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UNIFORMLY CONTINUOUS SEMIGROUPS OF SUBLINEAR TRANSITION OPERATORS

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Abstract. In this work we investigate uniformly continuous semigroups of sublinear transition operators on the Banach space of bounded real-valued functions on some countable set. We show how such a semigroup can be retrieved as the solution to an abstract Cauchy problem by showing that it is equal to the family of exponentials generated by a so-called bounded sublinear rate operator. We also show that given any bounded sublinear rate operator, the family of corresponding exponentials forms such a semigroup.

1. Introduction and main result

It is well-known—see for example [11, Theorem VIII.1.2] or [12, Theorem 3.7]—that a semigroup $(\mathrm{S}_t)_{t\in\mathbb{R}_{\geq 0}}$ of bounded linear operators on some Banach space $\mathfrak{B}$ is uniformly continuous—that is, continuous with respect to the operator norm—if and only if there is some bounded linear operator A such that

$$\mathrm{S}_t = e^{t\mathrm{A}} = \lim_{n\to+\infty}\Big(\mathrm{I}+\frac{t}{n}\mathrm{A}\Big)^n = \sum_{n=0}^{+\infty}\frac{t^k\mathrm{A}^k}{k!} \quad \text{for all } t\in\mathbb{R}_{\geq 0}; \tag{1.1}$$

whenever this is the case, this generator A is

$$\mathrm{A} = \lim_{t\searrow 0}\frac{\mathrm{S}_t-\mathrm{I}}{t},$$

and the semigroup $(e^{t\mathrm{A}})_{t\in\mathbb{R}_{\geq 0}}$ is the unique solution to the abstract Cauchy problem

$$\lim_{s\to t}\frac{\mathrm{S}_s-\mathrm{S}_t}{s-t} = \mathrm{A}\mathrm{S}_t \quad \text{for all } t\in\mathbb{R}_{\geq 0}, \quad \mathrm{S}_0=\mathrm{I}.$$

While we cannot imagine that this result has never been generalised to nonlinear operators, I haven't been able to surface a reference where this is done. Instead, most of the work on nonlinear operators seems to be focused on strongly continuous semigroups [1, 7, 22, 23].

In contrast, this work thoroughly investigates uniformly continuous semigroups of nonlinear operators, albeit only semigroups of so-called sublinear transition operators on $\mathrm{C_b}(\mathcal{X})$, with $\mathcal{X}$ a countable set equipped with the discrete metric. The main results in this work are the following. First, Theorem 3.1 establishes that if a 'sublinear rate operator' $\overline{\mathrm{Q}}\colon \mathrm{C_b}(\mathcal{X})\to\mathrm{C_b}(\mathcal{X})$—a nonlinear generalisation of the notion of a (linear) rate operator which can be thought of as the upper envelope of a uniformly bounded set of rate operators–is bounded, then

$$e^{t\overline{\mathrm{Q}}} = \lim_{n\to+\infty}\Big(\mathrm{I}+\frac{t}{n}\overline{\mathrm{Q}}\Big)^n$$

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is well-defined for all $t \in \mathbb{R}_{\geq 0}$, and $(e^{t\overline{\mathrm{Q}}})_{t\in\mathbb{R}_{\geq 0}}$ forms a uniformly continuous semigroup of sublinear transition operators [Proposition 3.4] that satisfies the abstract Cauchy problem [Proposition 3.6]

$$\lim_{s\to t}\frac{\mathrm{S}_s-\mathrm{S}_t}{s-t}=\overline{\mathrm{Q}}\mathrm{S}_t \quad \text{for all } t\in\mathbb{R}_{\geq 0}, \quad \mathrm{S}_0=\mathrm{I}.$$

Conversely, Theorem 4.5 tells us that any uniformly continuous semigroup of sublinear transition operators is generated by a bounded sublinear rate operator $\overline{\mathrm{Q}}$. For finite state spaces $\mathcal{X}$ this was already shown by De Bock [8, Propositions 8 and 10] and myself [13, Theorem 3.75], but in this work $\mathcal{X}$ is only assumed to be countable.

These results are important because semigroups of sublinear transition operators are crucial to the construction of sublinear (or imprecise) Markov processes [21, 9, 25, 24, 14, 31], which recently have received quite some attention in the fields of imprecise probabilities [34, 32] and robust mathematical finance [27]. In the present context, it suffices to understand that a sublinear Markov process is constructed from an initial sublinear expectation $\overline{E}_0$ and a semigroup of sublinear transition operators $(\overline{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}}$, much like how a Markov process is constructed from an initial distribution ν and a Markov semigroup $(\mathrm{T}_t)_{t\in\mathbb{R}_{\geq 0}}$ [17, Chapter 4, Theorem 1.1 & Proposition 1.6]. Obviously, the sublinear rate operator $\overline{\mathrm{Q}}$ is to the sublinear transition operators $(\overline{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}}$ what the (linear) rate operator Q is to the Markov semigroup $(\mathrm{T}_t)_{t\in\mathbb{R}_{\geq 0}}$: it simplifies the specification considerably.

1.1. **Structure.** The remainder of this article is structured as follows. In Section 2 we (i) introduce the Banach space of bounded (nonlinear) operators on $\mathscr{B}$; (ii) introduce the semigroups we are interested in; and (iii) establish some convenient properties of sublinear transition and rate operators. Section 3 examines how we can go from a sublinear rate operator to a (family of) sublinear transition operator(s), and investigates the properties of the resulting family $(e^{t\overline{\mathrm{Q}}})_{t\in\mathbb{R}_{\geq 0}}$. Section 4 deals with the other implication: there we start from a uniformly continuous sublinear transition semigroup and show that it must be generated by a bounded sublinear rate operator. Finally, Section 5 relates the present approach to similar existing ones in the setting of convex monotone semigroups, most notably Nendel's [25].

2. Operators and semigroups

Throughout the paper, we let $\mathcal{X}$ be a countable set, and we denote the linear vector space of bounded real-valued maps on $\mathcal{X}$ by $\mathscr{B} := \mathrm{C_b}(\mathcal{X})$, which is well-known to be a Banach space under the supremum norm $\|\cdot\|_\infty$. The bounded real-valued functions on $\mathcal{X}$ include the indicator functions: for any subset X of $\mathcal{X}$, the corresponding *indicator* $\mathbb{I}_X \in \mathscr{B}$ maps $x \in \mathcal{X}$ to 1 if $x \in X$ and to 0 otherwise; for any $x \in \mathcal{X}$, we shorten $\mathbb{I}_{\{x\}}$ to $\mathbb{I}_x$.

An operator, then, is a (possibly nonlinear) map from $\mathscr{B}$ to $\mathscr{B}$; let $\mathfrak{O}$ denote the set of operators. One example of an operator is the identity operator I, which maps any $f \in \mathscr{B}$ to itself. The identity operator I is not the only special operator that we will need: another important one is the *zero operator* O, which maps any $f \in \mathscr{B}$ to the zero function 0—here and in the remainder, for any constant $\alpha \in \mathbb{R}$ we write α for the function $\alpha\mathbb{I}_{\mathcal{X}}$. It will also be convenient to construct new operators through addition and scaling of operators, which are defined in the obvious pointwise manner. Composition of operators will also be essential: for any two operators A, B, we let

$$\mathrm{AB}\colon \mathscr{B}\to\mathscr{B}\colon f\mapsto \mathrm{A}(\mathrm{B}f).$$

2.1. **Banach space of bounded operators.** It is customary—see for example [22, Chapter 3]—to call a (possibly nonlinear) operator $\mathrm{A} \in \mathfrak{O}$ *Lipschitz* if

$$\|\mathrm{A}\|_{\mathrm{Lip}} := \sup\Big\{\frac{\|\mathrm{A}f-\mathrm{A}g\|_\infty}{\|f-g\|_\infty}\colon f,g\in\mathscr{B}, f\neq g\Big\} < +\infty;$$

we collect all Lipschitz operators in

$$\mathfrak{O}_{\mathrm{L}} := \Big\{\mathrm{A}\in\mathfrak{O}\colon \|\mathrm{A}\|_{\mathrm{Lip}} < +\infty\Big\}.$$

It is easy to see that $\|\cdot\|_{\mathrm{Lip}}$ is a seminorm on the real vector space $\mathfrak{O}_{\mathrm{L}}$, and that the derived function

$$\|\cdot\|_{\mathrm{L}}\colon \mathfrak{O}_{\mathrm{L}} \to \mathbb{R}_{\geq 0}\colon \mathrm{A} \mapsto \|\mathrm{A}\|_{\mathrm{L}} := \|\mathrm{A}0\|_\infty + \|\mathrm{A}\|_{\mathrm{Lip}}$$

is a norm on $\mathfrak{O}_{\mathrm{L}}$ such that $(\mathfrak{O}_{\mathrm{L}}, \|\cdot\|_{\mathrm{L}})$ is a Banach space [22, Lemma III.2.1 and Proposition III.2.1].

While we will deal with Lipschitz operators, the set $\mathfrak{O}_{\mathrm{L}}$ of Lipschitz operators is not the most convenient for our purposes. As will become clear, it is more convenient to consider the set of bounded operators, where a (possibly nonlinear) operator $\mathrm{A} \in \mathfrak{O}$ is *bounded* if

$$\|\mathrm{A}\|_{\mathrm{s}} := \sup\Big\{\frac{\|\mathrm{A}f\|_\infty}{\|f\|_\infty}\colon f \in \mathscr{B}, f \neq 0\Big\} < +\infty. \tag{2.1}$$

We collect all bounded operators in $\mathfrak{O}_{\mathrm{b}}$. Note that the identity operator is bounded because clearly $\|\mathrm{I}\|_{\mathrm{s}} = 1$.

Similar to how the Lipschitz seminorm $\|\cdot\|_{\mathrm{Lip}}$ gave rise to a norm $\|\cdot\|_{\mathrm{L}}$ on the set of Lipschitz operators $\mathfrak{O}_{\mathrm{L}}$, this seminorm $\|\cdot\|_s$ gives rise to a norm on the set $\mathfrak{O}_{\mathrm{b}}$ of bounded operators. The interested reader can find the proof of the following result, as well as some additional results and observations, in Section 6

Proposition 2.1. *The space space $\mathfrak{O}_{\mathrm{b}}$ of bounded operators is a Banach space when equipped with the norm*

$$\|\cdot\|_{\mathrm{b}}\colon \mathfrak{O}_{\mathrm{b}} \to \mathbb{R}_{\geq 0}\colon \mathrm{A} \mapsto \|\mathrm{A}0\|_\infty + \|\mathrm{A}\|_{\mathrm{s}}.$$

Furthermore, for any two bounded operators $\mathrm{A}, \mathrm{B} \in \mathfrak{O}_{\mathrm{b}}$, their composition AB is bounded as well, with

$$\|\mathrm{AB}\|_{\mathrm{b}} \leq \|\mathrm{A}\|_{\mathrm{b}}\|\mathrm{B}\|_{\mathrm{b}}. \tag{2.2}$$

We will be almost exclusively concerned with two types of operators: sublinear transition operators and sublinear rate operators.

2.2. **Sublinear transition operators.** Sublinear transition operators, which generalise the notion of transition operators [35, Chapter 9] (sometimes also called stochastic/transition matrices), go back to De Cooman & Hermans [5, Section 8], but they also go by other names; Denk et al. [9, Definition 5.1], for example, call them sublinear kernels.

Definition 2.2. *A* sublinear transition operator $\overline{\mathrm{T}}$ *is an operator such that*

- *T1.* $\overline{\mathrm{T}}(\lambda f) = \lambda \overline{\mathrm{T}} f$ *for all* $f \in \mathscr{B}$ *and* $\lambda \in \mathbb{R}_{\geq 0}$*;*
- *T2.* $\overline{\mathrm{T}}(f+g) \leq \overline{\mathrm{T}} f + \overline{\mathrm{T}} g$ *for all* $f, g \in \mathscr{B}$*;*
- *T3.* $\overline{\mathrm{T}} f \leq \sup f$ *for all* $f \in \mathscr{B}$*.*

A transition operator *is a sublinear transition operator that is linear.*

The three axioms for sublinear transition operators $\overline{\mathrm{T}}$ ensure that for all $x \in \mathcal{X}$, the corresponding component functional

$$[\overline{\mathrm{T}}\cdot](x)\colon \mathscr{B} \to \mathbb{R}\colon f \mapsto [\overline{\mathrm{T}} f](x)$$

is a coherent upper prevision/expectation in Walley's [34, Section 2.3.5] sense—see also [32]—or a sublinear expectation in that of Peng [27, Definition 1.1.1]. Hence, it follows from the well-known properties of coherent upper previsions—see for example [34, Section 2.6.1] or [32, Theorem 4.13]—that for any sublinear transition operator $\overline{\mathrm{T}}$,

- T4. $\overline{\mathrm{T}} f \leq \overline{\mathrm{T}} g$ for all $f, g \in \mathscr{B}$ such that $f \leq g$;
- T5. $\overline{\mathrm{T}}(f+\mu) = \mu + \overline{\mathrm{T}} f$ for all $f \in \mathscr{B}$ and $\mu \in \mathbb{R}$;
- T6. $\overline{\mathrm{T}}\mu = \mu$ for all $\mu \in \mathbb{R}_{\geq 0}$;
- T7. $-\overline{\mathrm{T}}(-f) \leq \overline{\mathrm{T}} f$ for all $f \in \mathscr{B}$;
- T8. $\|\overline{\mathrm{T}} f\|_\infty \leq \|f\|_\infty$ for all $f \in \mathscr{B}$;
- T9. $\|\overline{\mathrm{T}} f - \overline{\mathrm{T}} g\|_\infty \leq \|f-g\|_\infty$ for all $f, g \in \mathscr{B}$.

It follows immediately from (T9), (6.1), (T8) and (T6)—for $\mu = 1$ and $\mu = 0$—that for any sublinear transition operator $\overline{\mathrm{T}}$,

- T10. $\|\overline{\mathrm{T}}\|_{\mathrm{Lip}} = \|\overline{\mathrm{T}}\|_{\mathrm{L}} = 1$;
- T11. $\|\overline{\mathrm{T}}\|_{\mathrm{b}} = \|\overline{\mathrm{T}}\|_{\mathrm{s}} = 1$.

Since $\overline{\mathrm{T}}$ is bounded and Lipschitz, we know from Lemma 6.2 that

T12. $\|\overline{\mathrm{T}}\mathrm{A}-\overline{\mathrm{T}}\mathrm{B}\|_{\mathrm{b}} \leq \|\mathrm{A}-\mathrm{B}\|_{\mathrm{b}}$ for all bounded operators $\mathrm{A},\mathrm{B} \in \mathfrak{O}_{\mathrm{b}}$.

Rather than in a single sublinear transition operators, we'll be interested in a ' semigroup' of them. Semigroups of general operators have been investigated thoroughly [20, 4, 26, 23, 1, 12, 7].

Definition 2.3. *A* semigroup *is a family* $(\mathrm{S}_t)_{t\in\mathbb{R}_{\geq 0}}$ *of operators such that*

SG1. $\mathrm{S}_{s+t} = \mathrm{S}_s\mathrm{S}_t$ *for all* $s,t \in \mathbb{R}_{\geq 0}$*;*

SG2. $\mathrm{S}_0 = \mathrm{I}$.

As we will be exclusively concerned with semigroups $(\overline{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}}$ of sublinear transition operators, we'll briefly call these *sublinear transition semigroups*; in this context, the semigroup property (SG1) is often called the 'Chapman–Kolmogorov equation.'

It is customary to consider semigroups that are continuous in some sense. The most common notion of continuity is that of 'strong continuity', which means that

$$\lim_{s\to t} \mathrm{S}_s f = \mathrm{S}_t f \quad \text{for all } t \in \mathbb{R}_{\geq 0}, f \in \mathscr{B}.$$

However, in this work we'll work with a more restrictive notion of continuity that is known as 'uniform continuity'—curiously enough, and as mentioned in the Introduction, I haven't been able to surface existing work where this notion is used in the context of nonlinear operators.

Definition 2.4. *A semigroup* $(\mathrm{S}_t)_{t\in\mathbb{R}_{\geq 0}}$ *of bounded operators is said to be* uniformly continuous *if*

$$\lim_{s\to t} \mathrm{S}_s = \mathrm{S}_t \quad \textit{for all } t \in \mathbb{R}_{\geq 0}.$$

The following result regarding uniform continuity is fairly standard.

Lemma 2.5. *A semigroup* $(\mathrm{S}_t)_{t\in\mathbb{R}_{\geq 0}}$ *of bounded operators is uniformly continuous if and only if* $\lim_{\Delta\searrow 0} \mathrm{S}_\Delta = \mathrm{I}$.

Proof. The condition in the statement is clearly necessary for uniform continuity, so we only need to show that it's sufficient as well: for all $t \in \mathbb{R}_{\geq 0}$, we need to show that it implies that $\lim_{s\to t} \mathrm{S}_s = \mathrm{S}_t$. So fix some $t \in \mathbb{R}_{\geq 0}$. For the right-sided limit, note that for all $s \in \mathbb{R}_{\geq 0}$ such that $s > t$ and with $\Delta := s-t$, it follows from (SG1) and (2.2) that

$$\|\mathrm{S}_s - \mathrm{S}_t\|_{\mathrm{b}} = \|\mathrm{S}_\Delta\mathrm{S}_t - \mathrm{S}_t\|_{\mathrm{b}} = \|(\mathrm{S}_\Delta - \mathrm{I})\mathrm{S}_t\|_{\mathrm{b}} \leq \|\mathrm{S}_\Delta - \mathrm{I}\|_{\mathrm{b}}\|\mathrm{S}_t\|_{\mathrm{b}}.$$

For the left-sided limit, a similar argument but with $s < t$ and $\Delta := t-s$ shows that

$$\|\mathrm{S}_s - \mathrm{S}_t\|_{\mathrm{b}} = \|\mathrm{S}_s - \mathrm{S}_\Delta\mathrm{S}_s\|_{\mathrm{b}} = \|(\mathrm{I} - \mathrm{S}_\Delta)\mathrm{S}_s\|_{\mathrm{b}} \leq \|\mathrm{S}_\Delta - \mathrm{I}\|_{\mathrm{b}}\|\mathrm{S}_s\|_{\mathrm{b}}.$$

This inequality suffices once we've verified that there are some $M,\omega \in \mathbb{R}_{\geq 0}$ such that

$$\|\mathrm{S}_s\|_{\mathrm{b}} \leq Me^{s\omega} \quad \text{for all } s \in \mathbb{R}_{\geq 0},$$

and we can do so with the following standard argument—see, for example, Proposition 5.5 in [12]. As $\lim_{\Delta\searrow 0} \mathrm{S}_\Delta = \mathrm{I}$, there is some $\delta \in \mathbb{R}_{>0}$ and $M \in [1,+\infty[$ such that $\sup\{\|\mathrm{S}_\Delta\|_{\mathrm{b}} \colon \Delta < \delta\} \leq M$. If we let $\omega := \frac{1}{\delta}\ln M$, then for all $s \in \mathbb{R}_{>0}$, and with $n \in \mathbb{N}$ such that $s/n < \delta$,

$$\|\mathrm{S}_s\|_{\mathrm{b}} \leq \|\mathrm{S}_{\frac{s}{n}}\|_{\mathrm{b}}^n \leq M^n = Me^{(n-1)\ln M} = Me^{(n-1)\delta\omega} \leq Me^{s\omega}. \qquad \square$$

2.3. **Sublinear rate operators.** Sublinear rate operators go back to Škulj [33, Section 2.5]—see also [8, Definition 5] or [24, Definition 2.1 and Theorem 2.5]. They generalize the notion of rate (or intensity) matrices/operators by dropping the requirement of linearity in favour of that of sublinearity.

