

PROPERTIES OF THE DIRICHLET GREEN'S FUNCTION FOR LINEAR DIFFUSIONS ON A HALF LINE

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ABSTRACT. This article concerns the study of Green's functions for one dimensional diffusions with constant diffusion coefficient and linear time inhomogeneous drift. It is well known that the whole line Green's function is given by a Gaussian. Formulas for the Dirichlet Green's function on the half line are only known in special cases. The main object of study in the paper is the ratio of the Dirichlet to whole line Green's functions. Bounds, asymptotic behavior in the limit as the diffusion coefficient vanishes, and a log concavity result are obtained for this ratio. These results have been used in the proof of asymptotic behavior for a simple model of Ostwald ripening.

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1. INTRODUCTION

In this article we prove some results for diffusions on the line with affine time dependent drift, which are used in [5] to study the large time behavior of solutions to a *nonlinear nonlocal* diffusion problem occurring in the theory of Ostwald ripening [14]. This theory describes the time evolution of crystals in a solute, whereby smaller crystals dissolve and then deposit onto larger crystals. An important quantity is the *coarsening rate*, which is the rate of increase of the average crystal volume with time. Mean field models of Ostwald ripening were developed by Becker-Döring [3], Lifshitz-Slyozov [13] and Wagner [19]. The LSW model introduced in [13],

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and independently in [19], consists of a first order linear transport PDE on the half line with a linear constraint corresponding to conservation of volume, making the model nonlinear and nonlocal. The Becker-Döring (BD) model is a linear transport equation on the positive integers with a linear conservation of volume constraint. The transport equation on the positive integers can be interpreted as the discretization of a linear diffusion equation [18]. Furthermore, solutions of the BD model at large time are expected to be approximate solutions to the LSW model [15].

The LSW model has a family of self-similar solutions, which may be parametrized by a real number β with $0 < \beta \leq 1$. Already in [13, 19] it was conjectured that the only physically relevant self-similar solution is the $\beta = 1$ solution. Therefore the large time asymptotic coarsening rate in Ostwald ripening may be obtained from the $\beta = 1$ solution. The Carr-Penrose (CP) model introduced in [4] is a simplified version of the LSW model in which the transportation vector field is affine. It also has a family of self-similar solutions parametrized by β with $0 < \beta \leq 1$. It is shown in [4] that for each β with $0 < \beta \leq 1$ there exists a large class of initial data with compact support for the CP model which asymptotically converges to the self-similar solution with parameter β . The main result of [5] is that all solutions to a *diffusive* CP model with initial data of compact support asymptotically converge to the $\beta = 1$ self-similar solution of the CP model. Thus diffusion acts as a *selection principle* on the one parameter family of self-similar solutions of the CP model. The selection principle was initially established for a semi-classical approximation to the diffusive CP model [6].

The key difficulty in going from proving the selection principle for the semi-classical diffusive CP model to proving it for the diffusive model is controlling the ratio of the Dirichlet Green's function for the CP diffusion equation on the half line to the Green's function on the whole line, which is Gaussian. This is the subject of the present paper. The results most relevant for the diffusive CP problem are the log concavity property (1.17) of the ratio of Green's functions and the convergence (1.19) of the logarithmic derivative of the ratio as the diffusion constant goes to zero.

To specify the diffusion equation we are interested in, let $b : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function $(y, t) \rightarrow b(y, t)$, which is linear in the space variable y . For $\varepsilon > 0$, the terminal value problem

$$\frac{\partial u_\varepsilon(y, t)}{\partial t} + b(y, t) \frac{\partial u_\varepsilon(y, t)}{\partial y} + \frac{\varepsilon}{2} \frac{\partial^2 u_\varepsilon(y, t)}{\partial y^2} = 0, \quad y \in \mathbb{R}, \quad t < T, \quad (1.1)$$

$$u_\varepsilon(y, T) = u_T(y), \quad y \in \mathbb{R}, \quad (1.2)$$

has a unique solution u_ε which has the representation

$$u_\varepsilon(y, t) = \int_{-\infty}^{\infty} G_\varepsilon(x, y, t, T) u_T(x) dx, \quad y \in \mathbb{R}, \quad t < T, \quad (1.3)$$

where G_ε is the Green's function for the problem. The adjoint problem to (1.1), (1.2) is the initial value problem

$$\frac{\partial v_\varepsilon(x, t)}{\partial t} + \frac{\partial}{\partial x} [b(x, t) v_\varepsilon(x, t)] = \frac{\varepsilon}{2} \frac{\partial^2 v_\varepsilon(x, t)}{\partial x^2}, \quad x \in \mathbb{R}, \quad t > 0, \quad (1.4)$$

$$v_\varepsilon(x, 0) = v_0(x), \quad y \in \mathbb{R}. \quad (1.5)$$

The solution to (1.4), (1.5) is given by the formula

$$v_\varepsilon(x, T) = \int_{-\infty}^{\infty} G_\varepsilon(x, y, 0, T) v_0(y) dy, \quad x \in \mathbb{R}, T > 0. \quad (1.6)$$

Since the drift $b(\cdot, \cdot)$ is linear and G_ε is Gaussian, $(x, y) \rightarrow \log G_\varepsilon(x, y, t, T)$ is a quadratic function in (x, y) . Here we shall obtain properties of the corresponding Dirichlet Green's function $(x, y) \rightarrow G_{\varepsilon, D}(x, y, t, T)$ on the half line $x, y > 0$. Thus

$$u_{\varepsilon, D}(y, t) = \int_0^{\infty} G_{\varepsilon, D}(x, y, t, T) u_T(x) dx, \quad y > 0, t < T, \quad (1.7)$$

is the solution to (1.1), (1.2) in the domain $\{(y, t) : y > 0, t < T\}$ with Dirichlet boundary condition $u_{\varepsilon, D}(0, t) = 0, t < T$.

