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EXISTENCE OF NORMALIZED SOLUTIONS TO KIRCHHOFF-BOUSSINESQ EQUATIONS IN THE SUBCRITICAL AND SUPERCRITICAL REGIME

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ABSTRACT. In this article we study the existence of normalized solutions to the Kirchhoff-Boussinesq equation under the mass constraint $\|u\|_2 = c$. In the L^2-subcritical regime, we apply Ekeland's variational principle and concentration compactness method to minimize the energy functional on the mass-constrained manifold. In the L^2-supercritical regime, we introduce a Pohožaev-constrained minimization approach, combined with scaling arguments to recover compactness. To handle the additional difficulties posed by q-Laplacian, we treat distinct ranges of q separately.

1. INTRODUCTION

In this article, we study the existence of normalized solutions to Kirchhoff-Boussinesq equation

$$\begin{gathered} \Delta^2 u - \Delta_q u + \lambda u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = c^2, \\ u \in X := H^2(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N), \end{gathered} \tag{1.1}$$

where $N \geq 5$, $c > 0$, $1 < q < N$, and $2 < p < \min\{4^*, q^*\}$, with $4^* = \frac{2N}{N-4}$ and $q^* = \frac{qN}{N-q}$ as the critical Sobolev exponents. Here $\Delta^2 u$ is the biharmonic operator, $\Delta_q u := \operatorname{div}(|\nabla u|^{q-2}\nabla u)$ is the q-Laplace operator, and λ is a Lagrange multiplier enforcing the mass constraint $\int_{\mathbb{R}^N} |u|^2\,dx = c$. If $(u, \lambda) \in H^2(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N) \times \mathbb{R}$ satisfies (1.1), then u is called a normalized solution of (1.1).

Problem (1.1) originates from nonlinear plate theory and provides a fundamental framework for modeling the dynamics of plates and elastic structures. More specifically, consider the following nonlinear plate equation referred to as Kirchhoff-Boussinesq model

$$w_{tt} + kw_t + \Delta^2 w = \operatorname{div}(|\nabla w|^{p-2}\nabla w) + \sigma\Delta(w^2) - f(w) \quad x \in \Omega \subset \mathbb{R}^2, \tag{1.2}$$

This equation is complemented by appropriate boundary and initial conditions (see [16, 17]). Equation (1.2) also arises as a limiting case of the Mindlin-Timoshenko system, which accounts for transverse shear effects in plate dynamics (see [23, 24]).

Problem (1.1) is closely related to the fourth-order nonlinear Schrödinger equation model widely studied in nonlinear optics, Bose-Einstein condensates, and quantum mechanics. Its general form is

$$i\partial_t\psi - \gamma\Delta^2\psi + \beta\Delta\psi + f(\psi) = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^N, \tag{1.3}$$

where $\gamma > 0$, $\beta \in \mathbb{R}$, i is the imaginary unit, and $f(\psi)$ is the nonlinear term. Recent studies focus on normalized solutions of the stationary biharmonic NLS (1.3), derived from standing waves

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$\psi(x,t) = e^{i\lambda t}u(x)$, specifically

$$\begin{gathered} \gamma\Delta^2 u + \beta\Delta u + \lambda u = f(u) \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2\,dx = c. \end{gathered} \tag{1.4}$$

When $\beta \neq 0$, Bonheure et al. [8] established the existence of minimizers for (1.4) and explored their qualitative properties and orbital stability. As a natural extension, if $\beta = -1$ and $2 + \frac{8}{N} \leq p < 4^*$. In [9] they investigated the existence of ground states and the multiplicity of radial solutions. Later, for the case where $\gamma = 1$, $\beta \in \mathbb{R}$ and $2 < p \leq 2 + \frac{8}{N}$, Luo et al. [30] applied the profile decomposition approaches and demonstrated the existence of orbitally stable ground state solutions. Luo and Yang [31] worked in the radial space $H^2_{rad}(\mathbb{R}^N)$ and proved the existence of two solutions to problem (1.4) for c sufficiently small, where one solution is a local minimizer and the other is of the mountain-pass type. Additionally, Boussaïd et al. [10] revisited problem (1.4) under the conditions $\gamma > 0$, $\beta > 0$ and $2 < p \leq 2 + \frac{8}{N}$. By ruling out the vanishing property of some minimization sequence, they confirmed that the results obtained by Luo et al. [30] hold for all $c > 0$ and $\beta > 0$.

For the case $\beta = 0$, Bellazzini and Visciglia [6] investigated problem (1.4) with potentials

$$\begin{gathered} \Delta^2 u + V(x)u - Q(x)|u|^{p-2}u = \lambda u \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2\,dx = c, \end{gathered}$$

where $2 < p < 2 + \frac{8}{N}$ and $V(x), Q(x) \in L^\infty(\mathbb{R}^N)$. Assuming $Q(x) \geq 0$ a.e. $x \in \mathbb{R}^N$ and the existence of $\lambda_0 > 0$ such that $0 < \text{meas}\{Q(x) > \lambda_0\} < \infty$, they established the existence of ground state normalized solutions and discussed the orbital stability of the minimizers. They also considered the case where $V(x) \equiv 0$ and $Q(x) \equiv 1$, extending this work from $|u|^{p-2}u$ to more general L^2-subcritical nonlinearity $f(u)$ in Bellazzini and Siciliano [5]. Phan [34] further investigated the effect of an external potential $V(x)$ on L^2-critical nonlinearity

$$\begin{gathered} \Delta^2 u + V(x)u - a|u|^{8/N}u = \lambda u \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2\,dx = 1. \end{gathered} \tag{1.5}$$

Under appropriate conditions on $V(x)$, problem (1.5) admits at least one ground state solution when a lies within a specific interval. For problems involving Sobolev critical growth, Ma and Chang [32] focused on the case where $\gamma = 1$, $\beta = 0$ and $f(u) = \mu|u|^{p-2}u + |u|^{4^*-2}u$ with $2 < p < 2 + \frac{8}{N}$, they established the existence of a normalized ground state solution. Later, Liu and Zhang [29] further extended this analysis to the supercritical regime $2 + \frac{8}{N} < p < 4^*$. Their results confirmed the existence of normalized solutions when μ is sufficiently large.

Equation (1.1) is also related to the problems involving (p,q)-Laplace operator arising in reaction-diffusion systems,

$$\partial_t u - \Delta_p u - \Delta_q u = f(x,u) \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{1.6}$$

and has been widely studied because of applications in plasma physics, fluid dynamics, and biology [3, 18]. For the stationary case with $p = 2$, Baldelli and Yang [2] investigated the existence of normalized solutions to the $(2,q)$-Laplacian equation

$$\begin{gathered} -\Delta u - \Delta_q u = \lambda u + |u|^{p-2}u \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2\,dx = c. \end{gathered}$$

They analyzed the problem in various regimes. In the L^2-subcritical case, they obtained a ground state solution by solving a global minimization problem. In the L^2-critical case, they demonstrated several nonexistence results, extending these findings to the L^q-critical case. In the L^2-supercritical case, they established the existence of a ground state and infinitely many radial solutions. Later, Ding et al. [19] investigated the existence of normalized solutions to the $(2,q)$-Laplacian equation, particularly when the nonlinearity $g(u)$ exhibits strongly sublinear behavior

$$\begin{gathered} -\Delta u - \Delta_q u + \lambda u = g(u) \quad \text{in } \mathbb{R}^N \\ \int_{\mathbb{R}^N} |u|^2 dx = c. \end{gathered} \tag{1.7}$$

The nonlinearity $g : \mathbb{R} \to \mathbb{R}$ is continuous and the behaviour of g at the origin is allowed to be strongly sublinear, i.e., $\lim_{s\to 0} \frac{g(s)}{s} = -\infty$, which includes the logarithmic nonlinearity $g(s) = s \log s^2$. The authors considered a family of approximating problems that can be set in $H^1(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N)$ and the corresponding least-energy solutions. Then, they proved that such a family of solutions converges to a least-energy solution to problem (1.7).

When $p \neq 2$, Zhang et al. [42] studied the p-Laplacian equation with an L^p-norm constraint

$$\begin{gathered} -\Delta_p u = \lambda |u|^{p-2}u + \mu |u|^{q-2}u + g(u) \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^p dx = c^p, \end{gathered}$$

where $N > 1$, $c > 0$, $1 < p < q \leq \tilde{p}$, $\mu \in \mathbb{R}$ and $g \in C(\mathbb{R}, \mathbb{R})$ is odd. In the L^p-supercritical case, they applied the Schwarz rearrangement and Ekeland's variational principle to prove the existence of a positive radial ground state for suitable values of μ, and extended these results using minimax theorems. Additionally, they demonstrated the existence of infinitely many radial and nonradial sign-changing solutions for $N = 4$ or $N \geq 6$. Other related results can be found in [22, 26] and so on.

Recently, Cai and Rădulescu [12] investigated the (p,q)-Laplacian equation with an L^p-norm constraint

$$\begin{gathered} -\Delta_p u - \Delta_q u + \lambda |u|^{p-2}u = f(u) \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^p dx = c^p, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N). \end{gathered}$$

They established the existence of ground states and analyzed the behavior of the ground state energy E_c as the parameter $c > 0$ varied.

For the fixed frequency problem associated with (1.1), Sun et al. in [38] studied the following biharmonic equation with p-Laplacian

$$\Delta^2 u - \beta \Delta_p u + \lambda V(x) u = f(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.8}$$

and they obtained multiple solutions of (1.8) where $N \geq 1$, $\beta \in \mathbb{R}$, the potential $V(x)$ is a steep potential well, and $f(x,u)$ satisfies some subcritical conditions. Moreover, Sun and Wu [39, 40] established the existence of nontrivial solutions to (1.8) with the singular sign-changing potential $V(x)$, and $f(x,u) \equiv 0$ and $\beta < 0$. Additionally, Figueiredo and Carlos [21] showed the existence of solutions for the class of elliptic Kirchhoff-Boussinesq-type problems given by

$$\Delta^2 u - \Delta_p u + u = h(u) \quad \text{in } \mathbb{R}^N, \tag{1.9}$$

They proved the existence and multiplicity of nontrivial solutions. Meanwhile, Razani et al. [36] studied the anisotropic Kirchhoff-Boussinesq equations with exponential growth, where the existence of solutions was proved. For other related results, we refer the interesting readers to see [13, 20, 35, 37].

To the best of our knowledge, the existence of normalized solutions for (1.1) has not been studied before. The main purpose of this paper is to establish the existence of normalized solutions to (1.1), which can be obtained by searching for critical points of the following functional

$$I(u) = \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{q}\|\nabla u\|_q^q - \frac{1}{p}\|u\|_p^p$$

on the constraint

$$S_c := \left\{ u \in X := H^2(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N) : \|u\|_2 = c \right\}.$$

Notation. Throughout the paper we use

$$\delta_p = \frac{N(p-2)}{2p}, \quad \delta_q = \frac{N(q-2)}{2q}, \quad \alpha = \frac{Nq(p-2)}{p\,[q(N+2)-2N]}.$$

The following constants will appear in Theorems 1.1 and 1.2:

$$A_1 = \Big(\tfrac{p\delta_q-4}{p\delta_p} + \tfrac{p\delta_p-q(1+\delta_q)}{p\delta_p}\Big) 2^{\frac{p\delta_p}{4-p\delta_p}} \Big(\tfrac{C_{N,p}^p}{p}\Big)^{\frac{4}{4-p\delta_p}},$$

$$A_2 = \Big(\tfrac{4-2\delta_p}{\delta_p} + \tfrac{q}{\delta_p}\Big) 2^{\frac{p\delta_p}{4-p\delta_p}} \Big(\tfrac{C_{N,p}^p}{p}\Big)^{\frac{4}{4-p\delta_p}},$$

$$A_3 = \Big(\tfrac{1}{2} - \tfrac{2}{p\delta_p}\Big)\Big(\tfrac{2}{\delta_p\, C_{N,p}^p}\Big)^{\frac{4}{p\delta_p-4}},$$

$$A_4 = \tfrac{1}{q}\Big(1 - \tfrac{q(1+\delta_q)}{p\delta_p}\Big)\Big(\tfrac{1+\delta_q}{\delta_p\, K_{N,p}^p}\Big)^{\frac{q}{p\alpha-q}},$$

$$A_5 = \Big(\tfrac{2}{\delta_p} - 1\Big)\Big(\tfrac{2}{\delta_p\, C_{N,p}^p}\Big)^{\frac{4}{p\delta_p-4}},$$

$$A_6 = \Big(\tfrac{1+\delta_q}{\delta_p} - 1\Big)\Big(\tfrac{1+\delta_q}{\delta_p\, K_{N,p}^p}\Big)^{\frac{q}{p\alpha-q}}.$$

Our main results are stated as follows. In the L^2-subcritical case, we study a global minimization problem.