Definition 2.6. A *sublinear rate operator* $\overline{\mathrm{Q}}$ is an operator such that

Q1. $\overline{\mathrm{Q}}(\lambda f) = \lambda\overline{\mathrm{Q}}f$ for all $f \in \mathscr{B}$ and $\lambda \in \mathbb{R}_{\geq 0}$;

Q2. $\overline{\mathrm{Q}}(f+g) \leq \overline{\mathrm{Q}}f + \overline{\mathrm{Q}}g$ for all $f,g \in \mathscr{B}$;

Q3. $\overline{\mathrm{Q}}\mu = 0$ for all $\mu \in \mathbb{R}$;

Q4. $[\overline{\mathrm{Q}}f](x) \leq 0$ for all $f \in \mathscr{B}$ and $x \in \mathcal{X}$ such that $\sup f = f(x) \geq 0$.

A *rate operator* is a sublinear rate operator that is linear.

Axiom (Q4) is known as the *positive maximum principle*.[1] A trivial example of a sublinear rate operator is the zero operator O.

It is not difficult to show that for any sublinear rate operator $\overline{\mathrm{Q}}$,

Q5. $\overline{\mathrm{Q}}(f+\mu)=\overline{\mathrm{Q}}f$ for all $f\in\mathscr{B}$ and $\mu\in\mathbb{R}$;
Q6. $-\overline{\mathrm{Q}}(-f)\leq\overline{\mathrm{Q}}f$ for all $f\in\mathscr{B}$;
Q7. $[\overline{\mathrm{Q}}\mathbb{I}_x](x)\leq 0$ for all $x\in\mathcal{X}$.

Proof. For (Q5), we simply repeat De Bock's proof for [8, R6]: it follows from subadditivity (Q2) and (Q3) that

$$\overline{\mathrm{Q}}(f+\mu)\leq\overline{\mathrm{Q}}(f)+\overline{\mathrm{Q}}(\mu)=\overline{\mathrm{Q}}(f)=\overline{\mathrm{Q}}(f+\mu-\mu)\leq\overline{\mathrm{Q}}(f+\mu)+\overline{\mathrm{Q}}(-\mu)=\overline{\mathrm{Q}}(f+\mu).$$

For (Q6), observe that due to (Q3) and subadditivity (Q2),

$$0=\overline{\mathrm{Q}}(f-f)\leq\overline{\mathrm{Q}}f+\overline{\mathrm{Q}}(-f).$$

Property (Q7) follows immediately from the positive maximum principle (Q4) for $f=\mathbb{I}_x$. □

With a bit more work, we obtain the following simple yet important expression for the operator seminorm of a sublinear rate operator; this result generalizes Proposition 4 in [15] to the countable-state case, but the proof here differs quite a bit from the one there.

Proposition 2.7. *For any sublinear rate operator* $\overline{\mathrm{Q}}$*,*

$$\|\overline{\mathrm{Q}}\|_{\mathrm{s}}=2\sup\big\{[\overline{\mathrm{Q}}(1-\mathbb{I}_x)](x)\colon x\in\mathcal{X}\big\}=\sup\big\{[\overline{\mathrm{Q}}(1-2\mathbb{I}_x)](x)\colon x\in\mathcal{X}\big\}.$$

Proof. For all $x\in\mathcal{X}$, it follows from positive homogeneity [(Q1)] and (Q5) that

$$2[\overline{\mathrm{Q}}(1-\mathbb{I}_x)](x)=[\overline{\mathrm{Q}}(2-2\mathbb{I}_x)](x)=[\overline{\mathrm{Q}}(1-2\mathbb{I}_x)](x).$$

Since the supremum is positively homogeneous, this proves the second equality in the statement.

For the first equality in the statement, it follows from (6.1) that since $\overline{\mathrm{Q}}$ is positively homogeneous,

$$\|\overline{\mathrm{Q}}\|_{\mathrm{s}}=\sup\big\{\|\overline{\mathrm{Q}}f\|_\infty\colon f\in\mathscr{B},\|f\|_\infty=1\big\}=\sup\big\{|[\overline{\mathrm{Q}}f](x)|\colon f\in\mathscr{B},\|f\|_\infty=1,x\in\mathcal{X}\big\}. \tag{2.3}$$

Next, observe that for all $x\in\mathcal{X}$, it follows from (Q3), the sublinearity of $\overline{\mathrm{Q}}$ and (Q7) that

$$0=[\overline{\mathrm{Q}}1](x)\leq[\overline{\mathrm{Q}}(1-2\mathbb{I}_x)](x)+2[\overline{\mathrm{Q}}\mathbb{I}_x](x)\leq[\overline{\mathrm{Q}}(1-2\mathbb{I}_x)](x).$$

Because $\|1-2\mathbb{I}_x\|_\infty=1$, it follows from all this that

$$\|\overline{\mathrm{Q}}\|_{\mathrm{s}}\geq\sup\big\{[\overline{\mathrm{Q}}(1-2\mathbb{I}_x)](x)\colon x\in\mathcal{X}\big\}=2\sup\big\{[\overline{\mathrm{Q}}(1-\mathbb{I}_x)](x)\colon x\in\mathcal{X}\big\}.$$

In the remainder of this proof, we set out to show that

$$\|\overline{\mathrm{Q}}\|_{\mathrm{s}}\leq 2\sup\big\{[\overline{\mathrm{Q}}(1-\mathbb{I}_x)](x)\colon x\in\mathcal{X}\big\}, \tag{2.4}$$

since the previous two inequalities imply the first equality in the statement.

Fix any $g\in\mathscr{B}$ with $\|g\|_\infty=1$ and any $x\in\mathcal{X}$, and observe that

$$[\overline{\mathrm{Q}}g](x)=[\overline{\mathrm{Q}}(g-\inf g)](x)$$

due to (Q5). Let $h:=g-\inf g\geq 0$ and $\alpha:=\sup h$, and note that $h(x)\geq 0$ and $0\leq\alpha\leq 2\|g\|_\infty=2$—the latter because $\alpha=\sup g-\inf g$. Moreover, let $\tilde{h}_x:=h-\alpha(1-\mathbb{I}_x)-h(x)\mathbb{I}_x$. Since $\overline{\mathrm{Q}}$ is sublinear,

$$\begin{aligned}[\overline{\mathrm{Q}}g](x)=[\overline{\mathrm{Q}}h](x)&=[\overline{\mathrm{Q}}(\tilde{h}_x+\alpha(1-\mathbb{I}_x)+h(x)\mathbb{I}_x)](x)\\&\leq[\overline{\mathrm{Q}}\tilde{h}_x](x)+\alpha[\overline{\mathrm{Q}}(1-\mathbb{I}_x)](x)+h(x)[\overline{\mathrm{Q}}\mathbb{I}_x](x).\end{aligned}$$

As $\tilde{h}_x\leq 0$ and $\sup\tilde{h}_x=0=\tilde{h}_x(x)$ by construction, it follows from the positive maximum principle [(Q4)] that $[\overline{\mathrm{Q}}\tilde{h}_x](x)\leq 0$; since furthermore $[\overline{\mathrm{Q}}\mathbb{I}_x](x)\leq 0$ due to (Q7) and $\alpha\leq 2$ and $h(x)\geq 0$ by construction, we conclude that

$$[\overline{\mathrm{Q}}g](x)\leq\alpha[\overline{\mathrm{Q}}(1-\mathbb{I}_x)](x)\leq 2[\overline{\mathrm{Q}}(1-\mathbb{I}_x)](x). \tag{2.5}$$

[1] After [6, Section 1.2], see also [17, Chapter 4, Section 2] or [28, Lemma III.6.8].

For all $f \in \mathscr{B}$ with $\|f\|_\infty = 1$ and $x \in \mathcal{X}$, it follow from (2.5) (once for $g = -f$ and once for $g = f$) and (Q6) that

$$-2[\overline{\mathrm{Q}}(1-\mathbb{I}_x)](x) \leq -[\overline{\mathrm{Q}}(-f)](x) \leq [\overline{\mathrm{Q}}f](x) \leq 2[\overline{\mathrm{Q}}(1-\mathbb{I}_x)](x).$$

Together with (2.3), this implies the inequality in (2.4). □

One way to obtain a sublinear rate operator that is particularly relevant in applications—see for example [16, 30]—is as the upper envelope over some set of candidate rate operators. We refer the interested reader to Section 7 for a discussion of this approach.

2.4. **From sublinear transition operator to bounded sublinear rate operator and back again.** Rather than through the upper envelope of a bounded set of rate operators, one can also obtain a sublinear rate operators from a sublinear transition operator. The following result generalises De Bock's [8] Proposition 5 from the setting of finite $\mathcal{X}$ to that of countable $\mathcal{X}$.

Lemma 2.8. *Let $\overline{\mathrm{T}}$ be a sublinear transition operator, and fix some strictly positive real number $\lambda \in \mathbb{R}_{>0}$. Then the operator $\overline{\mathrm{Q}} := \lambda(\overline{\mathrm{T}} - \mathrm{I})$ is a bounded sublinear rate operator.*

Proof. Note that $\overline{\mathrm{Q}}$ is a bounded operator because $\mathfrak{O}_\mathrm{b}$ is a real vector space and $\overline{\mathrm{Q}}$ is defined as a linear combination of bounded operators. That $\overline{\mathrm{Q}}$ is sublinear—that is, satisfies (Q1) and (Q2)—follows immediately from the sublinearity of $\overline{\mathrm{T}}$ and the linearity of I. That $\overline{\mathrm{Q}}$ maps constants to zero—so satisfies (Q3)—follows from the fact that $\overline{\mathrm{T}}$ and I are constant preserving [(T6)]. Finally, it is obvious that $\overline{\mathrm{Q}}$ satisfies the positive maximum principle (Q4) due to (T3): for all $f \in \mathscr{B}$ and $x \in \mathcal{X}$ such that $f(x) = \sup f \geq 0$,

$$[\overline{\mathrm{Q}}f](x) = \lambda([\overline{\mathrm{T}}f](x) - f(x)) \leq \lambda(\sup f - f(x)) = 0.$$ □

We can also go the other way around as in Lemma 2.8: a suitable linear combination of the identity operator and a bounded sublinear transition operator gives a (automatically bounded) sublinear rate operator. The next result formalises this, and in doing so generalises De Bock's [8] Proposition 5—or the slightly improved version in [13, Lemma 3.72]—to the present, more general setting.

Lemma 2.9. *For any bounded sublinear rate operator Q and any $\Delta \in \mathbb{R}_{>0}$ such that $\Delta\|\overline{\mathrm{Q}}\|_\mathrm{b} \leq 2$, $\overline{\mathrm{T}} := \mathrm{I} + \Delta\overline{\mathrm{Q}}$ is a sublinear transition operator.*

Proof. That $\overline{\mathrm{T}}$ is a (bounded) sublinear operator—so an operator that satisfies (T1) and (T2)—follows immediately from the fact that I and $\overline{\mathrm{Q}}$ are sublinear bounded operators and that $\mathfrak{O}_\mathrm{b}$ is a real linear space, so it remains for us to verify that $\overline{\mathrm{T}}$ satisfies (T3). To this end, we fix some $x \in \mathcal{X}$ and $f \in \mathscr{B}$. Then it follows from (Q5) that

$$[\overline{\mathrm{T}}f](x) = f(x) + \Delta[\overline{\mathrm{Q}}f](x) = f(x) + \Delta[\overline{\mathrm{Q}}(f - f(x))](x).$$

With $f_x := f - f(x)$, $\alpha := \sup f_x = \sup f - f(x) \geq 0$ and $\tilde{f}_x := f_x - \alpha(1 - \mathbb{I}_x)$, it follows from this and the sublinearity of $\overline{\mathrm{Q}}$ that

$$\begin{aligned}[\overline{\mathrm{T}}f](x) = f(x) + \Delta[\overline{\mathrm{Q}}f_x](x) &= f(x) + \Delta[\overline{\mathrm{Q}}(\tilde{f}_x + \alpha(1-\mathbb{I}_x))](x) \\ &\leq f(x) + \Delta[\overline{\mathrm{Q}}\tilde{f}_x](x) + \alpha\Delta[\overline{\mathrm{Q}}(1-\mathbb{I}_x)](x).\end{aligned}$$

Since $\tilde{f}_x \leq 0$ and $\sup \tilde{f}_x = 0 = \tilde{f}_x(x)$ by construction, the positive maximum principle (Q4) tells us that $[\overline{\mathrm{Q}}\tilde{f}_x](x) \leq 0$, and therefore

$$[\overline{\mathrm{T}}f](x) \leq f(x) + \alpha\Delta[\overline{\mathrm{Q}}(1-\mathbb{I}_x)](x).$$

From (6.2) and Proposition 2.7 we know that $[\overline{\mathrm{Q}}(1-\mathbb{I}_x)](x) \leq \|\overline{\mathrm{Q}}\|_\mathrm{b}/2$, whence

$$[\overline{\mathrm{T}}f](x) \leq f(x) + \alpha\frac{\Delta\|\overline{\mathrm{Q}}\|_\mathrm{b}}{2}.$$

Since $\Delta\|\overline{\mathrm{Q}}\|_\mathrm{b} \leq 2$ by the assumptions in the statement and $\alpha = \sup f_x = \sup f - f(x)$ by definition, we conclude that

$$[\overline{\mathrm{T}}f](x) \leq f(x) + \sup f - f(x) \leq \sup f,$$

which is what we needed to prove. □

When combined with (T10), the previous lemma can be used to show that any bounded sublinear rate operator is Lipschitz, which is already known to be true in case $\mathcal{X}$ is finite [8, (R11) and (R12)]. This Lipschitz property will come in handy further on, which is why we establish it formally here.

Proposition 2.10. *Consider a bounded sublinear rate operator* $\overline{\mathrm{Q}}$*. Then*

- Q8. $\left\|\overline{\mathrm{Q}}f-\overline{\mathrm{Q}}g\right\|_\infty \le \|\overline{\mathrm{Q}}\|_{\mathrm{b}}\|f-g\|_\infty$ *for all* $f,g\in\mathscr{B}$*; and*
- Q9. $\left\|\overline{\mathrm{Q}}\mathrm{A}-\overline{\mathrm{Q}}\mathrm{B}\right\|_{\mathrm{b}} \le \|\overline{\mathrm{Q}}\|_{\mathrm{b}}\|\mathrm{A}-\mathrm{B}\|_{\mathrm{b}}$ *for all* $\mathrm{A},\mathrm{B}\in\mathfrak{O}_{\mathrm{b}}$.

Proof. Since the two properties in the statement are trivial if $\|\overline{\mathrm{Q}}\|_{\mathrm{b}}=0 \Leftrightarrow \overline{\mathrm{Q}}=\mathrm{O}$, we assume without loss of generality that $\|\overline{\mathrm{Q}}\|_{\mathrm{b}}>0$. For (Q8), we fix some $f,g\in\mathscr{B}$. Then with $\Delta\coloneqq 2/\|\overline{\mathrm{Q}}\|_{\mathrm{b}}$,

$$\|\overline{\mathrm{Q}}f-\overline{\mathrm{Q}}g\|_\infty=\frac{1}{\Delta}\|\Delta\overline{\mathrm{Q}}f-\Delta\overline{\mathrm{Q}}g\|_\infty\le\frac{1}{\Delta}\|(\mathrm{I}+\Delta\overline{\mathrm{Q}})f-(\mathrm{I}+\Delta\overline{\mathrm{Q}})g\|_\infty+\frac{1}{\Delta}\|f-g\|_\infty.$$

Now we know from Lemma 2.9 that $\mathrm{I}+\Delta\overline{\mathrm{Q}}$ is a sublinear transition operator, so it follows from the previous inequality and (T11) that

$$\|\overline{\mathrm{Q}}f-\overline{\mathrm{Q}}g\|_\infty\le\frac{2}{\Delta}\|f-g\|_\infty=\|\overline{\mathrm{Q}}\|_{\mathrm{b}}\|f-g\|_\infty,$$

which is the inequality we were after.

Property (Q9) follows immediately from (Q8) due to Lemma 6.2. □

3. Sublinear transition semigroup generated by a bounded sublinear rate operator

Now that we have gone over the preliminaries, it is time to get going on our first goal: define the operator exponential of a bounded rate operator through a Cauchy sequence of sublinear transition operators. After doing so in Section 3.1, we show that the family of operator exponentials is the solution to an abstract Cauchy problem in Section 3.2.

3.1. Exponential of a bounded sublinear rate operator.

We will follow the path outlined by Krak et al. [21, Section 7.3] for the case of a finite state space, who took inspiration from earlier work by De Bock [8] and Škulj [33]. The crucial idea is to combine Lemma 2.9 with the following observation: for any two sublinear transition operators $\overline{\mathrm{S}}$ and $\overline{\mathrm{T}}$, their composition $\overline{\mathrm{S}}\,\overline{\mathrm{T}}$ is again a sublinear transition operator. Henceforth, we will use this basic observation implicitly in order not to unnecessarily repeat ourselves. The combination of these two results leads to the following key result; it generalizes Corollary 7.10 in [21], but goes back to well-known ideas in the theory of (nonlinear) operator semigroups [4]—see also [1, Section III.1.2] or [29, Sections 30.18 to 30.28].

Theorem 3.1. *Consider a bounded sublinear rate operator* $\overline{\mathrm{Q}}$*, and fix some* $t\in\mathbb{R}_{\ge 0}$*. Then the sequence* $((\mathrm{I}+\frac{t}{n}\overline{\mathrm{Q}})^n)_{n\in\mathbb{N}}$ *of bounded operators is Cauchy, and its limit*

$$e^{t\overline{\mathrm{Q}}}\coloneqq\lim_{n\to+\infty}\left(\mathrm{I}+\frac{t}{n}\overline{\mathrm{Q}}\right)^n$$

is a sublinear transition operator.

To prove this result, we will rely on two intermediary results which generalise Lemmas E.4 and E.5 in [21], respectively; the proofs of these generalized results follow the proofs of the originals closely, whence I have relegated them to Section 8.

