{II}}), \tag{6.5}$$

in the sense of distribution. By direct calculation, we have

$$\begin{aligned}
-\int_\Omega \Delta u(x,y)v(x,y)\,dx\,dy &= -\int_\Omega \Delta\big(1 + |xy| - |x| - |y|\big)v(x,y)\,dx\,dy \\
&= 2\Big(\int_{-1}^1 \big(1-|x|\big)v(x,0)\,dx + \int_{-1}^1 \big(1-|y|\big)v(0,y)\,dy\Big)
\end{aligned}$$

$$
\begin{aligned}
&= 2\int_\Omega \big(1-|x|-|y|\big)v(x,y)\,d(\mu_0+\mu_1)\\
&= 2\int_\Omega \big(1-|x|-|y|\big)v(x,y)\,d\mu.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\lambda\int_\Omega u(x,y)v(x,y)\,d\mu &= \lambda\int_\Omega \big(1+|xy|-|x|-|y|\big)v(x,y)\,d(\mu_0+\mu_1)\\
&= \lambda\int_\Omega \big(1-|x|-|y|\big)v(x,y)\,d(\mu_0+\mu_1)\\
&= \lambda\int_\Omega \big(1-|x|-|y|\big)v(x,y)\,d\mu.
\end{aligned}
$$

Combining the above two equalities and (6.4), we obtain $\lambda = 2$. Therefore, if $u(x,y)\in \operatorname{dom}(\Delta_\mu)$, it is a 2-eigenfunction. Next, we prove that $u(x,y)\in\operatorname{dom}(\Delta_\mu)$. We divide the proof into three steps.

Step 1. We claim that $u(x,y)\in H_0^1(\Omega)$. To see this, let

$$
\xi_1(x,y)=\begin{cases}\operatorname{sgn}(x)(y-1), & y\ge 0,\\ -\operatorname{sgn}(x)(y+1) & y<0,\end{cases}
$$

$$
\xi_2(x,y)=\begin{cases}\operatorname{sgn}(y)(x-1), & x\ge 0,\\ -\operatorname{sgn}(y)(x+1), & x<0.\end{cases}
$$

By direct calculation, $(\xi_1(x,y),\xi_2(x,y))$ is the weak derivative of $u(x,y)$. Moreover, both $\xi_1(x,y)$ and $\xi_2(x,y)$ are in $L^2(\Omega)$. Thus, $u(x,y)\in H^1(\Omega)$. Note that the square domain Ω is a Lipschitz domain. According to [34, Theorem 3.33], we have $u(x,y)\in H_0^1(\Omega)$.

Step 2. We prove that $u(x,y)\in\operatorname{dom}(\mathcal{E})$. Let $\mathcal{I}$ be defined as in (2.2) and

$$
\tau(\mathbf{x})\in\mathcal{N} := \big\{u\in H_0^1(\Omega) : \|\mathcal{I}(u)\|_{L^2(\Omega,\mu)}=0\big\}.
$$

Write $\Omega=\cup_{i=1}^4\Omega_i$, where $\Omega_1=[0,1)\times[0,1)$, $\Omega_2=(-1,0)\times[0,1)$, $\Omega_3=(-1,0]\times(-1,0]$, and $\Omega_4=(0,1)\times(-1,0)$. By Step 1, we have

$$
(u,\tau)_{H_0^1(\Omega)}=\int_\Omega \nabla u\cdot\nabla\tau\,d\mathbf{x}=\int_\Omega(\xi_1,\xi_2)\cdot\nabla\tau\,d\mathbf{x}=\sum_{i=1}^4\int_{\Omega_i}(\xi_1,\xi_2)\cdot\nabla\tau\,d\mathbf{x}.
$$

Since $\tau\in\mathcal{N}$, we have

$$
\begin{aligned}
\int_{\Omega_1}(\xi_1,\xi_2)\cdot\nabla\tau\,d\mathbf{x} &= \int_0^1\int_0^1 (y-1,x-1)\cdot(\tau_x,\tau_y)\,dx\,dy\\
&= \int_0^1\int_0^1 (y-1)\tau_x+(x-1)\tau_y\,dx\,dy\\
&= \int_0^1(y-1)\,dy\int_0^1\tau_x\,dx+\int_0^1(x-1)\,dx\int_0^1\tau_y\,dy\\
&= \int_0^1(y-1)\tau(x,y)\Big|_0^1\,dy+\int_0^1(x-1)\tau(x,y)\Big|_0^1\,dx\\
&= -\int_0^1(y-1)\tau(0,y)\,dy-\int_0^1(x-1)\tau(x,0)\,dx=0.
\end{aligned}
$$

The last equality holds because $\tau = 0$ dx-a.e. on $\{0\} \times (-1,1)$ and $(-1,1) \times \{0\}$. Similarly, we obtain

$$\int_{\Omega_i} (\xi_1, \xi_2) \cdot \nabla \tau \, d\mathbf{x} = 0, \quad \text{for } i = 1,2,3,4.$$

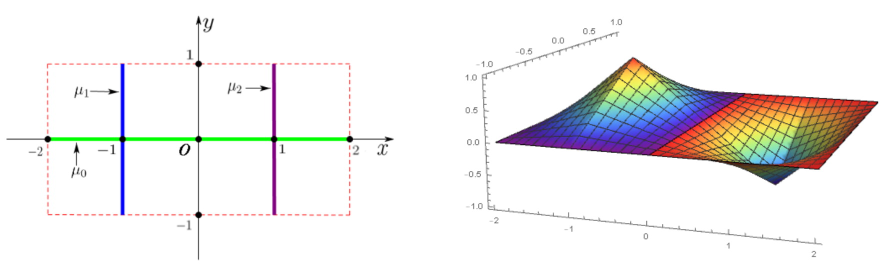
Hence, $\langle u, \tau \rangle_{H_0^1(\Omega)} = 0$, for $\tau \in \mathcal{N}$. Thus $u(x,y) \in \mathcal{N}^{\perp} = \operatorname{dom}(\mathcal{E})$.