Theorem 1.1. *Let $N \ge 5$, $c > 0$. Assume that*

$$2 < p < \bar{p}, \tag{1.10}$$

where $\bar{p}$ denotes the upper critical exponent threshold defined as $\bar{p} := \min\left\{q\left(1+\frac{2}{N}\right), 2+\frac{8}{N}\right\}$. Suppose that q satisfies

$$\text{either } \frac{2N}{N+2} < q \le 2 + \frac{4}{N+2}, \quad \text{or} \quad 2 + \frac{4}{N+2} < q < N. \tag{1.11}$$

Then $m(c) := \inf_{u\in S_c} I(u)$ is attained by some $u \in S_c$, which is a ground state of (1.1). Additionally, we have

1. $A_1 c^{\frac{2p(2-\delta_p)}{4-p\delta_p}} \le m(c) < 0$, $0 < \lambda_c \le A_2 c^{\frac{4(p-2)}{4-p\delta_p}}$;
2. $m(c) \to 0^-$ *and* $\lambda_c \to 0^+$ *as* $c \to 0^+$.

In the L^2-supercritical case, we consider a modified minimization problem $\sigma(c) := \inf_{u\in\mathcal{P}_c} I(u)$, where

$$\mathcal{P}_c := \left\{u \in S_c : P(u) := 2\|\Delta u\|_2^2 + (1+\delta_q)\|\nabla u\|_q^q - \delta_p\|u\|_p^p = 0\right\}.$$

We establish the existence of a ground state.

Theorem 1.2. *Let $N \ge 5$, $c > 0$, and*

$$\max\Big\{2 + \frac{8}{N}, q\Big(1+\frac{2}{N}\Big)\Big\} < p < \min\{q^*, 4^*\}, \tag{1.12}$$

$$\text{either} \quad \frac{2N^2+8N}{N^2+2N+8} < q < \frac{2N}{N-2}, \quad \text{or} \quad \frac{2N}{N-2} \le q < \min\Big\{N, \frac{2N^2}{N^2-2N-8}\Big\}. \tag{1.13}$$

Then $\sigma(c) := \inf_{u\in\mathcal{P}_c} I(u)$ is achieved by some $u \in \mathcal{P}_c$, and is a ground state of (1.1). Moreover, we have

1. $\sigma(c) \ge A_3 c^{-\frac{2p(2-\delta_p)}{p\delta_p-4}} + A_4 c^{-\frac{pq(1-\alpha)}{p\alpha-q}} > 0$, $\lambda_c \ge A_5 c^{-\frac{4(p-2)}{p\delta_p-4}} + A_6 c^{-\frac{pq(1-\alpha)+2(p\alpha-q)}{p\alpha-q}}$;
2. $\sigma(c) \to +\infty$ *and* $\lambda_c \to +\infty$ *as* $c \to 0^+$.

Remark 1.3. In the subcritical regime, $m(c) \to 0^-$ as $c \to 0^+$ indicates stable, low-energy bound states for small mass, consistent with minimal energy configurations in plate theory. In the supercritical regime, $\sigma(c) \to +\infty$ as $c \to 0^+$ suggests that maintaining a normalized ground state at vanishing mass requires infinite energy, reflecting the instability often found in supercritical nonlinearities.

Comments on Theorems 1.1 and 1.2: To prove Theorem 1.1, we first show that the global minimum $m(c)$ is attained. From a minimizing sequence $\{v_n\} \subset S_c$, we apply the Ekeland's variational principle to obtain $\{u_n\} \subset S_c$, a Palais-Smale sequence for $I|_{S_c}$, satisfying

$$\|u_n - v_n\|_X \to 0, \quad (I|_{S_c})'(u_n) \to 0 \quad (n \to \infty).$$

We then show that there exists $u_c \in S_c$ with $\nabla u_n \to \nabla u_c$ a.e. in $\mathbb{R}^N$ which is a key step for the Brézis-Lieb splitting. After ruling out vanishing we use the strict subadditivity $m(c) < m(c_1) + m(c_2)$ $(0 < c_1 < c)$ to exclude dichotomy and recover the compactness of $\{u_n\}$. Meanwhile, this yields the required bounds for $m(c)$ and λ_c.

Theorem 1.2 addresses the more delicate situation in which $I|_{S_c} = -\infty$. We first prove that the modified problem $\sigma(c)$ admits a minimizer. The set $\mathcal{P}_c$ is characterized by the mass constraint and the Pohozaev identity $P(u) = 0$, and we verify that $I|_{\mathcal{P}_c} > -\infty$. Next, we show that $\sigma(c)$ decreases strictly with c; the resulting monotonicity yields a minimizer.

The remainder of the paper is organized as follows: Section 2 presents preliminary results required for Theorems 1.1 and 1.2, while Sections 3 and 4 present their proofs.

2. Preliminaries

In this section we present lemmas required for proving Theorem 1.1. First, we recall the Gagliardo-Nirenberg inequality, which will be crucial in our analysis.

Lemma 2.1 ([33, Theorem in Lecture II])**.** *Let $N \geq 5$, $2 < p < 4^* = \frac{2N}{N-4}$, and $\delta_p = \frac{N(p-2)}{2p}$. There exists an optimal constant $\mathcal{C}_{N,p} > 0$, depending on N and p, such that for all $u \in H^2(\mathbb{R}^N)$,*

$$\|u\|_p \leq \mathcal{C}_{N,p} \|\Delta u\|_2^{\delta_p/2} \|u\|_2^{1-\frac{\delta_p}{2}}. \tag{2.1}$$

Furthermore, we frequently use the inequality

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \leq \Big(\int_{\mathbb{R}^N} |\Delta u|^2 \, dx \Big)^{1/2} \Big(\int_{\mathbb{R}^N} |u|^2 \, dx \Big)^{1/2} \quad \forall u \in H^2(\mathbb{R}^N). \tag{2.2}$$

Next we fix notation. For $1 \leq p < \infty$ we set $\|u\|_p := \big(\int_{\mathbb{R}^N} |u|^p dx \big)^{\frac{1}{p}}$, whereas for $p = \infty$ we define $\|u\|_\infty := \operatorname{ess\,sup}_{x \in \mathbb{R}^N} |u(x)|$. The Hilbert space $H^2(\mathbb{R}^N)$ consists of all $u \in L^2(\mathbb{R}^N)$ with $\nabla u, \Delta u \in L^2(\mathbb{R}^N)$ and is endowed with the norm $\|u\|_{H^2(\mathbb{R}^N)} := \big(\|u\|_2^2 + \|\nabla u\|_2^2 + \|\Delta u\|_2^2 \big)^{1/2}$. Inequality (2.2) implies that the seminorm

$$\|u\|_{H^2} := \Big(\int_{\mathbb{R}^N} |\Delta u|^2 + |u|^2 dx \Big)^{1/2}$$

is equivalent to $\|u\|_{H^2(\mathbb{R}^N)}$. We define $D^{1,q}(\mathbb{R}^N) := \{u \in L^{q^*}(\mathbb{R}^N) : \nabla u \in L^q(\mathbb{R}^N)\}$ and equip it with the norm $\|u\|_{D^{1,q}} := \|\nabla u\|_q$.

Lemma 2.2 ([1, Theorem 2.1])**.** *Let $N \geq 5$, $2 < p < q^*$, $\frac{2N}{N+2} < q < N$ and $\alpha = \frac{Nq(p-2)}{p[q(N+2)-2N]}$. Then there exists a constant $K_{N,p} > 0$ such that*

$$\|u\|_p \leq K_{N,p} \|\nabla u\|_q^\alpha \|u\|_2^{1-\alpha} \quad \forall u \in D^{1,q}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N).$$

Now, we introduce the work space $X := H^2(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N)$ and endow it with the norm $\|u\|_X := \|u\|_{H^2(\mathbb{R}^N)} + \|u\|_{D^{1,q}(\mathbb{R}^N)}$. It is clear that X is a reflexive Banach space. Throughout the paper X' denotes the dual of X, and $\langle \cdot, \cdot \rangle$ stands for the corresponding duality pairing.

Lemma 2.3. *Let $N \geq 5$, $1 < q < N$ and $2 < p < 4^*$. If $u \in X$ is a weak solution of* (1.1)*, then it satisfies the following Pohožaev identity*

$$2\|\Delta u\|_2^2 + (1 + \delta_q) \|\nabla u\|_q^q - \delta_p \|u\|_p^p = 0.$$

Proof. Since u is a weak solution, for any $\varphi \in X$, we have

$$\int_{\mathbb{R}^N} \Delta u \Delta \varphi \, dx + \int_{\mathbb{R}^N} |\nabla u|^{q-2} \nabla u \cdot \nabla \varphi \, dx + \lambda \int_{\mathbb{R}^N} u \varphi \, dx = \int_{\mathbb{R}^N} |u|^{p-2} u \varphi \, dx.$$

We begin by assuming that $u \in C_0^2(\mathbb{R}^N)$. First, choose the test function $x \cdot \nabla u$ and integrate over the ball $B_R = \{x \in \mathbb{R}^N : |x| < R\}$,

$$\int_{B_R} \Delta^2 u(x\cdot\nabla u)\,dx - \int_{B_R} \Delta_q u(x\cdot\nabla u)\,dx + \lambda\int_{B_R} u(x\cdot\nabla u)\,dx = \int_{B_R} |u|^{p-2}u(x\cdot\nabla u)\,dx.$$

