Electronic Journal of Differential Equations, Vol. 2025 (2025), No. 13, pp. 1–17.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2025.13

EXISTENCE OF SOLUTIONS TO FRACTIONAL P-LAPLACIAN PROBLEMS WITH ROBIN BOUNDARY CONDITIONS

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Abstract. This article studies the existence of solutions for the fractional p-Laplacian problem

$$\begin{gathered}(-\Delta)_p^s u = \lambda |u|^{q-2}u + \frac{|u|^{r-2}u}{|x|^\alpha}, \quad \text{in } \Omega,\\ N_{s,p}u(x) + \beta(x)|u|^{p-2}u = 0, \quad \text{in } \mathbb{R}^n \backslash \Omega,\end{gathered}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$ containing 0 with smooth boundary, $(-\Delta)_p^s$ denotes the fractional p-Laplace operator and $\lambda > 0$, $1 < q < p < r < p_\alpha^*$, p_α^* is the fractional critical Hardy-Sobolev exponent for $0 \le \alpha < ps < n$ and $0 < s < 1$. By using fibering maps and Nehari manifold, we obtain the existence of solution for Hardy-Sobolev subcritical and critical cases.

1. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^n$ containing 0 with smooth boundary. We consider the fractional p-Laplacian Robin problem

$$\begin{gathered}(-\Delta)_p^s u = \lambda |u|^{q-2}u + \frac{|u|^{r-2}u}{|x|^\alpha}, \quad \text{in } \Omega,\\ N_{s,p}u(x) + \beta(x)|u|^{p-2}u = 0, \quad \text{in } \mathbb{R}^n \backslash \Omega,\end{gathered} \tag{1.1}$$

where λ is a positive parameter, $0 < s < 1$, $0 \le \alpha < ps < n$, $1 < q < p < r < p_\alpha^*$ and p_α^* is the fractional critical Hardy-Sobolev exponent. The fractional p-Laplace operator $(-\Delta)_p^s$ is defined by

$$(-\Delta)_p^s u(x) = c_{n,s,p}\text{P.V.} \int_{\mathbb{R}^n} \frac{|u(y)-u(x)|^{p-2}(u(y)-u(x))}{|x-y|^{n+ps}}\,dy,$$

where $c_{n,s,p}$ is a suitable positive normalization constant only depending on n, s and p, while

$$N_{s,p}u(x) = c_{n,s,p} \int_\Omega \frac{|u(y)-u(x)|^{p-2}(u(y)-u(x))}{|x-y|^{n+ps}}\,dy$$

is the nonlocal normal derivative associated to $(-\Delta)_p^s$, see [6, 14] and [8] for its introduction in the case $p = 2$. Besides, $\beta(x)$ is a given nonnegative function. We

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2020 *Mathematics Subject Classification.* 35R11, 35S15, 35A15, 47G20.
Key words and phrases. Fractional p-Laplacian; Nehari manifold; Robin boundary; Hardy-Sobolev exponent.

Submitted September 28, 2024. Published February 18, 2025.

would like to point out that the Neumann operator $N_{s,2}u(x)$ recovers the classical Neumann condition as a limit case, and has a clear probabilistic and variational interpretation a well, see [8] for the details.

Recently, partial differential equations involving the fractional Laplacian operator $(-\Delta)^s$ with $s \in (0,1)$ has received a special attention, because its arises in a quite natural way in many different contexts, such as, among the others, the thin obstacle problem, optimization, anomalous diffusion, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, minimal surfaces, materials science and water waves, for more detail see [7]. In the framework of nonlocal problems, the following Brezis-Nirenberg type problem for the fractional p-Laplacian is considered

$$\begin{gathered}(-\Delta)_p^s u = \lambda |u|^{p-2}u + |u|^{p_s^*-2}u, \quad \text{in } \Omega,\\ u = 0, \quad \text{in } \mathbb{R}^n \backslash \Omega,\end{gathered} \tag{1.2}$$

where $s \in (0,1)$, $n > sp$, $\lambda > 0$ and $p_s^* = \frac{np}{n-sp}$ is the fractional critical Sobolev exponent. In [12] the authors proved, among other results, that the above problem has a nontrivial weak solution for all $\lambda > 0$ provided that $\frac{n^3+s^3p^3}{n(n+s)} > sp^2$ and Ω is the domain of class $C^{1,1}$.

The fractional p-Laplace elliptic problems with Hardy term have also been studied by many researchers. Chen-Mosconi-Squassina [5] studied the problem

$$\begin{gathered}(-\Delta)_p^s u = \lambda |u|^{q-2}u + \frac{|u|^{p_\alpha^*-2}u}{|x|^\alpha}, \quad \text{in } \Omega,\\ u = 0, \quad \text{in } \mathbb{R}^n \backslash \Omega,\end{gathered} \tag{1.3}$$

where $p \le q < \frac{np}{n-ps}$. By finding the minimizer of the corresponding energy functional on positive Nehari and sigh-changing Nehari sets, the existence of positive and sigh-changing least energy solutions for the above problem were established in [5].

Chen-Gui [4] studied the existence of multiple solutions for the fractional p-Kirchhoff problem

$$\begin{gathered}M\Big(\int_{R^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy\Big)(-\Delta)_p^s u = \lambda |u|^{q-2}u + \frac{|u|^{r-2}u}{|x|^\alpha}, \quad \text{in } \Omega,\\ u = 0, \quad \text{in } \mathbb{R}^n \backslash \Omega.\end{gathered} \tag{1.4}$$

It is worth pointing out that Mugnai-Pinamonti-Vecchi [13] considered the boundary value problem driven by the p-fractional Laplacian with nonlocal Robin boundary conditions

$$\begin{gathered}(-\Delta)_p^s u = f(x,u), \quad \text{in } \Omega,\\ N_{s,p}u(x) + \beta(x)|u|^{p-2}u = 0, \quad \text{in } \mathbb{R}^n \backslash \Omega,\end{gathered} \tag{1.5}$$

they provided necessary and sufficient conditions which ensure the existence of a unique positive solution for this problem.

Recently, a wide interest arised in p-fractional Laplacian with nonlocal Robin boundary value problem, see [3, 5, 9, 11] and the references therein.

In this article, we mainly focus on the existence of solution for fractional p-Laplacian Robin problem (1.1). To show our main result, we first give some notation. For any couple of functions (u,v) and $\mathcal{C}\Omega = \mathbb{R}^n \backslash \Omega$, we denote

$$H_{s,p}(u,v) \doteq \frac{c_{n,s,p}}{2} \int\int_{\mathbb{R}^{2n}\backslash(\mathcal{C}\Omega)^2} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+sp}}\,dx\,dy.$$

Next, we define the fractional Sobolev space, which can be suitably modeled to deal with fractional Robin boundary conditions. Precisely, given $\beta(x) \in L^\infty(\mathbb{R}^n\backslash\Omega)$, we define the function space

$$X_\beta^{s,p} \doteq \{u : \mathbb{R}^n \to \mathbb{R} \text{ measurable: } \|u\|_{X_\beta^{s,p}} < +\infty\},$$

where

$$\begin{aligned}\|u\|^p_{X_\beta^{s,p}} &\doteq \int_\Omega |u|^p dx + \int\int_{\mathbb{R}^{2n}\backslash(\mathcal{C}\Omega)^2} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dx\, dy + \int_{\mathbb{R}^n\backslash\Omega} |\beta(x)||u|^p dx \\ &= \|u\|^p_{L^p(\Omega)} + [u]^p_{s,p} + \|u\|^p_{L^p(\beta;\mathbb{R}^n\backslash\Omega)}\,.\end{aligned}$$

Observe that

$$[u]_{s,p} \doteq \Big(\int\int_{\mathbb{R}^{2n}\backslash(\mathcal{C}\Omega)^2} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dx\, dy\Big)^{1/p}$$

is strictly related to the Gagliardo seminorm

$$[u] = \Big(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dx\, dy\Big)^{1/p}.$$

We denote the fractional Hardy-Sobolev constant S_α by

$$S_\alpha = \inf_{u\in W^{s,p}(\Omega)\backslash\{0\}} \frac{\|u\|^p}{\|u\|^p_{L^{p^*_\alpha}(\Omega,|x|^{-\alpha}dx)}}$$

and $L^{p^*_\alpha}(\Omega,|x|^{-\alpha}dx)$ is the weighted $L^{p^*_\alpha}$ space with norm

$$\|u\|_{L^{p^*_\alpha}(\Omega,|x|^{-\alpha}dx)} = \Big(\int_\Omega \frac{|u|^{p^*_\alpha}}{|x|^\alpha} dx\Big)^{1/p^*_\alpha},$$

where $p^*_\alpha = \frac{(n-\alpha)p}{n-ps}$. When $\alpha = 0$, S_0 is the best fractional Sobolev constant. Moreover, $p^*_\alpha = \frac{(n-\alpha)p}{n-ps}$ arises from the general fractional Hardy-Sobolev inequality

$$\Big(\int_{\mathbb{R}^n} \frac{|u|^{p^*_\alpha}}{|x|^\alpha} dx\Big)^{1/p^*_\alpha} \le C(n,p,\alpha)\Big(\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\Big)^{1/p}, \quad \text{for } u \in W_0^{s,p}(\Omega).$$

Definition We say that $u \in X_\beta^{s,p}$ is a weak solution of (1.1) if

$$H_{s,p}(u,\varphi) + \int_{\mathbb{R}^n\backslash\Omega} \beta(x)|u|^{p-2}u\varphi dx = \lambda\int_\Omega |u|^{q-2}u\varphi dx + \int_\Omega \frac{|u|^{r-2}u\varphi}{|x|^\alpha} dx \tag{1.6}$$

for all $\varphi \in X_\beta^{s,p}$.