Lemma 3.2. *Consider some* $n\in\mathbb{N}$ *and some sublinear transition operators* $\overline{\mathrm{T}}_1$*, …,* $\overline{\mathrm{T}}_n$ *and* $\overline{\mathrm{S}}_1$*, …,* $\overline{\mathrm{S}}_n$*. Then*

$$\left\|\overline{\mathrm{T}}_1\cdots\overline{\mathrm{T}}_n-\overline{\mathrm{S}}_1\cdots\overline{\mathrm{S}}_n\right\|_{\mathrm{b}}\le\sum_{k=1}^n\left\|\overline{\mathrm{T}}_k-\overline{\mathrm{S}}_k\right\|_{\mathrm{b}}.$$

Lemma 3.3. *Consider a bounded sublinear rate operator $\overline{\mathrm{Q}}$. Then for all $\Delta \in \mathbb{R}_{>0}$ such that $\Delta\|\overline{\mathrm{Q}}\|_{\mathrm{b}} \leq 2$ and $\ell \in \mathbb{N}$,*

$$\Big\|\Big(\mathrm{I} + \frac{\Delta}{\ell}\overline{\mathrm{Q}}\Big)^{\ell} - (\mathrm{I} + \Delta\overline{\mathrm{Q}})\Big\|_{\mathrm{b}} \leq \Delta^2\|\overline{\mathrm{Q}}\|_{\mathrm{b}}^2.$$

Proof for Theorem 3.1. The statement holds trivially for $t = 0$, so without loss of generality we may assume that $t > 0$. Fix some $n, m \in \mathbb{N}$ such that $t\|\overline{\mathrm{Q}}\|_{\mathrm{b}} \leq 2\min\{n, m\}$. Then by the triangle inequality,

$$\begin{aligned}
&\Big\|\Big(\mathrm{I} + \frac{t}{n}\overline{\mathrm{Q}}\Big)^{n} - \Big(\mathrm{I} + \frac{t}{m}\overline{\mathrm{Q}}\Big)^{m}\Big\|_{\mathrm{b}} \\
&\leq \Big\|\Big(\mathrm{I} + \frac{t}{n}\overline{\mathrm{Q}}\Big)^{n} - \Big(\mathrm{I} + \frac{t}{nm}\overline{\mathrm{Q}}\Big)^{nm}\Big\|_{\mathrm{b}} + \Big\|\Big(\mathrm{I} + \frac{t}{nm}\overline{\mathrm{Q}}\Big)^{nm} - \Big(\mathrm{I} + \frac{t}{m}\overline{\mathrm{Q}}\Big)^{m}\Big\|_{\mathrm{b}}.
\end{aligned}$$

Now since $t\|\overline{\mathrm{Q}}\|_{\mathrm{b}} \leq 2n \leq 2nm$, it follows from Lemma 2.9, Lemma 3.2 (with $\overline{\mathrm{T}}_k = (\mathrm{I} + \frac{t}{n}\overline{\mathrm{Q}})$ and $\overline{\mathrm{S}}_k = (\mathrm{I} + \frac{t}{nm}\overline{\mathrm{Q}})^m$) and Lemma 3.3 (with $\Delta = \frac{t}{n}$ and $\ell = m$) that

$$\begin{aligned}
\Big\|\Big(\mathrm{I} + \frac{t}{n}\overline{\mathrm{Q}}\Big)^{n} - \Big(\mathrm{I} + \frac{t}{nm}\overline{\mathrm{Q}}\Big)^{nm}\Big\|_{\mathrm{b}} &\leq n\Big\|\Big(\mathrm{I} + \frac{t}{n}\overline{\mathrm{Q}}\Big) - \Big(\mathrm{I} + \frac{t}{nm}\overline{\mathrm{Q}}\Big)^{m}\Big\|_{\mathrm{b}} \\
&\leq n\Big(\frac{t}{n}\Big)^2\|\overline{\mathrm{Q}}\|_{\mathrm{b}}^2 \\
&= \frac{1}{n}t^2\|\overline{\mathrm{Q}}\|_{\mathrm{b}}^2.
\end{aligned}$$

A similar argument shows that

$$\Big\|\Big(\mathrm{I} + \frac{t}{m}\overline{\mathrm{Q}}\Big)^{m} - \Big(\mathrm{I} + \frac{t}{nm}\overline{\mathrm{Q}}\Big)^{nm}\Big\|_{\mathrm{b}} \leq \frac{1}{m}t^2\|\overline{\mathrm{Q}}\|_{\mathrm{b}}^2,$$

and therefore

$$\Big\|\Big(\mathrm{I} + \frac{t}{n}\overline{\mathrm{Q}}\Big)^{n} - \Big(\mathrm{I} + \frac{t}{m}\overline{\mathrm{Q}}\Big)^{m}\Big\|_{\mathrm{b}} \leq \Big(\frac{1}{n} + \frac{1}{m}\Big)t^2\|\overline{\mathrm{Q}}\|_{\mathrm{b}}^2.$$

From this, we infer that $((\mathrm{I} + \frac{t}{n}\overline{\mathrm{Q}})^n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Since $(\mathfrak{O}_{\mathrm{b}}, \|\cdot\|_{\mathrm{b}})$ is a Banach space [Proposition 2.1], this Cauchy sequence converges to a limit

$$e^{t\overline{\mathrm{Q}}} = \lim_{n\to+\infty}\Big(\mathrm{I} + \frac{t}{n}\overline{\mathrm{Q}}\Big)^{n}$$

in $\mathfrak{O}_{\mathrm{b}}$. That this limit $e^{t\overline{\mathrm{Q}}}$ is a sublinear transition operator follows from its definition as the limit of $((\mathrm{I} + \frac{t}{n}\overline{\mathrm{Q}})^n)_{n\in\mathbb{N}}$ because (i) we know from Lemma 2.9 that for sufficiently large n, $(\mathrm{I} + \frac{t}{n}\overline{\mathrm{Q}})$ and therefore $(\mathrm{I} + \frac{t}{n}\overline{\mathrm{Q}})^n$ is a sublinear transition operator; and (ii) the axioms (T1)–(T3) of sublinear transition operators are preserved under limits. □

With $\overline{\mathrm{Q}}$ a bounded sublinear rate operator and $t \in \mathbb{R}_{\geq 0}$, we call $e^{t\overline{\mathrm{Q}}}$ the *operator exponential* of $t\overline{\mathrm{Q}}$ because its defining limit expression mirrors one of the many limit expressions for the exponential of a real number or bounded linear operator (1.1). It is quite peculiar that we obtain Euler's limit expression, though, as it is not commonly used in the theory of (nonlinear) semigroups. In contrast, the limit expression that is usually encountered for the exponential of an operator A is—see, for example, [20, Theorem 11.3.2], [23, Chapter 4], [36, Chapter IX] or [29, Section 30.28]—of the form

$$e^{\mathrm{A}} = \lim_{n\to+\infty}\Big(\mathrm{I} - \frac{1}{n}\mathrm{A}\Big)^{-n},$$

which of course requires that the inverse of the operator on the right hand side is well defined for sufficiently large n. Note, also, that usually this definition is done pointwise, so through a limit in the ‘ original’ Banach space (here $\mathscr{B}$) instead of through a limit in a suitable Banach space of operators (here $\mathfrak{O}_{\mathrm{b}}$). One notable exception is the setting of semigroups of convex monotone operators [10, 19, 2], where the Euler expression—or, more generally, a Chernoff-like expression—is also used, but in the functional space (here $\mathscr{B}$) rather than the operator space; see also Section 5 further on.

3.2. **The exponential family.** Theorem 3.1 gives us a family $(e^{t\overline{\mathrm{Q}}})_{t\in\mathbb{R}_{\geq 0}}$ of sublinear transition operators starting from a bounded sublinear rate operator. As the motivation for this work is to use this family to construct a sublinear Markov process—see [14] for more details—we are particularly interested in whether this family forms a semigroup. The following result establishes that, quite nicely, this is always the case; the proof uses fairly standard arguments—see for example Theorem 2.5.3 in [4].

Proposition 3.4. *Consider a bounded sublinear rate operator $\overline{\mathrm{Q}}$. Then $(e^{t\overline{\mathrm{Q}}})_{t\in\mathbb{R}_{\geq 0}}$ is a uniformly continuous sublinear transition semigroup.*

Our proof for Proposition 3.4 makes use of the following continuity result, which will come in handy further on as well.

Lemma 3.5. *Consider a bounded sublinear rate operator $\overline{\mathrm{Q}}$. Then for all $s,t\in\mathbb{R}_{\geq 0}$,*

$$\left\|e^{s\overline{\mathrm{Q}}}-e^{t\overline{\mathrm{Q}}}\right\|_{\mathrm{b}}\leq |s-t|\|\overline{\mathrm{Q}}\|_{\mathrm{b}}.$$

Consequently, the function $e^{\cdot\overline{\mathrm{Q}}}\colon\mathbb{R}_{\geq 0}\to\mathfrak{O}_{\mathrm{b}}\colon t\mapsto e^{t\overline{\mathrm{Q}}}$ is Lipschitz continuous.

Proof. Fix some $s,t\in\mathbb{R}_{\geq 0}$ and observe that for all $n\in\mathbb{N}$,

$$\begin{aligned}\left\|e^{s\overline{\mathrm{Q}}}-e^{t\overline{\mathrm{Q}}}\right\|_{\mathrm{b}}&\leq\left\|e^{s\overline{\mathrm{Q}}}-\Big(\mathrm{I}+\frac{s}{n}\overline{\mathrm{Q}}\Big)^n\right\|_{\mathrm{b}}+\left\|e^{t\overline{\mathrm{Q}}}-\Big(\mathrm{I}+\frac{t}{n}\overline{\mathrm{Q}}\Big)^n\right\|_{\mathrm{b}}\\&\quad+\left\|\Big(\mathrm{I}+\frac{s}{n}\overline{\mathrm{Q}}\Big)^n-\Big(\mathrm{I}+\frac{t}{n}\overline{\mathrm{Q}}\Big)^n\right\|_{\mathrm{b}}.\end{aligned}$$

For the last term, it follows from Lemmas 2.9 and 3.2 that for all $n\in\mathbb{N}$ such that $t\|\overline{\mathrm{Q}}\|_{\mathrm{b}}/2\leq n$ and $s\|\overline{\mathrm{Q}}\|_{\mathrm{b}}/2\leq n$,

$$\left\|\Big(\mathrm{I}+\frac{s}{n}\overline{\mathrm{Q}}\Big)^n-\Big(\mathrm{I}+\frac{t}{n}\overline{\mathrm{Q}}\Big)^n\right\|_{\mathrm{b}}\leq n\left\|\Big(\mathrm{I}+\frac{s}{n}\overline{\mathrm{Q}}\Big)-\Big(\mathrm{I}+\frac{t}{n}\overline{\mathrm{Q}}\Big)\right\|_{\mathrm{b}}=|s-t|\|\overline{\mathrm{Q}}\|_{\mathrm{b}}.$$

By Theorem 3.1, the inequality in the statement now follows from all this by taking the limit for $n\to+\infty$ in the first inequality of this proof. □

Proof of Proposition 3.4. As the argument is a standard one, we only provide a sketch. First, it follows from Theorem 3.1 and some simple manipulations that

$$e^{nt\overline{\mathrm{Q}}}=\big(e^{t\overline{\mathrm{Q}}}\big)^n\quad\text{for all }t\in\mathbb{R}_{\geq 0},n\in\mathbb{N},\tag{3.1}$$

from which it follows that for all $p,q\in\mathbb{Q}_{\geq 0}$, and with $n_p,n_q\in\mathbb{Z}_{\geq 0}$ and $d\in\mathbb{N}$ such that $p=n_p/d$ and $q=n_q/d$,

$$e^{p\overline{\mathrm{Q}}}e^{q\overline{\mathrm{Q}}}=\big(e^{\frac{1}{d}\overline{\mathrm{Q}}}\big)^{n_p}\big(e^{\frac{1}{d}\overline{\mathrm{Q}}}\big)^{n_q}=\big(e^{\frac{1}{d}\overline{\mathrm{Q}}}\big)^{n_p+n_q}=e^{(p+q)\overline{\mathrm{Q}}}.\tag{3.2}$$

Because the function $e^{\cdot\overline{\mathrm{Q}}}\colon\mathbb{R}_{\geq 0}\to\mathfrak{O}_{\mathrm{b}}\colon t\mapsto e^{t\overline{\mathrm{Q}}}$ is Lipschitz continuous [Lemma 3.5] and $\mathbb{Q}_{\geq 0}$ is dense in $\mathbb{R}_{\geq 0}$, this equation extends to $\mathbb{R}_{\geq 0}$. □

Let us investigate the function

$$e^{\cdot\overline{\mathrm{Q}}}\colon\mathbb{R}_{\geq 0}\to\mathfrak{O}_{\mathrm{b}}\colon t\mapsto e^{t\overline{\mathrm{Q}}},$$

with $\overline{\mathrm{Q}}$ a bounded sublinear rate operator, a bit more. We already know from Lemma 3.5 that this function is (Lipschitz) continuous. The natural follow up question, then—at least to me—is whether this function $e^{t\overline{\mathrm{Q}}}$ is differentiable. The following result answers this question positively; in doing so, it generalizes Proposition 7.15 in [21] and Proposition 9 in [8] from the setting of finite state spaces to countable ones.

Proposition 3.6. *Consider a bounded sublinear rate operator $\overline{\mathrm{Q}}$. Then*

$$\lim_{s\to t}\frac{e^{s\overline{\mathrm{Q}}}-e^{t\overline{\mathrm{Q}}}}{s-t}=\overline{\mathrm{Q}}e^{t\overline{\mathrm{Q}}}\quad\text{for all }t\in\mathbb{R}_{\geq 0},$$

where for $t=0$ we only take the right-sided limit.

Proof. Let us prove an intermediary result first. Fix some $\Delta \in \mathbb{R}_{>0}$ such that $\Delta\|\overline{\mathrm{Q}}\|_{\mathrm{b}} \leq 2$. Then for all $n \in \mathbb{N}$,

$$\begin{aligned}\|\frac{e^{\Delta\overline{\mathrm{Q}}}-\mathrm{I}}{\Delta}-\overline{\mathrm{Q}}\|_{\mathrm{b}} &= \frac{1}{\Delta}\Big\|e^{\Delta\overline{\mathrm{Q}}}-(\mathrm{I}+\Delta\overline{\mathrm{Q}})\Big\|_{\mathrm{b}} \\ &\leq \frac{1}{\Delta}\|e^{\Delta\overline{\mathrm{Q}}}-(\mathrm{I}+\frac{\Delta}{n}\overline{\mathrm{Q}})^n\|_{\mathrm{b}}+\frac{1}{\Delta}\|(\mathrm{I}+\frac{\Delta}{n}\overline{\mathrm{Q}})^n-(\mathrm{I}+\Delta\overline{\mathrm{Q}})\|_{\mathrm{b}}.\end{aligned}$$

From Theorem 3.1 and Lemma 3.3, taking the limit superior for $n \to +\infty$ gives us that

$$\Big\|\frac{e^{\Delta\overline{\mathrm{Q}}}-\mathrm{I}}{\Delta}-\overline{\mathrm{Q}}\Big\|_{\mathrm{b}} \leq \Delta\|\overline{\mathrm{Q}}\|_{\mathrm{b}}^2. \tag{3.3}$$

For the right-sided limit, we fix some $s \in \mathbb{R}_{\geq 0}$ with $s > t$. Using the semigroup property (SG1) of $e^{\cdot\overline{\mathrm{Q}}}$ [Proposition 3.4], we find with $\Delta := s - t$ that

$$\Big\|\frac{e^{s\overline{\mathrm{Q}}}-e^{t\overline{\mathrm{Q}}}}{s-t}-\overline{\mathrm{Q}}e^{t\overline{\mathrm{Q}}}\Big\|_{\mathrm{b}} = \Big\|\Big(\frac{e^{\Delta\overline{\mathrm{Q}}}-\mathrm{I}}{\Delta}-\overline{\mathrm{Q}}\Big)e^{t\overline{\mathrm{Q}}}\Big\|_{\mathrm{b}} \leq \Big\|\frac{e^{\Delta\overline{\mathrm{Q}}}-\mathrm{I}}{\Delta}-\overline{\mathrm{Q}}\Big\|_{\mathrm{b}},$$

where for the inequality we used (2.2) and (T11). Since (3.3) holds for sufficiently small Δ, we conclude from this that

$$\lim_{s\searrow t}\frac{e^{s\overline{\mathrm{Q}}}-e^{t\overline{\mathrm{Q}}}}{s-t} = \overline{\mathrm{Q}}e^{t\overline{\mathrm{Q}}}.$$

The proof for the left-sided limit is similar—we need one extra step in the argument—and therefore omitted. □

From Lemma 3.5 and Propositions 3.6 and 2.10, we know that $e^{\cdot\overline{\mathrm{Q}}}$ belongs to $\mathrm{C}^1(\mathbb{R}_{\geq 0}, \mathfrak{O}_{\mathrm{b}})$, and that it is a solution of the abstract Cauchy problem

$$\lim_{s\to t}\frac{\mathrm{S}_s-\mathrm{S}_t}{s-t} = \overline{\mathrm{Q}}\mathrm{S}_t \quad \text{for all } t \in \mathbb{R}_{\geq 0}, \quad \mathrm{S}_0 = \mathrm{I}.$$

Even more, because the bounded rate operator $\overline{\mathrm{Q}}$ is Lipschitz [Proposition 2.10], it follows from the Cauchy–Lipschitz Theorem—see for example Theorem 7.3 in [3]—that $e^{\cdot\overline{\mathrm{Q}}}$ is the *unique* solution (in $\mathrm{C}^1(\mathbb{R}_{\geq 0}, \mathfrak{O}_{\mathrm{b}})$) to this abstract Cauchy problem. Our construction of the solution to this abstract Cauchy problem through an Euler-style limit expression differs from the construction by means of Picard iterates typically used in the proof of the Cauchy–Lipschitz Theorem; the benefit of using the Euler approximations $(\mathrm{I}+\frac{t}{n}\overline{\mathrm{Q}})^n$ is that they are guaranteed to be sublinear transition operators for sufficiently large n [by Lemma 2.9], while it's not easy to see—at least not to me—that the Picard iterates converge to a sublinear transition operator.

4. Uniformly continuous sublinear transition semigroups

The question now arises whether the converse of the main results in the previous section also hold: is every uniformly continuous sublinear transition semigroup $(\overline{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}}$ *generated* by a bounded sublinear rate operator $\overline{\mathrm{Q}}$, in the sense that

$$\overline{\mathrm{T}}_t = e^{t\overline{\mathrm{Q}}} \quad \text{for all } t \in \mathbb{R}_{\geq 0}?$$

In this section we set out to show that the answer to this question is positive.

Before we get into our investigation, let us take a closer look at the requirement of uniform continuity for sublinear transition semigroups.

Proposition 4.1. *A sublinear transition semigroup $\overline{\mathrm{T}}_\cdot$ is uniformly continuous if and only if*

$$\limsup_{t\searrow 0}\Big\|\frac{\overline{\mathrm{T}}_t-\mathrm{I}}{t}\Big\|_{\mathrm{b}} = 2\limsup_{t\searrow 0}\sup\Big\{\frac{1}{t}[\overline{\mathrm{T}}(1-\mathbb{I}_x)](x)\colon x \in \mathcal{X}\Big\} < +\infty.$$

The proof of this result is a bit long and not necessarily informative, but the interested reader can find it in Section 9. This result is relevant because it establishes that a sublinear transition semigroup $(\overline{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}}$ is uniformly continuous if and only if it has 'uniformly bounded rate' in the sense of [14, Definition 4], which is one of the two conditions on the sublinear transition semigroup—see Theorem 4 there—that ensures that it induces a sublinear Markov process with desirable properties.

We'll progress through a sequence of (intermediate) results in order to establish the main result, Theorem 4.5 further on. As a first step, we set out to establish the 'inverse' to Theorem 3.1: instead of defining the exponential of a bounded sublinear rate operator through a Cauchy sequence, we seek to define the natural logarithm of a sublinear transition semigroup through a Cauchy sequence. The way we will go about this is to generalise the following well-known limit expression for the natural logarithm: for any strictly positive real number $\alpha \in \mathbb{R}_{>0}$,

$$\ln \alpha = \lim_{n \to +\infty} n(\alpha^{\frac{1}{n}} - 1).$$

To translate this limit expression to the setting of bounded operators, we (i) replace α by $\overline{\mathrm{T}}_t$ and 1 by I, and (ii) observe that since $\overline{\mathrm{T}}_t = \big(\overline{\mathrm{T}}_{t/n}\big)^n$, we can think of $\overline{\mathrm{T}}_{t/n}$ as the—or an—n-th root of $\overline{\mathrm{T}}_t$. It still surprises me that this approach works, since never before have I seen this limit expression in the setting of operators.