Direct calculations yield

$$\begin{aligned}
&\int_{B_R} \Delta^2 u\,(x\cdot\nabla u)\,dx \\
&= \int_{B_R} \Delta u\cdot\Delta(x\cdot\nabla u)\,dx - \int_{\partial B_R} \Delta u\cdot\partial_\nu(x\cdot\nabla u)\,dS + \int_{\partial B_R} (x\cdot\nabla u)\cdot\partial_\nu(\Delta u)\,dS \\
&= \frac{4-N}{2}\int_{B_R} |\Delta u|^2\,dx + \frac{R}{2}\int_{\partial B_R} |\Delta u|^2\,dS - \int_{\partial B_R} \Delta u\cdot\partial_\nu(x\cdot\nabla u)\,dS + \int_{\partial B_R} (x\cdot\nabla u)\cdot\partial_\nu(\Delta u)\,dS,
\end{aligned}$$

$$\begin{aligned}
\int_{B_R} -\Delta_q u(x\cdot\nabla u)\,dx &= \int_{B_R} |\nabla u|^{q-2}\nabla u\cdot\nabla(x\cdot\nabla u)\,dx - \int_{\partial B_R} |\nabla u|^{q-2}(x\cdot\nabla u)(\nabla u\cdot\nu)\,dS \\
&= \frac{q-N}{q}\int_{B_R} |\nabla u|^q\,dx + \frac{R}{q}\int_{\partial B_R} |\nabla u|^q\,dS - \int_{\partial B_R} |\nabla u|^{q-2}(x\cdot\nabla u)(\nabla u\cdot\nu)\,dS,
\end{aligned}$$

$$\int_{B_R} u(x\cdot\nabla u)\,dx = \int_{B_R} x\cdot\nabla(\frac{u^2}{2})\,dx = -\frac{N}{2}\int_{B_R} u^2\,dx + \frac{R}{2}\int_{\partial B_R} u^2\,dS,$$

$$\int_{B_R} |u|^{p-2}u(x\cdot\nabla u)\,dx = \int_{B_R} x\cdot\nabla(\frac{|u|^p}{p})\,dx = -\frac{N}{p}\int_{B_R} |u|^p\,dx + \frac{R}{p}\int_{\partial B_R} |u|^p\,dS,$$

where ν is the unit outward normal on ∂B_R and dS represents the surface area element on ∂B_R. Therefore,

$$\begin{aligned}
&\frac{N-4}{2}\|\Delta u\|^2_{L^2(B_R)} + \frac{N-q}{q}\|\nabla u\|^q_{L^q(B_R)} + \frac{\lambda N}{2}\|u\|^2_{L^2(B_R)} - \frac{N}{p}\|u\|^p_{L^p(B_R)} \\
&= \frac{R}{2}\int_{\partial B_R} |\Delta u|^2\,dS - \int_{\partial B_R} \Delta u\cdot\partial_\nu(x\cdot\nabla u)\,dS + \int_{\partial B_R} (x\cdot\nabla u)\cdot\partial_\nu(\Delta u)\,dS \\
&\quad + \frac{R}{q}\int_{\partial B_R} |\nabla u|^q\,dS - \int_{\partial B_R} |\nabla u|^{q-2}(x\cdot\nabla u)(\nabla u\cdot\nu)\,dS \\
&\quad + \frac{R}{2}\int_{\partial B_R} u^2\,dS - \frac{R}{p}\int_{\partial B_R} |u|^p\,dS.
\end{aligned}$$

Let $R = R_n$ with $R_n \to \infty$ as $n \to \infty$, and the right-hand side of the above equation tends to zero. Thus, we obtain

$$\frac{N-4}{2}\|\Delta u\|_2^2 + \frac{N-q}{q}\|\nabla u\|_q^q + \frac{\lambda N}{2}\|u\|_2^2 = \frac{N}{p}\|u\|_p^p. \tag{2.3}$$

Next, by choosing $\varphi = u$ in (1.1), it follows that

$$\|\Delta u\|_2^2 + \|\nabla u\|_q^q + \lambda\|u\|_2^2 - \|u\|_p^p = 0. \tag{2.4}$$

By combining equations (2.3) and (2.4), we derive the Pohožaev identity

$$2\|\Delta u\|_2^2 + (1+\delta_q)\|\nabla u\|_q^q - \delta_p\|u\|_p^p = 0.$$

The proof is complete. □

At the end of this section, we recall some preliminary lemmas and inequalities that will be used later.

Lemma 2.4 ([15, Lemma 5], [25, Lemma 2.7])**.** *Assume $s > 1$. Let Ω be an open set in $\mathbb{R}^N$, α, β positive numbers, and $a(x,\xi) \in C(\Omega\times\mathbb{R}^N, \mathbb{R}^N)$ such that*

(1) $\alpha|\xi|^s \le a(x,\xi)\xi$ *for all* $(x,\xi) \in \Omega\times\mathbb{R}^N$,
(2) $|a(x,\xi)| \le \beta|\xi|^{s-1}$ *for all* $(x,\xi) \in \Omega\times\mathbb{R}^N$,
(3) $(a(x,\xi) - a(x,\eta))(\xi,\eta) > 0$ *for all* $(x,\xi) \in \Omega\times\mathbb{R}^N$ *with* $\xi \ne \eta$,
(4) $a(x,\gamma\xi) = \gamma|\gamma|^{p-2}a(x,\xi)$ *for all* $(x,\xi) \in \Omega\times\mathbb{R}^N$ *and* $\gamma \in \mathbb{R}\setminus\{0\}$.

Consider $\{u_n\}$, $u \in W^{1,s}(\Omega)$, *then* $\nabla u_n \to \nabla u$ *in* $L^s(\Omega)$ *if and only if*

$$\lim_{n\to\infty} \int_\Omega \left(a(x, \nabla u_n(x)) - a(x, \nabla u(x))\right)\left(\nabla u_n(x) - \nabla u(x)\right) dx = 0.$$

Lemma 2.5 ([7, Lemma 3]). *Let* $\{u_n\}$ *be a sequence in* S_c *which is bounded in* X. *Then the following statements are equivalent*

(1) $\|(I|_{S_c})'(u_n)\| \to 0$ *as* $n \to +\infty$;
(2) $I'(u_n) - \langle I'(u_n), u_n\rangle u_n \to 0$ *in* X' *as* $n \to +\infty$.

Lemma 2.6 ([41, Theorem 8.5]). *Let* X *be a Banach space and for every* $v \in S_c$, $G'(v) \neq 0$, *where* $G(v) = \|v\|_2^2 - c^2$. *If* $I \in C^1(X, \mathbb{R})$ *is bounded below on* S_c, $v \in S_c$ *and* $\varepsilon, \delta > 0$ *satisfying* $I(v) \leq \inf_{S_c} I + \varepsilon$, *there exists* $u \in S_c$ *such that*

$$I(u) \leq \inf_{S_c} I + 2\varepsilon,\ \|I'|_{S_c}(u)\|_{X'} = \sup_{\substack{\langle u,h\rangle=0\\ \|h\|_X=1}} |\langle I'(u), h\rangle| = \min_{\lambda\in\mathbb{R}} \|I'(u) - \lambda G'(u)\| \leq \frac{8\varepsilon}{\delta},\ \|u - v\|_X \leq 2\delta.$$

Lemma 2.7 ([11, Theorem 1]). *Let* $1 < p < \infty$ *and let* $\{u_n\} \subset L^p(\mathbb{R}^N)$ *be a bounded sequence converging to* u *almost everywhere. Then* $u_n \rightharpoonup u$ *(weakly) in* $L^p(\mathbb{R}^N)$.

Remark 2.8. *For any* $r > 1$, *there exists a constant* $C(r) > 0$ *such that for all* $x, y \in \mathbb{R}^N$ *with* $|x| + |y| \neq 0$, *it holds*

$$\langle |x|^{r-2}x - |y|^{r-2}y, x - y\rangle \geq C(r) \times \begin{cases} \frac{|x-y|^2}{(|x|+|y|)^{2-r}} & \text{if } 1 \leq r < 2,\\ |x - y|^r & \text{if } r \geq 2. \end{cases}$$

3. Proof of Theorem 1.1

In this section, we study the existence of a global minimizer in the mass subcritical case and consider the following constrained minimization problem

$$m(c) := \inf_{u\in S_c} I(u),$$

where $S_c := \{u \in X, \|u\|_2 = c\}$.

Lemma 3.1. *Let* $c > 0$, *and assume conditions* (1.10) *and* (1.11) *hold. Then* $-\infty < m(c) < 0$.

Proof. If $2 < p < q(1 + \frac{2}{N})$ and $\frac{2N}{N+2} < q \leq 2 + \frac{4}{N+2}$, then $p\delta_p < q(1 + \delta_q) \leq 4$. If $2 < p < 2 + \frac{8}{N}$ and $2 + \frac{4}{N+2} < q < N$, then $p\delta_p < 4 < q(1 + \delta_q)$. Therefore,

$$p\delta_p < \min\{4, q(1 + \delta_q)\}.$$

For any fixed $u \in S_c$, by applying Lemma 2.1, we obtain

$$\begin{aligned} I(u) &= \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{q}\|\nabla u\|_q^q - \frac{1}{p}\|u\|_p^p \\ &\geq \frac{1}{2}\|\Delta u\|_2^2 - \frac{1}{p}\|u\|_p^p \\ &\geq \frac{1}{2}\|\Delta u\|_2^2 - \frac{C_{N,p}^p}{p} c^{p(1-\frac{\delta_p}{2})}\|\Delta u\|_2^{\frac{p\delta_p}{2}}, \end{aligned} \tag{3.1}$$

which implies that $m(c) > -\infty$.

Next, we show that $m(c) < 0$. Consider the rescaled function $u_t(x) = t^{N/2}u(tx) \in S_c$, we deduce that

$$I(u_t) = \frac{t^4}{2}\|\Delta u\|_2^2 + \frac{t^{q(1+\delta_q)}}{q}\|\nabla u\|_q^q - \frac{t^{p\delta_p}}{p}\|u\|_p^p.$$

Since $p\delta_p < \min\{4, q(1 + \delta_q)\}$, for sufficiently small $t > 0$, we conclude that $m(c) \leq I(u_t) < 0$. This completes the proof. □

Lemma 3.2. *Suppose that* (1.10) *and* (1.11) *hold, then the map* $c \mapsto m(c)$ *is continuous for* $c > 0$.

Proof. Let $c > 0$ and $\{c_n\} \subset (0, +\infty)$ such that $c_n \to c$. We aim to show that $m(c_n) \to m(c)$. For each $n \in \mathbb{N}^+$, there exists $u_n \in S_{c_n}$ such that $m(c_n) \le I(u_n) < m(c_n) + \frac{1}{n}$. Hence, by Lemma 3.1, for n sufficiently large, we have $I(u_n) \le 0$. Then, we deduce from (3.1) that

$$\|\Delta u_n\|_2 \le \Big(\frac{2C_{N,p}^p}{p}\Big)^{\frac{2}{4-p\delta_p}} c_n^{\frac{p(2-\delta_p)}{4-p\delta_p}} \le \Big(\frac{2C_{N,p}^p}{p}\Big)^{\frac{2}{4-p\delta_p}} c^{\frac{p(2-\delta_p)}{4-p\delta_p}} + o_n(1). \tag{3.2}$$

On the other hand, it follows that

$$0 \ge I(u_n) = \frac{1}{2}\|\Delta u_n\|_2^2 + \frac{1}{q}\|\nabla u_n\|_q^q - \frac{1}{p}\|u_n\|_p^p \ge \frac{1}{q}\|\nabla u_n\|_q^q - \frac{1}{p}\|u_n\|_p^p,$$

which implies that

$$\frac{1}{q}\|\nabla u_n\|_q^q \le \frac{1}{p}\|u_n\|_p^p \le \frac{C_{N,p}^p}{p} c^{p(1-\frac{\delta_p}{2})}\|\Delta u\|_2^{\frac{p\delta_p}{2}} \le 2^{\frac{p\delta_p}{4-p\delta_p}}\big[\frac{C_{N,p}^p}{p}\big]^{\frac{4}{4-p\delta_p}} c^{\frac{2p(2-\delta_p)}{4-p\delta_p}} + o_n(1). \tag{3.3}$$

Therefore, the sequence $\{u_n\}$ is bounded in X. Next, considering $v_n := \frac{c}{c_n}u_n \in S_c$, we have

$$\begin{aligned} I(v_n) &= \frac{1}{2}\|\Delta v_n\|_2^2 + \frac{1}{q}\|\nabla v_n\|_q^q - \frac{1}{p}\|v_n\|_p^p \\ &= \frac{1}{2}\Big(\frac{c}{c_n}\Big)\|\Delta u_n\|_2^2 + \frac{1}{q}\Big(\frac{c}{c_n}\Big)^{q/2}\|\nabla u_n\|_q^q - \frac{1}{p}\Big(\frac{c}{c_n}\Big)^{p/2}\|u_n\|_p^p. \end{aligned}$$

The boundness of $\{u_n\}$ in X and the convergence $c_n \to c$ as $n \to +\infty$ imply that

$$\begin{aligned} &m(c) \le I(v_n) \\ &= I(u_n) + \frac{1}{2}\Big(\frac{c}{c_n} - 1\Big)\|\Delta u_n\|_2^2 + \frac{1}{q}\Big(\Big(\frac{c}{c_n}\Big)^{\frac{q}{2}} - 1\Big)\|\nabla u_n\|_q^q - \frac{1}{p}\Big(\Big(\frac{c}{c_n}\Big)^{\frac{p}{2}} - 1\Big)\|u_n\|_p^p \\ &= I(u_n) + o_n(1). \end{aligned}$$

Taking the limit as $n \to +\infty$, we obtain

$$m(c) \le I(v_n) = I(u_n) + o_n(1) \le m(c_n) + \frac{1}{n} + o_n(1),$$

which yields that $m(c) \le \lim_{n\to+\infty} \inf m(c_n)$. In a similar way, let $\{w_n\}$ be a minimizing sequence for $m(c)$, which can be proved that it is also bounded in X. Define $z_n := \frac{c_n}{c}w_n \in S_{c_n}$, then

$$m(c_n) \le I(z_n) = I(w_n) + o_n(1),$$

which leads to $\lim_{n\to+\infty} \sup m(c_n) \le m(c)$, thus completing the proof. □

Lemma 3.3. *Suppose that* (1.10) *and* (1.11) *hold. For any* $c_1 \in (0, c)$ *and* $c_2 = \sqrt{c^2 - c_1^2}$*, then* $m(c) < m(c_1) + m(c_2)$.