Formally, weak solutions of (1.1) coincide with the critical points of the functional

$$I_\lambda(u) \doteq \frac{1}{p}\|u\|^p_{X_\beta^{s,p}} + \frac{1}{p}\int_{\mathbb{R}^n\backslash\Omega} \beta(x)|u|^p dx - \frac{\lambda}{q}\int_\Omega |u|^q dx - \frac{1}{r}\int_\Omega \frac{|u|^r}{|x|^\alpha} dx. \tag{1.7}$$

We can see that

$$\langle I'_\lambda(u),\varphi\rangle = \|u\|^p_{X_\beta^{s,p}} + \int_{\mathbb{R}^n\backslash\Omega} \beta(x)|u|^p dx - \lambda\int_\Omega |u|^q dx - \int_\Omega \frac{|u|^r}{|x|^\alpha} dx. \tag{1.8}$$

Now we state our main results.

Theorem 1.1. *Let $0 \le \alpha < ps < n$ and $1 < q < p < r < p^*_\alpha$. Then there exists Λ such that problem (1.1) has at least two solutions for $\lambda \in (0, \frac{q}{p}\Lambda)$.*

Theorem 1.2. *Let $0 \le \alpha < ps < n$, $r = p_\alpha^*$, $q \ge \frac{n(p-1)}{n-ps}$, then there exists $\lambda^* > 0$ such that problem (1.1) has at least two solutions for $\lambda \in (0, \lambda^*)$.*

This article organized as follows: we give some preliminary results in Section 2. In Section 3, we prove Theorem 1.1 by variational approach. Section 4 gives the proof of Theorem 1.2.

2. Preliminaries

We want to collect several technical results needed in the upcoming sections and we will give some notations and properties of the Nehari manifold, which will be used to prove our main results. We define the manifold

$$N_\lambda = \{u \in X_0 \backslash \{0\} : \langle I'_\lambda(u), u\rangle = 0\}.$$

It is clear that all critical points of I_λ must lie on N_λ. We can see that $u \in N_\lambda$ if and only if $u \neq 0$ and

$$\|u\|^p_{X^{s,p}_\beta} + \int_{\mathbb{R}^n \backslash \Omega} \beta(x)|u|^p dx = \lambda \int_\Omega |u|^q dx + \int_\Omega \frac{|u|^r}{|x|^\alpha} dx.$$

Set

$$\Psi_\lambda(u) = \langle I'_\lambda(u), u\rangle.$$

Then for $u \in N_\lambda$, we have

$$\begin{aligned}
\langle \Psi'_\lambda(u), u\rangle &= p\|u\|^p_{X^{s,p}_\beta} + p\int_{\mathbb{R}^n \backslash \Omega} \beta(x)|u|^p dx - \lambda q \int_\Omega |u|^q dx - r\int_\Omega \frac{|u|^r}{|x|^\alpha} dx \\
&= (p-q)\|u\|^p_{X^{s,p}_\beta} + (p-q)\int_{\mathbb{R}^n \backslash \Omega} \beta(x)|u|^p dx - (r-q)\int_\Omega \frac{|u|^r}{|x|^\alpha} dx \\
&= (p-r)\|u\|^p_{X^{s,p}_\beta} + (p-r)\int_{\mathbb{R}^n \backslash \Omega} \beta(x)|u|^p dx - (q-r)\lambda \int_\Omega |u|^q dx.
\end{aligned}$$

Then N_λ can be divided into the following three parts

$$\begin{aligned}
N^+_\lambda &= \{u \in N_\lambda | \langle \Psi'_\lambda(u), \varphi\rangle > 0\}, \\
N^-_\lambda &= \{u \in N_\lambda | \langle \Psi'_\lambda(u), \varphi\rangle < 0\}, \\
N^0_\lambda &= \{u \in N_\lambda | \langle \Psi'_\lambda(u), \varphi\rangle = 0\}.
\end{aligned}$$

Lemma 2.1. *The space $X^{s,p}_\beta$ is a reflexive Banach space for every $1 < p < \infty$.*

Lemma 2.2. *The embedding $X^{s,p}_\beta \hookrightarrow L^q(\Omega)$ is compact for every $q \in [1, p^*)$, where $p^* = \frac{np}{n-ps}$ if $n < ps$, $p^* = \infty$ if $n \ge ps$.*

The proofs of Lemmas 2.1 and 2.2 are the same as that [13, Lemmas 2.1 and 2.2] respectively.

Lemma 2.3. *Suppose u_0 is a local minimizer of the functional I_λ on N_λ and $u_0 \notin N^0_\lambda$. Then u_0 is a critical point of I_λ.*

The proof of the above lemma is the same as that in Brown-Zhang [2, Theorem 2.3] and Chen-Gui [4, Theorem 2.1].

Let

$$\Lambda = \Big(\frac{(p-q)\widehat{C} S_\alpha^{r/p}}{r-q}\Big)^{(p-q)/(r-p)} |\Omega|^{-1/\gamma} S_\alpha^{q/p},$$

where $|\Omega|$ denotes the measure of Ω and

$$\gamma = \frac{p^*}{p^* - q}, \quad \widehat{C} = \Big(\int_\Omega \frac{1}{|x|^{\frac{\alpha p^*_\alpha}{r - p^*_\alpha}}} \Big)^{\frac{r-p^*_\alpha}{p^*_\alpha}}.$$

Lemma 2.4. *If $u_0 \in X^{s,p}_\beta \backslash \{0\}$, then there exists unique $t_0 > 0$ such that for any $\lambda \in (0, \Lambda)$, then there exist $t^+ > 0$ and $t^- > 0$ satisfying $t^+ u \in N^+_\lambda$, $t^- u \in N^-_\lambda$. Moreover,*

$$I_\lambda(t^+ u) = \inf_{0 \le t \le t_0} I_\lambda(tu), \quad I_\lambda(t^- u) = \sup_{t \ge 0} I_\lambda(tu).$$

Proof. Fix $u_0 \in X^{s,p}_\beta \backslash \{0\}$, we consider the map $\phi : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\phi(t) = t^{p-q} \|u\|^p_{X^{s,p}_\beta} + t^{p-q} \int_{\mathbb{R}^n \backslash \Omega} \beta(x) |u|^p dx - t^{r-q} \int_\Omega \frac{|u|^r}{|x|^\alpha} dx.$$

Obviously, $\phi(0) = 0$, $\lim_{t\to\infty} \phi(t) = -\infty$, $\phi'(t) = t^{p-q-1} g(t)$, where

$$g(t) = (p-q) \|u\|^p_{X^{s,p}_\beta} + (p-q) \int_{\mathbb{R}^n \backslash \Omega} \beta(x) |u|^p dx - t^{r-p} (r-q) \int_\Omega \frac{|u|^r}{|x|^\alpha} dx.$$

Hence

$$g'(t) = -t^{r-p-1} (r-q)(r-p) \int_\Omega \frac{|u|^r}{|x|^\alpha} dx < 0.$$

Then, we have $g(t)$ is strictly decreasing on $[0, +\infty)$ and $g(0) \ge 0$, $\lim_{t\to\infty} g(t) = -\infty$, so there exists a unique t_0 such that $g(t_0) = 0$. Then $\phi(t)$ is strictly increasing on $[0, t_0]$ and strictly decreasing on $(t_0, +\infty)$, which reaches the maximum at t_0. Now

$$\phi(t_0) = t_0^{-q} \Big(t_0^p \|u\|^p_{X^{s,p}_\beta} + t_0^p \int_{\mathbb{R}^n \backslash \Omega} \beta(x) |u|^p dx - t_0^r \int_\Omega \frac{|u|^r}{|x|^\alpha} dx \Big),$$

where

$$t_0 = \Big(\frac{(p-q) \|u\|^p_{X^{s,p}_\beta} + (p-q) \int_{\mathbb{R}^n \backslash \Omega} \beta(x) |u|^p dx}{(r-q) \int_\Omega \frac{|u|^r}{|x|^\alpha} dx} \Big)^{1/(r-p)}.$$

Using Hölder and Hardy-Sobolev inequalities, we obtain

$$\int_\Omega |u|^q dx \le |\Omega|^{1/\gamma} S_0^{-q/p} \|u\|^q_{X^{s,p}_\beta} \tag{2.1}$$

$$\int_\Omega \frac{|u|^r}{|x|^\alpha} dx \le S_\alpha^{-r/p} \widehat{C}^{-1} \|u\|^r_{X^{s,p}_\beta}. \tag{2.2}$$

Combining the definition of t_0 and (2.2), we have

$$t_0 > \Big(\frac{(p-q) \|u\|^p_{X^{s,p}_\beta}}{(r-q) \int_\Omega \frac{|u(x)|^r}{|x|^\alpha} dx} \Big)^{1/(r-p)} \ge \Big(\frac{p-q}{(r-q) \widehat{C}^{-1} S_\alpha^{-r/p}} \Big)^{1/(r-p)} \|u\|^{-1}_{X^{s,p}_\beta} \doteq t' \ge 0,$$

which implies that

$$\begin{aligned} \phi(t_0) \ge \phi(t') &> t'^{p-q} \|u\|^p_{X^{s,p}_\beta} - t'^{r-q} \int_\Omega \frac{|u|^r}{|x|^\alpha} dx \\ &\ge \Big(\frac{(p-q) \widehat{C} S_\alpha^{r/p}}{r-q} \Big)^{(p-q)/(r-p)} \|u\|^q_{X^{s,p}_\beta} \qquad (2.3) \\ &\ge \lambda \int_\Omega |u|^q dx \end{aligned}$$

for $\lambda \in (0,\Lambda)$ by (2.1). Consequently, there exist $t^+ > 0$ and $t^- > 0$ with $t^+ < t_0 < t^-$ such that $\phi(t^+) = \phi(t^-) = \lambda \int_\Omega |u(x)|^q dx$, which means $t^+u \in N_\lambda$, $t^-u \in N_\lambda$.

