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HÉNON-TYPE EQUATIONS INVOLVING THE BIHARMONIC OPERATOR AND A COORDINATE PRODUCT WEIGHT

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Abstract. We consider the Hénon-type equation

$$\Delta^2 u = [W(z)]^\ell f(u) \text{ in } B, \quad u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial B,$$

where B is the unit ball in $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, the weight function $W(z)$ behaves like $|x||y|$ for $x \in \mathbb{R}^{N_1}$, $y \in \mathbb{R}^{N_2}$, and the nonlinearity f is allowed to exhibit supercritical growth.We establish a new radial-type lemma adapted to the weight $|x|^{(N_1-2)/2}\,|y|^{(N_2-2)/2}$, which yields a weighted Sobolev embeddings for our functional framework into L^p spaces, with exponents $p > 1$, possibly within the supercritical range. Finally, we prove the existence of a weak solution to the problem.

1. Introduction

Radial-type inequalities have played a central role in the analysis of some supercritical elliptic problems. In this context, the problem

$$-\Delta u = |z|^\ell |u|^{p-2}u \text{ in } B, \quad u = 0 \text{ on } \partial B, \tag{1.1}$$

where $B \subset \mathbb{R}^N$ is the unit ball, $N \geq 3$, $\ell > 0$ and $p > 2$, was introduced by Michel Hénon [13] as a model for investigating spherically symmetric clusters of stars. Mathematically, its significance grew after Ni's paper [18], where the existence of positive weak solutions was established for $2 < p < 2^* + 2\ell/(N-2)$, with $2^* := 2N/(N-2)$. The crucial aspect there lies in obtaining a constant $C > 0$ such that

$$|u(z)| \leq \frac{\|\nabla u\|_{L^2(\Omega)}}{\sqrt{\omega_N (N-2)}|z|^{(N-2)/2}}, \quad z \in B,$$

for any radially symmetric $u \in C^1(B)$ vanishing in the boundary of B. With this inequality in hand, it is possible to embed the space of radial functions $H^1_{0,rad}(B)$ into Lebesgue spaces $L^s(B)$ with the number s beyond the critical Sobolev exponent 2^*.

The first aim of this paper is establishing a version of the above inequality that involves the $H^2_0(B)$ norm, when the space $\mathbb{R}^N$ has an specific decomposition.

To be more specific, we decompose $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ with $3 \leq N_2 \leq N_1$ and denote an arbitrary vector in $\mathbb{R}^N$ as $z = (x, y)$, with $x \in \mathbb{R}^{N_1}$ and $y \in \mathbb{R}^{N_2}$. For any $1 \leq p < +\infty$ and $\ell \geq 0$, we set

$$L^p_\ell(B) := \big\{u \in L^1_{\text{loc}}(B) : \int_B |u(z)|^p [W(z)]^\ell \,\mathrm{d}z < +\infty\big\},$$

where W satisfies

(A1) $W \in L^1_{\text{loc}}(B)$ and there exists $c_W > 0$ such that

$$0 < W(z) \leq c_W |x||y|, \quad \text{for a.e. } z \in B.$$

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This is a Banach space with the norm

$$\|u\|_{L^p_\ell(B)} := \Big(\int_B |u(z)|^p [W(z)]^\ell \, dz\Big)^{1/p}.$$

Denoting by $O(k)$ the group of real orthogonal $k \times k$ matrices, we say that $u : B \to \mathbb{R}$ is *coordinate-radial* if, for any $(x, y) \in B$, and $T_i \in O(N_i)$, $i = 1, 2$, it holds

$$u(x, y) = u(T_1 x, T_2 y).$$

With this definition in mind, we set

$$H^2_{0,x,y}(B) := \overline{C^\infty_{0,x,y}(B)}^{H^2_0(B)},$$

where

$$C^\infty_{0,x,y}(B) := \{u \in C^\infty_0(B) : u \text{ is coordinate-radial}\}.$$

Now we state our first main result.

Theorem 1.1. *There exists $C = C(N_1, N_2) > 0$ such that, for any $u \in H^2_{0,x,y}(B)$, it holds*

$$|u(x, y)| \le C \frac{\|\Delta u\|_{L^2(B)}}{|x|^{\frac{N_1-2}{2}} |y|^{\frac{N_2-2}{2}}}, \quad \text{for a.e. } (x, y) \in B. \tag{1.2}$$

The above theorem was inspired by some ideas contained in [18, p. 802] and [6, Corollary 4.1], which considered versions of the above inequality for the Laplacian and biharmonic operator, respectively, but with a power of $|(x, y)|$ depending on N in the denominator. We also learn from [14, Lemma 2.1], where it is proved that there exists $C_1 = C_1(N)$ such that

$$u(x, y) \le C_1 \frac{\|u\|^{1/2}_{L^2(\mathbb{R}^N)} \|\nabla_x u\|^{1/2}_{L^2(\mathbb{R}^N)}}{|x|^{\frac{N_1-1}{2}} |y|^{\frac{N_2}{2}}},$$

for any $u \in C^\infty_0(\mathbb{R}^N)$ such that $u(x, y) = \varphi(|x|, |y|)$, for φ non-increasing in $|y|$, and $N_1 \ge 2, N_2 \ge 1$.