Proof. For each $c_1 \in (0, c)$, let $\{u_n\} \subset S_{c_1}$ be a minimizing sequence for $m(c_1)$, that is, $I(u_n) \to m(c_1)$ as $n \to \infty$. Clearly, $u_n(\theta^{-\frac{2}{N}}x) \in S_{\theta c_1}$, by using $2 - \frac{8}{N} < 2$ and $2 - \frac{2q}{N} < 2$, for any $\theta > 1$, we have that

$$\begin{aligned} m(\theta c_1) - \theta^2 m(c_1) + o_n(1) = m(\theta c_1) - \theta^2 I(u_n) &\le I(u_n(\theta^{-\frac{2}{N}}x)) - \theta^2 I(u_n) \\ &= \frac{\theta^{2-\frac{8}{N}}}{2}\|\Delta u_n\|_2^2 + \frac{\theta^{2-\frac{2q}{N}}}{q}\|\nabla u_n\|_q^q - \theta^2\|u_n\|_p^p - \theta^2 I(u_n) \\ &= \frac{1}{2}\left(\theta^{2-\frac{8}{N}} - \theta^2\right)\|\Delta u_n\|_2^2 + \frac{1}{q}\left(\theta^{2-\frac{2q}{N}} - \theta^2\right)\|\nabla u_n\|_q^q \le 0, \end{aligned}$$

which yields that $m(\theta c_1) \le \theta^2 m(c_1)$ for all $\theta > 1$, and $m(\theta c_1) = \theta^2 m(c_1)$ holds if and only if $\|\Delta u_n\|_2 \to 0$ and $\|\nabla u_n\|_q \to 0$, by (2.1), we have that $\|u_n\|_p \to 0$ and

$$0 > m(c_1) = \lim_{n\to+\infty} I(u_n) = \frac{1}{2}\lim_{n\to+\infty}\|\Delta u_n\|_2^2 + \frac{1}{q}\lim_{n\to+\infty}\|\nabla u_n\|_q^q - \frac{1}{p}\lim_{n\to+\infty}\|u_n\|_p^p = 0,$$

which achieves a contradiction. Thus, there must hold true that

$$m(\theta c_1) < \theta^2 m(c_1) \quad \forall \theta > 1. \tag{3.4}$$

By a similar argument, we derive that

$$m(\theta c_2) < \theta^2 m(c_2) \quad \forall \theta > 1. \tag{3.5}$$

Finally, apply (3.4) with $\theta = \frac{c}{c_1} > 1$ and (3.5) with $\theta = \frac{c}{c_2} > 1$, we get

$$\frac{c_1^2}{c^2} m\Big(\frac{c}{c_1} c_1\Big) + \frac{c_2^2}{c^2} m\Big(\frac{c}{c_2} c_2\Big) < m(c_1) + m(c_2),$$

Because $c_2 = \sqrt{c^2 - c_1^2}$, we conclude that $m(c) < m(c_1) + m(c_2)$. □

Lemma 3.4. *Suppose that* (1.10) *and* (1.11) *hold. Let* $\{v_n\} \subset S_c$ *be a minimizing sequence for* $m(c)$*, there exists a sequence* $\{u_n\} \subset S_c$ *such that*

$$\|u_n - v_n\|_X \to 0 \quad \text{and} \quad (I|_{S_c})'(u_n) \to 0 \quad \text{as } n \to +\infty.$$

As a result, the sequence $\{u_n\}$ *is relatively compact in* X *up to translations, implying that* $m(c)$ *is attained.*

Proof. Since $\{v_n\} \subset S_c$ is a minimizing sequence for $m(c)$, by Lemma 2.6, there exists a new sequence $\{u_n\} \subset S_c$ such that $\|u_n - v_n\|_X \to 0$, which is also a Palais-Smale sequence for $I|_{S_c}$, that is, $I(u_n) \to m(c)$ and $(I|_{S_c})'(u_n) \to 0$. Thus, Lemma 2.5 shows that

$$I'(u_n) - \langle I'(u_n), u_n\rangle u_n \to 0 \text{ in } \ X'. \tag{3.6}$$

If $\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n(x)|^2\,dx = 0$, for any $R > 0$, then it follows from [28, Lemma I.1] that

$$\lim_{n\to\infty} \|u_n\|_r = 0 \quad \text{for } 2 < r < 4^*.$$

This leads to $\|u_n\|_p \to 0$ and

$$0 > m(c) = \lim_{n\to\infty} I(u_n) = \frac{1}{2}\lim_{n\to\infty} \|\Delta u_n\|_2^2 + \frac{1}{q}\lim_{n\to\infty} \|\nabla u_n\|_q^q - \frac{1}{p}\lim_{n\to\infty} \|u_n\|_p^p \geq 0,$$

which achieves a contradiction. Therefore, there exist $\delta > 0$ and a subsequence $\{y_n\} \subset \mathbb{R}^N$ such that

$$\int_{B_R(y_n)} |u_n(x)|^2\,dx \geq \frac{\delta}{2} > 0,$$

for some $R > 0$. By Lemma 3.2, we obtain that $\{u_n\}$ is bounded in X and thus, up to a subsequence, still denoted by $\{u_n\}$, we may assume that there exists $u_c \in X$ such that

$$\begin{gathered} u_n(\cdot + y_n) \rightharpoonup u_c \quad \text{in } X, \\ u_n(\cdot + y_n) \to u_c \quad \text{in } L^r_{\rm loc}(\mathbb{R}^N) \quad \text{for all } 2 \leq r < 4^*, \\ u_n(\cdot + y_n) \to u_c \quad \text{a.e. in } \mathbb{R}^N. \end{gathered} \tag{3.7}$$

Define $w_n(x) := u_n(\cdot + y_n) - u_c$, (3.7) implies that $w_n \rightharpoonup 0$ in X. Moreover, from the Brézis-Lieb Lemma [11], we obtain as $n \to +\infty$

$$\begin{gathered} \|u_n\|_2^2 = \|u_n(\cdot + y_n)\|_2^2 = \|w_n + u_c\|_2^2 = \|w_n\|_2^2 + \|u_c\|_2^2 + o_n(1), \\ \|\Delta u_n\|_2^2 = \|\Delta u_n(\cdot + y_n)\|_2^2 = \|\Delta w_n + \Delta u_c\|_2^2 = \|\Delta w_n\|_2^2 + \|\Delta u_c\|_2^2 + o_n(1), \\ \|u_n\|_p^p = \|u_n(\cdot + y_n)\|_p^p = \|w_n\|_p^p + \|u_c\|_p^p + o_n(1). \end{gathered} \tag{3.8}$$

Next, we let $\Phi \in C_0^\infty(\mathbb{R}^N)$ satisfy $0 \leq \Phi \leq 1$ and

$$\Phi(x) = \begin{cases} 1 & \text{if } x \in B_1(0), \\ 0 & \text{if } x \in \mathbb{R}^N \setminus B_2(0). \end{cases}$$

For any $R > 0$, define $\Psi_R(x) := \Phi\left(\frac{x}{R}\right)$ for $x \in \mathbb{R}^N$. It is easy to verify that $\Psi_R(x) \in X$. Now, let

$$P_n(x) := (\Delta u_n - \Delta u_c)(\Delta u_n - \Delta u_c) + \left(|\nabla u_n|^{q-2}\nabla u_n - |\nabla u_c|^{q-2}\nabla u_c\right)(\nabla u_n - \nabla u_c).$$

From Remark 2.8, it follows that $P_n(x) \geq 0$. Then, we get $\Psi_R = 1$ in B_R by the definition of $\Psi_R(x)$, so that

$$\int_{B_R} P_n dx = \int_{B_R} P_n \Psi_R dx \leq \int_{\mathbb{R}^N} P_n \Psi_R dx.$$

Since

$$\begin{aligned}
&\langle I'(u_n), (u_n - u_c)\Psi_R(x - y_n)\rangle = \langle I'(\tilde{u}_n), (\tilde{u}_n - u_c)\Psi_R\rangle \\
&= \int_{\mathbb{R}^N} \Delta\tilde{u}_n \Delta\left[(\tilde{u}_n - u_c)\Psi_R\right] dx + \int_{\mathbb{R}^N} |\nabla\tilde{u}_n|^{q-2}\nabla\tilde{u}_n \nabla\left[(\tilde{u}_n - u_c)\Psi_R\right] dx \\
&\quad - \int_{\mathbb{R}^N} |\tilde{u}_n|^{p-2}\tilde{u}_n(\tilde{u}_n - u_c)\Psi_R\, dx \\
&= \int_{\mathbb{R}^N} \Delta\tilde{u}_n[\Delta(\tilde{u}_n - u_c)]\Psi_R\, dx + 2\int_{\mathbb{R}^N} \Delta\tilde{u}_n \nabla(\tilde{u}_n - u_c)\nabla\Psi_R\, dx \\
&\quad + \int_{\mathbb{R}^N} \Delta\tilde{u}_n \Delta\Psi_R(\tilde{u}_n - u_c)\, dx + \int_{\mathbb{R}^N} |\nabla\tilde{u}_n|^{q-2}\nabla\tilde{u}_n \nabla(\tilde{u}_n - u_c)\Psi_R\, dx \\
&\quad + \int_{\mathbb{R}^N} |\nabla\tilde{u}_n|^{q-2}\nabla\tilde{u}_n(\tilde{u}_n - u_c)\nabla\Psi_R\, dx - \int_{\mathbb{R}^N} |\tilde{u}_n|^{p-2}\tilde{u}_n(\tilde{u}_n - u_c)\Psi_R\, dx,
\end{aligned}$$

it follows that

$$\begin{aligned}
&\int_{B_R} P_n\, dx \\
&\leq \int_{\mathbb{R}^N} \Big[(\Delta u_n - \Delta u_c)(\Delta u_n - \Delta u_c) + (|\nabla\tilde{u}_n|^{q-2}\nabla\tilde{u}_n - |\nabla u_c|^{q-2}\nabla u_c)(\nabla\tilde{u}_n - \nabla u_c)\Big]\Psi_R dx \\
&= \langle I'(\tilde{u}_n), [(\tilde{u}_n - u_c)\Psi_R]\rangle - \int_{\mathbb{R}^N} \Delta u_c \Delta(\tilde{u}_n - u_c)\Psi_R dx \\
&\quad - 2\int_{\mathbb{R}^N} \Delta\tilde{u}_n \nabla(\tilde{u}_n - u_c)\nabla\Psi_R dx - \int_{\mathbb{R}^N} \Delta\tilde{u}_n(\tilde{u}_n - u_c)\Delta\Psi_R dx \\
&\quad + \int_{\mathbb{R}^N} |\tilde{u}_n|^{p-2}\tilde{u}_n(\tilde{u}_n - u_c)\Psi_R dx - \int_{\mathbb{R}^N} |\nabla u_c|^{q-2}\nabla u_c(\nabla\tilde{u}_n - \nabla u_c)\Psi_R dx \\
&\quad - \int_{\mathbb{R}^N} |\nabla\tilde{u}_n|^{q-2}\nabla\tilde{u}_n(\tilde{u}_n - u_c)\nabla\Psi_R dx.
\end{aligned}$$