Proposition 4.2. *For a sublinear transition semigroup $(\overline{\mathrm{T}}_t)_{t \in \mathbb{R}_{\geq 0}}$ that is uniformly continuous and $t \in \mathbb{R}_{\geq 0}$, the sequence $\big(n(\overline{\mathrm{T}}_{t/n} - \mathrm{I})\big)_{n \in \mathbb{N}}$ is Cauchy in $\mathfrak{O}_{\mathrm{b}}$, and its limit*

$$\ln \overline{\mathrm{T}}_t \coloneqq \lim_{n \to +\infty} n(\overline{\mathrm{T}}_{t/n} - \mathrm{I})$$

is a bounded sublinear rate operator.

Our proof for Proposition 4.2 relies on Proposition 4.1 as well as on the following intermediary result, which establishes a convenient bound on $\|\overline{\mathrm{T}} - \mathrm{I} - n(\overline{\mathrm{T}}_{t/n} - \mathrm{I})\|_{\mathrm{b}}$.

Lemma 4.3. *Consider a sublinear transition operator $\overline{\mathrm{T}}$. Then for all $n \in \mathbb{N}$,*

$$\Big\|(\overline{\mathrm{T}}^n - \mathrm{I}) - n(\overline{\mathrm{T}} - \mathrm{I})\Big\|_{\mathrm{b}} \leq \frac{n(n-1)}{2} \|\overline{\mathrm{T}} - \mathrm{I}\|_{\mathrm{b}}^2.$$

Proof. Our proof will be one by induction. The statement is clearly satisfied for $n = 1$, so it remains for us to check the inductive step. So we suppose that the inquality in the statement holds for some $n \in \mathbb{N}$, and set out to show that

$$\Big\|(\overline{\mathrm{T}}^{n+1} - \mathrm{I}) - (n+1)(\overline{\mathrm{T}} - \mathrm{I})\Big\|_{\mathrm{b}} \leq \frac{(n+1)n}{2} \|\overline{\mathrm{T}} - \mathrm{I}\|_{\mathrm{b}}^2. \tag{4.1}$$

First, we rewrite the operator on the left-hand side of this inequality:

$$\begin{aligned}(\overline{\mathrm{T}}^{n+1} - \mathrm{I}) - (n+1)(\overline{\mathrm{T}} - \mathrm{I}) &= (\overline{\mathrm{T}}^{n+1} - \mathrm{I}) - n(\overline{\mathrm{T}} - \mathrm{I}) - (\overline{\mathrm{T}} - \mathrm{I}) \\ &= (\overline{\mathrm{T}}^n - \mathrm{I})\overline{\mathrm{T}} - n(\overline{\mathrm{T}} - \mathrm{I}).\end{aligned}$$

Adding and subtracting $n(\overline{\mathrm{T}} - \mathrm{I})\overline{\mathrm{T}}$ on the right-hand side then gives

$$(\overline{\mathrm{T}}^{n+1} - \mathrm{I}) - (n+1)(\overline{\mathrm{T}} - \mathrm{I}) = \big((\overline{\mathrm{T}}^n - \mathrm{I}) - n(\overline{\mathrm{T}} - \mathrm{I})\big)\overline{\mathrm{T}} + n(\overline{\mathrm{T}} - \mathrm{I})\overline{\mathrm{T}} - n(\overline{\mathrm{T}} - \mathrm{I}),$$

so we see that

$$\Big\|(\overline{\mathrm{T}}^{n+1} - \mathrm{I}) - (n+1)(\overline{\mathrm{T}} - \mathrm{I})\Big\|_{\mathrm{b}} \leq \Big\|\big((\overline{\mathrm{T}}^n - \mathrm{I}) - n(\overline{\mathrm{T}} - \mathrm{I})\big)\overline{\mathrm{T}}\Big\|_{\mathrm{b}} + \Big\|n(\overline{\mathrm{T}} - \mathrm{I})\overline{\mathrm{T}} - n(\overline{\mathrm{T}} - \mathrm{I})\mathrm{I}\Big\|_{\mathrm{b}}.$$

For the first term on the right-hand side of this inequality, it follows from (2.2), (T11) and the induction hypothesis that

$$\begin{aligned}\Big\|\big((\overline{\mathrm{T}}^n - \mathrm{I}) - n(\overline{\mathrm{T}} - \mathrm{I})\big)\overline{\mathrm{T}}\Big\|_{\mathrm{b}} &\leq \Big\|(\overline{\mathrm{T}}^n - \mathrm{I}) - n(\overline{\mathrm{T}} - \mathrm{I})\Big\|_{\mathrm{b}} \Big\|\overline{\mathrm{T}}\Big\|_{\mathrm{b}} \\ &= \Big\|(\overline{\mathrm{T}}^n - \mathrm{I}) - n(\overline{\mathrm{T}} - \mathrm{I})\Big\|_{\mathrm{b}} \\ &\leq \frac{n(n-1)}{2} \|\overline{\mathrm{T}} - \mathrm{I}\|_{\mathrm{b}}^2.\end{aligned}$$

For the second term, we recall from Lemma 2.8 that $n(\overline{\mathrm{T}}-\mathrm{I})$ is a bounded sublinear rate operator; as $\overline{\mathrm{T}}$ and I are both bounded operators, it therefore follows from (Q9) that

$$\Big\|n(\overline{\mathrm{T}}-\mathrm{I})\overline{\mathrm{T}}-n(\overline{\mathrm{T}}-\mathrm{I})\mathrm{I}\Big\|_{\mathrm{b}} \leq \Big\|n(\overline{\mathrm{T}}-\mathrm{I})\Big\|_{\mathrm{b}}\Big\|\overline{\mathrm{T}}-\mathrm{I}\Big\|_{\mathrm{b}} = n\|\overline{\mathrm{T}}-\mathrm{I}\|_{\mathrm{b}}^2.$$

Thus, we see that

$$\Big\|(\overline{\mathrm{T}}^{n+1}-\mathrm{I})-(n+1)(\overline{\mathrm{T}}-\mathrm{I})\Big\|_{\mathrm{b}} \leq \frac{n(n-1)}{2}\|\overline{\mathrm{T}}-\mathrm{I}\|_{\mathrm{b}}^2+n\|\overline{\mathrm{T}}-\mathrm{I}\|_{\mathrm{b}}^2 = \frac{(n+1)n}{2}\|\overline{\mathrm{T}}-\mathrm{I}\|_{\mathrm{b}}^2,$$

which verifies (4.1) and concludes our proof. □

Proof of Proposition 4.2. The statement holds trivially in case $t=0$, so we assume without loss of generality that $t>0$. On the one hand, it follows from the properties of $\|\cdot\|_{\mathrm{b}}$ and (T11)—once for $\overline{\mathrm{T}}$ and once for I—that

$$\Big\|\frac{\overline{\mathrm{T}}_s-\mathrm{I}}{s}\Big\|_{\mathrm{b}} = \frac{\Big\|\overline{\mathrm{T}}_s-\mathrm{I}\Big\|_{\mathrm{b}}}{s} \leq \frac{\Big\|\overline{\mathrm{T}}_s\Big\|_{\mathrm{b}}+\|\mathrm{I}\|_{\mathrm{b}}}{s} = \frac{2}{s} \quad \text{for all } s\in\mathbb{R}_{>0}.$$

On the other hand, since $(\overline{\mathrm{T}}_s)_{s\in\mathbb{R}_{\geq 0}}$ is uniformly continuous by assumption, it follows from Proposition 4.1 that there are some $\delta,\beta'\in\mathbb{R}_{>0}$ such that

$$\Big\|\frac{\overline{\mathrm{T}}_s-\mathrm{I}}{s}\Big\|_{\mathrm{b}} \leq \beta' \quad \text{for all } s\in[0,\delta[.$$

From these two inequalities, we infer that

$$\beta := \sup\Big\{\Big\|\frac{\overline{\mathrm{T}}_s-\mathrm{I}}{s}\Big\|_{\mathrm{b}} : s\in\mathbb{R}_{>0}\Big\} \leq \max\Big\{\frac{2}{\delta},\beta'\Big\} < +\infty.$$

Consequently, for all $k\in\mathbb{N}$,

$$\Big\|\overline{\mathrm{T}}_{\frac{t}{k}}-\mathrm{I}\Big\|_{\mathrm{b}} \leq \frac{t\beta}{k}. \tag{4.2}$$

Fix some $n,m\in\mathbb{N}$. Then

$$\begin{aligned}
&\Big\|n(\overline{\mathrm{T}}_{\frac{t}{n}}-\mathrm{I})-m(\overline{\mathrm{T}}_{\frac{t}{m}}-\mathrm{I})\Big\|_{\mathrm{b}}\\
&= \Big\|n\big(\overline{\mathrm{T}}_{\frac{t}{n}}-\mathrm{I}\big)-nm\big(\overline{\mathrm{T}}_{\frac{t}{nm}}-\mathrm{I}\big)+nm\big(\overline{\mathrm{T}}_{\frac{t}{nm}}-\mathrm{I}\big)-m\big(\overline{\mathrm{T}}_{\frac{t}{m}}-\mathrm{I}\big)\Big\|_{\mathrm{b}}\\
&\leq n\Big\|\big(\overline{\mathrm{T}}_{\frac{t}{n}}-\mathrm{I}\big)-m\big(\overline{\mathrm{T}}_{\frac{t}{nm}}-\mathrm{I}\big)\Big\|_{\mathrm{b}}+m\Big\|\big(\overline{\mathrm{T}}_{\frac{t}{m}}-\mathrm{I}\big)-n\big(\overline{\mathrm{T}}_{\frac{t}{nm}}-\mathrm{I}\big)\Big\|_{\mathrm{b}}.
\end{aligned}$$

From the semigroup property (SG1) of $(\overline{\mathrm{T}}_s)_{s\in\mathbb{R}_{\geq 0}}$, we infer that

$$\overline{\mathrm{T}}_{\frac{t}{n}} = \big(\overline{\mathrm{T}}_{\frac{t}{nm}}\big)^m \quad \text{and} \quad \overline{\mathrm{T}}_{\frac{t}{m}} = \big(\overline{\mathrm{T}}_{\frac{t}{nm}}\big)^n.$$

From these two inequalities, it follows from the preceding inequality, Lemma 4.3 and (4.2) that

$$\begin{aligned}
\Big\|n(\overline{\mathrm{T}}_{\frac{t}{n}}-\mathrm{I})-m(\overline{\mathrm{T}}_{\frac{t}{m}}-\mathrm{I})\Big\|_{\mathrm{b}} &\leq n\frac{m(m-1)}{2}\Big(\frac{t\beta}{nm}\Big)^2+m\frac{n(n-1)}{2}\Big(\frac{t\beta}{nm}\Big)^2\\
&= \frac{1}{2n}\frac{m(m-1)}{m^2}t^2\beta^2+\frac{1}{2m}\frac{n(n-1)}{n^2}t^2\beta^2\\
&< \frac{1}{2}\Big(\frac{1}{n}+\frac{1}{m}\Big)t^2\beta^2.
\end{aligned}$$

Since this inequality holds for arbitrary $n,m\in\mathbb{N}$, we can conclude that the sequence $(n(\overline{\mathrm{T}}_{t/n})-\mathrm{I})_{n\in\mathbb{N}}$ in $\mathfrak{O}_{\mathrm{b}}$ is Cauchy. As $(\mathfrak{O}_{\mathrm{b}},\|\cdot\|_{\mathrm{b}})$ is complete, this sequence converges to the bounded operator

$$\ln\overline{\mathrm{T}}_t = \lim_{n\to+\infty} n\Big(\overline{\mathrm{T}}_{\frac{t}{n}}-\mathrm{I}\Big).$$

To verify that the bounded operator $\ln\overline{\mathrm{T}}_t$ is a sublinear rate operator, it suffices to realise that (i) for all $n\in\mathbb{N}$, $n(\overline{\mathrm{T}}_{t/n}-\mathrm{I})$ is a bounded rate operator due to Lemma 2.8; and (ii) the axioms (Q1)–(Q4) of a rate operator are preserved when taking limits. □

Its limit expression already warrants calling $\ln \overline{\mathrm{T}}_t$ the ‘ (natural) operator logarithm of $\overline{\mathrm{T}}_t$,’ but the following result provides full justification: the operator logarithm is indeed the inverse of the operator exponential.

Proposition 4.4. *For any bounded sublinear rate operator* $\overline{\mathrm{Q}}$,

$$\ln e^{t\overline{\mathrm{Q}}} = t\overline{\mathrm{Q}} \quad \text{for all } t \in \mathbb{R}_{\geq 0}.$$

Conversely, for any uniformly continuous semigroup $(\overline{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}}$ *of sublinear transition operators,*

$$\overline{\mathrm{T}}_t = e^{\ln \overline{\mathrm{T}}_t} \quad \text{for all } t \in \mathbb{R}_{\geq 0}.$$

Proof. For the first part of the proof, recall from Proposition 3.4 and Lemma 3.5 that $(e^{s\overline{\mathrm{Q}}})_{s\in\mathbb{R}_{\geq 0}}$ is a uniformly continuous sublinear transition semigroup, so the operator logarithm is well defined. The equality for $t = 0$ holds trivially because $e^{0\overline{\mathrm{Q}}} = \mathrm{I}$, so we assume without loss of generality that $t > 0$. Fix some $\epsilon \in \mathbb{R}_{>0}$. Then it follows from Propositions 3.6 and 4.2—and the fact that $e^{0\overline{\mathrm{Q}}} = \mathrm{I}$—that there is some $n \in \mathbb{N}$ such that

$$\Big\| \frac{e^{\frac{t}{n}\overline{\mathrm{Q}}} - \mathrm{I}}{\frac{t}{n}} - \overline{\mathrm{Q}} \Big\|_{\mathrm{b}} < \frac{\epsilon}{2t} \quad \text{and} \quad \Big\| n(e^{\frac{t}{n}\overline{\mathrm{Q}}} - \mathrm{I}) - \ln e^{t\overline{\mathrm{Q}}} \Big\|_{\mathrm{b}} < \frac{\epsilon}{2}.$$

From this, it follows that

$$\begin{aligned} \|\ln e^{t\overline{\mathrm{Q}}} - t\overline{\mathrm{Q}}\|_{\mathrm{b}} &\leq \Big\| \ln e^{t\overline{\mathrm{Q}}} - n(e^{\frac{t}{n}\overline{\mathrm{Q}}} - \mathrm{I}) \Big\|_{\mathrm{b}} + \Big\| n(e^{\frac{t}{n}\overline{\mathrm{Q}}} - \mathrm{I}) - t\overline{\mathrm{Q}} \Big\|_{\mathrm{b}} \\ &= \Big\| \ln e^{t\overline{\mathrm{Q}}} - n(e^{\frac{t}{n}\overline{\mathrm{Q}}} - \mathrm{I}) \Big\|_{\mathrm{b}} + t\Big\| \frac{e^{\frac{t}{n}\overline{\mathrm{Q}}} - \mathrm{I}}{\frac{t}{n}} - \overline{\mathrm{Q}} \Big\|_{\mathrm{b}} \\ &< \epsilon. \end{aligned}$$

Since this holds for arbitrary $\epsilon \in \mathbb{R}_{>0}$, we have proven the first part of the statement.

For the second part of the statement, we again fix some $\epsilon \in \mathbb{R}_{>0}$ and $t \in \mathbb{R}_{\geq 0}$. Then by Theorem 3.1 and Proposition 4.2 there is some $n \in \mathbb{N}$ such that

$$\|\ln \overline{\mathrm{T}}_t\|_{\mathrm{b}} \leq 2n, \Big\| e^{\ln \overline{\mathrm{T}}_t} - \Big(\mathrm{I} + \frac{1}{n} \ln \overline{\mathrm{T}}_t\Big)^n \Big\|_{\mathrm{b}} < \frac{\epsilon}{2} \quad \text{and} \quad \Big\| \ln \overline{\mathrm{T}}_t - n\Big(\overline{\mathrm{T}}_{\frac{t}{n}} - \mathrm{I}\Big) \Big\|_{\mathrm{b}} < \frac{\epsilon}{2}.$$

Note furthermore that

$$\mathrm{I} + \frac{1}{n}\Big(n\Big(\overline{\mathrm{T}}_{\frac{t}{n}} - \mathrm{I}\Big)\Big) = \overline{\mathrm{T}}_{\frac{t}{n}};$$

we use that $(\overline{\mathrm{T}}_{\frac{t}{n}})^n = \overline{\mathrm{T}}_t$ because $(\overline{\mathrm{T}}_s)_{s\in\mathbb{R}_{\geq 0}}$ is a semigroup, to yield

$$\Big(\mathrm{I} + \frac{1}{n}\Big(n\big(\overline{\mathrm{T}}_{\frac{t}{n}} - \mathrm{I}\big)\Big)\Big)^n = \big(\overline{\mathrm{T}}_{\frac{t}{n}}\big)^n = \overline{\mathrm{T}}_t.$$

Since $\|\ln \overline{\mathrm{T}}_t\|_{\mathrm{b}} \leq 2n$ by our choice of n, Lemma 2.9 ensures that $\mathrm{I} + \frac{1}{n} \ln \overline{\mathrm{T}}_t$ is a sublinear transition operator; this means that we may invoke Lemma 3.2, to yield

$$\begin{aligned} &\Big\| \Big(\mathrm{I} + \frac{1}{n} \ln \overline{\mathrm{T}}_t\Big)^n - \Big(\mathrm{I} + \frac{1}{n}\Big(n\big(\overline{\mathrm{T}}_{\frac{t}{n}} - \mathrm{I}\big)\Big)\Big)^n \Big\|_{\mathrm{b}} \\ &\leq n\Big\| \Big(\mathrm{I} + \frac{1}{n} \ln \overline{\mathrm{T}}_t\Big) - \Big(\mathrm{I} + \frac{1}{n}\Big(n\big(\overline{\mathrm{T}}_{\frac{t}{n}} - \mathrm{I}\big)\Big)\Big) \Big\| \\ &= \Big\| \ln \overline{\mathrm{T}}_t - n\Big(\overline{\mathrm{T}}_{\frac{t}{n}} - \mathrm{I}\Big) \Big\|_{\mathrm{b}} \\ &< \frac{\epsilon}{2}. \end{aligned}$$

From all this, it follows that

$$\begin{aligned} \Big\| \overline{\mathrm{T}}_t - e^{\ln \overline{\mathrm{T}}_t} \Big\|_{\mathrm{b}} &= \Big\| \Big(\mathrm{I} + \frac{1}{n}\Big(n\big(\overline{\mathrm{T}}_{\frac{t}{n}} - \mathrm{I}\big)\Big)\Big)^n - e^{\ln \overline{\mathrm{T}}_t} \Big\|_{\mathrm{b}} \\ &\leq \Big\| \Big(\mathrm{I} + \frac{1}{n}\Big(n\big(\overline{\mathrm{T}}_{\frac{t}{n}} - \mathrm{I}\big)\Big)\Big)^n - \Big(\mathrm{I} + \frac{1}{n} \ln \overline{\mathrm{T}}_t\Big)^n \Big\|_{\mathrm{b}} \Big\| \Big(\mathrm{I} + \frac{1}{n} \ln \overline{\mathrm{T}}_t\Big)^n - e^{\ln \overline{\mathrm{T}}_t} \Big\|_{\mathrm{b}} \\ &< \epsilon. \end{aligned}$$

Since $\epsilon \in \mathbb{R}_{>0}$ was arbitrary, this shows that $\overline{\mathrm{T}}_t = e^{\ln \overline{\mathrm{T}}_t}$, as required. □

At long last, we are ready to provide a positive answer to the question posited at the beginning of this section: is every uniformly continuous sublinear transition semigroup generated by a bounded sublinear rate operator?