As a consequence of Theorem 1.1, we can prove an embedding result for the space $H^2_{0,x,y}(B)$. Actually, if we set

$$2^*_{\ell,N_1} := \frac{2N_1}{N_1 - 2} + \frac{2\ell}{N_1 - 2},$$

for any $N_1 > 2$ and $\ell > 0$, we have the following:

Theorem 1.2. *Let $N = N_1 + N_2$ with $3 \le N_2 \le N_1$. Then, for any $\ell \ge 0$ and $1 \le p < 2^*_{\ell,N_1}$, the embedding $H^2_{0,x,y}(B) \hookrightarrow L^p_\ell(B)$ is compact.*

It is important to analyze situations in which the above result allows us to consider exponents beyond the critical Sobolev exponent $2^{**} := 2N/(N-4)$ of the embedding $H^2(B)$ into the Lebesgue spaces. A simple computation shows that

$$2^{**} < 2^*_{\ell,N_1} \iff \ell > \frac{2(N_1 - N_2)}{N - 4},$$

and therefore we can consider supercritical growth. The most favorable situation occurs when $N_1 = N_2$, as in this case the exponent $2^*_{\ell,N_1}$ is supercritical for every $\ell > 0$. Nevertheless, even when $N_1 \ne N_2$, the condition stated above remains rather mild. Indeed, under the assumption $3 \le N_2 < N_1$, it is straightforward to verify that

$$\frac{2(N_1 - N_2)}{N - 4} < 2,$$

which in turn guarantees that supercritical growth may still occur for all $\ell \ge 2$.

The embeddings obtained in Theorem 1.2 are closely related to, and complement, the results presented in [6, Theorem 1.4 and Corollary 1.5]. For a detailed comparison, we refer to Remark 3.2, where it is shown that our result covers a strictly larger range of the parameters p and ℓ. We omit the full verification here, as it involves technical computations that are more suitably addressed in that specific remark.

We also observe that in the case $N_1 \le N_2$, analogous results can be derived by replacing the exponent $2^*_{\ell,N_1}$ with $2^*_{\ell,N_2}$ throughout the argument.

Sobolev embeddings like that in Theorem 1.2 can be used to derive the existence of solutions for nonlinear PDE's. As a model example, we follow Ni [18] and consider the problem

$$\begin{gathered}\Delta^2 u = [W(z)]^\ell f(u), \quad \text{in } B,\\ u = \frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial B,\end{gathered} \tag{P}$$

with the supercritical nonlinearity f satisfying

- (A2) $f \in C(\mathbb{R},\mathbb{R})$;
- (A3) there exists $c_f > 0$ and $p \in (2, 2^*_{\ell,N_1})$ such that
$$|f(s)| \le c_f \left(1 + |s|^{p-1}\right), \quad \forall s \in \mathbb{R};$$
- (A4) it holds $\lim_{s\to 0} f(s)/s = 0$;
- (A5) there exist $\mu > 2$ and $s_0 > 0$ such that
$$0 < \mu F(s) \le s f(s), \quad \forall\, |s| \ge s_0 > 0,$$
where $F(s) := \int_0^s f(t)dt$.

We shall prove the following result.

Theorem 1.3. *Let $\ell \ge 0$ and $p \in (2, 2^*_{\ell,N_1})$. Then (P) has non-zero weak solution in $H^2_{0,x,y}(B)$.*

Since the seminal contribution of Ni, numerous authors have investigated equation (1.1) from different viewpoints. Given the vastness of the literature, we do not aim to provide an exhaustive account; instead, we refer the reader to [3, 4, 5, 7, 11, 15, 16, 17, 9, 19, 20] and the references therein for a broader perspective. In particular, we highlight the contributions in [6, 12, 21, 22], which concern the biharmonic operator and have played a relevant role in motivating the present study.

To the best of our knowledge, this is the first investigation of the Hénon-type equation, referred to in (1.1), that simultaneously incorporates the biharmonic operator and the non-radial, coupled weight $|x||y|$. We note that even in the simpler case involving the Laplacian operator, equation (1.1) with the $|x||y|$ weight seems not to have been addressed. We could cite the related paper [8], where the authors considered (1.1) with weights of the form $|z_N|^\ell$ or $|y|^\ell$ (for $(x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_1}$), and obtained qualitative properties of solutions concentrating on spheres. So, the results established in this paper extend and complement the aforementioned works by handling the additional complexity introduced by both the higher-order operator and the coupled weight structure.

The rest of this article is organized as follows. In Section 2, we present some notations and preliminary results, including the proof of an integral identity in the space $H^2_{0,x,y}(B)$. In Section 3, we provide the proofs of Theorems 1.1 and 1.2. Finally, Section 4 is devoted to the study of problem (P).

2. Technical results

From now on, we shall denote by

$$D := \left\{(s,t) \in \mathbb{R}^2 \,:\, s^2 + t^2 < 1\right\}$$

the unitary ball in $\mathbb{R}^2$, and

$$D_+ := \left\{(s,t) \in D \,:\, s,\, t \ge 0\right\}.$$

Notice that, if $u \in H^1_0(B)$ is such that $u = u \circ (T_1, T_2)$, for any $T_i \in O(N_i)$, $i = 1, 2$, then the function

$$v(s,t) := u(se, tf), \quad (s,t) \in D$$

is well defined, where $e \in \mathbb{R}^{N_1}$ and $f \in \mathbb{R}^{N_2}$ satisfies $|e|_{\mathbb{R}^{N_1}} = |f|_{\mathbb{R}^{N_2}} = 1$. From the symmetry properties of u it is clear that v is radial in each of this components and $u(x, y) = v(|x|, |y|)$, for any $(x, y) \in B$.