If $tu \in N_\lambda$, then $\langle \Psi'_\lambda(tu), tu\rangle = t^{q+1}\phi'(t)$. According to $t^+u \in N_\lambda$, $t^-u \in N_\lambda$ and $\phi'(t^+) > 0$, $\phi'(t^-) < 0$, then $t^+u \in N_\lambda^+$ and $t^-u \in N_\lambda^-$. Since $\langle I'_\lambda(tu), tu\rangle = t^q(\phi(t) - \int_\Omega |u(x)|^q dx)$, we can see that $I_\lambda(t^-u) > I_\lambda(tu) > I_\lambda(t^+u)$ for $t \in [t^+, t^-]$, and $I_\lambda(tu) > I_\lambda(t^+u)$ for $t \in [0, t^+]$. Thus

$$I_\lambda(t^+u) = \inf_{0\le t\le t_0} I_\lambda(tu), \quad I_\lambda(t^-u) = \sup_{t\ge 0} I_\lambda(tu).$$

□

3. Proof of Theorem 1.1

Definition We say that $\{u_k\}$ is a $(PS)_c$ sequence in $X_\beta^{s,p}$ for I_λ, if $I_\lambda(u_k) \to c$ and $I'_\lambda(u_k) \to 0$ in $X_\beta^{-s,p}$ as $k \to \infty$. We say that I_λ satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence $\{u_k\}$ in $X_\beta^{s,p}$ has a strongly convergent subsequence.

Next, we prove some technical lemmas which will be very useful hereinafter.

Lemma 3.1. *If $\{u_k\} \subset X_\beta^{s,p}$ is a $(PS)_c$ sequence for I_λ, then $\{u_k\}$ is bounded in $X_\beta^{s,p}$.*

Proof. If $\{u_k\} \subset X_\beta^{s,p}$ is a $(PS)_c$ sequence for I_λ, then

$$I_\lambda(u_k) \to c, \quad I'_\lambda(u_k) \to 0 \quad \text{in } X_\beta^{-s,p} \text{ as } k \to \infty.$$

Namely,

$$\frac{1}{p}\|u_k\|^p_{X_\beta^{s,p}} + \frac{1}{p}\int_{\mathbb{R}^n\setminus\Omega} \beta(x)|u_k|^p dx - \frac{1}{q}\lambda \int_\Omega |u_k|^q dx - \frac{1}{r}\int_\Omega \frac{|u_k|^r}{|x|^\alpha} dx = c + o(1), \tag{3.1}$$

$$\|u_k\|^p_{X_\beta^{s,p}} + \int_{\mathbb{R}^n\setminus\Omega} \beta(x)|u_k|^p dx - \lambda \int_\Omega |u_k|^q dx - \int_\Omega \frac{|u_k|^r}{|x|^\alpha} dx = o(\|u_k\|_{X_\beta^{s,p}}). \tag{3.2}$$

By (3.1), (3.2), Hölder inequality and Sobolev inequality, we obtain

$$\begin{aligned} c + o(\|u_k\|_{X_\beta^{s,p}}) &= I_\lambda(u_k) - \frac{1}{p}\langle I'_\lambda(u_k), u_k\rangle \\ &= (\frac{1}{p} - \frac{1}{q})\lambda \int_\Omega |u_k|^q dx + (\frac{1}{p} - \frac{1}{r})\int_\Omega \frac{|u_k|^r}{|x|^\alpha} dx \\ &\ge (\frac{1}{p} - \frac{1}{q})|\Omega|^{1/\gamma} S_0^{-q/p}\|u_k\|^q_{X_\beta^{s,p}}. \end{aligned}$$

Hence $\{u_k\}$ is bounded in $X_\beta^{s,p}$. □

Lemma 3.2. *For any $\lambda \in (0,\Lambda)$, we have $N_\lambda^0 = \emptyset$.*

Proof. On the contrary, if $N_\lambda^0 \ne \emptyset$, then there exists $u \in N_\lambda^0$, this implies $\langle \Psi'(u), u\rangle = 0$, we can deduce that

$$\begin{aligned} (p-q)\|u\|^p_{X_\beta^{s,p}} &\le (p-q)\|u\|^p_{X_\beta^{s,p}} + (p-q)\int_{\mathbb{R}^n\setminus\Omega} \beta(x)|u|^p dx \\ &= (r-q)\int_\Omega \frac{|u|^r}{|x|^\alpha} dx, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned}(r-p)\|u\|^p_{X^{s,p}_\beta} &\le (r-p)\|u\|^p_{X^{s,p}_\beta} + (r-p)\int_{\mathbb{R}^n\setminus\Omega}\beta(x)|u|^p dx \\ &= (r-q)\lambda\int_\Omega |u|^q dx.\end{aligned} \tag{3.4}$$

By (2.2), we obtain

$$(r-q)\int_\Omega \frac{|u|^r}{|x|^\alpha}dx \le (r-q)S_\alpha^{-r/p}\widehat{C}^{-1}\|u\|^r_{X^{s,p}_\beta}. \tag{3.5}$$

From (3.3) and (3.5), we have

$$\|u\|_{X^{s,p}_\beta} \ge \Big(\frac{(p-q)S_\alpha^{r/p}\widehat{C}}{r-q}\Big)^{1/(r-p)}. \tag{3.6}$$

By (2.1), we have

$$(r-q)\lambda\int_\Omega |u|^q dx \le (r-q)\lambda|\Omega|^{1/\gamma}S_0^{-q/p}\|u\|^q_{X^{s,p}_\beta}. \tag{3.7}$$

Combining (3.4) and (3.6), it follows that

$$\|u\|_{X^{s,p}_\beta} \le \Big(\frac{(r-q)\lambda|\Omega|^{1/\gamma}S_0^{-q/p}}{r-p}\Big)^{1/(p-q)}. \tag{3.8}$$

Hence, by (3.6) and (3.8), we obtain $\lambda \ge \Lambda$, which is a contradiction. □

Lemma 3.3. *The energy functional I_λ is coercive and bounded from below on N_λ for all $\lambda > 0$.*

Proof. According to (2.1), for any $\lambda > 0$ and $u \in N_\lambda$, we can see that

$$\begin{aligned}&I_\lambda(u)\\ &= \frac{1}{p}\|u\|^p_{X^{s,p}_\beta} + \frac{1}{p}\int_{\mathbb{R}^n\setminus\Omega}\beta(x)|u|^p dx - \frac{\lambda}{q}\int_\Omega |u|^q dx - \frac{1}{r}\int_\Omega\frac{|u|^r}{|x|^\alpha}dx\\ &= (\frac{1}{p}-\frac{1}{r})\|u\|^p_{X^{s,p}_\beta} + (\frac{1}{p}-\frac{1}{r})\int_{\mathbb{R}^n\setminus\Omega}\beta(x)|u|^p dx - (\frac{1}{q}-\frac{1}{r})\lambda\int_\Omega |u|^q dx\\ &\ge (\frac{1}{p}-\frac{1}{r})\|u\|^p_{X^{s,p}_\beta} + (\frac{1}{p}-\frac{1}{r})\int_{\mathbb{R}^n\setminus\Omega}\beta(x)|u|^p dx - (\frac{1}{q}-\frac{1}{r})\lambda|\Omega|^{1/\gamma}S_0^{-q/p}\|u\|^q_{X^{s,p}_\beta}.\end{aligned}$$

Then I_λ is coercive and bounded from below on N_λ for $q < p < r$. □

From Lemmas 3.2 and 3.3, for each $\lambda \in (0,\Lambda)$, we know that $N_\lambda = N_\lambda^+ \cup N_\lambda^-$ and I_λ is coercive and bounded from below on N_λ^+ and N_λ^-. Therefore we can define

$$c_\lambda = \inf_{N_\lambda} I_\lambda, \quad c_\lambda^+ = \inf_{N_\lambda^+} I_\lambda, \quad c_\lambda^- = \inf_{N_\lambda^-} I_\lambda.$$

We have the following Lemma.