Theorem 4.5. *Let $(\overline{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}}$ be a sublinear transition semigroup. If this semigroup is uniformly continuous, then $\ln \overline{\mathrm{T}}_1$ is a bounded sublinear rate operator, and*

$$\overline{\mathrm{T}}_t = e^{t\ln \overline{\mathrm{T}}_1} \quad \text{for all } t\in\mathbb{R}_{\geq 0}.$$

Proof. Since $(\overline{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}}$ is a uniformly continuous sublinear transition semigroup, Proposition 4.2 guarantees that for all $t\in\mathbb{R}_{\geq 0}$, $\ln\overline{\mathrm{T}}_t$ is a bounded sublinear rate operator, while Proposition 4.4 ensures that

$$\overline{\mathrm{T}}_t = e^{\ln \overline{\mathrm{T}}_t} \quad \text{for all } t\in\mathbb{R}_{\geq 0}.$$

As $(e^{t\ln\overline{\mathrm{T}}_1})_{t\in\mathbb{R}_{\geq 0}}$ is uniformly continuous as well [Lemma 3.5], it suffices to show that $\overline{\mathrm{T}}_t = e^{\ln\overline{\mathrm{T}}_t} = e^{t\ln\overline{\mathrm{T}}_1}$ for all t in some dense subset $\mathcal{T}$ of $\mathbb{R}_{\geq 0}$, and we will do so for $\mathcal{T}=\mathbb{Q}_{\geq 0}$. That is, it suffices to show that

$$\ln\overline{\mathrm{T}}_q = q\ln\overline{\mathrm{T}}_1 \quad \text{for all } q\in\mathbb{Q}_{\geq 0}. \tag{4.3}$$

To this end, note that for all $t\in\mathbb{R}_{\geq 0}$ and $n\in\mathbb{N}$, it follows from Proposition 4.2 that

$$\ln\overline{\mathrm{T}}_{nt} = \lim_{k\to+\infty} nk\Big(\overline{\mathrm{T}}_{\frac{nt}{nk}} - \mathrm{I}\Big) = n\lim_{k\to+\infty} k\Big(\overline{\mathrm{T}}_{\frac{t}{k}} - \mathrm{I}\Big) = n\ln\overline{\mathrm{T}}_t. \tag{4.4}$$

Now fix some $q\in\mathbb{Q}_{\geq 0}$. Then there are some $n\in\mathbb{Z}_{\geq 0}$ and $d\in\mathbb{N}$ such that $q=n/d$, and (4.4) tells us that

$$\ln\overline{\mathrm{T}}_{\frac{n}{d}} = n\ln\overline{\mathrm{T}}_{\frac{1}{d}} \quad\text{and}\quad \ln\overline{\mathrm{T}}_1 = \ln\overline{\mathrm{T}}_{\frac{d}{d}} = d\ln\overline{\mathrm{T}}_{\frac{1}{d}}.$$

Because $d>0$, these equalities clearly imply the one in (4.3) for $q=n/d$, and this concludes our proof. □

5. Comparison to related work

Semigroups of nonlinear operators have received quite some attention in the setting of imprecise probabilities and nonlinear expectations. In the setting of imprecise probabilities, the setting has typically been that of finite state spaces $\mathcal{X}$ [33, 21, 8, 13, 31]. This work extends the approach of De Bock, Krak and Siebes [8, 21] to construct a sublinear transition semigroup from a bounded sublinear rate operator to countable state spaces.

In contrast, much more work has been done in the setting of nonlinear expectations; I'll briefly mention that which is most closely related to the results presented here. Nendel [24] studies semigroups of convex transition operators on $\mathscr{B}$ with $\mathcal{X}$ finite, but they only require strong continuity. In [25, Section 5], the same author considers semigroups of sublinear transition operators for countable state spaces, but their construction differs from the approach taken in this work; I'll discuss this in more detail in Section 5.1 further on. More generally, there's been quite some work on 'semigroups of convex (and monotone) operators' on a variety of functions spaces.

Denk, Kupper & Nendel [10] study strongly continuous semigroups of convex monotone operators on 'L^p-like spaces'. In the present setting, the relevant space is $\mathrm{C}_0(\mathcal{X})$, the space of functions 'vanishing at infinity', equipped with the supremum norm $\|\cdot\|_\infty$; note that $\mathrm{C}_0(\mathcal{X})=\mathscr{B}$ if $\mathcal{X}$ is finite but $\mathrm{C}_0(\mathcal{X})\subsetneq\mathscr{B}$ if $\mathcal{X}$ is countably infinite. They show that a uniformly continuous semigroup $(\mathrm{S}_t)_{t\in\mathbb{R}_{\geq 0}}$ of convex monotone operators from $\mathrm{C}_0(\mathcal{X})$ to $\mathrm{C}_0(\mathcal{X})$ is completely defined by the generator A whose domain consists of those $f\in\mathrm{C}_0(\mathcal{X})$ for which

$$\mathbb{R}_{\geq 0}\to\mathrm{C}_0(\mathcal{X})\colon t\mapsto \frac{\mathrm{S}_t f - \mathrm{S}_0 f}{t} \quad \text{converges in } \mathrm{C}_0(\mathcal{X}) \text{ for } t\searrow 0.$$

In contrast, in the present setting we investigated sublinear operators—which are convex and monotone—in the Banach space $(\mathfrak{O}_\mathrm{b}, \|\cdot\|_\mathrm{b})$ of bounded operators rather than in the functional space $(\mathscr{B},\|\cdot\|_\infty)$, and we did not require that these operators map the space $\mathrm{C}_0(\mathcal{X})$ of function vanishing at infinity to itself.

Much of the other work on strongly continuous semigroups of convex monotone operators equips the function space with the so-called 'mixed topology', see for example [19, 2]. In the present setting this amounts to (i) choosing some $\kappa \in \mathscr{B}$ with $\kappa(x) > 0$ for all $x \in \mathcal{X}$; (ii) letting $\|f\|_\kappa \coloneqq \sup_{x\in\mathcal{X}}|f(x)|\kappa(x)$ for all $f \in \mathbb{R}^{\mathcal{X}}$; and (iii) considering the space $\mathrm{C}_\kappa(\mathcal{X}) \coloneqq \{f \in \mathbb{R}^{\mathcal{X}} \colon \|f\|_\kappa < +\infty\}$. A sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{C}_\kappa(\mathcal{X})$ then converges to $f \in \mathrm{C}_\kappa(\mathcal{X})$ in the mixed topology if and only if $\sup_{n\in\mathbb{N}}\|f_n\|_\kappa < +\infty$ and $(f_n)_{n\in\mathbb{N}}$ converges to f pointwise. For $\kappa = 1$, this reduces to pointwise convergence on the set of bounded functions $\mathscr{B}$. However, the intended use of the mixed topology is to allow for functions that are unbounded, but grow at most as fast as $1/\kappa$. In this mixed topology, Blessing, Denk, Kupper & Nendel [2, Section 5] come up with a Chernoff-type method to generate a strongly continuous semigroup of convex monotone operators on $\mathrm{C}_\kappa(\mathcal{X})$, which is quite similar in spirit to the construction of the semigroup $(e^{t\overline{\mathrm{Q}}})_{t\in\mathbb{R}_{\geq 0}}$ from a bounded sublinear rate operator $\overline{\mathrm{Q}}$ in Theorem 3.1.

5.1. **Nendel's** [25] **Nisio semigroups.** As mentioned before, Nendel [25, Section 5] also considers semigroups of sublinear transition operators for countable state spaces, but the way they construct them differs a bit from the approach I've taken in this work. Their starting point is a family $\{(\mathrm{T}^\lambda_t)_{t\in\mathbb{R}_{\geq 0}} \colon \lambda \in \Lambda\}$ of Markov semigroups—that is, a set of semigroups $(\mathrm{T}^\lambda_t)_{t\in\mathbb{R}_{\geq 0}}$ of *linear* transition operators that are furthermore *upward continuous* in the following sense.

Definition 5.1. An operator $\mathrm{A} \in \mathfrak{O}$ is called (pointwise) *upward continuous*—sometimes also continuous from below—if for all $x \in \mathcal{X}$, the corresponding component functional $[\mathrm{A}\cdot](x) \colon \mathscr{B} \to \mathbb{R}$ is upward continuous, meaning that for any monotone sequence $(f_n)_{n\in\mathbb{N}} \in \mathscr{B}^{\mathbb{N}}$ that increases pointwise to some $f \in \mathscr{B}$,

$$\lim_{n\to+\infty}[\mathrm{A}f_n](x) = [\mathrm{A}f](x).$$

Nendel constructs the so-called *Nisio semigroup* $(\overline{\mathrm{S}}_t)_{t\in\mathbb{R}_{\geq 0}}$ as follows. First, for all $s \in \mathbb{R}_{\geq 0}$, the sublinear transition operator $\tilde{\mathrm{T}}_s$ is defined as the point-wise upper envelope of the family $(\mathrm{T}^\lambda_s)_{\lambda\in\Lambda}$ of upward continuous transition operators:

$$\tilde{\mathrm{T}}_s f(x) \coloneqq \sup\{\mathrm{T}^\lambda_s f(x) \colon \lambda \in \Lambda\}.$$

In the proof for their Proposition 5.2, Nendel [25] uses that $\tilde{\mathrm{T}}_s$ is upward continuous, which follows from the definition above and the upward continuity of the transition operators $(\mathrm{T}^\lambda_s)_{\lambda\in\Lambda}$. Second, for all $t \in \mathbb{R}_{\geq 0}$, $f \in \mathscr{B}$ and $x \in \mathcal{X}$, they let

$$\overline{\mathrm{S}}_t f(x) \coloneqq \sup\{\tilde{\mathrm{T}}_{t_1-t_0}\cdots\tilde{\mathrm{T}}_{t_n-t_{n-1}} f(x) \colon n \in \mathbb{Z}_{\geq 0}, 0 = t_0 < t_1 < \cdots t_n = t\}.$$

They then go on to show that this Nisio semigroup $(\overline{\mathrm{S}}_t)_{t\in\mathbb{R}_{\geq 0}}$ is the point-wise smallest semigroup that dominates $(\mathrm{T}^\lambda_\cdot)_{\lambda\in\Lambda}$: for any semigroup $(\mathrm{S}_t)_{t\in\mathbb{R}_{\geq 0}}$ such that

$$\mathrm{T}^\lambda_t f \leq \mathrm{S}_t f \quad \text{for all } \lambda \in \Lambda, t \in \mathbb{R}_{\geq 0}, f \in \mathscr{B},$$

they show that

$$\mathrm{T}^\lambda_t f \leq \overline{\mathrm{S}}_t f \leq \mathrm{S}_t f \quad \text{for all } \lambda \in \Lambda, t \in \mathbb{R}_{\geq 0}, f \in \mathscr{B}.$$

In light of this paper, an obvious question is whether there are necessary and sufficient conditions on the family $\{(\mathrm{T}^\lambda_t)_{t\in\mathbb{R}_{\geq 0}} \colon \lambda \in \Lambda\}$ of semigroups such that the corresponding Nisio semigroup $(\overline{\mathrm{S}}_t)_{t\in\mathbb{R}_{\geq 0}}$ is uniformly continuous. The following result establishes exactly such conditions.

Proposition 5.2. *Let $(\overline{\mathrm{S}}_t)_{t\in\mathbb{R}_{\geq 0}}$ be the Nisio semigroup corresponding to a family $\{(\mathrm{T}^\lambda_t)_{t\in\mathbb{R}_{\geq 0}} \colon \lambda \in \Lambda\}$ of semigroups of linear transition operators that are upward continuous. Then the Nisio semigroup $(\overline{\mathrm{S}}_t)_{t\in\mathbb{R}_{\geq 0}}$ is uniformly continuous if and only if $(\mathrm{T}^\lambda_t)_{t\in\mathbb{R}_{\geq 0}}$ is uniformly continuous for all $\lambda \in \Lambda$ and the set $\{\|\mathrm{Q}^\lambda\|_{\mathrm{b}} \colon \lambda \in \Lambda\}$ of the norms of the corresponding rate operators is bounded. Whenever this is the case, the generator $\overline{\mathrm{R}}$ of the Nisio semigroup is the point-wise upper envelope of the rate operators:*

$$\overline{\mathrm{R}}f(x) = \sup\{\mathrm{Q}^\lambda f(x) \colon \lambda \in \Lambda\} \quad \text{for all } f \in \mathscr{B}, x \in \mathcal{X}.$$

Proof. Let us establish necessity first, so assume that the Nisio semigroup $(\overline{\mathrm{S}}_t)_{t\in\mathbb{R}_{\geq 0}}$ is uniformly continuous. To this end, fix some $\lambda \in \Lambda$. For all $\Delta \in \mathbb{R}_{>0}$, $\mathrm{T}^\lambda_\Delta - \mathrm{I}$ and $\overline{\mathrm{S}}_\Delta - \mathrm{I}$ are sublinear transition operators by Lemma 2.8, so it follows from Proposition 2.7 and the definition of the Nisio semigroup that

$$\begin{aligned}\|\mathrm{T}^\lambda_\Delta - \mathrm{I}\|_{\mathrm{b}} &= 2\sup\bigl\{[(\mathrm{T}^\lambda_\Delta - \mathrm{I})(1-\mathbb{I}_x)](x) \colon x \in \mathcal{X}\bigr\} \\ &\le 2\sup\bigl\{[(\overline{\mathrm{S}}_\Delta - \mathrm{I})(1-\mathbb{I}_x)](x) \colon x \in \mathcal{X}\bigr\} \\ &= \|\overline{\mathrm{S}}_\Delta - \mathrm{I}\|_{\mathrm{b}}.\end{aligned}$$

Since the Nisio semigroup $(\overline{\mathrm{S}}_t)_{t\in\mathbb{R}_{\ge 0}}$ is uniformly continuous, it follows from this inequality and Lemma 2.5 that the semigroup $(\mathrm{T}^\lambda_t)_{t\in\mathbb{R}_{\ge 0}}$ is uniformly continuous. We still need to establish a universal bound on $\|\mathrm{Q}^\lambda\|_{\mathrm{b}}$. To this end, observe that for all $f \in \mathscr{B}$,

$$\mathrm{Q}^\lambda f = \lim_{\Delta\searrow 0} \frac{\mathrm{T}^\lambda_\Delta f - \mathrm{T}^\lambda_0 f}{\Delta} \le \lim_{\Delta\searrow 0} \frac{\overline{\mathrm{S}}_\Delta f - \overline{\mathrm{S}}_0 f}{\Delta} = \overline{\mathrm{R}} f,$$

where $\overline{\mathrm{R}}$ is the bounded sublinear rate operator that generates the uniformly continuous semigroup $(\overline{\mathrm{S}}_t)_{t\in\mathbb{R}_{\ge 0}}$ [Theorem 4.5], the equalities follow from Theorems 3.6, and the inequality follows from the definition of the Nisio semigroup. Since Q^λ is linear, it also follows from this inequality that $\mathrm{Q}^\lambda f \ge -\overline{\mathrm{R}}(-f)$. From these two inequalities, it follows immediately that $\|\mathrm{Q}^\lambda\|_{\mathrm{b}} \le \|\overline{\mathrm{R}}\|_{\mathrm{b}}$.

For the sufficiency, suppose that every upward continuous transition semigroup is uniformly continuous, meaning that $\mathrm{T}^\lambda = e^{t\mathrm{Q}^\lambda}$ for all $t \in \mathbb{R}_{\ge 0}$ and $\lambda \in \Lambda$, and that $\mathcal{Q} \coloneqq \{\mathrm{Q}^\lambda \colon \lambda \in \Lambda\}$ is uniformly bounded. Nendel [25, Remark 5.6] shows that for all $f \in \mathscr{B}$,

$$\mathbb{R}_{\ge 0} \to \mathscr{B} \colon t \mapsto \overline{\mathrm{S}}_t f \tag{5.1}$$

is the *unique* solution to the Cauchy problem

$$\lim_{s\to t} \frac{v(s) - v(t)}{s - t} = \overline{\mathrm{Q}}_{\mathcal{Q}} v(t) \quad \text{for all } t \in \mathbb{R}_{\ge 0}, \quad v(0) = f, \tag{5.2}$$

where $\overline{\mathrm{Q}}_{\mathcal{Q}}$ is the (point-wise) lower envelope of $\mathcal{Q} = \{\mathrm{Q}^\lambda \colon \lambda \in \Lambda\}$, defined in Section 7 by

$$\overline{\mathrm{Q}}_{\mathcal{Q}} f(x) = \sup\bigl\{\mathrm{Q} f(x) \colon \mathrm{Q} \in \mathcal{Q}\bigr\} = \sup\bigl\{\mathrm{Q}^\lambda f(x) \colon \lambda \in \Lambda\bigr\} \quad \text{for all } f \in \mathscr{B}, x \in \mathcal{X}.$$

Proposition 7.2 in Section 7 establishes that $\overline{\mathrm{Q}}_{\mathcal{Q}}$ is a bounded sublinear rate operator, so it follows from Proposition 3.6 that

$$\mathbb{R}_{\ge 0} \to \mathscr{B} \colon t \mapsto e^{t\overline{\mathrm{Q}}_{\mathcal{Q}}} f$$

solves the Cauchy problem in (5.2). Consequently, we conclude that the Nisio semigroup $(\overline{\mathrm{S}}_t)_{t\in\mathbb{R}_{\ge 0}}$ is uniformly continuous and generated by $\overline{\mathrm{R}} = \overline{\mathrm{Q}}_{\mathcal{Q}}$. □

6. Appendix: Proofs and additional results regarding the Banach space of bounded operators

Our proof for Proposition 2.1 relies on the following intermediary result.

Lemma 6.1. *The function* $\|\cdot\|_s \colon \mathfrak{O} \to \mathbb{R}_{\ge 0} \cup \{+\infty\}$ *as defined by* (2.1) *is an extended seminorm on* $\mathfrak{O}$. *Furthermore, for all* $\mathrm{A}, \mathrm{B} \in \mathfrak{O}$,

$$\|\mathrm{AB}\|_{\mathrm{s}} \le \|\mathrm{A}\|_{\mathrm{s}} \|\mathrm{B}\|_{\mathrm{s}}.$$

Proof. The function $\|\cdot\|_s$ is positive by definition, and it is clear that $\|\mathrm{O}\|_{\mathrm{s}} = 0$. That $\|\cdot\|_s$ is subadditive follows from the subadditivity of the supremum norm $\|\cdot\|_\infty$ and the subadditivity of the supremum, and $\|\cdot\|_{\mathrm{b}}$ inherits the absolute homogeneity of the supremum norm $\|\cdot\|_\infty$.

For the second part of the statement, note that

$$\begin{aligned}\|\mathrm{AB}\|_{\mathrm{s}} &= \sup\Bigl\{\frac{\|\mathrm{AB}f\|_\infty}{\|f\|_\infty} \colon f \in \mathscr{B}, f \ne 0\Bigr\} \\ &\le \sup\Bigl\{\frac{\|\mathrm{A}\|_{\mathrm{s}}\|\mathrm{B}f\|_\infty}{\|f\|_\infty} \colon f \in \mathscr{B}, f \ne 0\Bigr\} \\ &= \|\mathrm{A}\|_{\mathrm{s}}\|\mathrm{B}\|_{\mathrm{s}}.\end{aligned}$$

□

Proving Proposition 2.1 is now a matter of adapting Martin's [22, Section III.2] proof for the Lipschitz norm.