For the reader's convenience, we state and prove below an integral identity that was used in [2].

Lemma 2.1. *Let $N \geq 2$, and write $N = N_1 + N_2$, with $N_1, N_2 \in \mathbb{N}$. If $u \in L^1(B)$ and $u(x,y) = v(|x|,|y|)$ for some v defined on D_+, then*

$$\int_B u(z)\,\mathrm{d}z = \frac{4\pi^{N/2}}{\Gamma(\frac{N_1}{2})\Gamma(\frac{N_2}{2})}\int_{D_+} v(s,t)s^{N_1-1}t^{N_2-1}d(s,t),$$

where $\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}\,dt$ is the Gamma function.

Proof. By setting $\mathbb{R}^+ := [0,+\infty)$ and using polar coordinates [10, Theorem 2.9], we obtain

$$\begin{aligned}
\int_B u(z)\,\mathrm{d}z &= \int_{\mathbb{R}^N} v(|x|,|y|)\chi_B(z)\,\mathrm{d}z\\
&= \int_{\mathbb{R}^{N_1}}\Big[\int_{\mathbb{R}^{N_2}} v(|x|,|y|)\chi_B(z)\mathrm{d}y\Big]\mathrm{d}x\\
&= \int_{\mathbb{R}^{N_1}}\Big[\int_{\mathbb{R}^+}\int_{\mathbb{S}^{N_2-1}} v(|x|,|y't|)\chi_B(x,y't)t^{N_2-1}\mathrm{d}\sigma^2_{y'}\mathrm{d}t\Big]\mathrm{d}x\\
&= \int_{\mathbb{R}^+}\int_{\mathbb{S}^{N_2-1}}\Big[\int_{\mathbb{R}^{N_1}} v(|x|,t)\chi_B(x,y't)t^{N_2-1}\mathrm{d}x\Big]\mathrm{d}\sigma^2_{y'}\mathrm{d}t\\
&= \int_{\mathbb{R}^+}\int_{\mathbb{S}^{N_2-1}}\Big[\int_{\mathbb{R}^+}\int_{\mathbb{S}^{N_1-1}} v(|x's|,t)\chi_B(x's,y't)s^{N_1-1}t^{N_2-1}\mathrm{d}\sigma^1_{x'}\mathrm{d}s\Big]\mathrm{d}\sigma^2_{y'}dt\\
&= \int_{\mathbb{R}^+\times\mathbb{R}^+}\Big[\int_{\mathbb{S}^{N_2-1}}\int_{\mathbb{S}^{N_1-1}} v(s,t)\chi_B(x's,y't)s^{N_1-1}t^{N_2-1}\mathrm{d}\sigma^1_{x'}\mathrm{d}\sigma^2_{y'}\Big]\mathrm{d}(s,t),
\end{aligned}$$

where $\mathbb{S}^{N_i-1}$ is the surface of the unit ball $B^i \subset \mathbb{R}^{N_i}$ and $d\sigma^i$ is the surface element.

Since for each $(x',y') \in \mathbb{S}^{N_1-1}\times\mathbb{S}^{N_2-1}$ and $s,t \geq 0$ it holds

$$\chi_B(x's,y't) = \chi_{D_+}(s,t),$$

we obtain

$$\begin{aligned}
\int_B u(z)\,\mathrm{d}z &= \int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\Big[\int_{\mathbb{S}^{N_2-1}}\int_{\mathbb{S}^{N_1-1}} v(s,t)\chi_B(x's,y't)s^{N_1-1}t^{N_2-1}\mathrm{d}\sigma^1_{x'}\mathrm{d}\sigma^2_{y'}\Big]\mathrm{d}s\mathrm{d}t\\
&= \omega_{N_1}\omega_{N_2}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} v(s,t)\chi_{D_+}(s,t)s^{N_1-1}t^{N_2-1}\mathrm{d}s\mathrm{d}t\\
&= \omega_{N_1}\omega_{N_2}\int_{\mathbb{R}^+\times\mathbb{R}^+} v(s,t)\chi_{D_+}(s,t)s^{N_1-1}t^{N_2-1}\mathrm{d}(s,t)\\
&= \omega_{N_1}\omega_{N_2}\int_{D_+} v(s,t)s^{N_1-1}t^{N_2-1}\mathrm{d}(s,t),
\end{aligned}$$

with $\omega_{N_i} := \sigma^i(\mathbb{S}^{N_i-1})$, $i = 1,2$. As proved in [10, Proposition 2.54], we have that

$$\omega_{N_i} = \frac{2\pi^{\frac{N_i}{2}}}{\Gamma(\frac{N_i}{2})}, \quad i = 1,2.$$

The proof is complete. $\square$

We present and prove in what follows two auxiliary results which will be essential in the proof of Theorem 1.1.