Step 3. We prove that $u(x,y) \in \operatorname{dom}(\Delta_\mu)$. Combining [25, Proposition 2.2] and (6.5), we have

$$\begin{aligned}
\int_\Omega \nabla u \cdot \nabla v \, d\mathbf{x} &= -\int_\Omega \Delta u \cdot v \, d\mathbf{x} \\
&= 2\Big[\int_{-1}^{1} \big(1 - |x|\big) v(x,0)\, dx + \int_{-1}^{1} \big(1 - |y|\big) v(0,y)\, dy\Big] \\
&= 2 \int_\Omega \big(1 - |x| - |y|\big) v(x,y)\, d(\mu_0 + \mu_1) \\
&= 2 \int_\Omega \big(1 + |xy| - |x| - |y|\big) v(x,y)\, d\mu.
\end{aligned}$$

Moreover, $f(x,y) := 2(1 + |xy| - |x| - |y|) = 2u(x,y) \in L^2(\Omega, \mu)$. Therefore, by [25, Proposition 2.2] again, $u(x,y) \in \operatorname{dom}(\Delta_\mu)$. □

Using the method of Example 6.4, we can construct the following two examples.



(a) (b)

FIGURE 3. Measures and eigenfunctions in Example 6.5.

Example 6.5. Let $\Omega = (-2,2) \times (-1,1)$. Write $\Omega_1 = (-2,0] \times (-1,1)$ and $\Omega_2 = [0,2) \times (-1,1)$. Let measure $\mu = \mu_0 + \mu_1 + \mu_2$ be defined on Ω, μ_0, μ_1 and μ_2 are 1-dimensional Lebesgue measures on $[-2,2] \times \{0\}$, $\{-1\} \times [-1,1]$ and $\{1\} \times [-1,1]$, respectively, as shown in Figure 3(a). Then

$$u(x,y) = \begin{cases} 1 + |(x+1)y| - |x+1| - |y|, & (x,y) \in \Omega_1, \\ -1 - |(x-1)y| + |x-1| + |y|, & (x,y) \in \Omega_2 \end{cases}$$

is a 2-eigenfunction that satisfies equation (1.2). $u(x,y)$ is continuous, and the nodal line of u divides the domain Ω into 2 subdomains (see Figure 3(b)). We omit the proof as it can be obtained by the argument in Example 6.4.

Example 6.6. Let $\Omega = (0,2n) \times (-1,1)$, $\Omega_i = (2(i-1), 2i] \times (-1,1)$, $i = 1, \dots, n-1$, $\Omega_n = [2(n-1), 2n) \times (-1,1)$, and $\Omega = \cup_{i=1}^n \Omega_i$. Let $\mu = \mu_0 + \mu_1 + \dots + \mu_n$ be defined on Ω, where μ_0 is the 1-dimensional Lebesgue measure on $[0,2n] \times \{0\}$, and

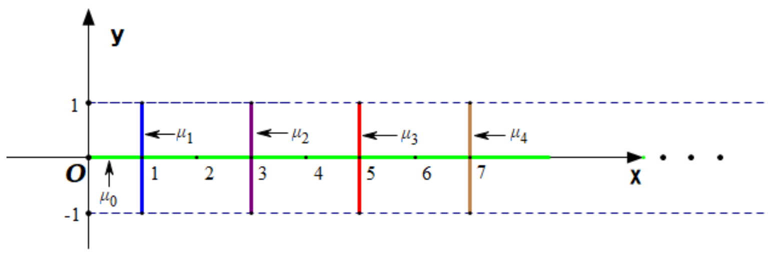
μ_i is the 1-dimensional Lebesgue measure on $\{2i-1\} \times [-1,1]$, as shown in Figure 4(a). For $i = 1, \dots, n$, let

$$\psi_i(x,y) := \begin{cases} (-1)^{i-1}\big(1 + \big|\big(x-(2i-1)\big)y\big| - |x-(2i-1)| - |y|\big), & (x,y) \in \Omega_i, \\ 0, & (x,y) \in \Omega \backslash \Omega_i. \end{cases}$$

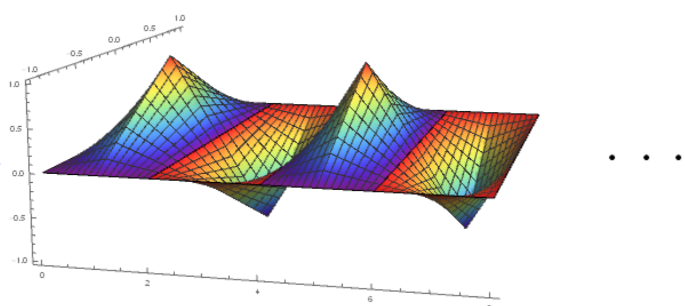
We define

$$u(x,y) := \sum_{i=1}^{n} \psi_i(x,y).$$

Then $u(x,y)$ is a 2-eigenfunction satisfying equation (1.2). $u(x,y)$ is continuous, and the nodal lines of the function divide Ω into n subdomains (see Figure 4(b). The method is the same as that in the proof of Example 6.4; we omit it.



(a)



(b)

FIGURE 4. Measures and eigenfunctions in Example 6.6.

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