Proof of Proposition 2.1. First, it is clear that $\mathfrak{O}_\mathrm{b}$ is a real vector space since addition and scaling clearly preserve finiteness of the operator seminorm $\|\cdot\|_s$. Second, it follows from Lemma 6.1 that $\|\cdot\|_s$ is a seminorm on $\mathfrak{O}_\mathrm{b}$. Furthermore, it is easy to see that $\|\mathrm{A}\|_\mathrm{s}=0$ if and only if $\mathrm{A}f=0$ for all $f\in\mathscr{B}$ such that $f\neq 0$; whenever this is the case, $\|\mathrm{A}\|_\mathrm{b}=0$ if and only if furthermore $\mathrm{A}0=0$, which can only be if $\mathrm{A}=\mathrm{O}$. This proves that $\|\cdot\|_\mathrm{b}$ is a norm.

A standard argument now shows that $(\mathfrak{O}_\mathrm{b},\|\cdot\|_\mathrm{b})$ is complete. Fix any Cauchy sequence $(\mathrm{A}_n)_{n\in\mathbb{N}}\in(\mathfrak{O}_\mathrm{b})^\mathbb{N}$. Then for all $f\in\mathscr{B}$, $(\mathrm{A}_nf)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete space $(\mathscr{B},\|\cdot\|_\infty)$, so $\lim_{n\to+\infty}\mathrm{A}_nf$ exists. The operator

$$\mathrm{A}_\mathrm{lim}\colon\mathscr{B}\to\mathscr{B}\colon f\mapsto\lim_{n\to+\infty}\mathrm{A}_nf$$

is bounded because the Cauchy sequence $(\mathrm{A}_n)_{n\in\mathbb{N}}$ is bounded [18, Lemma 1.17]:

$$\begin{aligned}\|\mathrm{A}_\mathrm{lim}\|_\mathrm{s}&=\sup\Big\{\frac{\|\lim_{n\to+\infty}\mathrm{A}_nf\|_\infty}{\|f\|_\infty}\colon f\in\mathscr{B},f\neq0\Big\}\\&\leq\sup\Big\{\frac{\sup\big\{\|\mathrm{A}_n\|_\mathrm{b}\colon n\in\mathbb{N}\big\}\|f\|_\infty}{\|f\|_\infty}\colon f\in\mathscr{B},f\neq0\Big\}\\&=\sup\big\{\|\mathrm{A}_n\|_\mathrm{b}\colon n\in\mathbb{N}\big\}<+\infty.\end{aligned}$$

To see that $(\mathrm{A}_n)_{n\in\mathbb{N}}$ converges to A_lim, we fix any $\epsilon\in\mathbb{R}_{>0}$. Because $(\mathrm{A}_n)_{n\in\mathbb{N}}$ is Cauchy, there is some $N\in\mathbb{N}$ such that for all $n,m\geq N$,

$$\|\mathrm{A}_n-\mathrm{A}_m\|_\mathrm{b}=\|\mathrm{A}_n0-\mathrm{A}_m0\|_\infty+\|\mathrm{A}_n-\mathrm{A}_m\|_\mathrm{s}<\frac{1}{2}\epsilon.$$

On the one hand, we infer from this that for all $n\geq N$

$$\begin{aligned}\|\mathrm{A}_\mathrm{lim}0-\mathrm{A}_n0\|_\infty&\leq\limsup_{m\to+\infty}\|\mathrm{A}_\mathrm{lim}0-\mathrm{A}_m0\|_\infty+\|\mathrm{A}_m0-\mathrm{A}_n0\|_\infty\\&=\limsup_{m\to+\infty}\|\mathrm{A}_m0-\mathrm{A}_n0\|_\infty\\&<\frac{1}{2}\epsilon.\end{aligned}$$

On the other hand, we infer from this that for all $n\geq N$ and $f\in\mathscr{B}$,

$$\begin{aligned}\|\mathrm{A}_\mathrm{lim}f-\mathrm{A}_nf\|_\infty&\leq\limsup_{m\to+\infty}\|\mathrm{A}_\mathrm{lim}f-\mathrm{A}_mf\|_\infty+\|\mathrm{A}_mf-\mathrm{A}_nf\|_\infty\\&=\limsup_{m\to+\infty}\|\mathrm{A}_mf-\mathrm{A}_nf\|_\infty\\&<\frac{1}{2}\epsilon\|f\|_\infty.\end{aligned}$$

From these two observations, it follows that for all $n\geq N$,

$$\|\mathrm{A}_\mathrm{lim}-\mathrm{A}_n\|_\mathrm{b}=\|\mathrm{A}_\mathrm{lim}0-\mathrm{A}_n0\|_\infty+\|\mathrm{A}_\mathrm{lim}-\mathrm{A}_n\|_\mathrm{s}<\epsilon.$$

Since this holds for all $\epsilon\in\mathbb{R}_{>0}$, we conclude that the Cauchy sequence $(\mathrm{A}_n)_{n\in\mathbb{N}}$ converges to a limit A_lim in $\mathfrak{O}_\mathrm{b}$, as required.

Finally, for the second part of the statement, it follows immediately from the definitions of $\|\cdot\|_s$ and $\|\cdot\|_\mathrm{b}$ and Lemma 6.1 that

$$\begin{aligned}\|\mathrm{AB}\|_\mathrm{b}&=\|\mathrm{AB}0\|_\infty+\|\mathrm{AB}\|_\mathrm{s}\\&\leq\|\mathrm{A}\|_\mathrm{s}\|\mathrm{B}0\|_\infty+\|\mathrm{A}\|_\mathrm{s}\|\mathrm{B}\|_\mathrm{s}\\&=\|\mathrm{A}\|_\mathrm{s}\|\mathrm{B}\|_\mathrm{b}\\&\leq\|\mathrm{A}\|_\mathrm{b}\|\mathrm{B}\|_\mathrm{b}.\end{aligned}$$

□

6.1. **Additional results.** While we'll predominantly deal with $\|\cdot\|_{\mathrm{b}}$, the Lipschitz norm $\|\cdot\|_{\mathrm{Lip}}$ will also be of use at some point further on due to the following result.

Lemma 6.2. *Consider bounded operators* $\mathrm{A}, \mathrm{B}, \mathrm{C} \in \mathfrak{O}_{\mathrm{b}}$. *If* A *is Lipschitz, then*

$$\|\mathrm{AB} - \mathrm{AC}\|_{\mathrm{b}} \leq \|\mathrm{A}\|_{\mathrm{Lip}} \|\mathrm{B} - \mathrm{C}\|_{\mathrm{b}}.$$

Proof. It suffices to observe that for all $f \in \mathscr{B}$,

$$\|\mathrm{AB}f - \mathrm{AC}f\|_{\infty} \leq \|\mathrm{A}\|_{\mathrm{Lip}} \|\mathrm{B}f - \mathrm{C}f\|_{\infty}. \qquad \square$$

Another link between the set of bounded operators and that of Lipschitz operators is the following: any Lipschitz operator $\mathrm{A} \in \mathfrak{O}_{\mathrm{L}}$ with $\mathrm{A}0 = 0$ is automatically bounded, as clearly

$$\begin{aligned}
\|\mathrm{A}\|_{\mathrm{s}} &= \sup\Big\{\frac{\|\mathrm{A}f\|_{\infty}}{\|f\|_{\infty}} \colon f \in \mathscr{B}, f \neq 0\Big\} \\
&= \sup\Big\{\frac{\|\mathrm{A}f - \mathrm{A}0 + \mathrm{A}0\|_{\infty}}{\|f - 0\|_{\infty}} \colon f \in \mathscr{B}, f \neq 0\Big\} \\
&\leq \sup\Big\{\frac{\|\mathrm{A}f - \mathrm{A}0\|_{\infty}}{\|f - 0\|_{\infty}} \colon f \in \mathscr{B}, f \neq 0\Big\} \\
&\leq \|\mathrm{A}\|_{\mathrm{Lip}}.
\end{aligned}$$

One particular class of operators that map 0 to 0 are the positively homogeneous ones: an operator $\mathrm{A} \in \mathfrak{O}$ is *positively homogeneous* if $\mathrm{A}(\lambda f) = \lambda \mathrm{A}f$ for all $\lambda \in \mathbb{R}_{\geq 0}$ and $f \in \mathscr{B}$. For any positively homogeneous operator $\mathrm{A} \in \mathfrak{O}$ and any $f \in \mathscr{B} \setminus \{0\}$,

$$\frac{1}{\|f\|_{\infty}} \mathrm{A}f = \mathrm{A}\Big(\frac{1}{\|f\|_{\infty}} f\Big) \quad \text{with } \Big\|\frac{1}{\|f\|_{\infty}} f\Big\|_{\infty} = 1;$$

consequently,

$$\|\mathrm{A}\|_{\mathrm{s}} = \sup\Big\{\|\mathrm{A}f\|_{\infty} \colon f \in \mathscr{B}, \|f\|_{\infty} = 1\Big\}; \tag{6.1}$$

since $\mathrm{A}0 = 0$ due to positive homogeneity, it follows from this equality that if A is bounded,

$$\|\mathrm{A}\|_{\mathrm{b}} = \|\mathrm{A}\|_{\mathrm{s}} = \sup\Big\{\|\mathrm{A}f\|_{\infty} \colon f \in \mathscr{B}, \|f\|_{\infty} = 1\Big\}. \tag{6.2}$$

This is in accordance with the operator norm for positively homogeneous operators used in [21, Eqn. (1)] and [8, Eqn. (4)], as well as with the standard norm for linear—additive and homogeneous—operators [29, Section 23.1].

6.2. **Convergence of sequences of sublinear transition operators.** Thanks to Lemma 2.9, the crucial notion of convergence is actually that of convergence for sequences of sublinear transition operators. So consider some sequence $(\overline{\mathrm{T}}_n)_{n \in \mathbb{N}}$ of upper transition operators and some upper transition operator $\overline{\mathrm{T}}$. If the state space $\mathcal{X}$ is finite, uniform convergence (that is, according to $\|\cdot\|_{\mathrm{b}}$) is equal to strong or pointwise convergence (that is, according to $\|\cdot\|_{\infty}$): this sequence converges to $\overline{\mathrm{T}}$ if and only if for all $f \in \mathscr{B}$, $(\overline{\mathrm{T}}_n f)_{n \in \mathbb{N}}$ converges to $\overline{\mathrm{T}}f$ [8, Proposition 3]. When $\mathcal{X}$ is countably infinite, this equivalence between uniform and strong convergence does not necessarily hold; the following is a counterexample.

Example 6.3. Let $\mathcal{X} \coloneqq \mathbb{N}$, and for all $n \in \mathbb{N}$, let

$$\overline{\mathrm{T}}_n \colon \mathscr{B} \to \mathscr{B} \colon f \mapsto \max\{f(k) \colon k \leq n\}.$$

Then for all $f \in \mathscr{B}$, $(\overline{\mathrm{T}}_n f)_{n \in \mathbb{N}}$ converges to $\sup f$. However, $(\overline{\mathrm{T}}_n)_{n \in \mathbb{N}}$ does not converge to

$$\overline{\mathrm{T}} \colon \mathscr{B} \to \mathscr{B} \colon f \mapsto \sup f.$$

Indeed, for all $n \in \mathbb{N}$, let $A_n \coloneqq \{m \in \mathbb{N} \colon m > n\}$, then $\|\mathbb{I}_{A_n}\|_{\infty} = 1$, $\overline{\mathrm{T}}_n \mathbb{I}_{A_n} = 0$ and $\overline{\mathrm{T}}\mathbb{I}_{A_n} = 1$, whence

$$\|\overline{\mathrm{T}}_n - \overline{\mathrm{T}}\|_{\mathrm{b}} \geq \|\overline{\mathrm{T}}_n \mathbb{I}_{A_n} - \overline{\mathrm{T}}\mathbb{I}_{A_n}\|_{\infty} = 1.$$

Because $\overline{\mathrm{T}}_n - \overline{\mathrm{T}}$ is Lipschitz [(T10)] and maps 0 to 0 [(T6)], it follows from the discussion in Section 6.1 that if $(\overline{\mathrm{T}}_n)_{n\in\mathbb{N}}$ converges to $\overline{\mathrm{T}}$ in $(\mathfrak{O}_{\mathrm{L}}, \|\cdot\|_{\mathrm{L}})$, then it also converges to $\overline{\mathrm{T}}$ in $(\mathfrak{O}_{\mathrm{b}}, \|\cdot\|_{\mathrm{b}})$. The following example, for which I'm indebted to Arne Decadt, shows that—quite peculiarly—the converse does not hold.

Example 6.4. Let $\mathcal{X} \coloneqq \{1,2,3\}$. For all $\epsilon \in [0,1]$, let $\overline{\mathrm{T}}_\epsilon$ be the upper transition operator that maps $f \in \mathscr{B}$ to

$$\overline{\mathrm{T}}_\epsilon f \colon \mathcal{X} \to \mathbb{R} \colon x \mapsto \begin{cases} \max\{\epsilon f(1) + (1-\epsilon) f(2), f(3)\} & \text{if } x = 1, \\ f(x) & \text{otherwise.} \end{cases}$$

We set out to show that for all $\epsilon \in]0,1]$,

$$\|\overline{\mathrm{T}}_\epsilon - \overline{\mathrm{T}}_0\|_{\mathrm{L}} = 2 \quad \text{and} \quad \|\overline{\mathrm{T}}_\epsilon - \overline{\mathrm{T}}_0\|_{\mathrm{b}} = 2\epsilon.$$

To obtain the desired counterexample, it then suffices to take $(\overline{\mathrm{T}}_{1/n})_{n\in\mathbb{N}}$ and $\overline{\mathrm{T}}_0$. So fix any $\epsilon \in]0,1]$.

Let us look at the Lipschitz norm first. On the one hand, it follows from (T10) that

$$\|\overline{\mathrm{T}}_\epsilon - \overline{\mathrm{T}}_0\|_{\mathrm{L}} \le \|\overline{\mathrm{T}}_\epsilon\|_{\mathrm{L}} + \|\overline{\mathrm{T}}_0\|_{\mathrm{L}} = 2.$$

On the other hand, consider the functions $f_\epsilon, g_\epsilon \in \mathscr{B}$ given by

$$\big(f_\epsilon(1), f_\epsilon(2), f_\epsilon(3)\big) \coloneqq \Big(1 + \frac{2}{\epsilon}, -1, 1\Big) \quad \text{and} \quad \big(g_\epsilon(1), g_\epsilon(2), g_\epsilon(3)\big) \coloneqq \Big(2 + \frac{2}{\epsilon}, 0, 0\Big).$$

Then $\|f_\epsilon - g_\epsilon\|_\infty = 1$ and

$$\begin{aligned} [\overline{\mathrm{T}}_\epsilon f_\epsilon](1) &= \max\{\epsilon + 2 - (1-\epsilon), 1\} = 1 + 2\epsilon \\ [\overline{\mathrm{T}}_\epsilon g_\epsilon](1) &= \max\{2\epsilon + 2, 0\} = 2\epsilon + 2, \\ [\overline{\mathrm{T}}_0 f_\epsilon](1) &= \max\{-1, 1\} = 1 \\ [\overline{\mathrm{T}}_0 g_\epsilon](1) &= \max\{0, 0\} = 0, \end{aligned}$$

and therefore

$$2 \ge \|\overline{\mathrm{T}}_\epsilon - \overline{\mathrm{T}}_0\|_{\mathrm{L}} \ge \frac{\|(\overline{\mathrm{T}}_\epsilon - \overline{\mathrm{T}}_0) f_\epsilon - (\overline{\mathrm{T}}_\epsilon - \overline{\mathrm{T}}_0) g_\epsilon\|_\infty}{\|f_\epsilon - g_\epsilon\|_\infty} = |1 + 2\epsilon - 2\epsilon - 2 - 1 + 0| = 2,$$

as required.

For $\|\cdot\|_{\mathrm{b}}$, observe that as $\overline{\mathrm{T}}_\epsilon - \overline{\mathrm{T}}_0$ is positively homogeneous [(T1)], it follows from (6.2) that

$$\|\overline{\mathrm{T}}_\epsilon - \overline{\mathrm{T}}_0\|_{\mathrm{b}} = \sup\Big\{\big|[\overline{\mathrm{T}}_\epsilon f](1) - [\overline{\mathrm{T}}_0 f](1)\big| \colon f \in \mathscr{B}, \|f\|_\infty = 1\Big\},$$

So fix any $f \in \mathscr{B}$ with $\|f\|_\infty = 1$. We distinguish four cases.

(1) If $\epsilon f(1) + (1-\epsilon) f(2) \ge f(3)$ and $f(2) \ge f(3)$, then

$$\big|[\overline{\mathrm{T}}_\epsilon f](1) - [\overline{\mathrm{T}}_0 f](1)\big| = |\epsilon f(1) + (1-\epsilon) f(2) - f(2)| = \epsilon |f(1) - f(2)| \le 2\epsilon.$$

(2) If $\epsilon f(1) + (1-\epsilon) f(2) \ge f(3)$ and $f(2) < f(3)$, then

$$\begin{aligned} \big|[\overline{\mathrm{T}}_\epsilon f](1) - [\overline{\mathrm{T}}_0 f](1)\big| &= |\epsilon f(1) + (1-\epsilon) f(2) - f(3)| \\ &= \epsilon f(1) + (1-\epsilon) f(2) - f(3) \\ &= \epsilon (f(1) - f(2)) + f(2) - f(3) \\ &< \epsilon |f(1) - f(2)| \\ &\le 2\epsilon. \end{aligned}$$

(3) If $\epsilon f(1) + (1-\epsilon) f(2) < f(3)$ and $f(2) < f(3)$, then

$$\big|[\overline{\mathrm{T}}_\epsilon f](1) - [\overline{\mathrm{T}}_0 f](1)\big| = |f(3) - f(3)| = 0.$$

(4) If $\epsilon f(1) + (1-\epsilon) f(2) < f(3)$ and $f(2) \geq f(3)$, then

$$\begin{aligned}\big|[\overline{\mathrm{T}}_\epsilon f](1) - [\overline{\mathrm{T}}_0 f](1)\big| &= |f(3) - f(2)| = f(2) - f(3) \\ &< f(2) - \epsilon f(1) - (1-\epsilon) f(2) = \epsilon\Big(f(2) - f(1)\Big) \\ &\leq 2\epsilon.\end{aligned}$$

From this, and with $g \in \mathscr{B}$ such that $(g(1), g(2), g(3)) \coloneqq (1, -1, -1)$, it follows that

$$2\epsilon \geq \|\overline{\mathrm{T}}_\epsilon - \overline{\mathrm{T}}_0\|_{\mathrm{b}} \geq \|\overline{\mathrm{T}}_\epsilon g - \overline{\mathrm{T}}_0 g\|_\infty = |\epsilon - (1-\epsilon) + 1| = 2\epsilon,$$

as required.