We deduce from (3.7) and Hölder's inequality that

$$\Big|\int_{\mathbb{R}^N} \tilde{u}_n(\tilde{u}_n - u_c)\Psi_R dx\Big| \leq \Big(\int_{\mathbb{R}^N} \tilde{u}_n^2 dx\Big)^{1/2}\Big(\int_{\mathbb{R}^N} (\tilde{u}_n - u_c)^2\Psi_R^2 dx\Big)^{1/2} \to 0.$$

Thus, (3.6) implies that $\langle I'(\tilde{u}_n), [(\tilde{u}_n - u_c)\Psi_R]\rangle \to 0$ as $n \to +\infty$. By the definition of Ψ_R, there exists a constant C such that $|\nabla\Psi_R(x)| \leq \frac{C}{R}$, $|\Delta\Psi_R(x)| \leq \frac{C}{R^2}$, $\forall x \in \mathbb{R}^N$. Applying Hölder's inequality again, we obtain

$$\Big|\int_{\mathbb{R}^N} \Delta\tilde{u}_n \nabla(\tilde{u}_n - u_c)\nabla\Psi_R\, dx\Big| \leq \frac{C}{R}\|\Delta\tilde{u}_n\|_2\|\nabla(\tilde{u}_n - u_c)\|_2,$$

$$\Big|\int_{\mathbb{R}^N} \Delta\tilde{u}_n(\tilde{u}_n - u_c)\Delta\Psi_R\, dx\Big| \leq \frac{C}{R^2}\|\Delta\tilde{u}_n\|_2\|\tilde{u}_n - u_c\|_2,$$

$$\begin{aligned}
\Big|\int_{\mathbb{R}^N} |\nabla\tilde{u}_n|^{q-2}\nabla\tilde{u}_n(\tilde{u}_n - u_c)\nabla\Psi_R\, dx\Big| &\leq \Big(\int_{\mathbb{R}^N} |\nabla\tilde{u}_n|^{(q-1)q'}\, dx\Big)^{1/q'}\Big(\int_{\mathbb{R}^N} |\tilde{u}_n - u_c|^q|\nabla\Psi_R|^q\, dx\Big)^{1/q} \\
&\leq \frac{C}{R}\|\nabla\tilde{u}_n\|_{L^q}^{q-1}\|\tilde{u}_n - u_c\|_{L^q}.
\end{aligned}$$

We define the functional

$$f(\nu) := \int_{\mathbb{R}^N} |\nabla u_c|^{q-2}\nabla u_c \nabla\nu\Psi_R dx,$$

for every $\nu \in X$. From [4], we see that f is a linear functional in X', implying that as $n \to +\infty$,

$$\int_{\mathbb{R}^N} |\nabla u_c|^{q-2} \nabla u_c \nabla \tilde{u}_n \Psi_R dx \to \int_{\mathbb{R}^N} |\nabla u_c|^q \Psi_R dx.$$

By the fact that $\Delta \tilde{u}_n \rightharpoonup \Delta u_c$ in H^2 and $\Delta u_c \Psi_R \in L^2$, we conclude that $n \to +\infty$

$$\int_{\mathbb{R}^N} \Delta u_c \Delta \tilde{u}_n \Psi_R dx \to \int_{\mathbb{R}^N} |\Delta u_c|^2 \Psi_R dx.$$

Since $H^2(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for $2 < p < 4^*$ and $\{\tilde{u}_n\}$ is bounded in X, it follows that there exists $M > 0$ such that $\|\tilde{u}_n\|_{p^*} \le M$, so

$$\|\tilde{u}_n^p\|_{L^{\frac{p^*}{p}}} = \Big[\int_{\mathbb{R}^N} (\tilde{u}_n^p)^{\frac{p^*}{p}} dx \Big]^{p/p^*} = \|\tilde{u}_n\|_{p^*}^p \le M^p,$$

which means $\{|\tilde{u}_n|^p\}$ is bounded in $L^{\frac{p^*}{p}}(\mathbb{R}^N)$. It follows from (3.7) and Lemma 2.7 that $|\tilde{u}_n|^p \rightharpoonup |u_c|^p$ in $L^{\frac{p^*}{p}}(\mathbb{R}^N)$. Therefore, as $n \to +\infty$

$$\int_{\mathbb{R}^N} |\tilde{u}_n|^p \Psi_R \to \int_{\mathbb{R}^N} |u_c|^p \Psi_R.$$

Similarly, we can prove $\{|\tilde{u}_n|^{p-1}\}$ is bounded in $L^{\frac{p^*}{p-1}}(\mathbb{R}^N)$. Thus, we are able to show that as $n \to +\infty$

$$\int_{\mathbb{R}^N} |\tilde{u}_n|^{p-2} \tilde{u}_n u_c \Psi_R \to \int_{\mathbb{R}^N} |u_c|^p \Psi_R.$$

Thus, we arrive to $\lim_{n\to\infty} \int_{B_R} P_n dx \le 0$. Since $P_n \ge 0$, we conclude that $\lim_{n\to\infty} \int_{B_R} P_n dx = 0$. By Lemma 2.4, up to subsequences, it holds $\nabla u_n \to \nabla u_c$ a.e. in $\mathbb{R}^N$. Thus, from the Brézis-Lieb Lemma, it follows that

$$\|\nabla u_n\|_q^q = \|\nabla u_n(\cdot + y_n)\|_q^q = \|\nabla w_n\|_q^q + \|\nabla u_c\|_q^q + o_n(1). \tag{3.9}$$

Finally, we claim that $u_n(\cdot + y_n) \to u_c \not\equiv 0$ in $L^2(\mathbb{R}^N)$, which implies that $\|w_n\|_2 \to 0$ in $L^2(\mathbb{R}^N)$. Denote $\overline{c} = \|u_c\|_2 > 0$. If $\overline{c} = c$, the proof is completed by (3.8). If $\overline{c} < c$. Using (3.8) and (3.9), we obtain

$$m(c) = I(u_n) + o_n(1) = I(u_n(\cdot + y_n)) + o_n(1) = I(w_n) + I(u_c) + o_n(1) \ge m(\|w_n\|_2) + m(\overline{c}). \tag{3.10}$$

By Lemma 3.2, it follows that

$$m(c) \ge m(c_2) + m(\overline{c}),$$

where $c_2 = \|w_n\|_2 = \sqrt{c^2 - \overline{c}^2} > 0$, this contradicts with Lemma 3.3. Thus, we conclude that $\|u_c\|_2 = c$ and hence $w_n \to 0$ in $L^2(\mathbb{R}^N)$. By Lemma 2.1, it immediately follows that

$$\|w_n\|_p \le \mathcal{C}_{N,p} \|\Delta w_n\|_2^{\delta_p/2} \|w_n\|_2^{1-\frac{\delta_p}{2}} \to 0.$$

Thus, one deduce that

$$\lim_{n\to\infty} \inf I(w_n) = \lim \inf_{n\to\infty} \frac{1}{2} \|\Delta w_n\|_2^2 + \frac{1}{q} \|\nabla w_n\|_q^q \ge 0. \tag{3.11}$$

On the other hand, using $\|u_c\|_2 = c$, we obtain

$$m(c) = I(w_n) + I(u_c) + o_n(1) \ge I(w_n) + m(c) + o_n(1),$$

which yields

$$\lim_{n\to\infty} \sup I(w_n) \le 0. \tag{3.12}$$

Therefore, we conclude that $\|\Delta w_n\|_2 \to 0$ and $\|\nabla w_n\|_q \to 0$ from (3.11) and (3.12). Combined with $\|w_n\|_2 \to 0$, this shows that $w_n \to 0$ in X. $\square$

Proof of Theorem 1.1. From Lemma 3.4, it follows that $m(c)$ is attained by some $u_c \in S_c$. By the Lagrange multiplier rule, there exists a pair (λ_c, u_c) that satisfies (1.1)

$$\|\Delta u_c\|_2^2 + \|\nabla u_c\|_q^q + \lambda_c\|u_c\|_2^2 - \|u_c\|_p^p = 0.$$

Using the fact that $\frac{2}{\delta_p} - 1 > 0$ and $\frac{1+\delta_p}{\delta_p} - 1 > 0$, it follows from Lemma 2.3 that

$$\lambda_c c^2 = \|u_c\|_p^p - \|\Delta u_c\|_2^2 - \|\nabla u_c\|_q^q = \Big(\frac{2}{\delta_p} - 1\Big)\|\Delta u_c\|_2^2 + \Big(\frac{1+\delta_p}{\delta_p} - 1\Big)\|\nabla u_c\|_q^q > 0, \tag{3.13}$$

which shows that $\lambda_c > 0$, indicating the existence of a nontrivial solution to (1.1). Similarly to get (3.2) and (3.3), we obtain

$$\|\Delta u_c\|_2 \le \Big(\frac{2C_{N,p}^p}{p}\Big)^{\frac{2}{4-p\delta_p}} c^{\frac{p(2-\delta_p)}{4-p\delta_p}},$$

$$\frac{1}{q}\|\nabla u_c\|_q^q \le \frac{1}{p}\|u_c\|_p^p \le \frac{C_{N,p}^p}{p} c^{p(1-\delta_p)}\|\nabla u_c\|_2^{p\delta_p} \le 2^{\frac{p\delta_p}{4-p\delta_p}}[\frac{C_{N,p}^p}{p}]^{\frac{4}{4-p\delta_p}} c^{\frac{2p(2-\delta_p)}{4-p\delta_p}}.$$

From (1.10) and (1.11), we have $\frac{1}{2} - \frac{2}{p\delta_p} < 0$, $\frac{1}{q} - \frac{1+\delta_q}{p\delta_p} < 0$ and $2 - \delta_p > 0$. By Lemma 2.3, we deduce that

$$\begin{aligned}
0 > m(c) = I(u_c) &= \frac{1}{2}\|\Delta u_c\|_2^2 + \frac{1}{q}\|\nabla u_c\|_q^q - \frac{1}{p}\|u_c\|_p^p \\
&= \Big(\frac{1}{2} - \frac{2}{p\delta_p}\Big)\|\Delta u_c\|_2^2 + \Big(\frac{1}{q} - \frac{1+\delta_q}{p\delta_p}\Big)\|\nabla u_c\|_q^q \\
&\ge \frac{p\delta_q - 4}{p\delta_p} 2^{\frac{p\delta_p}{4-p\delta_p}}[\frac{C_{N,p}^p}{p}]^{\frac{4}{4-p\delta_p}} c^{\frac{2p(2-\delta_p)}{4-p\delta_p}} + \frac{p\delta_p - q(1+\delta_q)}{p\delta_p} 2^{\frac{p\delta_p}{4-p\delta_p}}[\frac{C_{N,p}^p}{p}]^{\frac{4}{4-p\delta_p}} c^{\frac{2p(2-\delta_p)}{4-p\delta_p}}.
\end{aligned}$$

Therefore,

$$m(c) \ge A_1 c^{\frac{2p(2-\delta_p)}{4-p\delta_p}},$$

where

$$A_1 = \Big(\frac{p\delta_q - 4}{p\delta_p} + \frac{p\delta_p - q(1+\delta_q)}{p\delta_p}\Big) 2^{\frac{p\delta_p}{4-p\delta_p}}[\frac{C_{N,p}^p}{p}]^{\frac{4}{4-p\delta_p}} < 0.$$

This fact also indicates that $m(c) \to 0^-$ as $c \to 0^+$.