Lemma 3.4. *(1) If $\lambda \in (0,\Lambda)$, then $c_\lambda \le c_\lambda^+ < 0$,*
(2) If $\lambda \in (0, \frac{q}{p}\Lambda)$, then $c_\lambda^- > 0$.

Proof. (1) Let $u \in N_\lambda^+$, then $\langle \Psi_\lambda'(u), u\rangle > 0$, which means that

$$\frac{p-q}{r-q}\|u\|^p_{X^{s,p}_\beta} + \frac{p-q}{r-q}\int_{\mathbb{R}^n\setminus\Omega}\beta(x)|u|^p dx > \int_\Omega\frac{|u|^r}{|x|^\alpha}dx.$$

Then

$$\begin{aligned} I_\lambda(u) &= \frac{1}{p}\|u\|^p_{X^{s,p}_\beta} + \frac{1}{p}\int_{\mathbb{R}^n\setminus\Omega}\beta(x)|u|^p dx - \frac{\lambda}{q}\int_\Omega |u|^q dx - \frac{1}{r}\int_\Omega \frac{|u|^r}{|x|^\alpha}dx \\ &= (\frac{1}{p}-\frac{1}{q})\|u\|^p_{X^{s,p}_\beta} + (\frac{1}{p}-\frac{1}{q})\int_{\mathbb{R}^n\setminus\Omega}\beta(x)|u|^p dx - (\frac{1}{r}-\frac{1}{q})\int_\Omega \frac{|u|^r}{|x|^\alpha}dx \\ &< (\frac{1}{p}-\frac{1}{q})\|u\|^p_{X^{s,p}_\beta} + (\frac{1}{p}-\frac{1}{q})\int_{\mathbb{R}^n\setminus\Omega}\beta(x)|u|^p dx \\ &\quad + (\frac{1}{q}-\frac{1}{r})\Big(\frac{p-q}{r-q}\|u\|^p_{X^{s,p}_\beta} + \frac{p-q}{r-q}\int_{\mathbb{R}^n\setminus\Omega}\beta(x)|u|^p dx\Big) \\ &= \frac{p-q}{q}(\frac{1}{r}-\frac{1}{p})\Big(\|u\|^p_{X^{s,p}_\beta} + \int_{\mathbb{R}^n\setminus\Omega}\beta(x)|u|^p dx\Big) < 0. \end{aligned} \tag{3.9}$$

Thus $c_\lambda \le c^+_\lambda < 0$ follows from the definition of c_λ and c^+_λ.

(2) Similarly, we assume that $u \in N^-_\lambda$, then we can deduce that $\langle \Psi'_\lambda(u), u\rangle < 0$, which implies that

$$\frac{r-p}{r-q}\|u\|^p_{X^{s,p}_\beta} + \frac{r-p}{r-q}\int_{\mathbb{R}^n\setminus\Omega}\beta(x)|u|^p dx > \lambda\int_\Omega |u|^q dx,$$

and

$$\frac{p-q}{r-q}\|u\|^p_{X^{s,p}_\beta} < \frac{p-q}{r-q}\|u\|^p_{X^{s,p}_\beta} + \frac{p-q}{r-q}\int_{\mathbb{R}^n\setminus\Omega}\beta(x)|u|^p dx < \int_\Omega \frac{|u|^r}{|x|^\alpha}dx.$$

By (2.2), we obtain

$$\|u\|_{X^{s,p}_\beta} \ge \Big(\frac{(p-q)S_\alpha^{r/p}\widehat{C}}{r-q}\Big)^{1/(r-p)}.$$

From (2.1), we find that

$$\begin{aligned} &I_\lambda(u) \\ &= \frac{1}{p}\|u\|^p_{X^{s,p}_\beta} + \frac{1}{p}\int_{\mathbb{R}^n\setminus\Omega}\beta(x)|u|^p dx - \frac{\lambda}{q}\int_\Omega |u|^q dx - \frac{1}{r}\int_\Omega \frac{|u|^r}{|x|^\alpha}dx \\ &= (\frac{1}{p}-\frac{1}{r})\|u\|^p_{X^{s,p}_\beta} + (\frac{1}{p}-\frac{1}{r})\int_{\mathbb{R}^n\setminus\Omega}\beta(x)|u|^p dx - (\frac{1}{q}-\frac{1}{r})\lambda\int_\Omega |u|^q dx \\ &\ge (\frac{1}{p}-\frac{1}{r})\|u\|^p_{X^{s,p}_\beta} - (\frac{1}{q}-\frac{1}{r})\lambda|\Omega|^{1/\gamma}S_0^{-q/p}\|u\|^q_{X^{s,p}_\beta} \\ &= \|u\|^q_{X^{s,p}_\beta}\Big((\frac{1}{p}-\frac{1}{r})\|u\|^{p-q}_{X^{s,p}_\beta} - (\frac{1}{q}-\frac{1}{r})\lambda|\Omega|^{1/\gamma}S_0^{-q/p}\Big) \\ &> \|u\|^q_{X^{s,p}_\beta}\Big((\frac{1}{p}-\frac{1}{r})\Big(\frac{(p-q)S_\alpha^{r/p}\widehat{C}}{r-q}\Big)^{(p-q)/(r-p)} - (\frac{1}{q}-\frac{1}{r})\lambda|\Omega|^{1/\gamma}S_0^{-q/p}\Big) > 0 \end{aligned}$$

for $\lambda \in (0, \frac{q}{p}\Lambda)$, which implies that $c^-_\lambda > 0$. □

Lemma 3.5. *Assume that $\lambda \in (0,\Lambda)$. Then for each $u \in N_\lambda$, there exist $\varepsilon > 0$ and a differentiable map $h : B(0,\varepsilon) \subset X^{s,p}_\beta \to \mathbb{R}^+$, with $h = 1$ such that $h(w)(u-w) \in$*

N_λ and

$$\langle h'(0), w\rangle = \frac{p\Lambda(u,w) + p\int_{\mathbb{R}^n\setminus\Omega}\beta(x)|u|^{p-2}uw\,dx - q\int_\Omega |u|^{q-2}uw\,dx - r\int_\Omega \frac{|u|^{r-2}uw}{|x|^\alpha}dx}{(p-q)\|u\|^p_{X^{s,p}_\beta} + (p-q)\int_{\mathbb{R}^n\setminus\Omega}\beta(x)|u|^{p-2}uw\,dx - (r-q)\int_\Omega \frac{|u|^{r-2}uw}{|x|^\alpha}dx}, \tag{3.10}$$

where

$$\Lambda(u,w) = \int\int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+sp}}\,dx\,dy$$

for each $w \in X^{s,p}_\beta$.

Proof. For $u \in N_\lambda$, we define the map $f : \mathbb{R}^+ \times X^{s,p}_\beta \to \mathbb{R}$ as follows

$$\begin{aligned} f(\xi, w) &= \langle I'_\lambda(\xi(u-w)), \xi(u-w)\rangle \\ &= \xi^p\|u-w\|^p_{X^{s,p}_\beta} + \xi^p\int_{\mathbb{R}^n\setminus\Omega}\beta(x)|u-w|^p dx \\ &\quad -\xi^q\int_\Omega |u-w|^q dx - \xi^r\int_\Omega \frac{|u-w|^r}{|x|^\alpha}dx \end{aligned} \tag{3.11}$$

for $\xi \in R^+$, $w \in X^{s,p}_\beta$. Then we know $f(1,0) = \langle I'_\lambda(u), u\rangle$. In addition combining with Lemma 3.2, we obtain

$$\begin{aligned} \frac{df(1,0)}{d\xi} &= p\|u\|^p_{X^{s,p}_\beta} + p\int_{\mathbb{R}^n\setminus\Omega}\beta(x)|u|^p dx - \lambda q\int_\Omega |u|^q dx - r\int_\Omega \frac{|u|^r}{|x|^\alpha}dx \\ &= (p-q)\|u\|^p_{X^{s,p}_\beta} + (p-q)\int_{\mathbb{R}^n\setminus\Omega}\beta(x)|u|^p dx - (r-q)\int_\Omega \frac{|u|^r}{|x|^\alpha}dx \neq 0. \end{aligned} \tag{3.12}$$

Using the Implicit Function Theorem, there exist $\varepsilon > 0$ and a C^1 map $h : B(0,\varepsilon) \subset X^{s,p}_\beta \to \mathbb{R}^+$ with $\xi = h(w)$ and $h(0) = 1$ such that

$$\langle h'(0), w\rangle = \frac{p\Lambda(u,w) + p\int_{\mathbb{R}^n\setminus\Omega}\beta(x)|u|^{p-2}uw\,dx - q\int_\Omega |u|^{q-2}uw\,dx - r\int_\Omega \frac{|u|^{r-2}uw}{|x|^\alpha}dx}{(p-q)\|u\|^p_{X^{s,p}_\beta} + (p-q)\int_{\mathbb{R}^n\setminus\Omega}\beta(x)|u|^p dx - (r-q)\int_\Omega \frac{|u|^r}{|x|^\alpha}dx},$$

and $f(h(w), w) = 0$ for all $w \in B(0,\varepsilon)$. Hence,

$$\langle I'_\lambda(h(w)(u-w)), h(w)(u-w)\rangle = 0.$$

It implies that $h(w)(u-w) \in N_\lambda$. □

In Lemma 3.5, we replace $u \in N_\lambda$ by $u \in N^-_\lambda$ and ξ by ξ^-, then the conclusion still holds. Moreover, the proof is similar to that in Lemma 3.5.