7. Appendix: Sublinear rate operators as upper envelopes

For any set $\mathcal{Q}$ of rate operators, its corresponding *pointwise upper envelope*

$$\overline{\mathrm{Q}}_{\mathcal{Q}} \colon \mathscr{B} \to \overline{\mathbb{R}}^{\mathcal{X}}$$

maps any $f \in \mathscr{B}$ to

$$\overline{\mathrm{Q}}_{\mathcal{Q}} f \colon \mathcal{X} \to \mathbb{R} \cup \{+\infty\} \colon x \mapsto [\overline{\mathrm{Q}}_{\mathcal{Q}} f](x) \coloneqq \sup\big\{[\mathrm{Q}f](x) \colon \mathrm{Q} \in \mathcal{Q}\big\}.$$

From this definition, it is easy to see that $\overline{\mathrm{Q}}_{\mathcal{Q}}$ is an operator—that is, that it has $\mathscr{B}$ as codomain—if and only if

$$\sup\Big\{\big|\sup\big\{[\mathrm{Q}f](x) \colon \mathrm{Q} \in \mathcal{Q}\big\}\big| \colon x \in \mathcal{X}\Big\} < +\infty \quad \text{for all } f \in \mathscr{B}. \tag{7.1}$$

Whenever this is the case, $\overline{\mathrm{Q}}_{\mathcal{Q}}$ turns out to be a sublinear rate operator.

Lemma 7.1. *Consider a set $\mathcal{Q}$ of rate operators. Then the corresponding pointwise upper envelope $\overline{\mathrm{Q}}_{\mathcal{Q}}$ is an operator if and only if* (7.1) *holds; whenever this is the case, $\overline{\mathrm{Q}}_{\mathcal{Q}}$ is a sublinear rate operator.*

Proof. The necessity and sufficiency of (7.1) follows immediately from the definition of $\overline{\mathrm{Q}}_{\mathcal{Q}}$. That $\overline{\mathrm{Q}}_{\mathcal{Q}}$ is a sublinear rate operator follows immediately from its definition as a pointwise supremum: $\overline{\mathrm{Q}}_{\mathcal{Q}}$ is sublinear and satisfies (Q3) and (Q4) because every rate operator $\mathrm{Q} \in \mathcal{Q}$ is linear and satisfies (Q3) and (Q4). □

It suffices for (7.1) that $\mathcal{Q}$ is uniformly bounded with respect to the operator seminorm $\|\cdot\|_s$, in the sense that $\sup\{\|\mathrm{Q}\|_{\mathrm{s}} \colon \mathrm{Q} \in \mathcal{Q}\} < +\infty$. In fact, this sufficient condition also ensures that $\overline{\mathrm{Q}}_{\mathcal{Q}}$ is a bounded operator.

Proposition 7.2. *Consider a set $\mathcal{Q}$ of rate operators. Then the corresponding upper envelope $\overline{\mathrm{Q}}_{\mathcal{Q}}$ is a bounded operator if and only if $\mathcal{Q}$ is uniformly bounded with respect to $\|\cdot\|_s$, in which case $\overline{\mathrm{Q}}_{\mathcal{Q}}$ is a sublinear rate operator and*

$$\|\overline{\mathrm{Q}}_{\mathcal{Q}}\|_{\mathrm{b}} = \sup\big\{\|\mathrm{Q}\|_{\mathrm{b}} \colon \mathrm{Q} \in \mathcal{Q}\big\}.$$

Proof. For the sufficiency, assume that $\beta \coloneqq \sup\{\|\mathrm{Q}\|_{\mathrm{s}} \colon \mathrm{Q} \in \mathcal{Q}\} < +\infty$. To use this to our advantage, we observe that for all $f \in \mathscr{B}$ and $\mathrm{Q} \in \mathcal{Q}$,

$$-\beta\|f\|_\infty \leq \|\mathrm{Q}\|_{\mathrm{s}}\|f\|_\infty \leq -\mathrm{Q}(-f) = \mathrm{Q}f \leq \|\mathrm{Q}\|_{\mathrm{s}}\|f\|_\infty \leq \beta\|f\|_\infty.$$

These inequalities imply that (7.1) is satisfied, so we know from Lemma 7.1 that $\overline{\mathrm{Q}}_{\mathcal{Q}}$ is a sublinear rate operator. It now follows from Proposition 2.7, the definition of $\overline{\mathrm{Q}}_{\mathcal{Q}}$, (6.2) and Proposition 2.7 that

$$\begin{aligned}\|\overline{\mathrm{Q}}_{\mathcal{Q}}\|_{\mathrm{s}} &= \sup\big\{[\overline{\mathrm{Q}}_{\mathcal{Q}}(1 - 2\mathbb{I}_x)](x) \colon x \in \mathcal{X}\big\} \\ &= \sup\Big\{\sup\big\{[\mathrm{Q}(1 - 2\mathbb{I}_x)](x) \colon \mathrm{Q} \in \mathcal{Q}\big\} \colon x \in \mathcal{X}\Big\} \\ &= \sup\Big\{\sup\big\{[\mathrm{Q}(1 - 2\mathbb{I}_x)](x) \colon x \in \mathcal{X}\big\} \colon \mathrm{Q} \in \mathcal{Q}\Big\} \\ &= \sup\big\{\|\mathrm{Q}\|_{\mathrm{s}} \colon \mathrm{Q} \in \mathcal{Q}\big\}.\end{aligned}$$

Since by assumption $\mathcal{Q}$ is uniformly bounded with respect to $\|\cdot\|_s$, we infer from these equalities that $\overline{\mathrm{Q}}_{\mathcal{Q}}$ is a bounded operator, as required. As $\overline{\mathrm{Q}}_{\mathcal{Q}}$ is bounded and positively homogeneous, it also follows immediately from this equality and (6.2) (twice) that

$$\|\overline{\mathrm{Q}}_{\mathcal{Q}}\|_{\mathrm{b}} = \|\overline{\mathrm{Q}}_{\mathcal{Q}}\|_{\mathrm{s}} = \sup\big\{\|\mathrm{Q}\|_{\mathrm{s}} \colon \mathrm{Q} \in \mathcal{Q}\big\} = \sup\big\{\|\mathrm{Q}\|_{\mathrm{b}} \colon \mathrm{Q} \in \mathcal{Q}\big\}.$$

For the necessity, suppose that $\overline{\mathrm{Q}}_{\mathcal{Q}}$ is a bounded operator. Then we know from Lemma 7.1 that $\overline{\mathrm{Q}}_{\mathcal{Q}}$ is a sublinear rate operator. Hence, in a reversal of the argument in the first part of this proof, it follows from (6.2), Proposition 2.7, the definition of $\overline{\mathrm{Q}}_{\mathcal{Q}}$ and again Proposition 2.7 that

$$\sup\big\{\|\mathrm{Q}\|_{\mathrm{s}} \colon \mathrm{Q} \in \mathcal{Q}\big\} = \|\overline{\mathrm{Q}}_{\mathcal{Q}}\|_{\mathrm{s}}.$$

Since $\overline{\mathrm{Q}}_{\mathcal{Q}}$ is a bounded operator by assumption, we may conclude from this equality that $\mathcal{Q}$ is uniformly bounded for $\|\cdot\|_s$. □

We can also go the other way around, so from a sublinear rate operator $\overline{\mathrm{Q}}$ to the corresponding set of dominated rate operators

$$\mathcal{Q}_{\overline{\mathrm{Q}}} \coloneqq \big\{\mathrm{Q} \in \mathfrak{Q} \colon (\forall f \in \mathscr{B})\ \mathrm{Q}f \leq \overline{\mathrm{Q}}f\big\},$$

where $\mathfrak{Q}$ denotes the set of all rate operators. The next results establish some properties of this set $\mathcal{Q}_{\overline{\mathrm{Q}}}$.

Definition 7.3. *A set $\mathcal{Q}$ of rate operators is* separately specified *if for any selection $(\mathrm{Q}_x)_{x\in\mathcal{X}}$ in $\mathcal{Q}$, there is a rate operator $\mathrm{Q} \in \mathcal{Q}$ such that $[\mathrm{Q}f](x) = [\mathrm{Q}_x f](x)$ for all $f \in \mathscr{B}$ and $x \in \mathcal{X}$.*

Proposition 7.4. *Consider an upper rate operator $\overline{\mathrm{Q}}$. Then the set $\mathcal{Q}_{\overline{\mathrm{Q}}}$ of dominated rate operators is non-empty, convex and separately specified.*

Proof. That $\mathcal{Q}_{\overline{\mathrm{Q}}}$ is non-empty follows almost immediately from the Hahn–Banach Theorem—see for example [3, Theorem 1.1] or [29, Theorem 12.31.(HB3)]. To see why, recall that $\mathscr{B}$ is a real vector space, and observe that the set $\mathcal{C} \subseteq \mathscr{B}$ of constant functions is a linear subspace of $\mathscr{B}$ and that $q \colon \mathcal{C} \to \mathbb{R} \colon \mu \mapsto 0$ is a linear functional on $\mathcal{C}$. For all $x \in \mathcal{X}$, the component functional $p_x \colon \mathscr{B} \to \mathbb{R} \colon f \mapsto [\overline{\mathrm{Q}}f](x)$ is sublinear and dominates q, so by the Hahn–Banach Theorem there is a linear functional q_x on $\mathscr{B}$ that extends q and is dominated by p_x, whence

$$-[\overline{\mathrm{Q}}(-f)](x) \leq -q_x(-f) = q_x(f) \leq [\overline{\mathrm{Q}}f](x). \tag{7.2}$$

Consider now the operator $\mathrm{Q} \colon \mathscr{B} \to \mathscr{B}$ defined by

$$[\mathrm{Q}f](x) \coloneqq q_x(f) \quad \text{for all } f \in \mathscr{B}, x \in \mathcal{X};$$

since $\overline{\mathrm{Q}}f, -\overline{\mathrm{Q}}(-f) \in \mathscr{B}$, (7.2) ensures that $\mathrm{Q}f \in \mathscr{B}$. It is now clear that by construction, Q is a linear operator that maps constant functions $\mu \in \mathcal{C}$ to 0 and satisfies the positive maximum principle [as it is dominated by $\overline{\mathrm{Q}}$]. In other words, $\mathrm{Q} \in \mathcal{Q}_{\overline{\mathrm{Q}}}$, so $\mathcal{Q}_{\overline{\mathrm{Q}}}$ is indeed non-empty.

To see that $\mathcal{Q}_{\overline{\mathrm{Q}}}$ is convex, it suffices to realise that (i) the convex combination of two rate operators is again a rate operator, and (ii) if two rate operators are dominated by $\overline{\mathrm{Q}}$, then so is their convex combination. To see that $\mathcal{Q}_{\overline{\mathrm{Q}}}$ is separately specified, it suffices to realise that all requirements on rate operators and the requirement of domination are pointwise for $x \in \mathcal{X}$. □

Lemma 7.5. *Consider a sublinear rate operator $\overline{\mathrm{Q}}$. Then*

$$\sup\big\{\|\mathrm{Q}\|_{\mathrm{s}} \colon \mathrm{Q} \in \mathcal{Q}_{\overline{\mathrm{Q}}}\big\} = \|\overline{\mathrm{Q}}\|_{\mathrm{s}},$$

so $\overline{\mathrm{Q}}$ is a bounded operator if and only if $\mathcal{Q}_{\overline{\mathrm{Q}}}$ is uniformly bounded. Whenever this is the case, $\mathcal{Q}_{\overline{\mathrm{Q}}}$ is closed with respect to $\|\cdot\|_{\mathrm{b}}$.

Proof. The first part of the statement follows almost immediately Proposition 2.7 (twice):

$$\begin{aligned}
\sup\big\{\|\mathrm{Q}\|_{\mathrm{s}} \colon \mathrm{Q} \in \mathcal{Q}_{\overline{\mathrm{Q}}}\big\} &= \sup\big\{[\mathrm{Q}(1 - 2\mathbb{I}_x)](x) \colon \mathrm{Q} \in \mathcal{Q}_{\overline{\mathrm{Q}}}, x \in \mathcal{X}\big\} \\
&= \sup\big\{[\overline{\mathrm{Q}}(1 - 2\mathbb{I}_x)](x) \colon x \in \mathcal{X}\big\} \\
&= \|\overline{\mathrm{Q}}\|_{\mathrm{s}}.
\end{aligned}$$

In the remainder of this proof, we show that $\mathcal{Q}_{\overline{\mathrm{Q}}}$ is closed in $(\mathfrak{O}_{\mathrm{b}}, \|\cdot\|_{\mathrm{b}})$. So we fix any sequence $(\mathrm{Q}_n)_{n\in\mathbb{N}}$ that converges to some $\mathrm{A} \in \mathfrak{O}_{\mathrm{b}}$, in the sense that $\lim_{n\to+\infty}\|\mathrm{A}-\mathrm{Q}_n\|_{\mathrm{b}} = 0$, and set out to show that $\mathrm{A} \in \mathcal{Q}_{\overline{\mathrm{Q}}}$. Fix any $f \in \mathscr{B}$ and $x \in \mathcal{X}$, and observe that because $\mathcal{Q}_{\overline{\mathrm{Q}}}$ is uniformly bounded, so is $([\mathrm{Q}_n f](x))_{n\in\mathbb{N}}$ because for all $n \in \mathbb{N}$,

$$\big|[\mathrm{Q}_n f](x)\big| \le \|\mathrm{Q}_n f\|_\infty \le \|\mathrm{Q}_n\|_{\mathrm{s}}\|f\|_\infty \le \sup\{\|\mathrm{Q}_m\|_{\mathrm{s}} \colon m \in \mathbb{N}\}\|f\|_\infty.$$

Furthermore, the assumption that $\lim_{n\to+\infty}\|\mathrm{A}-\mathrm{Q}_n\|_{\mathrm{b}} = 0$ implies that

$$0 \le \lim_{n\to+\infty}\big|[\mathrm{A}f](x) - [\mathrm{Q}_n f](x)\big| \le \lim_{n\to+\infty}\|\mathrm{A}f - \mathrm{Q}_n f\|_\infty \le \lim_{n\to\infty}\|\mathrm{A}-\mathrm{Q}_n\|_{\mathrm{b}}\|f\|_\infty = 0.$$

From this, we conclude that

$$[\mathrm{A}f](x) = \lim_{n\to+\infty}[\mathrm{Q}_n f](x) \quad \text{for all } f \in \mathscr{B}, x \in \mathcal{X}.$$

Because every Q_n is a rate operator, we infer from this realisation that (i) A is linear, (ii) A maps constant functions to 0 [(Q3)], and (iii) A satisfies the positive maximum principle [(Q4)]; consequently, A is a rate operator. Since every Q_n iominated by $\overline{\mathrm{Q}}$, it also follows from the equality above that the rate operator A is dominated by $\overline{\mathrm{Q}}$, or equivalently, belongs to $\mathcal{Q}_{\overline{\mathrm{Q}}}$. ☐

8. Appendix: Proofs for results in Section 3.1

This appendix contains the proofs for the two intermediary lemmas which we rely on in our proof for Theorem 3.1.

Proof of Lemma 3.2. Our proof will be one by induction, and basically repeats the one given by Krak et al. [21, Proof for Lemma E.4]. For the induction base $n = 1$, the inequality in the statement is trivial. For the inductive step, we assume that the inequality in the statement holds for $n = \ell$, and set out to verify that it then also holds for $n = \ell + 1$. To this end, observe that

$$\begin{aligned}
&\Big\|\overline{\mathrm{T}}_1\cdots\overline{\mathrm{T}}_{\ell+1} - \overline{\mathrm{S}}_1\cdots\overline{\mathrm{S}}_{\ell+1}\Big\|_{\mathrm{b}} \\
&\le \Big\|\overline{\mathrm{T}}_1\cdots\overline{\mathrm{T}}_\ell\overline{\mathrm{T}}_{\ell+1} - \overline{\mathrm{T}}_1\cdots\overline{\mathrm{T}}_\ell\overline{\mathrm{S}}_{\ell+1}\Big\|_{\mathrm{b}} + \Big\|\overline{\mathrm{T}}_1\cdots\overline{\mathrm{T}}_\ell\overline{\mathrm{S}}_{\ell+1} - \overline{\mathrm{S}}_1\cdots\overline{\mathrm{S}}_\ell\overline{\mathrm{S}}_{\ell+1}\Big\|_{\mathrm{b}}.
\end{aligned}$$

For the first term, $\overline{\mathrm{T}}_1\cdots\overline{\mathrm{T}}_\ell$ is a sublinear transition operator and $\overline{\mathrm{T}}_{\ell+1}$ and $\overline{\mathrm{S}}_{\ell+1}$ are bounded operators, so it follows from (T12) that

$$\Big\|\overline{\mathrm{T}}_1\cdots\overline{\mathrm{T}}_\ell\overline{\mathrm{T}}_{\ell+1} - \overline{\mathrm{T}}_1\cdots\overline{\mathrm{T}}_\ell\overline{\mathrm{S}}_{\ell+1}\Big\|_{\mathrm{b}} \le \Big\|\overline{\mathrm{T}}_{\ell+1} - \overline{\mathrm{S}}_{\ell+1}\Big\|_{\mathrm{b}}.$$

To bound the second term, we use (2.2) (with $\mathrm{A} = \overline{\mathrm{T}}_1\cdots\overline{\mathrm{T}}_\ell - \overline{\mathrm{S}}_1\cdots\overline{\mathrm{S}}_\ell$ and $\mathrm{B} = \overline{\mathrm{S}}_{\ell+1}$) and (T11) and invoke the induction hypothesis:

$$\begin{aligned}
\Big\|\overline{\mathrm{T}}_1\cdots\overline{\mathrm{T}}_\ell\overline{\mathrm{S}}_{\ell+1} - \overline{\mathrm{S}}_1\cdots\overline{\mathrm{S}}_\ell\overline{\mathrm{S}}_{\ell+1}\Big\|_{\mathrm{b}} &\le \Big\|\overline{\mathrm{T}}_1\cdots\overline{\mathrm{T}}_\ell - \overline{\mathrm{S}}_1\cdots\overline{\mathrm{S}}_\ell\Big\|_{\mathrm{b}}\Big\|\overline{\mathrm{S}}_{\ell+1}\Big\|_{\mathrm{b}} \\
&\le \Big\|\overline{\mathrm{T}}_1\cdots\overline{\mathrm{T}}_\ell - \overline{\mathrm{S}}_1\cdots\overline{\mathrm{S}}_\ell\Big\|_{\mathrm{b}} \\
&\le \sum_{k=1}^{\ell}\Big\|\overline{\mathrm{T}}_k - \overline{\mathrm{S}}_k\Big\|_{\mathrm{b}}.
\end{aligned}$$

From all this we infer that

$$\Big\|\overline{\mathrm{T}}_1\cdots\overline{\mathrm{T}}_{\ell+1} - \overline{\mathrm{S}}_1\cdots\overline{\mathrm{S}}_{\ell+1}\Big\|_{\mathrm{b}} \le \sum_{k=1}^{\ell+1}\Big\|\overline{\mathrm{T}}_k - \overline{\mathrm{S}}_k\Big\|_{\mathrm{b}},$$

which is precisely the inequality in the statement for $n = \ell + 1$. ☐

Proof of Lemma 3.3. Our proof follows that of Krak et al. [21, Proof for Lemma E.5] closely, so it will be one by induction over ℓ. The statement holds trivially for the induction base $\ell = 1$. For the inductive step, we assume that the inequality in the statement holds for some $\ell = k$ and all $\Delta \in \mathbb{R}_{>0}$ such that $\Delta\|\overline{\mathrm{Q}}\|_{\mathrm{b}} \leq 2$, and set out to verify this inequality for $\ell = k+1$ and some $\Delta \in \mathbb{R}_{>0}$ such that $\Delta\|\overline{\mathrm{Q}}\|_{\mathrm{b}} \leq 2$. Then with $\delta := \Delta/(k+1)$,

$$\Big(\mathrm{I} + \delta\overline{\mathrm{Q}}\Big)^{k+1} - (\mathrm{I} + (k+1)\delta\overline{\mathrm{Q}}) = \Big(\mathrm{I} + \delta\overline{\mathrm{Q}}\Big)^{k} + \delta\overline{\mathrm{Q}}\Big(\mathrm{I} + \delta\overline{\mathrm{Q}}\Big)^{k} - \Big(\mathrm{I} + k\delta\overline{\mathrm{Q}}\Big) - \delta\overline{\mathrm{Q}}.$$