Similarly, from (3.13), we have $\lambda_c \le A_2 c^{\frac{4(p-2)}{4-p\delta_p}}$, where

$$A_2 = \Big(\frac{4-2\delta_p}{\delta_p} + \frac{q}{\delta_p}\Big) 2^{\frac{p\delta_p}{4-p\delta_p}}[\frac{C_{N,p}^p}{p}]^{\frac{4}{4-p\delta_p}} > 0.$$

This fact leads to $\lambda_c \to 0^+$ as $c \to 0^+$. □

4. Proof of Theorem 1.2

This section is devoted to proving the existence of a normalized ground state solution in the L^2-supercritical case. We first observe that the following conditions hold

$$2 + \frac{8}{N} < p < \min\{q^*, 4^*\},\ 1 < q \le \frac{2N+8}{N+2} \implies q(1+\delta_q) \le 4 < p\delta_p,$$

$$q(1 + \frac{2}{N}) < p < \min\{q^*, 4^*\},\ \frac{2N+8}{N+2} < q < N \implies 4 < q(1+\delta_q) < p\delta_p,$$

which yields

$$\max\{4, q(1+\delta_q)\} < p\delta_p.$$

Let $u \in S_c$ be fixed. We then define $u_t(x) = t^{N/2}u(tx) \in S_c$, and it follows that

$$m(c) = \inf_{u\in S_c} I(u) \le I(u_t) = \frac{t^4}{2}\|\Delta u\|_2^2 + \frac{t^{q(1+\delta_q)}}{q}\|\nabla u\|_q^q - \frac{t^{p\delta_p}}{p}\|u\|_p^p \to -\infty \quad \text{as } t \to +\infty.$$

In this case, $I(u)$ is unbounded from below on S_c, so the global minimization method cannot be used to find critical points of $I|_{S_c}$. Thus, we consider a modified minimization problem

$$\sigma(c) := \inf_{u\in\mathcal{P}_c} I(u),$$

where

$$\mathcal{P}_c = \left\{u \in S_c : P(u) := 2\|\Delta u\|_2^2 + (1+\delta_q)\|\nabla u\|_q^q - \delta_p\|u\|_p^p = 0\right\}.$$

By the following lemma, we see that $I(u)$ is bounded from below on $\mathcal{P}_c$.

Lemma 4.1. *Let $c > 0$, and assume that conditions* (1.12) *and* (1.13) *hold. Then, I is coercive on $\mathcal{P}_c$, and*

$$\sigma(c) := \inf_{u\in\mathcal{P}_c} I(u) > 0.$$

Proof. For any $u \in \mathcal{P}_c$, by Lemma 2.1 and 2.2, we obtain

$$2\|\Delta u\|_2^2 \le 2\|\Delta u\|_2^2 + (1+\delta_q)\|\nabla u\|_q^q = \delta_p\|u\|_p^p \le \delta_p\mathcal{C}_{N,p}^p\|\Delta u\|_2^{\frac{p\delta_p}{2}} c^{p(1-\frac{\delta_p}{2})},$$
$$(1+\delta_q)\|\nabla u\|_q^q \le 2\|\Delta u\|_2^2 + (1+\delta_q)\|\nabla u\|_q^q = \delta_p\|u\|_p^p \le \delta_p K_{N,p}^p\|\nabla u\|_q^{p\alpha}c^{p(1-\alpha)}.$$

Consequently, we can deduce the lower-bounds for $\|\Delta u\|_2$, $\|\nabla u\|_q$ and $\|u\|_p^p$ as follows

$$\begin{gathered}\|\Delta u\|_2 \ge \Big[\frac{2}{\delta_p\mathcal{C}_{N,p}^p}\Big]^{\frac{2}{p\delta_p-4}} c^{-\frac{p(2-\delta_p)}{p\delta_p-4}}, \quad \|\nabla u\|_q \ge \Big[\frac{1+\delta_q}{\delta_p K_{N,p}^p}\Big]^{\frac{1}{p\alpha-q}} c^{-\frac{p(1-\alpha)}{p\alpha-q}}, \\ \|u\|_p^p \ge \frac{2\big[\frac{2}{\delta_p\mathcal{C}_{N,p}^p}\big]^{\frac{4}{p\delta_p-4}} c^{-\frac{2p(2-\delta_p)}{p\delta_p-4}} + (1+\delta_q)\big[\frac{1+\delta_q}{\delta_p K_{N,p}^p}\big]^{\frac{q}{p\alpha-q}} c^{-\frac{pq(1-\alpha)}{p\alpha-q}}}{\delta_p}.\end{gathered} \tag{4.1}$$

For any fixed $u \in \mathcal{P}_c$, we can rewrite $I(u)$ as follows

$$\begin{aligned} I(u) &= \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{q}\|\nabla u\|_q^q - \frac{1}{p}\|u\|_p^p \\ &= \Big(\frac{1}{2} - \frac{2}{p\delta_p}\Big)\|\Delta u\|_2^2 + \Big(\frac{1}{q} - \frac{1+\delta_q}{p\delta_p}\Big)\|\nabla u\|_q^q. \end{aligned} \tag{4.2}$$

For every sequence $\{u_k\} \subset \mathcal{P}_c$ such that $\|u_k\|_X \to +\infty$, we deduce from $\max\{4, q(1+\delta_q)\} < p\delta_p$ that $I(u_k) \to +\infty$. Hence, I is coercive on $\mathcal{P}_c$. By (4.1) and (4.2), we also conclude that $\sigma(c) > 0$. Thus the proof is complete. ☐

Lemma 4.2. *Assume that* (1.12) *and* (1.13) *are satisfied. Then, for any $u \in S_c$ and $u_t(x) = t^{N/2}u(tx)$, there exists a unique $t_0 > 0$ such that $I(u_{t_0}) = \max_{t>0} I(u_t)$ and $u_{t_0} \in \mathcal{P}_c$. In particular, the following results hold*

(1) $t_0 < 1 \iff P(u) < 0$;
(2) $t_0 = 1 \iff P(u) = 0$.

Proof. For any $u \in S_c$, since $u_t(x) = t^{N/2}u(tx) \in S_c$, we define the function

$$h(t) = I(u_t) = \frac{t^4}{2}\|\Delta u\|_2^2 + \frac{t^{q(1+\delta_q)}}{q}\|\nabla u\|_q^q - \frac{t^{p\delta_p}}{p}\|u\|_p^p \quad \forall t > 0.$$

Differentiating $h(t)$ with respect to t, we have

$$h'(t) = \frac{2t^4\|\Delta u\|_2^2 + (1+\delta_q)t^{q(1+\delta_q)}\|\nabla u\|_q^q - \delta_p t^{p\delta_p}\|u\|_p^p}{t} = \frac{P(u_t)}{t}.$$

By $\max\{4, q(1+\delta_q)\} < p\delta_p$, it follows that $h'(t) > 0$ for $t > 0$ small enough, and $\lim_{t\to+\infty} h'(t) = -\infty$. Therefore, $h(t)$ has a unique maximum at some point $t_0 > 0$, see [27]. Moreover, since $h'(t_0) = \frac{P(u_{t_0})}{t_0} = 0$, we can infer that $u_{t_0} \in \mathcal{P}_c$. Thus, we conclude that $I(u_{t_0}) = \max_{t>0} I(u_t)$ and $P(u_{t_0}) = 0$. Next, we show that $P(u) < 0 \Rightarrow t_0 < 1$. Assume that $t_0 \ge 1$, since $h'(t_0) = 0$ and $P(u) < 0$, we obtain

$$\begin{aligned} 0 &= 2t_0^{4-p\delta_p}\|\Delta u\|_2^2 + (1+\delta_q)t_0^{q(1+\delta_q)-p\delta_p}\|\nabla u\|_q^q - \delta_p\|u\|_p^p \\ &\le 2\|\Delta u\|_2^2 + (1+\delta_q)\|\nabla u\|_q^q - \delta_p\|u\|_p^p \end{aligned}$$

$$= P(u) < 0,$$

which is a contradiction. Thus, $P(u) < 0 \Rightarrow t_0 < 1$ is proved. If $P(u) = 0$, it is easy to verify that neither $t_0 > 1$ nor $t_0 < 1$ can not occur. Hence, $P(u) = 0 \Rightarrow t_0 = 1$.

Finally, we prove that $t_0 < 1 \Rightarrow P(u) < 0$ and $t_0 = 1 \Rightarrow P(u) = 0$. If $t_0 < 1$, we have

$$\begin{aligned} 0 &= 2t_0^{4-p\delta_p}\|\Delta u\|_2^2 + (1+\delta_q)t_0^{q(1+\delta_q)-p\delta_p}\|\nabla u\|_q^q - \delta_p\|u\|_p^p \\ &> 2\|\Delta u\|_2^2 + (1+\delta_q)\|\nabla u\|_q^q - \delta_p\|u\|_p^p \\ &= P(u). \end{aligned}$$

This implies $t_0 < 1 \Rightarrow P(u) < 0$. When $t_0 = 1$, it can be easily shown that $t_0 = 1 \Rightarrow P(u) = 0$. □

Lemma 4.3. *Suppose that* (1.12) *and* (1.13) *hold. Then, every minimizer of $I|_{\mathcal{P}_c}$ is a critical point of $I|_{\mathcal{S}_c}$.*

Proof. Let u be a minimizer of $I|_{\mathcal{P}_c}$. Then, we have $P(u) = 2\|\Delta u\|_2^2 + (1+\delta_q)\|\nabla u\|_q^q - \delta_p\|u\|_p^p = 0$. According to [14, Corollary 4.1.2], there exist two Lagrange multipliers λ and μ such that

$$I'(u) - \lambda u - \mu P'(u) = 0 \quad \text{in } X'.$$

That is,

$$(1-4\mu)\Delta^2 u - [1 - \mu q(1+\delta_q)]\Delta_q u + (\mu p\delta_p - 1)|u|^{p-2}u - \lambda u = 0.$$

Similarly to the proof of Lemma 2.3, we can derive that

$$2(1-4\mu)\|\Delta u\|_2^2 + [1 - q(1+\delta_q)\mu](1+\delta_q)\|\nabla u\|_q^q + (\mu p\delta_p - 1)\delta_p\|u\|_p^p = 0. \tag{4.3}$$

Recalling that $P(u) = 0$, thus (4.3) can be reduced to

$$\mu\Big\{8\|\Delta u\|_2^2 + q(1+\delta_q)^2\|\nabla u\|_q^q - p\delta_p^2\|u\|_p^p\Big\} = 0.$$

Using $P(u) = 0$ once more, we obtain

$$\mu\Big\{2(4 - p\delta_p)\|\Delta u\|_2^2 + (1+\delta_q)[q(1+\delta_q) - p\delta_p]\|\nabla u\|_q^q\Big\} = 0. \tag{4.4}$$

Owing to $p\delta_p > 4$ and $p\delta_p > q(1+\delta_q)$, we can conclude that $\mu = 0$ from (4.4). Consequently, $I'(u) - \lambda u = 0$ in X'. This indicates that $P(u) = 0$ in $\mathcal{P}_c$ is a natural constraint. □

Lemma 4.4. *Suppose that* (1.12) *and* (1.13) *hold. If $c_2 > c_1 > 0$, then $\sigma(c_2) < \sigma(c_1)$.*

Proof. From Lemma 4.1, we deduce that $\sigma(c) > 0$ for any $c > 0$. By using Lemma 4.2, there exists a sequence $\{u_n\} \subset \mathcal{P}_{c_1}$ such that

$$\sigma(c_1) \le I(u_n) = \max_{t>0} I(t^{N/2}u_n(tx)) < \sigma(c_1) + \frac{1}{n}. \tag{4.5}$$

For each $u_n \in \mathcal{P}_{c_1}$, we have

$$\Big(\frac{1}{2} - \frac{2}{p\delta_p}\Big)\|\Delta u_n\|_2^2 + \Big(\frac{1}{q} - \frac{1+\delta_q}{p\delta_p}\Big)\|\nabla u_n\|_q^q = I(u_n) \le \sigma(c_1) + 1. \tag{4.6}$$

By $p\delta_p > 4$ and $p\delta_p > q(1+\delta_q)$, we infer that $\{u_n\}$ is bounded in X. From $\max\{2+\frac{8}{N}, q(1+\frac{2}{N})\} < p < \min\{q^*, 4^*\}$, we can deduce the following facts.