Proposition 3.6.

(1) *If $\lambda \in (0, \Lambda)$, then there exists a $(PS)_{c_\lambda}$ sequence $\{u_k\} \subset N_\lambda$ for I_λ.*
(2) *If $\lambda \in (0, \frac{q}{p}\Lambda)$, then there exists a $(PS)_{c^-_\lambda}$ sequence $\{u_k\} \subset N^-_\lambda$ for I_λ.*

Proof. (1) By Ekeland's Variational Principle, there exists a minimizing sequence $\{u_k\} \subset N_\lambda$ such that

$$I_\lambda(u_k) < c_\lambda + \frac{1}{k}, \quad I_\lambda(u_k) < I_\lambda(u) + \frac{1}{k}\|u - u_k\|_{X_\beta^{s,p}}, \quad \forall u \in N_\lambda. \tag{3.13}$$

Using that $c_\lambda < 0$, we obtain

$$I_\lambda(u_k) = (\frac{1}{p} - \frac{1}{r})\|u_k\|^p_{X_\beta^{s,p}} + (\frac{1}{p} - \frac{1}{r})\int_{\mathbb{R}^n\setminus\Omega} \beta(x)|u_k|^p dx - (\frac{1}{q} - \frac{1}{r})\lambda\int_\Omega |u_k|^q dx < \frac{c_\lambda}{2}.$$

This yields

$$\frac{c_\lambda qr}{2(q-r)} < \lambda\int_\Omega |u_k|^q dx < \lambda|\Omega|^{1/\gamma} S_0^{-q/p}\|u_k\|^q_{X_\beta^{s,p}}, \tag{3.14}$$

$$\begin{aligned}(\frac{1}{p} - \frac{1}{r})\|u_k\|^p_{X_\beta^{s,p}} &< (\frac{1}{q} - \frac{1}{r})\lambda\int_\Omega |u_k|^q dx \\ &< (\frac{1}{q} - \frac{1}{r})\lambda|\Omega|^{1/\gamma} S_0^{-q/p}\|u_k\|^q_{X_\beta^{s,p}}.\end{aligned} \tag{3.15}$$

By (3.14) and (3.15), we have

$$\|u_k\|_{X_\beta^{s,p}} > \Big(\frac{c_\lambda qr S_0^{q/p}}{2(q-r)\lambda|\Omega|^{1/\gamma}}\Big)^{1/q}, \quad \|u_k\|_{X_\beta^{s,p}} < \Big(\frac{(r-q)p\lambda|\Omega|^{1/\gamma}}{(r-p)qS_0^{q/p}}\Big)^{1/(p-q)}. \tag{3.16}$$

Next we claim that

$$\|I'_\lambda(u_k)\|_{X_\beta^{-s,p}} \to 0 \quad \text{as } k \to \infty.$$

The proof of this claim is similar to [4, Proposition 3.1], hence we omit it here. From Lemma 3.5 ($u \in N_\lambda^-$), using the same arguments, we obtain (2) of Proposition 3.6. □

Theorem 3.7. *If $\lambda \in (0, \Lambda)$, $1 < q < p < r < p_\alpha^*$, then there exists $u_1 \in N_\lambda^+$ and satisfies*

(1) $I_\lambda(u_1) = c_\lambda = c_\lambda^+ < 0$,
(2) *u_1 is a solution of the problem* (1.1).

Proof. (1) First, we prove $I_\lambda(u_1) = c_\lambda$. Since

$$\begin{aligned}I_\lambda(u_1) &= \frac{1}{p}\|u_1\|^p_{X_\beta^{s,p}} + \frac{1}{p}\int_{\mathbb{R}^n\setminus\Omega} \beta(x)|u_1|^p dx - \frac{\lambda}{q}\int_\Omega |u_1|^q dx - \frac{1}{r}\int_\Omega \frac{|u_1|^r}{|x|^\alpha} dx \\ &= (\frac{1}{p} - \frac{1}{r})\|u_1\|^p_{X_\beta^{s,p}} + (\frac{1}{p} - \frac{1}{r})\int_{\mathbb{R}^n\setminus\Omega} \beta(x)|u_1|^p dx - (\frac{1}{q} - \frac{1}{r})\lambda\int_\Omega |u_1|^q dx \\ &\le \lim_{k\to\infty}\inf\Big((\frac{1}{p} - \frac{1}{r})\|u_k\|^p_{X_\beta^{s,p}} + (\frac{1}{p} - \frac{1}{r})\int_{\mathbb{R}^n\setminus\Omega} \beta(x)|u_k|^p dx \\ &\quad - (\frac{1}{q} - \frac{1}{r})\lambda\int_\Omega |u_k|^q dx\Big) \\ &= \lim_{k\to\infty}\inf I_\lambda(u_k) = c_\lambda.\end{aligned}$$

It follows that $I_\lambda(u_1) = c_\lambda$.

Then we claim that $c_\lambda = c_\lambda^+$ for $u_1 \in N_\lambda^+$. By $I'_\lambda(u_1) = 0$ and Lemma 3.2, we have $u_1 \in N_\lambda^+ \cup N_\lambda^-$. Assume that $u_1 \in N_\lambda^-$, and combining with Lemma 2.4, there exist $t^- > 0$ and $t^+ > 0$ with $t^- > t^+$ such that $t^-u_1 \in N_\lambda^-$, $t^+u_1 \in N_\lambda^+$. In particular $t^+ < t^- = 1$. Since $\frac{dI_\lambda(t^+u_1)}{dt} = 0$, $\frac{d^2 I_\lambda(t^+u_1)}{dt^2} > 0$, there exists $t \in (t^+, 1]$ such that

$$c_\lambda \leq I_\lambda(t^+u_1) < I_\lambda(tu_1) = c_\lambda,$$

which is a contradiction, so $u_1 \in N_\lambda^+$. Then $c_\lambda = I_\lambda(u_1) \geq c_\lambda^+$, this together with the definitions of c_λ and we have $c_\lambda = c_\lambda^+$. Hence we finish the proof of $I_\lambda(u_1) = c_\lambda = c_\lambda^+$.

(2) By (1) of Proposition 3.6, there exists a bounded minimizing sequence $\{u_k\} \subset N_\lambda$ such that

$$\lim_{k\to\infty} I_\lambda(u_k) = c_\lambda \leq c_\lambda^+ < 0, \quad I'_\lambda(u_k) = o_k(1).$$

From Lemma 3.1, we know that $\{u_k\}$ is bounded in $X_\beta^{s,p}$. Then there exists $u_1 \in X_\beta^{s,p}$ such that, up to a subsequence, $u_k \rightharpoonup u_1$ weakly in $X_\beta^{s,p}$ and $u_k \to u_1$ strongly in $L^\theta(\Omega, |X|^{-\alpha})$ for any $\theta \in [1, p_\alpha^*)$ and $0 \leq \alpha < ps$. In particular, we have

$$\lambda \int_\Omega |u_k|^q dx \to \lambda \int_\Omega |u_1|^q dx, \quad \int_\Omega \frac{|u_k|^r}{|x|^\alpha} dx \to \int_\Omega \frac{|u_1|^r}{|x|^\alpha} dx \quad \text{as } k \to \infty.$$

Moreover, for all $\phi \in X_\beta^{s,p}$,

$$o(1) = \langle I'_\lambda(u_k), \phi\rangle = \langle I'_\lambda(u_1), \phi\rangle + o(1).$$

Thus, $u_1 \in N_\lambda$ is a nonzero solution of the problem (1.1) and $I_\lambda(u_1) \geq c_\lambda$. ☐

Theorem 3.8. *If $\lambda \in (0, \frac{q}{p}\Lambda)$, $1 < q < p < r < p_\alpha^*$, then the functional I_λ has a minimizer $u_2 \in N_\lambda^-$ and satisfies*

1. *$I_\lambda(u_2) = c_\lambda^-$,*
2. *u_2 is a solution of the problem (1.1).*

Proof. By Proposition 3.6 (2), there exists a bounded minimizing sequence $\{u_k\} \subset N_\lambda^-$ such that

$$\lim_{k\to\infty} I_\lambda(u_k) = c_\lambda^-, \quad I'_\lambda(u_k) = o_k(1).$$

As in the proof of Theorem 3.7, there exists $u_2 \in N_\lambda^-$ such that $I_\lambda(u_2) = c_\lambda^-$ and u_2 is a solution of the problem (1.1). ☐

Proof of Theorem 1.1. By Theorems 3.7 and 3.8, we know that for $0 < \lambda < \frac{q}{p}\Lambda$, then problem (1.1) has two solutions $u_1 \in N_\lambda^+$ and $u_2 \in N_\lambda^-$ in $X_\beta^{s,p}$. Since $N_\lambda^+ \cap N_\lambda^- = \emptyset$, these two solutions are distinct. ☐

4. Proof of Theorem 1.2

This section we consider the multiplicity of solutions for the critical case. We need the following lemmas.