The drifts $b(\cdot, \cdot)$ we consider are of the form

$$b(y, t) = A(t)y - 1, \quad \text{where } A : \mathbb{R} \rightarrow \mathbb{R} \text{ is a continuous function,} \quad (1.8)$$

but the methods of the paper may be extended to more general linear drifts. In the case $A(\cdot) \equiv 0$ there are simple explicit formulas for G_ε and $G_{\varepsilon, D}$. These are given by

$$G_\varepsilon(x, y, t, T) = \frac{1}{\sqrt{2\pi\varepsilon(T-t)}} \exp\left[-\frac{(x+T-t-y)^2}{2\varepsilon(T-t)}\right], \quad (1.9)$$

$$G_{\varepsilon, D}(x, y, t, T) = \left\{1 - \exp\left[-\frac{2xy}{\varepsilon(T-t)}\right]\right\} G_\varepsilon(x, y, t, T). \quad (1.10)$$

For non-trivial $A(\cdot)$, we write

$$G_{\varepsilon, D}(x, y, t, T) = \left\{1 - \exp\left[-\frac{q_\varepsilon(x, y, t, T)}{\varepsilon}\right]\right\} G_\varepsilon(x, y, t, T), \quad (1.11)$$

and study the properties of the function q_ε . When $A(\cdot) \equiv 0$ we have from (1.10) that the function $(x, y) \rightarrow q_\varepsilon(x, y, t, T)$ is independent of ε and bilinear. One can also obtain explicit formulas for q_ε in some other cases of linear drift, in particular for the Ornstein-Uhlenbeck process where $b(y, t) = -\gamma y$ with constant γ (see [17, Prop. 20] and Remark 3.4 of the present paper). However there appears not to be an explicit formula for q_ε in the case of the general function $A(\cdot)$.

We are able to obtain linear bounds on the function $x \rightarrow q_\varepsilon(x, y, t, T)$, $x > 0$, and its first two x derivatives, which are uniform in $\varepsilon > 0$, when the function $A(\cdot)$ is assumed to be non-negative:

Theorem 1.1. *Assume the function $A(\cdot)$ of (1.8) is continuous and non-negative, and q_ε is defined by (1.11). Then there exists a continuous positive function $\alpha_1(\cdot)$ and continuous non-negative functions $\beta_1(\cdot), \beta_2(\cdot)$, with domain $\{(t, T) : t, T \in \mathbb{R}, t < T\}$, which bound q_ε as follows:*

$$\alpha_1(t, T)xy \leq q_\varepsilon(x, y, t, T) \leq [\alpha_1(t, T)x + \beta_1(t, T)]y, \quad x, y > 0, t < T, \quad (1.12)$$

$$\lim_{x \rightarrow \infty} \{[\alpha_1(t, T)x + \beta_1(t, T)]y - q_\varepsilon(x, y, t, T)\} = 0, \quad y > 0, t < T, \quad (1.13)$$

$$q_\varepsilon(x, y, t, T) \leq [\alpha_1(t, T)y + \beta_2(t, T)]x, \quad x, y > 0, t < T, \quad (1.14)$$

$$\alpha_1(t, T)y \leq \frac{\partial q_\varepsilon(x, y, t, T)}{\partial x} \leq \alpha_1(t, T)y + \beta_2(t, T), \quad x, y > 0, t < T, \quad (1.15)$$

$$\lim_{x \rightarrow \infty} \left\{ \frac{\partial q_\varepsilon(x, y, t, T)}{\partial x} - \alpha_1(t, T)y \right\} = 0, \quad y > 0, t < T, \quad (1.16)$$

$$\text{The function } x \rightarrow q_\varepsilon(x, y, t, T), \quad x, y > 0, t < T, \text{ is concave.} \quad (1.17)$$

Since $q_\varepsilon(0, y, t, T) = 0$ the lower bound in (1.12) is implied by the lower bound in (1.15). Similarly the upper bound in (1.15) implies the upper bound in (1.14). However our proof of (1.15) in Proposition 5.1 uses the inequalities (1.12), (1.14), which have previously been established in Proposition 3.3. The proof of (1.13) is given in Proposition 5.2, the proof of (1.16) in Proposition 5.4, and the proof of (1.17) in Theorem 5.7.

Theorem 1.1 tells us that the graph of the function $x \rightarrow q_\varepsilon(x, y, t, T)$, $x > 0$, lies between two parallel lines and is asymptotic to the upper line at large x . The graph also lies in a wedge formed by two lines through the origin and is concave. This geometric picture gives us a rather precise understanding of the global behavior of the function $x \rightarrow q_\varepsilon(x, y, 0, T)$, $x > 0$. The significance of the upper bounds (1.14), (1.15) can be understood by considering the situation when $\varepsilon \rightarrow 0$. The function $[x, T] \rightarrow q_\varepsilon(x, y, 0, T)$, $x, T > 0$, is a solution to the Hamilton-Jacobi-Bellman equation (2.22). One expects then that $\lim_{\varepsilon \rightarrow 0} q_\varepsilon(x, y, t, T) = q_0(x, y, t, T)$ exists, and in the case $t = 0$ is a solution to the Hamilton-Jacobi equation (2.29). In §2 we show that the limit does exist and $q_0(x, y, 0, T)$ is given by the variational formula (2.28) corresponding to the Hamilton-Jacobi equation. In §4 we study this variational problem in great detail, establishing in particular that if $A(\cdot)$ is continuous and non-negative then the function $x \rightarrow q_0(x, y, t, T)$ is differentiable in a neighborhood of $x = 0$ and $\partial q_0(0, y, t, T)/\partial x = \alpha_1(t, T)y + \beta_2(t, T)$. More precisely we have the following theorem.