Lemma 2.2. *If $(s_0,t_0) \in D$, then there exists $\tilde{t}$ such that $|t_0| < |\tilde{t}| \leq 1$ and $(s_0,\tilde{t}) \in \partial D$. Also, for each $\beta \in (|t_0|,|\tilde{t}|)$, there exists s_β such that $|s_0| < |s_\beta| < 1$ and $(s_\beta,\beta) \in \partial D$.*

Proof. Let $(s_0,t_0) \in D$, define the vertical line $r_1(t) := (|s_0|,t)$ and notice that $|r_1(\tilde{t})| = 1$ for $\tilde{t} = \pm\sqrt{1-s_0^2}$, that is, $(s_0,\tilde{t}) \in \partial D$. Notice that $|t_0| < |\tilde{t}| < 1$. Analogously, fixing $|t_0| < \beta < |\tilde{t}|$ and defining the horizontal line $r_2(\eta) := (\eta,\beta)$, we can notice that $|r_2(s_\beta)| = 1$ for $s_\beta = \pm\sqrt{1-\beta^2}$ and $|s_0| < |s_\beta| < 1$, because $s_0^2 + \beta^2 < s_0^2 + \tilde{t}^2 = 1$ (see Figure 1). $\square$

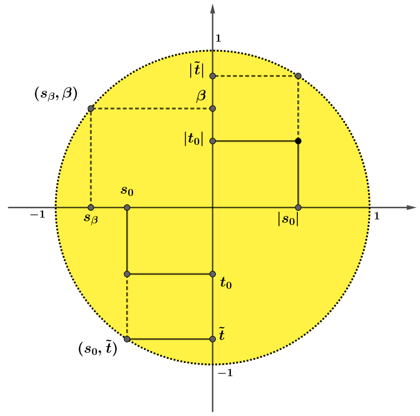

FIGURE 1. Construction of $(s_0,\tilde t)$ and (s_β,β)

Lemma 2.3. *Let $N=N_1+N_2$ with $N_1,N_2\geq 3$, and $\varphi\in C_0^2(D)$ be such that*
$$\varphi(s_1,t_1)=\varphi(s_2,t_2),\quad \text{if } (|s_1|,|t_1|)=(|s_2|,|t_2|).$$
Then, for some constant $C=C(N_1,N_2)>0$ independent of φ, it holds
$$|\varphi(s_0,t_0)|\leq C\frac{\big(\int_B|\partial_{st}\varphi(|x|,|y|)|^2\,\mathrm{d}z\big)^{1/2}}{|s_0|^{\frac{N_1-2}{2}}|t_0|^{\frac{N_2-2}{2}}},\quad \forall\,(s_0,t_0)\in D.$$

Proof. Given $(s_0,t_0)\in D$, we can use Lemma 2.2 to obtain $\tilde t$ such that $|t_0|<|\tilde t|<1$ and $(s_0,\tilde t)\in\partial D$. From the equation
$$-\varphi(s_0,t_0)=\varphi(s_0,|\tilde t|)-\varphi(s_0,|t_0|)=\int_{|t_0|}^{|\tilde t|}\partial_t\varphi(s_0,\beta)\,\mathrm{d}\beta,$$
we obtain
$$|\varphi(s_0,t_0)|\leq\int_{|t_0|}^{|\tilde t|}|\partial_t\varphi(s_0,\beta)|\,\mathrm{d}\beta. \tag{2.1}$$
Now, for each $\beta\in(|t_0|,|\tilde t|)$, we can use Lemma 2.2 again to choose s_β such that $|s_0|<|s_\beta|<1$ and $(s_\beta,\beta)\in\partial D$. Notice that, as $\varphi(s_1,t)=\varphi(s_2,t)$, if $|s_1|=|s_2|$ then $\partial_t\varphi(s_1,t)=\partial_t\varphi(s_2,t)$. So, as the support of φ and $\partial_t\varphi$ are in D
$$\partial_t\varphi(s_0,\beta)=-[\partial_t\varphi(|s_\beta|,\beta)-\partial_t\varphi(|s_0|,\beta)]=-\int_{|s_0|}^{|s_\beta|}\partial_{st}\varphi(\alpha,\beta)\,\mathrm{d}\alpha,$$
which implies
$$|\partial_t\varphi(s_0,\beta)|\leq\int_{|s_0|}^{|s_\beta|}|\partial_{st}\varphi(\alpha,\beta)|\,\mathrm{d}\alpha\leq\int_{|s_0|}^{1}|\partial_{st}\varphi(\alpha,\beta)|\,\mathrm{d}\alpha, \tag{2.2}$$
for any $\beta\in(|t_0|,|\tilde t|)$. Finally, combining (2.1) with (2.2), we obtain
$$|\varphi(s_0,t_0)|\leq\int_{|t_0|}^{|\tilde t|}\int_{|s_0|}^{1}|\partial_{st}\varphi(\alpha,\beta)|\,\mathrm{d}\alpha\,\mathrm{d}\beta\leq\int_{|t_0|}^{1}\int_{|s_0|}^{1}|\partial_{st}\varphi(\alpha,\beta)|\,\mathrm{d}\alpha\,\mathrm{d}\beta.$$
If we define $I_{t_0,s_0}:=(|t_0|,1)\times(|s_0|,1)$, we can use Hölder's inequality to obtain
$$\begin{aligned}\int_{I_{t_0,s_0}}|\partial_{st}\varphi(\alpha,\beta)|\,\mathrm{d}(\alpha,\beta)&=\int_{I_{t_0,s_0}}|\partial_{st}\varphi(\alpha,\beta)|\alpha^{\frac{N_1-1}{2}}\beta^{\frac{N_2-1}{2}}\alpha^{\frac{1-N_1}{2}}\beta^{\frac{1-N_2}{2}}\,\mathrm{d}(\alpha,\beta)\\&\leq\Big(\int_{I_{t_0,s_0}}|\partial_{st}\varphi(\alpha,\beta)|^2\alpha^{N_1-1}\beta^{N_2-1}\,\mathrm{d}(\alpha,\beta)\Big)^{1/2}\\&\quad\times\Big(\int_{I_{t_0,s_0}}\alpha^{1-N_1}\beta^{1-N_2}\,\mathrm{d}(\alpha,\beta)\Big)^{1/2}.\end{aligned}$$