Lemma 4.1. *Let $r = p_\alpha^*$, $\{u_k\} \subset X_\beta^{s,p}$ be a sequence such that $I_\lambda(u_k) \to c_*$ with*

$$c_* < c_\Lambda = (\frac{1}{p} - \frac{1}{r}) S_\alpha^{r/(r-p)} - \bar{c}\frac{r-q}{r}\Big(\frac{r-p}{pq}\Big)^{q/(q-r)}\Big(\frac{(p-q)\lambda}{pq}\Big)^{r/(r-q)}$$

and $I'_\lambda(u_k) \to 0$ in $X_\beta^{-s,p}$. Then there exists a strongly convergent subsequence.

Proof. By Lemma 3.1, we know that $\{u_k\}$ is bounder in $X_\beta^{s,p}$, up to a subsequence, denote by itself, there exists $u \in X_\beta^{s,p}$ such that $u_k \rightharpoonup u_0$ weakly in $X_\beta^{s,p}$ and $u_k \to u_0$ strongly in $L^\gamma(\Omega, |x|^{-\alpha}dx)$ for any $\gamma \in [1, p_\alpha^*)$ and $0 \le \alpha < ps < n$. Now from [9, Theorem 1.1], we can assume that there exist two positive measure μ, ν on $\mathbb{R}^n$ and at most countable set $\{x_j\}_{j\in J} \subseteq \bar{\Omega}$ such that

$$\int_{\mathbb{R}^n} \frac{|u_k(x)-u_k(y)|^p}{|x-y|^{n+ps}} dy \rightharpoonup \mu, \quad \mu \ge \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dy + \sum_{j\in J} \mu_j \delta_{x_j}, \tag{4.1}$$

$$\frac{|u_k|^{p_\alpha^*}}{|x|^\alpha} \rightharpoonup \nu, \quad \nu = \frac{|u|^{p_\alpha^*}}{|x|^\alpha} \nu_j \delta_{x_j}, \tag{4.2}$$

$$\mu_j \ge S_\alpha \nu_j^{p/p_\alpha^*}, \quad \forall j \in J. \tag{4.3}$$

Next we claim that $J = \emptyset$. By contradiction, suppose that $J \ne \emptyset$, then there exists $j \in J$, for this x_j, define $\varphi_{\delta,j}(x) = \varphi(\frac{x-x_j}{\delta})$, where $x \in \mathbb{R}^n$, $\varphi \in C_0^\infty(\mathbb{R}^n)$ is a smooth cut off function, that is $\varphi = 1$ in $B(0,1)$ and $\varphi = 0$ in $\mathbb{R}^n \backslash B(0,2)$. Since $u_k\varphi_{\delta,j}$ is bounded in $X_\beta^{s,p}$, we have that $\langle I'_\lambda(u_k), u_k\varphi_{\delta,j}\rangle \to 0$ as $k \to \infty$. Then

$$\begin{aligned} &\int_{\mathbb{R}^{2n}} \frac{|u_k(x)-u_k(y)|^{p-2}(u_k(x)-u_k(y))(u_k(x)\varphi_{\delta,j}(x)-u_k(y)\varphi_{\delta,j}(y))}{|x-y|^{n+ps}}\,dx\,dy \\ &= \int_{\mathbb{R}^{2n}} \frac{u_k(x)|u_k(x)-u_k(y)|^{p-2}(u_k(x)-u_k(y))(\varphi_{\delta,j}(x)-\varphi_{\delta,j}(y))}{|x-y|^{n+ps}}\,dx\,dy \\ &+ \int_{\mathbb{R}^{2n}} \frac{\varphi_{\delta,j}(y)|u_k(x)-u_k(y)|^p}{|x-y|^{n+ps}}\,dx\,dy + \int_{\mathbb{R}^n} \beta(x)u_k(x)^p\varphi_{\delta,j}(x)dx \\ &= \lambda\int_\Omega |u_k(x)|^q\varphi_{\delta,j}(x)dx + \int_\Omega \frac{|u_k(x)|^{p_\alpha^*}\varphi_{\delta,j}(x)}{|x|^\alpha}dx + o(1). \end{aligned} \tag{4.4}$$

Now using Hölder inequality and that u_k is bounded in $X_\beta^{s,p}$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^{2n}} \frac{u_k(x)|u_k(x)-u_k(y)|^{p-2}(u_k(x)-u_k(y))(\varphi_{\delta,j}(x)-\varphi_{\delta,j}(y))}{|x-y|^{n+ps}}\,dx\,dy \\ &\le C\Big(\int_{\mathbb{R}^{2n}} \frac{|u_k(x)|^p|\varphi_{\delta,j}(x)-\varphi_{\delta,j}(y)|^p}{|x-y|^{n+ps}}\,dx\,dy\Big)^{1/p}, \end{aligned} \tag{4.5}$$

where C is a positive constant. From [15, Lemma 2.3], it holds that

$$\lim_{\delta\to0}\lim_{k\to\infty}\int_{\mathbb{R}^{2n}} \frac{|u_k(x)|^p|\varphi_{\delta,j}(x)-\varphi_{\delta,j}(y)|^p}{|x-y|^{n+ps}}\,dx\,dy = 0. \tag{4.6}$$

From (4.1) and (4.2), we have

$$\lim_{\delta\to0}\lim_{k\to\infty}\int_{\mathbb{R}^{2n}} \frac{\varphi_{\delta,j}(y)|u_k(x)-u_k(y)|^p}{|x-y|^{n+ps}}\,dx\,dy \ge \mu_j, \tag{4.7}$$

$$\lim_{\delta\to0}\lim_{k\to\infty}\int_\Omega \frac{|u_k(x)|^{p_\alpha^*}\varphi_{\delta,j}(x)}{|x|^\alpha}dx = \nu_j, \tag{4.8}$$

$$\lim_{\delta\to0}\lim_{k\to\infty}\lambda\int_\Omega |u_k(x)|^q\varphi_{\delta,j}(x)dx = 0. \tag{4.9}$$

From (4.4)-(4.9), we have

$$\nu_j \ge \mu_j. \tag{4.10}$$

Combining (4.3) with (4.10), we obtain

$$\nu_j \geq S_\alpha^{p_\alpha^*/(p_\alpha^*-p)}. \tag{4.11}$$

and

$$\begin{aligned} c_* &= \lim_{k\to\infty} (I_\lambda(u_k) - \frac{1}{p}\langle I'_\lambda(u_k), u_k\rangle) \\ &= \lim_{k\to\infty} \Big((\frac{1}{p} - \frac{1}{q})\lambda \int_\Omega |u_k(x)|^q dx + (\frac{1}{p} - \frac{1}{r}) \int_\Omega \frac{|u_k(x)|^{p_\alpha^*}}{|x|^\alpha} dx \Big) \\ &\geq (\frac{1}{p} - \frac{1}{q})\lambda \int_\Omega |u(x)|^q dx + (\frac{1}{p} - \frac{1}{r}) \int_\Omega \frac{|u(x)|^{p_\alpha^*}}{|x|^\alpha} dx + (\frac{1}{p} - \frac{1}{r})\nu_j \\ &\geq (\frac{1}{p} - \frac{1}{r}) S_\alpha^{r/(r-p)} - \bar{c}\frac{r-q}{r}\Big(\frac{r-p}{pq}\Big)^{q/(q-r)} \Big(\frac{(p-q)\lambda}{pq}\Big)^{r/(r-q)}, \end{aligned} \tag{4.12}$$

where

$$\begin{aligned} &(\frac{1}{q} - \frac{1}{p})\lambda \int_\Omega |u(x)|^q dx \\ &\leq (\frac{1}{q} - \frac{1}{p})\lambda \Big(\int_\Omega (\frac{|u(x)|^q}{|x|^{\alpha q/r}})^{r/q} dx \Big)^{q/r} \Big(\int_\Omega |x|^{\alpha q/r \cdot r/(r-q)} dx \Big)^{(r-q)/r} \\ &= \Big(\frac{r}{q}(\frac{1}{p} - \frac{1}{r}) \Big)^{q/r} \Big(\int_\Omega \frac{|u(x)|^r}{|x|^\alpha} dx \Big)^{q/r} \Big(\frac{r}{q}(\frac{1}{p} - \frac{1}{r}) \Big)^{-q/r} \\ &\quad \times (\frac{1}{q} - \frac{1}{p})\lambda \Big(\int_\Omega |x|^{\alpha q/(r-q)} dx \Big)^{(r-q)/r} \\ &\leq (\frac{1}{p} - \frac{1}{r}) \int_\Omega \frac{|u(x)|^r}{|x|^\alpha} dx + \bar{c}\frac{r-q}{r} \Big(\frac{r}{q}\frac{r-p}{pr} \Big)^{q/(q-r)} \Big(\frac{(p-q)\lambda}{pq} \Big)^{r/(r-q)} \end{aligned} \tag{4.13}$$

by Hölder inequality, Young inequality, and $\bar{c} = \int_\Omega |x|^{\alpha q/(r-q)} dx$. According to the definition of c_Λ, we have $c_* > c_\Lambda$, which is a contradiction. Hence $J = \emptyset$, which implies $\frac{|u_k|^{p_\alpha^*}}{|x|^\alpha} \to \frac{|u|^{p_\alpha^*}}{|x|^\alpha}$. Therefore, $\langle I'_\lambda(u_k) - I'_\lambda(u), u_k - u\rangle \to 0$ as $k \to \infty$. By the well-known Simon inequalities:

$$\begin{aligned} &|\alpha - \beta|^m \\ &\leq \begin{cases} C'_m(|\alpha|^{m-2}\alpha - |\beta|^{m-2}\beta)(\alpha - \beta), & \text{for } m \geq 2, \\ C''_m\Big((|\alpha|^{m-2}\alpha - |\beta|^{m-2}\beta)(\alpha - \beta)\Big)^{m/2} (|\alpha|^m + |\beta|^m)^{(2-m)/2}, & \text{for } 1 < m < 2, \end{cases} \end{aligned}$$

where $\alpha, \beta \in R^n$, C'_m, C''_m are positive constants depending only on m. Then, we obtain $u_k \to u$ strongly in $X_\beta^{s,p}$ as $k \to \infty$. □