Theorem 1.2. *Assume the function $A(\cdot)$ of (1.8) is continuous and q_ε is defined by (1.11). Then there is a continuous function $[x, y, t, T] \rightarrow q_0(x, y, t, T)$ with domain $\{[x, y, t, T] : x, y \geq 0, t, T \in \mathbb{R}, t < T\}$ such that $\lim_{\varepsilon \rightarrow 0} q_\varepsilon(x, y, t, T) = q_0(x, y, t, T)$ for all $x, y \geq 0, t < T$. If $A(\cdot)$ is non-negative then the function $x \rightarrow q_0(x, y, t, T)$ is differentiable at $x = 0$ and*

$$\frac{\partial q_0(x, y, t, T)}{\partial x} \Big|_{x=0} = \alpha_1(t, T)y + \beta_2(t, T). \quad (1.18)$$

Furthermore, for any $t \in \mathbb{R}, T_0 > 0$, there are constants $C_1, C_2 > 0$, depending only on T_0 and $\sup_{t \leq s \leq t+T_0} A(s)$, such that

$$\left| \frac{\partial q_\varepsilon(0, y, t, T)}{\partial x} - [\alpha_1(t, T)y + \beta_2(t, T)] \right| \leq \frac{\varepsilon C_2 (T - t)^2}{y^2} \quad (1.19)$$

for $y \geq C_1(T - t)^2$, $T - t \leq T_0$, provided $\varepsilon \leq (T - t)^3$.

In Theorem 2.6 we prove that $\lim_{\varepsilon \rightarrow 0} q_\varepsilon(x, y, t, T) = q_0(x, y, t, T)$ exists and is the solution to the variational problem (2.28). In Proposition 4.2 we show if $A(\cdot)$ is non-negative that $q_0(x, y, 0, T)$ may be obtained by the method of characteristics in a subdomain of $\{[x, T] : x > 0, T > 0\}$, which includes a neighborhood of the boundary $\{[0, T] : T > 0\}$. The function $[x, T] \rightarrow q_0(x, y, 0, T)$ is then a classical solution of the Hamilton-Jacobi equation (2.29) in this region, and the derivative $\partial q_0(x, y, 0, T)/\partial x$ is given by the formula (4.26).

We use the methods of stochastic control theory to prove that $\lim_{\varepsilon \rightarrow 0} q_\varepsilon = q_0$ and (1.19). The Bellman equation (2.22) corresponds to the stochastic variational problem (2.26). The proofs of $\lim_{\varepsilon \rightarrow 0} q_\varepsilon = q_0$ and (1.19) are then obtained by comparing the solution of this stochastic variational problem to the solution of the classical variational problem (2.28). The proof of (1.19) is given in Proposition

6.2, and uses in a crucial way the regularity of the function $x \rightarrow q_0(x, y, 0, T)$ in a neighborhood of $x = 0$.

The ratio of Green's functions $G_{\varepsilon, D}(x, y, t, T)/G_{\varepsilon}(x, y, t, T)$ given in (1.11) is the probability that a generalized Brownian bridge, beginning at y at time t and ending at x at time T , lies entirely in the positive half line. We may therefore try to estimate $q_{\varepsilon}(x, y, t, T)$ by comparing this generalized bridge to the standard Brownian bridge. This method of bridge comparison is used in the proof of Proposition 3.2. However for the most part we use the fact that the bridge process is a Gaussian Markov process with the linear drift (2.15) in order to estimate $q_{\varepsilon}(x, y, t, T)$. There is a considerable literature on the study of bridges. In [8, 9] Conforti et al study bridges associated to diffusions with a gradient drift, using the fact that it is the *reciprocal characteristics* which determine the bridge uniquely. In particular, diffusions with differing drifts may have the same bridge processes. A simple example of this is the case of the drift (1.8) with $A(\cdot) \equiv 0$. The bridge process associated with the constant drift is the same as the Brownian bridge. In Proposition [12, Prop. 3] formulas for first passage time for diffusions with time-inhomogeneous drift are given. However in these cases there needs to be a relation between the graph of the boundary and the drift and diffusion coefficients of the process. The first passage time for the half line is given in terms of the Dirichlet Green's function by the function $t \rightarrow \int_0^{\infty} G_{\varepsilon, D}(x, y, t, T) dx$, $t < T$. An alternative approach to understanding the limit $\lim_{\varepsilon \rightarrow 0} q_{\varepsilon}$ may be taken using the techniques of large deviation theory [11]. This is the approach in Baldi et al [1, 2], which considers the asymptotic behavior of the ratio of Green's functions in the limit $T - t \rightarrow 0$ for time homogeneous diffusions.