As $|s_0|, |t_0| < 1$,

$$\begin{aligned}\int_{I_{t_0,s_0}} \alpha^{1-N_1}\beta^{1-N_2}\,\mathrm{d}(\alpha,\beta) &= \frac{1}{(N_1-2)(N_2-2)}\Big(\frac{1}{|s_0|^{N_1-2}}-1\Big)\Big(\frac{1}{|t_0|^{N_2-2}}-1\Big)\\ &\le \frac{1}{(N_1-2)(N_2-2)|s_0|^{N_1-2}|t_0|^{N_2-2}}.\end{aligned}$$

Therefore, for $C_1 := [(N_1-2)(N_2-2)]^{-1/2}$ we have that

$$\begin{aligned}|\varphi(s_0,t_0)| &\le C_1 \frac{\big(\int_{I_{t_0,s_0}} |\partial_{st}\varphi(\alpha,\beta)|^2\alpha^{N_1-1}\beta^{N_2-1}\,\mathrm{d}(\alpha,\beta)\big)^{1/2}}{|s_0|^{\frac{N_1-2}{2}}|t_0|^{\frac{N_2-2}{2}}}\\ &\le C_1 \frac{\big(\int_{(0,1)\times(0,1)} |\partial_{st}\varphi(\alpha,\beta)|^2\alpha^{N_1-1}\beta^{N_2-1}\,\mathrm{d}(\alpha,\beta)\big)^{1/2}}{|s_0|^{\frac{N_1-2}{2}}|t_0|^{\frac{N_2-2}{2}}}\\ &= C_1 \frac{\big(\int_{D_+} |\partial_{st}\varphi(\alpha,\beta)|^2\alpha^{N_1-1}\beta^{N_2-1}\,\mathrm{d}(\alpha,\beta)\big)^{1/2}}{|s_0|^{\frac{N_1-2}{2}}|t_0|^{\frac{N_2-2}{2}}},\end{aligned}$$

because, as the support of φ is on D, then the same happens to $\partial_{st}\varphi$. Now, we apply Lemma 2.1 to obtain

$$|\varphi(s_0,t_0)| \le C \frac{\big(\int_B |\partial_{st}\varphi(|x|,|y|)|^2\,\mathrm{d}z\big)^{1/2}}{|s_0|^{\frac{N_1-2}{2}}|t_0|^{\frac{N_2-2}{2}}},$$

where

$$C = \sqrt{\frac{\Gamma(\frac{N_1}{2})\Gamma(\frac{N_2}{2})}{4\pi^{N/2}(N_1-2)(N_2-2)}},$$

which completes the proof. □

3. Proof of Theorems 1.1 and 1.2

We start this section by proving our version of the radial lemma.

Proof of Theorem 1.1. It suffices to prove the result for a function $u \in C^\infty_{0,x,y}(B)$. To that end, we consider a function φ such that $u(x,y) = \varphi(|x|,|y|)$ and $\varphi(s_1,t_1) = \varphi(s_2,t_2)$ whenever $(|s_1|,|t_1|) = (|s_2|,|t_2|)$.

For each $i = 1,\dots,N_1$ and $j = 1,\dots,N_2$, it holds

$$u_{x_iy_j}(x,y) = \partial_{st}\varphi(|x|,|y|)\frac{x_i}{|x|}\frac{y_j}{|y|}.$$

Squaring this identity and summing over i and j gives

$$\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}[u_{x_iy_j}(x,y)]^2 = [\partial_{st}\varphi(|x|,|y|)]^2. \tag{3.1}$$

Since

$$\int_B \Delta_x u\Delta_y u\,\mathrm{d}z = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\int_B u_{x_ix_i}u_{y_jy_j}\,\mathrm{d}z = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\int_B [u_{x_iy_j}(x,y)]^2\,\mathrm{d}z$$

we can use identity (3.1) to get

$$\int_B \Delta_x u\Delta_y u\,\mathrm{d}z = \int_B [\partial_{st}\varphi(|x|,|y|)]^2\,\mathrm{d}z.$$