In [1] the existence and properties of solutions for the minimization problem (1.6) when $\alpha = 0$, were investigated. For $0 \leq \alpha < ps < n$, from [10, Theorem 1.1], there exists a minimizer for S_α, for every minimizer U_α, there exist $x_0 \in \mathbb{R}^n$ and a non-increasing $u : \mathbb{R}^+ \to \mathbb{R}$ such that $U_\alpha(x) = u(|x - x_0|)$. Next we fix a radially symmetric decreasing minimizer $U_\alpha = U_\alpha(r)$ for S_α, multiplying U_α by a positive constant if necessary, we may assume that

$$(-\Delta)_p^s U_\alpha = \frac{U_\alpha^{p_\alpha^*-1}}{|x|^\alpha}, \quad \text{in } \mathbb{R}^n.$$

Lemma 4.2 ([10])**.** *There exist $c_1, c_2 > 0$ and $\kappa > 1$ such that*

$$\frac{c_1}{r^{(n-ps)/(p-1)}} \le U_\alpha(r) \le \frac{c_2}{r^{(n-ps)/(p-1)}}, \quad \frac{U_\alpha(\kappa r)}{U_\alpha(r)} \le \frac{1}{2} \quad \text{for all } r \ge 1.$$

For each $\delta \ge \varepsilon > 0$. Let

$$m_{\varepsilon,\delta} = \frac{U_{\alpha,\varepsilon}(\delta)}{U_{\alpha,\varepsilon}(\delta) - U_{\alpha,\varepsilon}(\kappa\delta)},$$

and

$$g_{\varepsilon,\delta}(t) = \begin{cases} 0, & \text{if } 0 \le t \le U_{\alpha,\varepsilon}(\kappa\delta), \\ m_{\varepsilon,\delta}^p(t - U_{\alpha,\varepsilon}(\kappa\delta)), & \text{if } U_{\alpha,\varepsilon}(\kappa\delta) \le t \le U_{\alpha,\varepsilon}(\delta), \\ t + U_{\alpha,\varepsilon}(\delta)(m_{\varepsilon,\delta}^{p-1} - 1), & \text{if } t \ge U_{\alpha,\varepsilon}(\delta). \end{cases} \tag{4.14}$$

The functions $g_{\varepsilon,\delta}$ and $G_{\varepsilon,\delta}$ are nondecreasing and absolutely continuous. Consider now the radially symmetric nonincreasing function $u_{\alpha,\varepsilon,\delta}(r) = G_{\varepsilon,\delta}(U_{\alpha,\varepsilon}(r))$, which satisfies

$$u_{\alpha,\varepsilon,\delta}(r) = \begin{cases} U_{\alpha,\varepsilon}(r), & \text{if } r \le \delta, \\ 0, & \text{if } r \ge \kappa\delta. \end{cases} \tag{4.15}$$

Lemma 4.3 ([10])**.** *There exists $\widetilde{C} > 0$ such that for any $0 < 2\varepsilon \le \delta < \kappa^{-1}\delta_\Omega$, it holds*

$$\int_{\mathbb{R}^{2n}} \frac{|u_{\alpha,\varepsilon,\delta}(x) - u_{\alpha,\varepsilon,\delta}(y)|^p}{|x-y|^\alpha}\,dx\,dy \le S_\alpha^{(n-\alpha)/(ps-\alpha)} + \widetilde{C}(\frac{\varepsilon}{\delta})^{(n-ps)/(p-1)},$$

$$\int_{\mathbb{R}^n} \frac{|u_{\alpha,\varepsilon,\delta}^{p_\alpha^*}|}{|x|^\alpha} dx \ge S_\alpha^{(n-\alpha)/(ps-\alpha)} - \widetilde{C}(\frac{\varepsilon}{\delta})^{(n-\alpha)/(p-1)}.$$

Moreover, for each $1 < q < p_\alpha^*$, there exists $C_q > 0$ such that

$$\int_{R^n} u_{\alpha,\varepsilon,\delta}(x)^q \ge C_q \begin{cases} \varepsilon^{n - \frac{n-ps}{p}q} |\log\frac{\varepsilon}{\delta}|, & \text{if } q = \frac{n(p-1)}{n-ps}, \\ \varepsilon^{\frac{n-ps}{n(p-1)}q} \delta^{n - \frac{n-ps}{p-1}q}, & \text{if } q < \frac{n(p-1)}{n-ps}, \\ \varepsilon^{n - \frac{n-ps}{p}q}, & \text{if } q > \frac{n(p-1)}{n-ps}. \end{cases} \tag{4.16}$$

Lemma 4.4. *Assume that $0 \le \alpha < ps < n$ and $q \ge \frac{n(p-1)}{(n-ps)}$. Then there exist $\widehat{\lambda} > 0$ and $u_0 \in X_\beta^{s,p} \backslash \{0\}$ such that $\sup_{t\ge0} I_\lambda(tu_0) < c_\Lambda$ for all $0 < \lambda < \widehat{\lambda}$, where c_Λ is the constant given in Lemma 4.1. In particular, $c_\Lambda^- < c_\Lambda$ for all λ satisfying $0 < \lambda < \widehat{\lambda}$.*

Proof. Let $u_0 = u_{\alpha,\varepsilon,\delta}$, which is defined in Lemma 4.2, we consider the function

$$\begin{aligned} f(t) &= I_\lambda(tu_0) \\ &= \frac{1}{p}t^p\|u_0\|_{X_\beta^{s,p}}^p + \frac{1}{p}t^p \int_{\mathbb{R}^n\backslash\Omega} \beta(x)|u_0|^p dx - \frac{\lambda}{q}t^q \int_\Omega |u_0|^q dx - \frac{1}{r}t^r \int_\Omega \frac{|u_0|^r}{|x|^\alpha} dx, \end{aligned}$$

with

$$\widetilde{f}(t) = \frac{1}{p}t^p\|u_0\|_{X_\beta^{s,p}}^p + \frac{1}{p}t^p \int_{\mathbb{R}^n\backslash\Omega} \beta(x)|u_0|^p dx - \frac{1}{r}t^r \int_\Omega \frac{|u_0|^r}{|x|^\alpha} dx,$$

for all $t > 0$, then there exists

$$t_* = \Big(\frac{\|u_0\|_{X_\beta^{s,p}}^p + \int_{\mathbb{R}^n\backslash\Omega} \beta(x)|u_0|^p dx}{\int_\Omega \frac{|u_0|^r}{|x|^\alpha} dx}\Big)^{1/(r-p)} > 0$$

such that $\widetilde{f}'(t_*) = 0$ and $\widetilde{f}(t_*) \geq \widetilde{f}(t)$.

Next we have

$$\begin{aligned}
&\sup_{t\geq 0} \widetilde{f}(t) \\
&= \widetilde{f}(t_*) \\
&= \frac{1}{p} t_*^p \|u_0\|^p_{X^{s,p}_\beta} + \frac{1}{p} t_*^p \int_{\mathbb{R}^n\setminus\Omega} \beta(x)|u_0|^p dx - \frac{1}{r} t_*^r \int_\Omega \frac{|u_0|^r}{|x|^\alpha} dx \\
&= (\frac{1}{p} - \frac{1}{r}) \frac{\Big(\|u_0\|^p_{X^{s,p}_\beta} + \int_{\mathbb{R}^n\setminus\Omega} \beta(x)|u_0|^p dx\Big)^{r/(r-p)}}{\Big(\int_\Omega \frac{|u_0|^r}{|x|^\alpha} dx\Big)^{p/(r-p)}} \qquad (4.17) \\
&\leq (\frac{1}{p} - \frac{1}{r}) \frac{\Big(S_\alpha^{(n-\alpha)/(ps-\alpha)} + \widetilde{C}(\frac{\varepsilon}{\delta})^{(n-ps)/(p-1)} + \int_{\mathbb{R}^n\setminus\Omega} \beta(x)|u_0|^p dx\Big)^{r/(r-p)}}{\Big(S_\alpha^{(n-\alpha)/(ps-\alpha)} - \widetilde{C}(\frac{\varepsilon}{\delta})^{(n-\alpha)/(p-1)}\Big)^{p/(r-p)}} \\
&\leq (\frac{1}{p} - \frac{1}{r}) S_\alpha^{r/(r-p)} + \widetilde{C}(\frac{\varepsilon}{\delta})^{(n-ps)/(p-1)}.
\end{aligned}$$