2. REPRESENTATION AND CONVERGENCE OF THE FUNCTION q_{ε}

For any $t \in \mathbb{R}$ let $Y_{\varepsilon}(s)$, $s > t$, be the solution to the initial value problem for the stochastic differential equation (SDE)

$$dY_{\varepsilon}(s) = b(Y_{\varepsilon}(s), s)ds + \sqrt{\varepsilon}dB(s), \quad Y_{\varepsilon}(t) = y, \quad (2.1)$$

where $B(\cdot)$ is Brownian motion. Then the Green's function $G_{\varepsilon}(\cdot, y, t, T)$ defined by (1.3) is the probability density for the random variable $Y_{\varepsilon}(T)$. In the case when the function $(y, t) \rightarrow b(y, t)$ is linear in y it is easy to see that (2.1) can be explicitly solved. The solution to (2.1) with $b(y, t) = A(t)y - 1$ as in (1.8) is given by

$$\begin{aligned} Y_{\varepsilon}(s) = & \exp \left[\int_t^s A(s')ds' \right] y - \int_t^s \exp \left[\int_{s'}^s A(s'')ds'' \right] ds' \\ & + \sqrt{\varepsilon} \int_t^s \exp \left[\int_{s'}^s A(s'')ds'' \right] dB(s'). \end{aligned} \quad (2.2)$$

Hence the random variable $Y_{\varepsilon}(T)$ conditioned on $Y_{\varepsilon}(0) = y$ is Gaussian with mean $m_{1,A}(T)y - m_{2,A}(T)$ and variance $\varepsilon\sigma_A^2(T)$, where

$$m_{1,A}(T) = \exp \left[\int_0^T A(s')ds' \right], \quad m_{2,A}(T) = \int_0^T \exp \left[\int_s^T A(s')ds' \right] ds, \quad (2.3)$$

$$\sigma_A^2(T) = \int_0^T \exp \left[2 \int_s^T A(s')ds' \right] ds. \quad (2.4)$$

The Green's function $G_\varepsilon(x, y, 0, T)$ is therefore explicitly given by the formula

$$G_\varepsilon(x, y, 0, T) = \frac{1}{\sqrt{2\pi\varepsilon\sigma_A^2(T)}} \exp\left[-\frac{\{x + m_{2,A}(T) - m_{1,A}(T)y\}^2}{2\varepsilon\sigma_A^2(T)}\right]. \quad (2.5)$$

It is useful to recall that the ratio $G_{\varepsilon,D}/G_\varepsilon$ is a probability for a generalized Brownian bridge process. Thus

$$\frac{G_{\varepsilon,D}(x, y, 0, T)}{G_\varepsilon(x, y, 0, T)} = P\left(\inf_{0 < s < T} Y_\varepsilon(s) > 0 \mid Y_\varepsilon(0) = y, Y_\varepsilon(T) = x\right), \quad (2.6)$$

where $Y_\varepsilon(\cdot)$ is the solution to the SDE (2.1). The process $Y_\varepsilon(\cdot)$ of (2.1), conditioned on $Y_\varepsilon(0) = y, Y_\varepsilon(T) = x$ is Gaussian and has variance independent of x, y . We may obtain a formula by extending the functions $m_{1,A}, m_{2,A}, \sigma_A^2$ of (2.3), (2.4), defined with respect to the interval $[0, T]$, to any interval $[t, T]$ with $t < T$. Thus we define $m_{1,A}(t, T), m_{2,A}(t, T)$ by

$$m_{1,A}(t, T) = \exp\left[\int_t^T A(s') ds'\right], \quad m_{2,A}(t, T) = \int_t^T \exp\left[\int_s^T A(s') ds'\right] ds, \quad (2.7)$$

and $\sigma_A^2(t, T)$ by

$$\sigma_A^2(t, T) = \int_t^T \exp\left[2\int_s^T A(s') ds'\right] ds. \quad (2.8)$$

The variance of $Y_\varepsilon(s)$ is then given by the formula

$$\text{Var}[Y_\varepsilon(s) \mid Y_\varepsilon(0) = y, Y_\varepsilon(T) = x] = \varepsilon\sigma_A^2(s)\sigma_A^2(s, T)/\sigma_A^2(T). \quad (2.9)$$

More generally, the covariance of $Y_\varepsilon(\cdot)$ is also independent of x, y and is given by the formula

$$\text{Covar}[Y_\varepsilon(s_1), Y_\varepsilon(s_2) \mid Y_\varepsilon(0) = y, Y_\varepsilon(T) = x] = \varepsilon\Gamma_A(s_1, s_2), \quad 0 \leq s_1, s_2 \leq T, \quad (2.10)$$

where the symmetric function $\Gamma : [0, T] \times [0, T] \rightarrow \mathbb{R}$ is defined by

$$\Gamma_A(s_1, s_2) = \frac{m_{1,A}(s_1, s_2)\sigma_A^2(s_1)\sigma_A^2(s_2, T)}{\sigma_A^2(T)}, \quad 0 \leq s_1 \leq s_2 \leq T. \quad (2.11)$$

Let $y_{\text{class}}(s)$, $0 \leq s \leq T$, be the path going from y at $s = 0$ to x at $s = T$ defined by

$$\begin{aligned} \sigma_A^2(T)y_{\text{class}}(s) &= xm_{1,A}(s, T)\sigma_A^2(s) + ym_{1,A}(s)\sigma_A^2(s, T) \\ &\quad + m_{1,A}(s, T)m_{2,A}(s, T)\sigma_A^2(s) - m_{2,A}(s)\sigma_A^2(s, T). \end{aligned} \quad (2.12)$$

Then the mean of $Y_\varepsilon(\cdot)$ conditioned on $Y_\varepsilon(0) = y, Y_\varepsilon(T) = x$ is given by the formula

$$E[Y_\varepsilon(s) \mid Y_\varepsilon(0) = y, Y_\varepsilon(T) = x] = y_{\text{class}}(s), \quad 0 \leq s \leq T. \quad (2.13)$$

In the case $A(\cdot) \equiv 0$ the process $Y_\varepsilon(\cdot)$, conditioned on $Y_\varepsilon(0) = y, Y_\varepsilon(T) = x$ is the standard Brownian Bridge (BB) from y at time 0 to x at time T .