Hence, from Lemma 2.3 we obtain

$$|u(x,y)| \le C\frac{\big(\int_B \Delta_x u\Delta_y u\ \mathrm{d}z\big)^{1/2}}{|x|^{\frac{N_1-2}{2}}|y|^{\frac{N_2-2}{2}}}, \quad \text{for a.e. } (x,y)\in B. \tag{3.2}$$

Using this inequality with

$$|\Delta u|^2 = |\Delta_x u + \Delta_y u|^2 = |\Delta_x u|^2 + 2\Delta_x u \Delta_y u + |\Delta_y u|^2,$$

we conclude that (1.2) holds. □

Remark 3.1. Consider the bilinear form

$$B_{x,y}[u,v] := \int_B \Delta_x u(z) \Delta_y u(z)\, \mathrm{d}z, \quad \forall\, u, v \in C^\infty_{0,x,y}(B).$$

Using the Divergence Theorem, it is easy to see that B is symmetric. Moreover, by (3.2), it is also positive definite. Since $C^\infty_{0,x,y}(B)$ is dense in $H^2_{0,x,y}(B)$, we conclude that $B_{x,y}$ can be taken as an inner product in $H^2_{0,x,y}(B)$ with associated norm given by

$$\|u\|_{x,y} := \Big(\int_B \Delta_x u(z) \Delta_y u(z)\, \mathrm{d}z \Big)^{1/2}.$$

We have proved our radial-type result, we can obtain embedding properties for the space $H^2_{0,x,y}(B)$.

Proof of Theorem 1.2. We first prove that the embedding $H^2_{0,x,y}(B) \hookrightarrow L^p_\ell(B)$ is continuous, for any $1 \le p < 2^*_{\ell,N_1}$. To that end, we select $u \in H^2_{0,x,y}(B)$ and use (A1) and (1.2) to obtain

$$\begin{aligned}
\|u\|^p_{L^p_\ell(B)} &= \int_B |u(z)|^p [W(z)]^\ell\, \mathrm{d}z \\
&\le c_W^\ell \int_B |u(z)|^p (|x||y|)^\ell\, \mathrm{d}z \\
&\le c_W^\ell \|u\|^p_{H^2_0(B)} \int_B |x|^{-\frac{(N_1-2)p}{2}} |y|^{-\frac{(N_2-2)p}{2}} (|x||y|)^\ell\, \mathrm{d}z \\
&\le c_W^\ell \|u\|^p_{H^2_0(B)} \int_{B^1}\int_{B^2} |x|^{\left[\ell - \frac{(N_1-2)p}{2}\right]} |y|^{\left[\ell - \frac{(N_2-2)p}{2}\right]}\, \mathrm{d}x\, \mathrm{d}y \\
&= c_W^\ell \|u\|^p_{H^2_0(B)} \Big(\int_{B^1} |x|^{[\ell - \frac{(N_1-2)p}{2}]}\, \mathrm{d}x \Big) \Big(\int_{B^2} |y|^{\left[\ell - \frac{(N_2-2)p}{2}\right]}\, \mathrm{d}y \Big),
\end{aligned}$$

where we have used the inclusion $B \subset B^1 \times B^2$ (recall that B^i is the unit ball of $\mathbb{R}^{N_i}$). Thus

$$\|u\|^p_{L^p_\ell(B)} \le C_1 \|u\|^p_{H^2_0(B)} \Big(\int_0^1 r^{[\ell - \frac{(N_1-2)p}{2} + N_1 - 1]}\, \mathrm{d}r \Big) \Big(\int_0^1 r^{[\ell - \frac{(N_2-2)p}{2} + N_2 - 1]}\, \mathrm{d}r \Big),$$

with $C_1 := c_W^\ell \omega_{N_1}\omega_{N_2}$. Since $p \in [1, 2^*_{\ell,N_1})$ and $N_2 \le N_1$, the integrals above are finite, and thus the continuity of the embedding is established.

To prove the compactness, we first point out that, since $H^2_{0,x,y}(B)$ is a subspace of $W^{2,2}_0(B)$, the Rellich-Kondrachov Theorem assures that it is compactly embedded in $L^1(B)$. Consider $\beta \in (0,1)$ to be chosen. For $u \in H^2_{0,x,y}(B)$, we can use Hölder's inequality with exponents $s = \beta^{-1}$ and $s' = (1-\beta)^{-1}$ to obtain

$$\begin{aligned}
\|u\|^p_{L^p_\ell(B)} &= \int_B |u(z)|^\beta |u(z)|^{p-\beta} [W(z)]^\ell\, \mathrm{d}z \\
&\le \|u\|^\beta_{L^1(B)} \Big(\int_B |u(z)|^{\frac{p-\beta}{1-\beta}} [W(z)]^{\frac{\ell}{1-\beta}}\, \mathrm{d}z \Big)^{1-\beta}
\end{aligned}$$

or, equivalently,

$$\|u\|^p_{L^p_\ell(B)} \le \|u\|^\beta_{L^1(B)} \cdot \|u\|^{p-\beta}_{L^{q_\beta}_{\ell_\beta}(B)} \tag{3.3}$$

with

$$\ell_\beta := \frac{\ell}{1-\beta}, \quad q_\beta := \frac{p-\beta}{1-\beta}.$$