Then, we prove that $\sup_{t\geq 0} I_\lambda(tu_0) < c_\Lambda$ in two cases $0 \leq t \leq \tau_1$ and $t \geq \tau_1$ for $\tau_1 \in (0,1)$. First, we have

$$\sup_{0\geq t\leq \tau_1} I_\lambda(tu_0) < c_\Lambda.$$

Then, from (4.17) and Lemma 4.3, we obtain

$$\begin{aligned}
\sup_{t\geq\tau_1} I_\lambda(tu_0) &= \sup_{t\geq\tau_1} \Big(\widetilde{f}(t) - \frac{1}{q} t^q \lambda \int_\Omega |u_0|^q dx\Big) \\
&\leq (\frac{1}{p} - \frac{1}{r}) S_\alpha^{\frac{r}{r-p}} + \widetilde{C}(\frac{\varepsilon}{\delta})^{\frac{n-ps}{p-1}} - \frac{1}{q}\tau_1^q \lambda \int_\Omega |u_0|^q dx.
\end{aligned} \qquad (4.18)$$

Hence, we cam compute that

$$\sup_{t\geq\tau_1} I_\lambda(tu_0) \leq (\frac{1}{p} - \frac{1}{r}) S_\alpha^{\frac{r}{r-p}} + \widetilde{C}(\frac{\varepsilon}{\delta})^{\frac{n-ps}{p-1}} - \widetilde{C}\lambda \begin{cases} \varepsilon^{n-\frac{n-ps}{p}q} |log\frac{\varepsilon}{\delta}|, & \text{if } q = \frac{n(p-1)}{n-ps}, \\ \varepsilon^{\frac{n-ps}{n(p-1)}q} \delta^{n-\frac{n-ps}{p-1}q}, & \text{if } q < \frac{n(p-1)}{n-ps}, \\ \varepsilon^{n-\frac{n-ps}{p}q}, & \text{if } q > \frac{n(p-1)}{n-ps}. \end{cases}$$

Let $\varepsilon = (\lambda^{\frac{p}{p-q}})^{\frac{p-1}{n-ps}} \in (0, \frac{\delta}{2})$¿ Then we have

$$\begin{aligned}
&\sup_{t\geq\tau_1} I_\lambda(tu_0) \\
&\leq (\frac{1}{p} - \frac{1}{r}) S_\alpha^{\frac{r}{r-p}} + \widetilde{C}\lambda^{\frac{p}{p-q}} - \widetilde{C}\lambda \begin{cases} (\lambda^{\frac{p}{p-q}})^{\frac{n(p-1)}{(n-ps)p}} |log(\lambda^{\frac{p}{p-q}})^{\frac{p-1}{n-ps}}|, & \text{if } q = \frac{n(p-1)}{n-ps}, \\ \Big((\lambda^{\frac{p}{p-q}})^{\frac{p-1}{n-ps}}\Big)^{n-\frac{n-ps}{p}q}, & \text{if } q > \frac{n(p-1)}{n-ps}. \end{cases}
\end{aligned}$$

If $q > \frac{n(p-1)}{n-ps}$, then

$$1 + \frac{p}{p-q}\frac{p-1}{n-ps}\Big(n - \frac{n-ps}{p}q\Big) < \frac{p}{p-q},$$

hence, we can find $\delta_2 > 0$ such that for $0 < \lambda < \delta_2$,

$$\widetilde{C}\lambda^{\frac{p}{p-q}} - \widetilde{C}\lambda\Big((\lambda^{\frac{p}{p-q}})^{\frac{p-1}{n-ps}}\Big)^{n-\frac{n-ps}{p}q} < -\bar{c}\frac{r-q}{r}\Big(\frac{r-p}{pq}\Big)^{q/(q-r)}\Big(\frac{(p-q)\lambda}{pq}\Big)^{r/(r-q)}.$$

If $q=\frac{n(p-1)}{n-ps}$, we can find $\delta_3>0$ such that for $0<\lambda<\delta_3$,

$$\begin{aligned}&\widetilde{C}\lambda^{\frac{p}{p-q}}-\widetilde{C}\lambda(\lambda^{\frac{p}{p-q}})^{\frac{n(p-1)}{(n-ps)p}}|\log(\lambda^{\frac{p}{p-q}})^{\frac{p-1}{n-ps}}|\\&<-\bar{c}\frac{r-q}{r}\Big(\frac{r-p}{pq}\Big)^{q/(q-r)}\Big(\frac{(p-q)\lambda}{pq}\Big)^{r/(r-q)}.\end{aligned}$$

Since $|log(\lambda^{\frac{p}{p-q}})^{\frac{p-1}{n-ps}}|\to\infty$ as $\lambda\to 0$, and $\lambda(\lambda^{\frac{p}{p-q}})^{\frac{n(p-1)}{(n-ps)p}}\sim\lambda^{\frac{p}{p-q}}$. Then taking

$$\widehat{\delta}=\min\{\delta_1,\delta_2,\delta_3,(\frac{\delta}{2})^{\frac{n-ps}{p-1}}\}>0,$$

we derive that

$$\sup_{t\geq 0}I_\lambda(tu_0)<c_\Lambda,\quad \text{for } \lambda\in(0,\widehat{\delta}).$$

From the above inequality and Lemma 2.4, there exists $t^->0$ such that $t^-u_0\in N_\lambda^-$ and

$$c_\lambda^-\leq I_\lambda(t^-u_0)\leq\sup_{t\geq 0}I_\lambda(tu_0)<c_\Lambda,$$

for all $\lambda\in(0,\widehat{\delta})$. □

Theorem 4.5. *There exists $\Lambda_1>0$ such that for $0<\lambda<\Lambda_1$ and $r=p_\alpha^*$, the functional I_λ has a minimizer $u_3\in N_\lambda^+$ and satisfies*

(1) $I_\lambda(u_3)=c_\lambda=c_\lambda^+<0$,
(2) *u_3 is a solution of the problem* (1.1).

Proof. Set $\Lambda_1=\min\{\frac{q}{p}\Lambda,\widehat{\delta}\}$. Then $c_\Lambda>0$. From Lemma 3.4, we obtain $c_\lambda\leq c_\lambda^+<0$, then $c_\lambda<c_\Lambda$. By Proposition 3.6 (1), for all $0<\lambda<\Lambda_1$, there exists a bounded minimizing sequence $\{u_k\}\subset N_\lambda$ such that

$$\lim_{k\to\infty}I_\lambda(u_k)=c_\lambda\leq c_\lambda^+,I'_\lambda(u_k)=o(1)\quad\text{in } X_\beta^{-s,p}.$$

Then there exists $u_3\in X_\beta^{s,p}$ such that, up to a subsequence, $u_k\rightharpoonup u_3$ weakly in $X_\beta^{s,p}$. By Lemma 4.1 and $c_\lambda<c_\Lambda$, we obtain $u_k\to u_3$ strongly in $X_\beta^{s,p}$.

As in the proof of Theorem 3.7, we can obtain $u_3\in N_\lambda^+$, $I_\lambda(u_3)=c_\lambda=c_\lambda^+$ and u_3 is a solution of the problem (1.1). □

Theorem 4.6. *There exists $\Lambda_2>0$ such that for $0<\lambda<\Lambda_2$ and $r=p_\alpha^*$, the functional I_λ has a minimizer $u_4\in N_\lambda^-$ and satisfies*

(1) $I_\lambda(u_4)=c_\lambda^-$,
(2) *u_4 is a solution of the problem* (1.1).

Proof. Set $\Lambda_2=\min\{\frac{q}{p}\Lambda,\widehat{\delta}\}$. By Lemma 4.4, it is easy to get $c_\lambda^-<c_\Lambda$. By Proposition 3.6 (2), for all $0<\lambda<\Lambda_2$, there exists a bounded minimizing sequence $\{u_k\}\subset N_\lambda^-$ such that

$$\lim_{k\to\infty}I_\lambda(u_k)=c_\lambda^-,I'_\lambda(u_k)=o(1)\quad\text{in } X_\beta^{-s,p}.$$

By the same argument as in the proof of Theorem 4.5, there exists $u_4\in N_\lambda^-$ such that $I_\lambda(u_4)=c_\lambda^-$ and u_4 is a solution of problem (1.1). □

Proof of Theorem 1.2. Taking $\lambda^*=\Lambda_2$, by Theorems 4.5 and 4.6, for all $\lambda\in(0,\lambda^*)$, problem (1.1) has two solutions $u_3\in N_\lambda^+$ and $u_4\in N_\lambda^-$ in $X_\beta^{s,p}$. In addition $N_\lambda^+\bigcap N_\lambda^-=\emptyset$, then the two solutions u_3 and u_4 are distinct. □

Acknowledgments. This work is supported by the Hubei Provincial Natural Science Foundation of China (No. 2022CFC016) and by the Doctoral Scientific Research Foundation of Hubei University of Education.

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