It is well known that the conditioned process $Y_\varepsilon(\cdot)$ is also Markovian. In [6] we showed that it is the solution to an SDE with a linear drift depending on x and with initial condition $Y_\varepsilon(0) = y$ (see [6, (4.43), (4.46)]). The SDE in this case is run *forwards* in time. Here we observe that the conditioned process is also the solution of an SDE with a linear drift depending on y , which is run *backwards* in time. Denoting by $X_\varepsilon(s)$, $0 < s < T$, the solution to this SDE with terminal condition $X_\varepsilon(T) = x$, we have that

$$dX_\varepsilon(s) = \lambda(X_\varepsilon(s), y, s) ds + \sqrt{\varepsilon} dB(s), \quad 0 < s < T, \quad X_\varepsilon(T) = x. \quad (2.14)$$

The function $\lambda(x, y, s)$ is given by the formula

$$\lambda(x, y, s) = \left[A(s) + \frac{1}{\sigma_A^2(s)} \right] x - 1 + \frac{m_{2,A}(s)}{\sigma_A^2(s)} - \frac{m_{1,A}(s)y}{\sigma_A^2(s)}. \quad (2.15)$$

Integrating (2.14), (2.15), we have that

$$X_\varepsilon(s) = y_{\text{class}}(s) - \sqrt{\varepsilon} \frac{\sigma_A^2(s)}{m_{1,A}(s)} Z(s), \quad \text{with } Z(s) = \int_s^T \frac{m_{1,A}(s') dB(s')}{\sigma_A^2(s')}, \quad (2.16)$$

where $y_{\text{class}}(\cdot)$ is given in (2.12). Note that the drift $\lambda(\cdot)$ is independent of ε and $\lambda(x, y, s)$ becomes singular as $s \rightarrow 0$. The singularity is necessary in order to ensure that $X_\varepsilon(0) = y$ with probability 1,

We define now

$$v_\varepsilon(x, y, T) = P\left(\inf_{0 \leq s \leq T} X_\varepsilon(s) < 0 \mid X_\varepsilon(T) = x\right). \quad (2.17)$$

Comparing (2.6), (2.17) we see that

$$v_\varepsilon(x, y, T) = 1 - \frac{G_{\varepsilon,D}(x, y, 0, T)}{G_\varepsilon(x, y, 0, T)}. \quad (2.18)$$

The function v_ε is a solution to the PDE

$$\frac{\partial v_\varepsilon(x, y, T)}{\partial T} = -\lambda(x, y, T) \frac{\partial v_\varepsilon(x, y, T)}{\partial x} + \frac{\varepsilon}{2} \frac{\partial^2 v_\varepsilon(x, y, T)}{\partial x^2}, \quad T > 0, x > 0, \quad (2.19)$$

with initial and boundary conditions

$$\lim_{x \rightarrow 0} v_\varepsilon(x, y, T) = 1, \quad T > 0, \quad \lim_{T \rightarrow 0} v_\varepsilon(x, y, T) = 0, \quad x > 0. \quad (2.20)$$

Next we set q_ε to be

$$q_\varepsilon(x, y, T) = -\varepsilon \log v_\varepsilon(x, y, T). \quad (2.21)$$

Then q_ε is a solution to the PDE

$$\frac{\partial q_\varepsilon(x, y, T)}{\partial T} = -\lambda(x, y, T) \frac{\partial q_\varepsilon(x, y, T)}{\partial x} - \frac{1}{2} \left[\frac{\partial q_\varepsilon(x, y, T)}{\partial x} \right]^2 + \frac{\varepsilon}{2} \frac{\partial^2 q_\varepsilon(x, y, T)}{\partial x^2}, \quad (2.22)$$

for $T > 0$ and $x > 0$, with initial and boundary conditions

$$q_\varepsilon(0, y, T) = 0, \quad T > 0, \quad q_\varepsilon(x, y, 0) = +\infty, \quad x > 0. \quad (2.23)$$

In the case $A(\cdot) \equiv 0$ it is easy to see that the drift λ and solution q_ε to (2.22), (2.23) are given by the formulae

$$\lambda(x, y, s) = \frac{(x-y)}{s}, \quad q_\varepsilon(x, y, T) = \frac{2xy}{T}. \quad (2.24)$$

Evidently (2.24) is consistent with (1.10).

The PDE (2.22) is the Hamilton-Jacobi-Bellman (HJB) equation for a stochastic control problem. Thus consider solutions $X_\varepsilon(\cdot)$ to the SDE

$$dX_\varepsilon(s) = \mu_\varepsilon(X_\varepsilon(s), y, s) ds + \sqrt{\varepsilon} dB(s), \quad (2.25)$$

run *backwards* in time with controller $\mu_\varepsilon(\cdot)$ and given terminal data. For $x, y, T > 0$ define $q_\varepsilon(x, y, T)$ by

$$q_\varepsilon(x, y, T) = \min_{\mu_\varepsilon} E \left[\frac{1}{2} \int_\tau^T [\mu_\varepsilon(X_\varepsilon(s), y, s) - \lambda(X_\varepsilon(s), y, s)]^2 ds : X_\varepsilon(T) = x, 0 < \tau < T, X_\varepsilon(\cdot) > 0, X_\varepsilon(\tau) = 0 \right], \quad (2.26)$$

where the function $\lambda(\cdot)$ is given by (2.15). The class of controllers $\mu_\varepsilon(\cdot)$ in (2.25) are those which have the property that paths $X_\varepsilon(s)$, $s < T$, with $X_\varepsilon(T) = x > 0$, exit the half line $(0, \infty)$ before time 0 with probability 1. The HJB equation for q_ε is then given by (2.22), with initial and boundary conditions (2.23). The optimal controller $\mu_\varepsilon^*(\cdot)$ in (2.25), (2.26) is given by the formula

$$\mu_\varepsilon^*(x, y, T) = \lambda(x, y, T) + \frac{\partial q_\varepsilon(x, y, T)}{\partial x}. \quad (2.27)$$