Since $p > 1$, it is clear that $q_\beta \ge 1$ for any $\beta \in (0,1)$. Moreover,

$$\lim_{\beta \to 0^+} \Big(q_\beta - 2^*_{\ell_\beta, N_1} \Big) = \lim_{\beta \to 0^+} \Big(\frac{p-\beta}{1-\beta} - \frac{2N_1}{N_1 - 2} - \frac{2\ell}{(1-\beta)(N_1-2)} \Big)$$

$$= \big(p - 2^*_{\ell,N_1}\big) < 0,$$

and therefore $q_\beta \in [1, 2^*_{\ell_\beta,N_1})$, for some $\beta \in (0,1)$ sufficiently close to 0. Considering this choice for β, we can apply the embedding proved in the first part to find $C_2 > 0$ such that

$$\|u\|_{L^{q_\beta}_{\ell_\beta}(B)} \le C_2 \|u\|_{H^2_0(B)}.$$

Combining this estimate with (3.3), we obtain

$$\|u\|^p_{L^p_\ell(B)} \le C_3 \|u\|^{\beta_0}_{L^1(B)} \|u\|^{p-\beta_0}_{H^2_0(B)}, \quad \forall u \in H^2_{0,x,y}(B).$$

If $(u_n) \subset H^2_{0,x,y}(B)$ is a bounded sequence, we can extract a subsequence (still denoted by (u_n)) such that $u_n \rightharpoonup u$ weakly in $H^2_{0,x,y}(B)$. By the compact embedding of $H^2_{0,x,y}(B)$ into $L^1(B)$, we also have that $u_n \to u$ strongly in $L^1(B)$. Therefore,

$$\|u_n - u\|^p_{L^p_\ell(B)} \le C_3 \|u_n - u\|^\beta_{L^1(B)} \|u_n - u\|^{p-\beta}_{H^2_0(B)} \le C_4 \|u_n - u\|^\beta_{L^1(B)} \to 0,$$

which shows that $u_n \to u$ in $L^p_\ell(B)$, completing the proof. □

Remark 3.2. In [6], among other results, the authors established certain embeddings of the space $H^2_{0,x,y}(B)$ into weighted L^p spaces with weight $|z|^\gamma$, for $\gamma > 0$. Since $[|x||y|]^\ell \le |z|^{2\ell}$ for any $z = (x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, we may consider $3 \le N_2 \le N_1$ and apply [6, Corollary 1.5] to obtain results analogous to those in Theorem 1.2 under the following conditions:

$$1 \le p < \frac{4(\ell+2)}{6}, \quad \text{if } N_1 = N_2 = 3 \text{ and } \ell > 1,$$
$$1 \le p < \frac{2(N_1+1)}{N_1-3}, \quad \text{if } 3 \le N_2 < N_1 \text{ and } \ell > \ell_* := \frac{N_1^2 + 4N_2 + 3}{2(N_1-3)}.$$

To verify that our results are sharper, we first consider the case $N_1 = N_2 = 3$. In this situation, Theorem 1.2 ensures the embedding for any $\ell > 0$ and $1 \le p < 6 + 2\ell$. Since

$$\frac{4(\ell+2)}{6} < 6 + 2\ell,$$

for all $\ell > 0$, we conclude that our result encompasses a strictly wider range. A similar improvement occurs when $3 \le N_2 < N_1$, because

$$\frac{2(N_1+1)}{N_1-3} < 2^*_{\ell,N_1} \iff \ell > \frac{2(N_1-1)}{N_1-3},$$

and a straightforward computation shows that the inequality on the right-hand side is always satisfied whenever $\ell > \ell_*$.

4. Application

In this section, we prove Theorem 1.3. We first notice that, given $\varepsilon > 0$, we can use (A2)–(A4) to obtain

$$|F(s)| \le \varepsilon |s|^2 + C_1 |s|^p, \quad \forall\, s \in \mathbb{R}.$$

Thus, for any $u \in H^2_{0,x,y}(B)$, we can use Theorem 1.2 to guarantee that

$$\int_B F(u)[W(z)]^\ell \,\mathrm{d}z \le \varepsilon \|u\|_{L^2_\ell(B)} + C_1 \|u\|_{L^p_\ell(B)} < +\infty\,. \tag{4.1}$$

So, the functional

$$I(u) := \frac{1}{2}\|u\|^2_{H^2_0(B)} - \int_B F(u)[W(z)]^\ell \,\mathrm{d}z, \quad u \in H^2_{0,x,y}(B),$$

is well defined. Moreover, standard computations shows that $I \in C^1(H^2_{0,x,y}(B), \mathbb{R})$ and its critical points are the weak solution of problem (P). We are ready to obtain a solution for our PDE.