(1) When $2 + \frac{8}{N} \ge q\left(1 + \frac{2}{N}\right)$ and $q^* < 4^*$, thus $2 + \frac{8}{N} < p < q^*$, and we deduce that

$$\begin{cases} 2 + \frac{8}{N} < q^*, \\ 2 + \frac{8}{N} \ge q(1 + \frac{2}{N}), \\ q^* < 4^* \end{cases} \Longrightarrow \frac{2N^2 + 8N}{N^2 + 2N + 8} < q \le \frac{2N+8}{N+2}.$$

(2) When $2 + \frac{8}{N} < q(1 + \frac{2}{N})$ and $q^* < 4^*$, we have that $q(1 + \frac{2}{N}) < p < q^*$ and

$$\begin{cases} q(1 + \frac{2}{N}) < q^*, \\ 2 + \frac{8}{N} < q(1 + \frac{2}{N}), \\ q^* < 4^*, \end{cases} \Longrightarrow \frac{2N+8}{N+2} < q < \frac{2N}{N-2}.$$

(3) When $2+\frac{8}{N} \geq q(1+\frac{2}{N})$ and $q^* \geq 4^*$, then $2+\frac{8}{N} < p < 4^*$ and

$$\begin{gathered} 2+\frac{8}{N} < 4^*, \\ 2+\frac{8}{N} \geq q(1+\frac{2}{N}), \\ q^* \geq 4^*, \end{gathered}$$

cannot occur.

(4) When $2+\frac{8}{N} < q(1+\frac{2}{N})$ and $q^* \geq 4^*$, we have that $q(1+\frac{2}{N}) < p < 4^*$ and

$$\begin{cases} q(1+\frac{2}{N}) < 4^*, \\ 2+\frac{8}{N} < q(1+\frac{2}{N}), \\ q^* \geq 4^*, \end{cases} \implies \frac{2N}{N-2} \leq q < \min\big\{N, \frac{2N^2}{N^2-2N-8}\big\}.$$

Now, for $c_2 > c_1 > 0$, we prove $\sigma(c_2) < \sigma(c_1)$ in two cases.

Case (i): $\frac{2N^2+8N}{N^2+2N+8} < q < \frac{2N}{N-2}$ and $\max\big\{2+\frac{8}{N}, q\left(1+\frac{2}{N}\right)\big\} < p < q^*$, this implies that

$$2\delta_q - 2 < 0 \quad \text{and} \quad N+p-\frac{Np}{q} > 0. \tag{4.7}$$

We denote

$$\theta := \Big(\frac{c_2}{c_1}\Big)^{\frac{1}{1+\delta_q}} > 1 \quad \text{and} \quad v_n(x) := \theta^{\frac{q-N}{q}} u_n(\theta^{-1}x).$$

Direct computations yield

$$\begin{gathered} \|v_n\|_2^2 = \theta^{N+2-\frac{2N}{q}}\|u_n\|_2^2 = \theta^{2(1+\delta_q)}c_1^2 = c_2^2, \quad \|\nabla v_n\|_q^q = \|\nabla u_n\|_q^q, \\ \|\Delta v_n\|_2^2 = \theta^{N-2-\frac{2N}{q}}\|\Delta u_n\|_2^2 = \theta^{2\delta_q-2}\|\Delta u_n\|_2^2, \quad \|v_n\|_p^p = \theta^{N+p-\frac{Np}{q}}\|u_n\|_p^p. \end{gathered} \tag{4.8}$$

By Lemma 4.2, there exists $t_n > 0$ such that $t_n^{N/2} v_n(t_n x) \in \mathcal{P}_{c_2}$ and

$$\sigma(c_2) \leq I(t_n^{N/2} v_n(t_n x)) = \max_{t>0} I(t^{N/2} v_n(tx)).$$

From (4.7) and $\theta > 1$, we have

$$\theta^{2\delta_q-2} - 1 < 0 \quad \text{and} \quad 1 - \theta^{N+p-\frac{Np}{q}} < 0.$$

Notice that $\{u_n\}$ is bounded in X and

$$P(t_n^{N/2} v_n(t_n x)) = 2t_n^4 \theta^{2\delta_q-2}\|\Delta u_n\|_2^2 + (1+\delta_q)t_n^{q(1+\delta_q)}\|\nabla u_n\|_q^q - \delta_p t_n^{p\delta_p}\theta^{N+p-\frac{Np}{q}}\|u_n\|_p^p = 0,$$

thus there exists a positive constant M such that $t_n \geq M$. Hence, by (4.1) and (4.8), we deduce that there exists a constant $C > 0$ such that

$$\begin{aligned} \sigma(c_2) \leq I(t_n^{N/2} v_n(t_n x)) &= \frac{t_n^4}{2}\|\Delta v_n\|_2^2 + \frac{t_n^{q(1+\delta_q)}}{q}\|\nabla v_n\|_q^q - \frac{t_n^{p\delta_p}}{p}\|v_n\|_p^p \\ &= \frac{t_n^4}{2}\theta^{2\delta_q-2}\|\Delta u_n\|_2^2 + \frac{t_n^{q(1+\delta_q)}}{q}\|\nabla u_n\|_q^q - \frac{t_n^{p\delta_p}}{p}\theta^{N+p-\frac{Np}{q}}\|u_n\|_p^p \\ &= I(t_n^{N/2} u_n(t_n x)) + \frac{t_n^4}{2}\left(\theta^{2\delta_q-2} - 1\right)\|\Delta u_n\|_2^2 + \frac{t_n^{p\delta_p}}{p}\Big(1 - \theta^{N+p-\frac{Np}{q}}\Big)\|u_n\|_p^p \\ &\leq I(u_n) + \frac{t_n^4}{2}\left(\theta^{2\delta_q-2} - 1\right)\|\Delta u_n\|_2^2 + \frac{t_n^{p\delta_p}}{p}\Big(1 - \theta^{N+p-\frac{Np}{q}}\Big)\|u_n\|_p^p \\ &\leq \sigma(c_1) + \frac{1}{n} - C, \end{aligned}$$

which leads to $\sigma(c_2) < \sigma(c_1)$ for n sufficiently large.

Case (ii): $\frac{2N}{N-2} \leq q < \min\big\{N, \frac{2N^2}{N^2-2N-8}\big\}$ and $q(1+\frac{2}{N}) < p < 4^*$, this yields that

$$\frac{q(1-\delta_q)}{2} \leq 0 \quad \text{and} \quad \frac{p(2-\delta_p)}{2} > 0. \tag{4.9}$$

We denote

$$\gamma := \frac{c_2}{c_1} > 1 \quad \text{and} \quad w_n(x) := \gamma^{\frac{4-N}{4}} u_n\big(\gamma^{-1/2}x\big).$$

Directly computations give

$$\begin{aligned} \|w_n\|_2^2 = \gamma^2\|u_n\|_2^2 = c_2^2, \quad \|\nabla w_n\|_q^q = \gamma^{\frac{N}{2}+\frac{q(2-N)}{4}}\|\nabla u_n\|_q^q = \gamma^{\frac{q(1-\delta_q)}{2}}\|\nabla u_n\|_q^q, \\ \|\Delta w_n\|_2^2 = \|\Delta u_n\|_2^2, \quad \|w_n\|_p^p = \gamma^{\frac{N}{2}+\frac{p(4-N)}{4}}\|u_n\|_p^p = \gamma^{\frac{p(2-\delta_p)}{2}}\|u_n\|_p^p. \end{aligned} \tag{4.10}$$

By Lemma 4.2, we obtain that there exists $t_n > 0$ such that $t_n^{\frac{N}{2}} w_n(t_n x) \in \mathcal{P}_{c_2}$ and

$$\sigma(c_2) \le I(t_n^{N/2} w_n(t_n x)) = \max_{t>0} I(t^{N/2} w_n(tx)).$$

By (4.9) and $\gamma > 1$, we have

$$\gamma^{\frac{q(1-\delta_q)}{2}} - 1 \le 0 \text{and} 1 - \gamma^{\frac{p(2-\delta_p)}{2}} < 0.$$

Similarly, we infer that $\{u_n\}$ is bounded in X and $P(t_n^{N/2} w_n(t_n x)) = 0$, by (4.1) and (4.10), there exists a constant $C' > 0$ such that

$$\begin{aligned} \sigma(c_2) &\le I(t_n^{N/2} w_n(t_n x)) \\ &= \frac{t_n^4}{2}\|\Delta w_n\|_2^2 + \frac{t_n^{q(1+\delta_q)}}{q}\|\nabla w_n\|_q^q - \frac{t_n^{p\delta_p}}{p}\|w_n\|_p^p \\ &= \frac{t_n^4}{2}\|\Delta u_n\|_2^2 + \frac{t_n^{q(1+\delta_q)}}{q}\gamma^{\frac{q(1-\delta_q)}{2}}\|\nabla u_n\|_q^q - \frac{t_n^{p\delta_p}}{p}\gamma^{\frac{p(2-\delta_p)}{2}}\|u_n\|_p^p \\ &= I(t_n^{N/2} u_n(t_n x)) + \frac{t_n^{q(1+\delta_q)}}{q}\Big(\gamma^{\frac{q(1-\delta_q)}{2}} - 1\Big)\|\nabla u_n\|_q^q + \frac{t_n^{p\delta_p}}{p}\Big(1 - \gamma^{\frac{p(2-\delta_p)}{2}}\Big)\|u_n\|_p^p \\ &\le I(u_n) + \frac{t_n^{q(1+\delta_q)}}{q}\Big(\gamma^{\frac{q(1-\delta_q)}{2}} - 1\Big)\|\nabla u_n\|_q^q + \frac{t_n^{p\delta_p}}{p}\Big(1 - \gamma^{\frac{p(2-\delta_p)}{2}}\Big)\|u_n\|_p^p \\ &\le \sigma(c_1) + \frac{1}{n} - C'. \end{aligned}$$

This means that $\sigma(c_2) < \sigma(c_1)$ for n sufficiently large. □

Lemma 4.5. *Suppose that* (1.12) *and* (1.13) *hold. Then, the* $\sigma(c) = \inf_{u\in\mathcal{P}_c} I(u)$ *is attained.*