The zero noise limit $\varepsilon \rightarrow 0$ of (2.25), (2.26) yields the classical variational formula

$$q_0(x, y, T) = \min \left\{ \frac{1}{2} \int_\tau^T \left[\frac{dx(s)}{ds} - \lambda(x(s), y, s) \right]^2 ds : \right. \\ \left. 0 < \tau < T, x(T) = x, x(\cdot) > 0, x(\tau) = 0 \right\}. \quad (2.28)$$

At least formally, the function q_0 is the solution to the Hamilton-Jacobi (HJ) equation

$$\frac{\partial q_0(x, y, T)}{\partial T} = -\lambda(x, y, T) \frac{\partial q_0(x, y, T)}{\partial x} - \frac{1}{2} \left[\frac{\partial q_0(x, y, T)}{\partial x} \right]^2, \quad T > 0, x > 0, \quad (2.29)$$

with initial and boundary conditions

$$q_0(0, y, T) = 0, \quad T > 0, \quad q_0(x, y, 0) = +\infty, \quad x > 0. \quad (2.30)$$

When $A(\cdot) \equiv 0$ the function $q_0(x, y, T) = 2xy/T$ is a classical C^1 solution to (2.29), (2.30). However in general we can only expect q_0 to be a viscosity solution of the HJ equation (see [10, Chapter 10]).

To obtain an upper bound on q_ε by q_0 plus a constant which vanishes as $\varepsilon \rightarrow 0$, we observe that the variational problem (2.28) for *fixed* τ with $0 < \tau < T$, without the positivity constraint $x(\cdot) > 0$, is quadratic with a linear constraint, which may be easily solved. The Euler-Lagrange equation for the minimization problem with fixed τ is

$$\left\{ \frac{d}{ds} + \frac{\partial \lambda(x(s), y, s)}{\partial x} \right\} \left[\frac{dx(s)}{ds} - \lambda(x(s), y, s) \right] = 0. \quad (2.31)$$

The minimizing trajectory is then the solution to (2.31) with initial and terminal conditions $x(\tau) = 0$, $x(T) = x$. To obtain a formula for this trajectory we observe that the solution to the equation

$$\frac{d\phi(s)}{ds} + \frac{\partial \lambda(x(s), y, s)}{\partial x} \phi(s) = 0, \quad (2.32)$$

is given by the formula

$$\phi(s) = \frac{C_1 m_{1,A}(s)}{\sigma_A^2(s)} \quad \text{where } C_1 \text{ is a constant.} \quad (2.33)$$

We then need to obtain the solution to

$$\frac{dx(s)}{ds} - \lambda(x(s), y, s) = \phi(s), \quad (2.34)$$

with initial and terminal conditions $x(\tau) = 0$, $x(T) = x$, and this determines the constant C_1 in (2.33). Observe that the function $x(s) = y_{\text{class}}(s)$, $0 < s < T$, where $y_{\text{class}}(\cdot)$ is defined by (2.12), is the solution to (2.34) with terminal condition

$x(T) = x$ in the case $\phi(\cdot) \equiv 0$. Let $y_p(\cdot)$ be the solution to the terminal value problem

$$\frac{dy_p(s)}{ds} - \left[A(s) + \frac{1}{\sigma_A^2(s)} \right] y_p(s) + \frac{m_{1,A}(s)}{\sigma_A^2(s)} = 0, \quad y_p(T) = 0. \tag{2.35}$$

The solution to (2.34) with initial and terminal conditions $x(\tau) = 0, x(T) = x$, is then $x(s) = y_{\text{class}}(s) - C_1 y_p(s)$, where C_1 is chosen so that $x(\tau) = 0$. The solution to (2.35) is given by

$$\sigma_A^2(T) y_p(s) = m_{1,A}(s) \sigma_A^2(s, T). \tag{2.36}$$

We have then from (2.35), (2.36) that

$$\begin{aligned} \sigma_A^2(T) x(s) &= x m_{1,A}(s, T) \sigma_A^2(s) + [y - \gamma(\tau)] m_{1,A}(s) \sigma_A^2(s, T) \\ &\quad + m_{1,A}(s, T) m_{2,A}(s, T) \sigma_A^2(s) - m_{2,A}(s) \sigma_A^2(s, T). \end{aligned} \tag{2.37}$$

where $\gamma(\tau)$ is chosen so that $x(\tau) = 0$. The optimal controller $\mu_{0,\tau}^*$ for the variational problem with fixed τ is obtained by evaluating $dx(s)/ds$ at $s = T$. Thus

$$\begin{aligned} \mu_{0,\tau}^*(x, y, T) &= \lambda(x, y, T) + \frac{\gamma(\tau) m_{1,A}(T)}{\sigma_A^2(T)} \\ &= \lambda(x, y, T) + \frac{m_{1,A}(T)}{\sigma_A^2(T)} [y + g_{1,A}(\tau, T)x + g_{2,A}(\tau, T)], \end{aligned} \tag{2.38}$$

where the functions $g_{1,A}, g_{2,A}$ are given by the formulae

$$g_{1,A}(s, T) = \frac{m_{1,A}(s, T) \sigma_A^2(s)}{m_{1,A}(s) \sigma_A^2(s, T)}, \quad s < T, \tag{2.39}$$

$$g_{2,A}(s, T) = \frac{m_{1,A}(s, T) m_{2,A}(s, T) \sigma_A^2(s) - m_{2,A}(s) \sigma_A^2(s, T)}{m_{1,A}(s) \sigma_A^2(s, T)}, \quad s < T. \tag{2.40}$$