It follows from this and the induction hypothesis that

$$\begin{aligned}\Big\|(\mathrm{I} + \delta\overline{\mathrm{Q}})^{k+1} - (\mathrm{I} + (k+1)\delta\overline{\mathrm{Q}})\Big\|_{\mathrm{b}} &\leq \Big\|(\mathrm{I} + \delta\overline{\mathrm{Q}})^{k} - (\mathrm{I} + k\delta\overline{\mathrm{Q}})\Big\|_{\mathrm{b}} + \delta\Big\|\overline{\mathrm{Q}}(\mathrm{I} + \delta\overline{\mathrm{Q}})^{k} - \overline{\mathrm{Q}}\Big\|_{\mathrm{b}} \\ &\leq k^2\delta^2\|\overline{\mathrm{Q}}\|_{\mathrm{b}}^2 + \delta\Big\|\overline{\mathrm{Q}}(\mathrm{I} + \delta\overline{\mathrm{Q}})^{k} - \overline{\mathrm{Q}}\Big\|_{\mathrm{b}}.\end{aligned}$$

Next, we note that $\overline{\mathrm{Q}} = \overline{\mathrm{Q}}\mathrm{I}^k$, invoke Proposition 2.10 (with $\mathrm{A} = (\mathrm{I} + \delta\overline{\mathrm{Q}})^k$ and $\mathrm{B} = \mathrm{I}^k$) and then Lemma 3.2 (with $\overline{\mathrm{T}}_k = (I + \delta\overline{\mathrm{Q}})$ and $\overline{\mathrm{S}}_k = \mathrm{I}$), to yield

$$\begin{aligned}\Big\|\Big(\mathrm{I} + \delta\overline{\mathrm{Q}}\Big)^{k+1} - (\mathrm{I} + (k+1)\delta\overline{\mathrm{Q}})\Big\|_{\mathrm{b}} &\leq k^2\delta^2\|\overline{\mathrm{Q}}\|_{\mathrm{b}}^2 + \delta\Big\|\overline{\mathrm{Q}}\Big\|_{\mathrm{b}}\Big\|(\mathrm{I} + \delta\overline{\mathrm{Q}})^{k} - \mathrm{I}^k\Big\|_{\mathrm{b}} \\ &\leq k^2\delta^2\|\overline{\mathrm{Q}}\|_{\mathrm{b}}^2 + k\delta\Big\|\overline{\mathrm{Q}}\Big\|_{\mathrm{b}}\Big\|\mathrm{I} + \delta\overline{\mathrm{Q}} - \mathrm{I}\Big\|_{\mathrm{b}} \\ &= k^2\delta^2\|\overline{\mathrm{Q}}\|_{\mathrm{b}}^2 + k\delta^2\|\overline{\mathrm{Q}}\|_{\mathrm{b}}^2.\end{aligned}$$

Since $k^2 + k \leq (k+1)^2$, it follows from this that indeed

$$\Big\|\Big(\mathrm{I} + \delta\overline{\mathrm{Q}}\Big)^{k+1} - (\mathrm{I} + (k+1)\delta\overline{\mathrm{Q}})\Big\|_{\mathrm{b}} \leq (k+1)^2\delta^2\|\overline{\mathrm{Q}}\|_{\mathrm{b}} = \Delta^2\Big\|\overline{\mathrm{Q}}\Big\|_{\mathrm{b}}.$$ □

9. Appendix: Proof of Proposition 4.1

Proposition 4.1 generalises Lemma 3.100 in my doctoral dissertation [13] from the setting of finite $\mathcal{X}$ to that of countable $\mathcal{X}$. The proof that we are about to go through is a rather straightforward generalisation of the proof of the aforementioned result, with some minor modifications.

Proof of Proposition 4.1. The equality in the statement follows immediately from Lemma 2.8, (6.1) and Proposition 2.7.

Because of Lemma 2.5, the inequality in the statement clearly implies that $(\overline{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}}$ is uniformly continuous. The proof of the converse implication—so starting from uniform continuity—is more involved; in fact, our proof will be one by contrapositive: we assume that

$$\limsup_{t\searrow 0}\Big\|\frac{\overline{\mathrm{T}}_t - \mathrm{I}}{t}\Big\|_{\mathrm{b}} = +\infty, \tag{9.1}$$

and set out to prove that then $(\overline{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}}$ is not uniformly continuous, which due to (T11) and Lemma 2.5 means that

$$\limsup_{t\searrow 0}\|\overline{\mathrm{T}}_t - \mathrm{I}\|_{\mathrm{b}} > 0,$$

or more formally, that

$$(\exists\epsilon \in \mathbb{R}_{>0})(\forall\delta \in \mathbb{R}_{>0})(\exists t \in]0,\delta[)\ \Big\|\overline{\mathrm{T}}_t - \mathrm{I}\Big\|_{\mathrm{b}} \geq \epsilon. \tag{9.2}$$

We fix some $\epsilon \in]0,1[$, some $\epsilon_1 \in]0, 1-\epsilon[$ and some arbitrary $\delta \in \mathbb{R}_{>0}$. Since $\lim_{\alpha\to+\infty} e^{-\alpha} = 0$ and $0 < 1-\epsilon-\epsilon_1$ by construction, we can moreover pick some $\lambda \in \mathbb{R}_{>0}$ such that $e^{-\lambda\delta} < 1-\epsilon-\epsilon_1$. There is some $N_{\epsilon_1} \in \mathbb{N}$ such that

$$\Big|e^{-\lambda\delta} - \Big(1 - \frac{\lambda\delta}{n+1}\Big)^n\Big| < \epsilon_1 \quad \text{for all } n \geq N_{\epsilon_1}. \tag{9.3}$$

Let us use our contrapositive assumption: it follows from (9.1) that there is some $\Delta \in]0, \min\{1/\lambda, \delta/N_{\epsilon_1}\}[$ such that $\lambda\Delta \leq \|\overline{\mathrm{T}}_\Delta - \mathrm{I}\|_{\mathrm{b}}$. With n the unique natural number such that $n\Delta < \delta \leq (n+1)\Delta$, our restrictions on Δ guarantee that $n \geq N_{\epsilon_1}$ and $\lambda\Delta < 1$.

Let $\beta := \|\overline{\mathrm{T}}_\Delta - \mathrm{I}\|_{\mathrm{b}}/2$. If $\beta \geq \epsilon/2$, then we have clearly verified (9.2) because δ was arbitrary, $\Delta \in]0,\delta[$ by construction and $\|\overline{\mathrm{T}}_\Delta - \mathrm{I}\|_{\mathrm{b}} = 2\beta \geq \epsilon$.

The case $\beta < \epsilon/2 < 1/2$ is quite more involved. Since $\lambda\Delta \leq 2\beta < 1$ by construction,

$$1 - \lambda\Delta \geq 1 - 2\beta \Rightarrow (1-\lambda\Delta)^n \geq (1-2\beta)^n \Rightarrow 1 - (1-\lambda\Delta)^n \leq 1 - (1-2\beta)^n; \tag{9.4}$$

similarly, because $0 \leq \frac{\lambda\delta}{n+1} \leq \lambda\Delta < 1$,

$$1 - \Big(1 - \frac{\lambda\delta}{n+1}\Big)^n \leq 1 - \Big(1 - \lambda\Delta\Big)^n. \tag{9.5}$$

To continue, we fix an arbitrary $\epsilon_2 \in \mathbb{R}_{>0}$ such that $\beta - \epsilon_2 > 0$; then since $\overline{\mathrm{T}}_\Delta - \mathrm{I}$ is a bounded sublinear rate operator [Lemma 2.8], it follows from (6.2) and Proposition 2.7 that there is some $x \in \mathcal{X}$ such that

$$\beta - \epsilon_2 < \big[\overline{\mathrm{T}}_\Delta(1 - \mathbb{I}_x)\big](x) \leq \beta \tag{9.6}$$

and for all other $y \in \mathcal{X} \setminus \{x\}$,

$$\big[\overline{\mathrm{T}}_\Delta(1 - \mathbb{I}_y)\big](y) \leq \beta. \tag{9.7}$$

It follows from (T7), (T4), (T5) and (9.7) that for all other $y \in \mathcal{X} \setminus \{x\}$,

$$\big[\overline{\mathrm{T}}_\Delta(1 - \mathbb{I}_x)\big](y) \geq -\big[\overline{\mathrm{T}}_\Delta(-1 + \mathbb{I}_x)\big](y) \geq -\big[\overline{\mathrm{T}}_\Delta(-\mathbb{I}_y)\big](y) = 1 - \big[\overline{\mathrm{T}}_\Delta(1 - \mathbb{I}_y)\big](y) \geq 1 - \beta.$$

Thus, we have shown that

$$\overline{\mathrm{T}}_\Delta(1 - \mathbb{I}_x) \geq \beta - \epsilon_2 + (1 - 2\beta)(1 - \mathbb{I}_x).$$

It follows from the semigroup property (SG1) of $(\overline{\mathrm{T}}_s)_{s \in \mathbb{R}_{\geq 0}}$, the previous inequality, some properties of $\overline{\mathrm{T}}_\Delta$—in particular (T4), (T5) and (T1) (which we may invoke because $\beta < 1/2$ whence $1 - 2\beta \geq 0$)—and again the previous inequality that

$$\begin{aligned}
\overline{\mathrm{T}}_{2\Delta}(1 - \mathbb{I}_x) &= \overline{\mathrm{T}}_\Delta \overline{\mathrm{T}}_\Delta(1 - \mathbb{I}_x) \\
&\geq \overline{\mathrm{T}}_\Delta\big(\beta - \epsilon_2 + (1 - 2\beta)(1 - \mathbb{I}_x)\big) \\
&= \beta - \epsilon_2 + (1 - 2\beta)\overline{\mathrm{T}}_\Delta(1 - \mathbb{I}_x) \\
&\geq \beta - \epsilon_2 + (1 - 2\beta)\Big(\beta - \epsilon_2 + (1 - 2\beta)(1 - \mathbb{I}_x)\Big) \\
&= \Big(\beta - \epsilon_2\Big)\big(1 + (1 - 2\beta)\big) + (1 - 2\beta)^2(1 - \mathbb{I}_x).
\end{aligned}$$

We apply this same trick $n - 2$ times more, to yield

$$\overline{\mathrm{T}}_{n\Delta}(1 - \mathbb{I}_x) \geq \Big(\beta - \epsilon_2\Big)\Big(\sum_{k=0}^{n-1}(1 - 2\beta)^k\Big) + (1 - 2\beta)^n(1 - \mathbb{I}_x).$$

Evaluating the functions on both sides of the equality in x and using the well-known expression for the partial sum of a geometric series, we find that

$$\big[\overline{\mathrm{T}}_{n\Delta}(1 - \mathbb{I}_x)\big](x) \geq \Big(\beta - \epsilon_2\Big)\frac{1 - (1 - 2\beta)^n}{1 - (1 - 2\beta)} = \frac{\beta - \epsilon_2}{2\beta}(1 - (1 - 2\beta)^n).$$

Since $\beta - \epsilon_2 > 0$, it follows from this, (9.4) and (9.5) that

$$\big[\overline{\mathrm{T}}_{n\Delta}(1 - \mathbb{I}_x)\big](x) \geq \frac{\beta - \epsilon_2}{2\beta}\Big(1 - \Big(1 - \frac{\lambda\delta}{n+1}\Big)^n\Big);$$

since $n \geq N_{\epsilon_1}$ by construction, we can also invoke (9.3), to yield

$$\big[\overline{\mathrm{T}}_{n\Delta}(1 - \mathbb{I}_x)\big](x) \geq \frac{\beta - \epsilon_2}{2\beta}\Big(1 - e^{-\lambda\delta} - \epsilon_1\Big) > \frac{\beta - \epsilon_2}{2\beta}\epsilon,$$

where for the second inequality we used that $e^{-\lambda\delta} < 1 - \epsilon - \epsilon_1$. Since this inequality holds for arbitrarily small ϵ_2, we may infer from it that

$$\big[\overline{\mathrm{T}}_{n\Delta}(1 - \mathbb{I}_x)\big](x) \geq \frac{1}{2}\epsilon.$$

Because $\overline{\mathrm{T}}_{n\Delta} - \mathrm{I}$ is a bounded sublinear rate operator, we conclude from this, Lemma 2.8 and Proposition 2.7 that

$$\Big\|\overline{\mathrm{T}}_{n\Delta} - \mathrm{I}\Big\|_{\mathrm{b}} \geq \epsilon.$$

Since $\delta \in \mathbb{R}_{>0}$ was arbitrary and we've ensured that $n\Delta \in]0, \delta[$, we've verified (9.2). $\square$

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References

[1] Viorel Barbu; *Nonlinear semigroups and differential equations in Banach spaces*, Noordhoff International Publishing, Leyderreygers.texen, 1976.

[2] Jonas Blessing, Robert Denk, Michael Kupper, Max Nendel; *Convex monotone semigroups and their generators with respect to* Γ*-convergence*, Journal of Functional Analysis **288** (2025), no. 8, 110841.

[3] Haim Brezis; *Functional analysis, sobolev spaces and partial differential equations*, Springer, New York, 2011.

[4] Paul R. Chernoff; *Product formulas, nonlinear semigroups, and addition of unbounded operators*, Memoirs of the American Mathematical Society, no. 140, American Mathematical Society, Providence, Rhode Island, 1974.

[5] Gert de Cooman Filip Hermans; *Imprecise probability trees: Bridging two theories of imprecise probability*, Artificial Intelligence **172** (2008), no. 11, 1400–1427.

[6] Philippe Courrège; *Sur la forme intégro-différentielle des opérateurs de c_k^∞ dans c satisfaisant au principe du maximum*, Séminaire Brelot-Choquet-Deny (Théorie du Potentiel) **10** (1965/66), no. 1.

[7] M. G. Crandall and T. M. Liggett; *Generation of semi-groups of nonlinear transformations on general banach spaces*, American Journal of Mathematics **93** (1971), no. 2, 265–298.

[8] Jasper De Bock; *The limit behaviour of imprecise continuous-time Markov chains*, Journal of Nonlinear Science **27** (2017), no. 1, 159–196.

[9] Robert Denk, Michael Kupper, Max Nendel; *Kolmogorov-type and general extension results for nonlinear expectations*, Banach Journal of Mathematical Analysis **12** (2018), no. 3, 515–540.

[10] Robert Denk, Michael Kupper, Max Nendel; *Convex semigroups on l^p-like spaces*, Journal of Evolution Equations **21** (2021), 2491–2521.

[11] Nelson Dunford Jacob T. Schwartz; *Linear operators*, Interscience Publishers, New York, 1958.

[12] Klaus-Jochen Engel, Rainer Nagel; *One-parameter semigroups for linear evolution equations*, Springer, New York, 2000.

[13] Alexander Erreygers; *Markovian imprecise jump processes: Foundations, algorithms and applications*, Ph.D. thesis, Ghent University, 2021, Available at https://users.ugent.be/ aerreyge/.

[14] Alexander Erreygers; *Convex expectations for countable-state uncertain processes with càdlàg sample paths*, International Journal of Approximate Reasoning **175** (2024).

[15] Alexander Erreygers, Jasper De Bock; *Imprecise continuous-time Markov chains: Efficient computational methods with guaranteed error bounds*, Proceedings of the Tenth International Symposium on Imprecise Probability: Theories and Applications (Lugano, Switzerland), Proceedings of Machine Learning Research, vol. 62, PMLR, 2017, pp. 145–156.

[16] Alexander Erreygers, Cristina Rottondi, Giacomo Verticale, Jasper De Bock; *Imprecise Markov models for scalable and robust performance evaluation of flexi-grid spectrum aladdress policies*, **66** (2018), no. 11, 5401–5414.

[17] Stewart N. Ethier, Thomas G. Kurtz; *Markov processes*, Wiley, Hoboken, New Jersey, 1986.

[18] Marián Fabian, Petr Habala, Petr Hájek, Vicente Montesinos, Václav Zizler; *Banach space theory*, Springer, New York, 2011.

[19] Ben Goldys, Max Nendel, Michael Röckner; *Operator semigroups in the mixed topology and the infinitesimal description of markov processes*, Journal of Differential Equations **412** (2024), 23–86.

[20] Einar Hille, Ralph S. Phillips; *Functional analysis and semi-groups*, American Mathematical Society, Providence, Rhode Island, 1957.

[21] Thomas Krak, Jasper De Bock, Arno Siebes; *Imprecise continuous-time Markov chains*, International Journal of Approximate Reasoning **88** (2017), 452–528.

[22] Robert H. Martin; *Nonlinear operators and differential equations in banach spaces*, Wiley, New York, 1976.

[23] Isao Miyadera; *Nonlinear semigroups*, American Mathematical Society, Providence, Rhode Island, 1992.

[24] Max Nendel; *Markov chains under nonlinear expectation*, Mathematical Finance **31** (2020), no. 1, 474–507.

[25] Max Nendel; *On nonlinear expectations and Markov chains under model uncertainty*, International Journal of Approximate Reasoning **130** (2021), 226–245.

[26] Amnon Pazy; *Semigroups of linear operators and applications to partial differential equations*, Springer, New York, 1983.

[27] Shige Peng; *Nonlinear expectations and stochastic calculus under uncertainty*, Springer, Berlin, 2019.

[28] L. Chris G. Rogers, David Williams; *Diffusions, markov processes and martingales*, 2 ed., vol. 1, Wiley, Chichester, 1994.

[29] Eric Schechter; *Handbook of analysis and its foundations*, Academic Press, San Diego, 1997.

[30] Matthias Troffaes, Thomas Krak, Henna Bains; *Two-state imprecise Markov chains for statistical modelling of two-state non-Markovian processes*, Proceedings of the Eleventh International Symposium on Imprecise

Probabilities: Theories and Applications (Ghent, Belgium), Proceedings of Machine Learning Research, vol. 103, PMLR, 2019, pp. 394–403.
[31] Matthias C. M. Troffaes; *A nonstandard approach to stochastic processes under probability bounding*, Proceedings of the Thirteenth International Symposium on Imprecise Probability: Theories and Applications, Proceedings of Machine Learning Research, vol. 215, PMLR, 2023, pp. 450–460.
[32] Matthias C. M. Troffaes, Gert de Cooman; *Lower previsions*, Wiley, Chichester, 2014.
[33] Damjan Škulj; *Efficient computation of the bounds of continuous time imprecise Markov chains*, Applied Mathematics and Computation **250** (2015), 165–180.
[34] Peter Walley; *Statistical reasoning with imprecise probabilities*, Chapman and Hall, London, 1991.
[35] Peter Whittle; *Probability via expectation*, 4 ed., Springer, New York, 2000.
[36] Kôsaku Yosida; *Functional analysis*, 6 ed., Springer, Berlin, 1995, Reprint of the 1980 Edition.

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