Proof. Let $\{u_n\}$ be a minimizing sequence for $\sigma(c)$. From (4.6), we have that $\{u_n\}$ is bounded in X. Therefore, up to a subsequence, there exists $\tilde{u} \in X$ such that $u_n \rightharpoonup \tilde{u}$ as $n \to \infty$. If for any $R > 0$, the vanishing occurs

$$\delta := \lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n(x)|^2\,dx = 0,$$

from [28, Lemma I.1], we obtain that $\|u_n\|_r \to 0$ for $2 < r < 4^*$. Since $P(u_n) = 2\|\Delta u_n\|_2^2 + (1+\delta_q)\|\nabla u_n\|_q^q - \delta_p\|u_n\|_p^p = 0$, it follows that $\|\Delta u_n\|_2 \to 0$ and $\|\nabla u_n\|_q \to 0$. Consequently, we have $\sigma(c) = 0$, which contradicts with Lemma 4.1. Hence, vanishing of $\{u_n\}$ can not occur, there exist $\delta > 0$ and a subsequence $\{y_n\} \subset \mathbb{R}^N$ such that

$$\int_{B_R(y_n)} |u_n(x)|^2\,dx \ge \frac{\delta}{2} > 0,$$

for some $R > 0$. Let $\tilde{u}_n := u_n(\cdot + y_n)$, and then $\{\tilde{u}_n\}$ is also a bounded minimizing sequence for $\sigma(c)$ in X and

$$\int_{B_R(0)} |\tilde{u}_n|^2 \mathrm{d}x \geqslant \frac{\delta}{2} > 0 \text{ for } n \in \mathbb{N}^+ \text{ large enough.}$$

It follows that

$$\tilde{u}_n \rightharpoonup \tilde{u} \not\equiv 0 \text{ in } X, \quad \tilde{u}_n \to \tilde{u} \text{ in } L^r_{\rm loc}(\mathbb{R}^N) \text{ for } 2 \le r < 4^*, \quad \tilde{u}_n \to \tilde{u} \text{ a.e. in } \mathbb{R}^N. \tag{4.11}$$

Similarly to the proof of Lemma 3.4, we may assume that there exist sequences $\lambda_n, \mu_n \in \mathbb{R}$ such that

$$I'(\tilde{u}_n) - \lambda_n \tilde{u}_n - \mu_n P'(\tilde{u}_n) \to 0.$$

By a similar argument as Lemma 4.3, owing to $P(\tilde{u}_n) = 0$ and $p\delta_p > \max\{4, q(1+\delta_q)\}$, we can get that $\mu_n \to 0$. Thus, we conclude that

$$I'(\tilde{u}_n) - \lambda_n \tilde{u}_n \to 0 \quad \text{in} \quad X'.$$

Since $\tilde{u}_n \in S_c$, and by Lemma 2.6, we have

$$\left(I|_{S_c}\right)'(\tilde{u}_n) = I'(\tilde{u}_n) - \lambda_n \tilde{u}_n \to 0 \quad \text{in} \quad X',$$

which means that $\{\tilde{u}_n\}$ is a Palais-Smale sequence for $I|_{S_c}$, since $\{\tilde{u}_n\}$ is bounded in X, $I(\tilde{u}_n) \to \sigma(c)$, and $\left(I|_{S_c}\right)'(\tilde{u}_n) \to 0$. Similar to the proof of (3.9) in Lemma 3.4, we can prove that $\nabla \tilde{u}_n(x) \to \nabla \tilde{u}(x)$ a.e. in $\mathbb{R}^N$.

We now claim that $\|\tilde{u}\|_2 = c$. Otherwise, if $\|\tilde{u}\|_2 = c_1 < c$, then Lemma 4.4 shows that $\sigma(c) < \sigma(c_1)$. By using (4.11), we know that $\|\Delta \tilde{u}\|_2^2 \le \liminf_{n\to\infty} \|\Delta \tilde{u}_n\|_2^2 \le \lim_{n\to\infty} \|\Delta \tilde{u}_n\|_2^2$ and $\|\nabla \tilde{u}\|_q^q \le \liminf_{n\to\infty} \|\nabla \tilde{u}_n\|_q^q \le \lim_{n\to\infty} \|\nabla \tilde{u}_n\|_q^q$. From Lemma 4.2, there exists $\tau_0 \in (0,1]$ such that

$$\tau_0^{N/2} \tilde{u}(\tau_0 x) \in \mathcal{P}_{c_1}, \quad P(\tau_0^{N/2} \tilde{u}(\tau_0 x)) = 0.$$

We can infer that

$$\begin{aligned}
&\sigma(c) < \sigma(c_1) \\
&\le I(\tau_0^{N/2}\tilde{u}(\tau_0 x)) = \Big(\frac{1}{2} - \frac{2}{p\delta_p}\Big)\tau_0^4 \|\Delta \tilde{u}\|_2^2 + \Big(\frac{1}{q} - \frac{1+\delta_q}{p\delta_p}\Big)\tau_0^{q(1+\delta_q)} \|\nabla \tilde{u}\|_q^q \\
&\le \Big(\frac{1}{2} - \frac{2}{p\delta_p}\Big)\|\Delta \tilde{u}\|_2^2 + \Big(\frac{1}{q} - \frac{1+\delta_q}{p\delta_p}\Big)\|\nabla \tilde{u}\|_q^q \\
&\le \Big(\frac{1}{2} - \frac{2}{p\delta_p}\Big)\lim_{n\to\infty} \|\Delta \tilde{u}_n\|_2^2 + \Big(\frac{1}{q} - \frac{1+\delta_q}{p\delta_p}\Big)\lim_{n\to\infty} \|\nabla \tilde{u}_n\|_q^q \\
&= \lim_{n\to\infty} I(\tilde{u}_n) = \sigma(c),
\end{aligned}$$

which leads to a contradiction. So, it must have $c_1 = c$ and $\tau_0 = 1$. That is, $\|\tilde{u}\|_2 = c$. Moreover, denote $w_n = \tilde{u}_n - \tilde{u}$, by Lemma 2.1, it follows that

$$\|w_n\|_p \le \mathcal{C}_{N,p} \|\Delta w_n\|_2^{\delta_p/2} \|w_n\|_2^{1-\frac{\delta_p}{2}} \to 0,$$

which yields that

$$\lim_{n\to\infty} \inf I(w_n) = \lim \inf_{n\to\infty} \frac{1}{2}\|\Delta w_n\|_2^2 + \frac{1}{q}\|\nabla w_n\|_q^q \ge 0. \tag{4.12}$$

By (3.8) and (3.9), we obtain

$$\sigma(c) = I(\tilde{u}_n) + o_n(1) = I(w_n) + I(\tilde{u}) + o_n(1) \ge I(w_n) + \sigma(c) + o_n(1),$$

which leads to

$$\lim_{n\to\infty} \sup I(w_n) \le 0. \tag{4.13}$$

Then we conclude that $\|\Delta w_n\|_2 \to 0$ and $\|\nabla w_n\|_q \to 0$ from (4.12) and (4.13). This shows that $w_n \to 0$ in X and thus $\tilde{u}$ is a minimizer for $\sigma(c)$. □

Proof of Theorem 1.2. From Lemma 4.5, we know that $\sigma(c)$ is attained by some $u_c \in S_c$. Suppose $v \in S_c$ is a critical point of $I|_{S_c}$, Lemma 2.3 implies that $v \in \mathcal{P}_c$. Thus, we have $I(v) \ge \sigma(c) = I(u_c)$. This indicates that u_c is a ground state solution of (1.1). Next, by the Lagrange multiplier rule, there exists a pair (λ_c, u_c) that satisfies (1.1)

$$\int_{\mathbb{R}^d} \big(\Delta u_c \Delta \varphi + |\nabla u_c|^{q-2} \nabla u_c \nabla \varphi - |u_c|^{p-2} u_c \varphi + \lambda_c u_c \varphi\big)\, dx = 0 \quad \forall \varphi \in X.$$

From $\frac{2}{\delta_p} - 1 > 0$ and $\frac{1+\delta_q}{\delta_p} - 1 > 0$, we have

$$\lambda_c c^2 = \|u_c\|_p^p - \|\Delta u_c\|_2^2 - \|\nabla u_c\|_q^q = \Big(\frac{2}{\delta_p} - 1\Big)\|\Delta u_c\|_2^2 + \Big(\frac{1+\delta_q}{\delta_p} - 1\Big)\|\nabla u_c\|_q^q > 0, \tag{4.14}$$

which implies that $\lambda_c > 0$. In the same way as in (4.1), we can obtain

$$\|\Delta u_c\|_2 \ge \Big[\frac{2}{\delta_p \mathcal{C}_{N,p}^p}\Big]^{\frac{2}{p\delta_p - 4}} c^{-\frac{p(2-\delta_p)}{p\delta_p - 4}} \quad \text{and} \quad \|\nabla u_c\|_q \ge \Big[\frac{1+\delta_q}{\delta_p K_{N,p}^p}\Big]^{\frac{1}{p\alpha - q}} c^{-\frac{p(1-\alpha)}{p\alpha - q}}.$$

The conditions (1.12) and (1.13) ensure that

$$\frac{p(2-\delta_p)}{p\delta_p - 4} > 0, \quad \frac{pq(1-\alpha)}{p\alpha - q} > 0, \quad \frac{1}{2} - \frac{2}{p\delta_p} > 0, \quad 1 - \frac{q(1+\delta_q)}{p\delta_p} > 0.$$

Since $u_c \in \mathcal{P}_c$, we have

$$\begin{aligned}
\sigma(c) = I(u_c) &= \frac{1}{2}\|\Delta u_c\|_2^2 + \frac{1}{q}\|\nabla u_c\|_q^q - \frac{1}{p}\|u_c\|_p^p \\
&= \Big(\frac{1}{2} - \frac{2}{p\delta_p}\Big)\|\Delta u_c\|_2^2 + \Big(\frac{1}{q} - \frac{1+\delta_q}{p\delta_p}\Big)\|\nabla u_c\|_q^q \\
&\ge \Big(\frac{1}{2} - \frac{2}{p\delta_p}\Big)\Big[\frac{2}{\delta_p \mathcal{C}_{N,p}^p}\Big]^{\frac{4}{p\delta_p - 4}} c^{-\frac{2p(2-\delta_p)}{p\delta_p - 4}} + \frac{1}{q}\Big(1 - \frac{q(1+\delta_q)}{p\delta_p}\Big)\Big[\frac{1+\delta_q}{\delta_p K_{N,p}^p}\Big]^{\frac{q}{p\alpha - q}} c^{-\frac{pq(1-\alpha)}{p\alpha - q}}.
\end{aligned}$$

Therefore, we have

$$\sigma(c) \ge A_3 c^{-\frac{2p(2-\delta_p)}{p\delta_p - 4}} + A_4 c^{-\frac{pq(1-\alpha)}{p\alpha - q}},$$

where

$$A_3 = \Big(\frac{1}{2} - \frac{2}{p\delta_p}\Big)\Big[\frac{2}{\delta_p \mathcal{C}_{N,p}^p}\Big]^{\frac{4}{p\delta_p - 4}} > 0 \quad A_4 = \frac{1}{q}\Big(1 - \frac{q(1+\delta_q)}{p\delta_p}\Big)\Big[\frac{1+\delta_q}{\delta_p K_{N,p}^p}\Big]^{\frac{q}{p\alpha - q}} > 0,$$

which implies that $\sigma(c) \to +\infty$ as $c \to 0^+$. Similarly, from (4.14), we can derive that

$$\lambda_c \ge A_5 c^{-\frac{4(p-2)}{p\delta_p - 4}} + A_6 c^{-\frac{pq(1-\alpha) + 2(p\alpha - q)}{p\alpha - q}},$$

where

$$A_5 = \Big(\frac{2}{\delta_p} - 1\Big)\Big[\frac{2}{\delta_p \mathcal{C}_{N,p}^p}\Big]^{\frac{4}{p\delta_p - 4}} > 0, \quad A_6 = \Big(\frac{1+\delta_q}{\delta_p} - 1\Big)\Big[\frac{1+\delta_q}{\delta_p K_{N,p}^p}\Big]^{\frac{q}{p\alpha - q}} > 0.$$

Thus, we also conclude that $\lambda_c \to +\infty$ as $c \to 0^+$. This completes the proof. □

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