Lemma 2.1. *Let $\tau > 0$ and $X_\varepsilon(s), s > \tau$, be the solution to the SDE (2.25) with μ_ε given by $\mu_\varepsilon(x, y, s) = \mu_{0,\tau}^*(x, y, s), s > \tau$. For $x > 0, T > \tau$ let $\tau_{\varepsilon,x,T}$ be the first exit time from the interval $(0, \infty)$ of $X_\varepsilon(\cdot)$ with terminal condition $X_\varepsilon(T) = x$. Then $\tau_{\varepsilon,x,T} > \tau$ with probability 1 and*

$$q_\varepsilon(x, y, T) \leq E \left[\frac{1}{2} \int_{\tau_{\varepsilon,x,T}}^T [\mu_\varepsilon(X_\varepsilon(s), y, s) - \lambda(X_\varepsilon(s), y, s)]^2 ds \mid X_\varepsilon(T) = x \right]. \tag{2.41}$$

Proof. Since the function $A(\cdot)$ is continuous, we have from (2.38)-(2.40) that

$$\mu_\varepsilon(x, y, s) = \left[\frac{1}{s - \tau} + A_\tau(s) \right] x + (s - \tau) B_\tau(s), \quad s > \tau, \tag{2.42}$$

where A_τ, B_τ are continuous functions on the closed interval $[\tau, \infty)$. Let m_{1,A_τ} be defined as in (2.7). The solution to (2.25) with μ_ε as in (2.42) and terminal condition $X_\varepsilon(T) = x$ is given by

$$X_\varepsilon(s) = \frac{s - \tau}{(T - \tau) m_{1,A_\tau}(s, T)} [X_{\text{class}}(s) - \sqrt{\varepsilon} Z(s)], \quad \tau < s < T, \tag{2.43}$$

where $X_{\text{class}}(\cdot), Z(\cdot)$ are given by the formulae

$$X_{\text{class}}(s) = x - (T - \tau) \int_s^T m_{1,A_\tau}(s', T) B_\tau(s') ds', \tag{2.44}$$

$$Z(s) = (T - \tau) \int_s^T \frac{m_{1,A_\tau}(s', T)}{s' - \tau} dB(s'). \tag{2.45}$$

Since $Z(\cdot)$ is by a change of variable equivalent to Brownian motion, the reflection principle applies to it. Hence for any $a > 0$, $\tau < s < T$,

$$P\left(\sup_{s < s' < T} Z(s') > a\right) = 2P(Z(s) > a). \tag{2.46}$$

From (2.45) we see that $Z(s)$ is Gaussian with mean zero. The variance $\text{Var}[Z(s)]$ satisfies the inequality

$$\frac{c_1(T - \tau)(T - s)}{s - \tau} \leq \text{Var}[Z(s)] \leq \frac{C_1(T - \tau)(T - s)}{s - \tau}, \quad \tau < s < T, \tag{2.47}$$

for some positive constants c_1, C_1 . From (2.46), (2.47) we conclude that

$$P\left(\sup_{s < s' < T} Z(s') > a\right) \geq 1 - C_2 a \frac{\sqrt{s - \tau}}{\sqrt{(T - \tau)(T - s)}}, \quad \tau < s < T, \tag{2.48}$$

where $C_2 > 0$ is a constant. We also have from (2.44) there is a constant C_3 such that $\sup_{\tau < s < T} X_{\text{class}}(s) \leq C_3$. Choosing $a = \varepsilon^{-1/2}C_3$ in (2.48), we conclude from (2.43), (2.48) that $\lim_{s \rightarrow \tau} P(\tau_{\varepsilon, x, T} > s) = 1$. Hence $\tau_{\varepsilon, x, T} > \tau$ with probability 1.

To prove (2.41) we first observe from Ito's lemma that the mapping $s \rightarrow M(s)$ on the interval $\tau < s \leq T$, where

$$\begin{aligned} M(s) &= q_\varepsilon(x, y, T) - q_\varepsilon(X_\varepsilon(s), y, s) - \int_s^T \frac{\partial q_\varepsilon(X_\varepsilon(s'), y, s')}{\partial s'} \\ &\quad + \frac{\partial q_\varepsilon(X_\varepsilon(s'), y, s')}{\partial x} \mu_\varepsilon(X_\varepsilon(s'), y, s') - \frac{\varepsilon}{2} \frac{\partial^2 q_\varepsilon(X_\varepsilon(s'), y, s')}{\partial x^2} ds' \end{aligned} \tag{2.49}$$

is a (backwards in time) stochastic integral. From (2.22) we see that $M(s)$ can be written as

$$\begin{aligned} M(s) &= q_\varepsilon(x, y, T) - q_\varepsilon(X_\varepsilon(s), y, s) \\ &\quad - \int_s^T \frac{\partial q_\varepsilon(X_\varepsilon(s'), y, s')}{\partial x} [\mu_\varepsilon(X_\varepsilon(s'), y, s') - \lambda(X_\varepsilon(s'), y, s')] \\ &\quad - \frac{1}{2} \left[\frac{\partial q_\varepsilon(X_\varepsilon(s'), y, s')}{\partial x} \right]^2 ds'. \end{aligned} \tag{2.50}$$

For $K > 0$ and $0 < x < K$ let $\tau_{\varepsilon, x, T, K}$ be the first exit time of $X_\varepsilon(s)$, $s < T$, with $X_\varepsilon(T) = x$ from the interval $(0, K)$. By the optional sampling theorem we have that $E[M(s \vee \tau_{\varepsilon, x, T, K})] = 0$ for all s in the interval $\tau < s \leq T$. Hence on using the Schwarz inequality in (2.50) we have that

$$\begin{aligned} &q_\varepsilon(x, y, T) \\ &\leq E[q_\varepsilon(X_\varepsilon(s \vee \tau_{\varepsilon, x, T, K}), y, s \vee \tau_{\varepsilon, x, T, K})] \\ &\quad + E\left[\frac{1}{2} \int_{s \vee \tau_{\varepsilon, x, T, K}}^T [\mu_\varepsilon(X_\varepsilon(s), y, s) - \lambda(X_\varepsilon(s), y, s)]^2 ds \mid X_\varepsilon(T) = x\right], \end{aligned} \tag{2.51}$$

for $\tau < s \leq T$.