Proof of Theorem 1.3. By using (4.1), (A1), and Theorem 1.2 again, we obtain

$$\begin{aligned} I(u) &\geq \frac{1}{2}\|u\|^2_{H_0^2(B)} - \varepsilon\|u\|_{L^2_\ell(B)} - C_1\|u\|_{L^p_\ell(B)} \\ &\geq \frac{1}{2}\|u\|^2_{H_0^2(B)}\Big(1 - C_2\varepsilon - 2C_3\|u\|^{p-2}_{H_0^2(B)}\Big). \end{aligned}$$

Since $p > 2$, we can choose $\varepsilon > 0$ sufficiently small to obtain constants $\rho, \beta > 0$ such that

$$I(u) \geq \beta, \quad \forall u \in H^2_{0,x,y}(B),\ \|u\|_{H_0^2(B)} = \rho.$$

Moreover, using (A2), (A3), and (A5), we obtain $C_4 > 0$ such that

$$F(x,s) \geq C_4|s|^\mu - C_4, \quad \forall (x,s) \in B \times \mathbb{R}.$$

Then, choosing a positive function $u_0 \in H^2_{0,x,y}(B)$, we find that

$$I(su_0) \leq \frac{s^2}{2}\|u_0\|^2_{H_0^2(B)} - C_4 s^\mu \int_B |u_0|^\mu [W(z)^\ell]\,\mathrm{d}z - C_5.$$

Since $\mu > 2$, it follows that $\lim_{s\to+\infty} I(su_0) = -\infty$. Then there exists $e \in H^2_{0,x,y}(B)$ such that $I(e) \leq 0$ and $\|e\|_{H_0^2(B)} > \rho$.

According to the above considerations, we can define

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)) \geq \beta > 0,$$

with $\Gamma := \{\gamma \in C([0,1], H^2_{0,x,y}(B)) : \gamma(0) = 0, \gamma(1) = e\}$, and invoke the Mountain Pass Theorem [1] to obtain a sequence $(u_n) \subset H^2_{0,x,y}(B)$ such that

$$\lim_{n\to+\infty} I(u_n) = c > 0, \quad \lim_{n\to+\infty} I'(u_n) = 0. \tag{4.2}$$

From the above convergences and (A5), we obtain

$$\begin{aligned} C_6 &\geq I(u_n) - \frac{1}{\mu} I'(u_n)(u_n) \\ &\geq \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\|u_n\|^2_{H_0^2(B)} + \int_{\{|u_n|\leq s_0\}} \big[\frac{1}{\mu} f(u_n)u_n - F(u_n)\big][W(z)]^\ell\,\mathrm{d}z, \end{aligned}$$

and hence it follows from $\mu > 2$ and (A2) that the sequence (u_n) is bounded.

Let $u \in H^2_{0,x,y}(B)$ be the weak limit of a subsequence of (u_n). We aim to prove that, along a subsequence, $u_n \to u$ strongly in $H^2_{0,x,y}(B)$. If this is true, we can use the regularity of I and (4.2) to conclude that $u \neq 0$ is a weak solution of problem (P).

To prove the strong convergence, we compute

$$I'(u_n)(u_n - u) = \|u_n\|^2_{H_0^2(B)} - A_n - \int_B \Delta u_n \Delta u\,\mathrm{d}z, \tag{4.3}$$

where

$$A_n := \int_B f(u_n)(u_n - u)[W(z)]^\ell\,\mathrm{d}z.$$

Using the boundedness of B, (A1), and (A3), we obtain

$$|A_n| \leq c_f \int_B (1 + |u_n|^{p-1})|u_n - u|[W(z)]^\ell\,\mathrm{d}z \leq c_f(D_n + E_n),$$

where

$$D_n := \int_B |u_n - u|[W(z)]^\ell\,\mathrm{d}z, \quad E_n := \int_B |u_n|^{p-1}|u_n - u|[W(z)]^\ell\,\mathrm{d}z.$$

Clearly, $D_n \to 0$ as $n \to +\infty$, because W is bounded in B and $u_n \to u$ in $L^1(B)$. Furthermore, using Hölder's inequality with exponents p and $p' = p/(p-1)$, we obtain

$$\begin{aligned} E_n &= \int_B \Big(|u_n|^{p-1}[W(z)]^{\frac{\ell}{p'}}\Big)\Big(|u_n - u|[W(z)]^{\frac{\ell}{p}}\Big)\,\mathrm{d}z \\ &\leq \Big(\int_B |u_n|^{p'(p-1)}[W(z)]^\ell\,\mathrm{d}z\Big)^{1/p'}\Big(\int_B |u_n - u|^p[W(z)]^\ell\,\mathrm{d}z\Big)^{1/p} \end{aligned}$$

$$
\begin{aligned}
&= \Big(\int_B |u_n|^p [W(z)]^\ell \,\mathrm{d}z \Big)^{1/p'} \Big(\int_B |u_n - u|^p [W(z)]^\ell \,\mathrm{d}z \Big)^{1/p} \\
&= \|u_n\|_{L^p_\ell(B)}^{\frac{p}{p'}} \|u_n - u\|_{L^p_\ell(B)} \\
&\le C_7 \|u_n\|_{H^2_0(B)}^{\frac{p}{p'}} \|u_n - u\|_{L^p_\ell(B)}.
\end{aligned}
$$

Since the embedding $H^2_{0,x,y}(B) \hookrightarrow L^p_\ell(B)$ is compact, we conclude that $E_n \to 0$.

Altogether, these estimates show that $A_n \to 0$ as $n \to +\infty$. Therefore, using the second convergence in (4.2), (4.3), and the weak convergence, we obtain

$$
\lim_{n\to+\infty} \|u_n\|^2_{H^2_0(B)} = \|u\|^2_{H^2_0(B)}.
$$

This equality and the weak convergence imply that $u_n \to u$ strongly in $H^2_{0,x,y}(B)$, completing the proof. □

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