Observe that for $\tau < s \leq T$ we have $s \vee \tau_{\varepsilon, x, T, K} \geq \tau_{\varepsilon, x, T, K} \geq \tau_{\varepsilon, x, T} > \tau$ with probability 1. Hence the second expectation on the RHS of (2.51) is bounded above by the expectation on the RHS of (2.41). Thus it is sufficient to show that the first term on the RHS of (2.51) converges to 0 as $s \rightarrow \tau$ and $K \rightarrow \infty$. We first consider the limit $s \rightarrow \tau$. Since $\tau_{\varepsilon, x, T, K} > \tau$ with probability 1, we have by path continuity

of $X_\varepsilon(\cdot)$ that $\lim_{s \rightarrow \tau} X_\varepsilon(s \vee \tau_{\varepsilon,x,T,K}) = X_\varepsilon(\tau_{\varepsilon,x,T,K})$ with probability 1. Using the continuity of the function q_ε , it follows by dominated convergence that

$$\begin{aligned} & \lim_{s \rightarrow \tau} E[q_\varepsilon(X_\varepsilon(s \vee \tau_{\varepsilon,x,T,K}), y, s \vee \tau_{\varepsilon,x,T,K})] \\ &= E[q_\varepsilon(X_\varepsilon(\tau_{\varepsilon,x,T,K}), y, \tau_{\varepsilon,x,T,K})] \\ &\leq \sup_{\tau < s \leq T} q_\varepsilon(K, y, s)P(X_\varepsilon(\tau_{\varepsilon,x,T,K}) = K). \end{aligned} \tag{2.52}$$

From (2.46) and (2.47) we have for some constants $C_2, c_2 > 0$ that

$$P\left(\inf_{s < s' < T} Z(s') < -a\right) \leq \frac{C_2}{a\sqrt{s-\tau}} \exp[-c_2 a^2(s-\tau)], \quad a > 0. \tag{2.53}$$

We have from (2.43) there is a constant $c_3 > 0$ such that for all large K ,

$$P(X_\varepsilon(\tau_{\varepsilon,x,T,K}) = K) \leq P\left(\inf_{\tau < s \leq T} (s-\tau)Z(s) < -c_3 K/\sqrt{\varepsilon}\right). \tag{2.54}$$

From (2.53) we see that

$$\begin{aligned} & P\left(\inf_{\tau < s \leq T} (s-\tau)Z(s) < -c_3 K/\sqrt{\varepsilon}\right) \\ &\leq \sum_{n=0}^\infty P\left(\inf_{2^n < (T-\tau)/(s-\tau) \leq 2^{n+1}} Z(s) < -\frac{2^n c_3 K}{\sqrt{\varepsilon}(T-\tau)}\right) \\ &\leq C_2 \sum_{n=0}^\infty \exp\left[-\frac{c_2 c_3^2 K^2 2^{n-1}}{\sqrt{\varepsilon}(T-\tau)}\right], \end{aligned} \tag{2.55}$$

if K is large. From our bound (3.9) on q_ε we conclude from (2.54), (2.55) that

$$\lim_{K \rightarrow \infty} \sup_{\tau < s \leq T} q_\varepsilon(K, y, s)P(X_\varepsilon(\tau_{\varepsilon,x,T,K}) = K) = 0,$$

whence (2.41) follows from (2.51). □

Lemma 2.2. For x, y, T positive one has $\limsup_{\varepsilon \rightarrow 0} [q_\varepsilon(x, y, T) - q_0(x, y, T)] = 0$.

Proof. We first observe that a minimizing $\tau = \tau_{0,x,T}$ (which may not be unique) in (2.28) satisfies $0 < \tau_{0,x,T} < T$. To see this we use that fact that the minimizing trajectory for fixed τ is given by (2.43), (2.44) with $\varepsilon = 0$. From (2.15) we have that $\lambda(x, y, s) \simeq (x-y)/s$ as $s \rightarrow 0$, whence the minimum action integral diverges as $\tau \rightarrow 0$. When τ is close to T we have from (2.42), (2.43) that $\mu_{0,\tau}^*(X_0(s), y, s) \simeq x/(T-\tau)$, whence the minimum action integral again diverges as $\tau \rightarrow T$.

From (2.43)-(2.45) we have that

$$X_\varepsilon(s) - X_0(s) = -\frac{s-\tau}{(T-\tau)m_{1,A_\tau}(s,T)}\sqrt{\varepsilon}Z(s), \quad \tau < s \leq T. \tag{2.56}$$

From (2.15), (2.41), (2.42) we have that

$$\begin{aligned} & q_\varepsilon(x, y, T) \\ &\leq \sqrt{\varepsilon}E\left[\int_{\tau_{\varepsilon,x,T}}^T f_\tau(s)Z(s) ds\right] + C(\tau)\varepsilon E\left[\frac{1}{2}\int_{\tau_{\varepsilon,x,T}}^T Z(s)^2 ds\right] \\ &+ E\left[\frac{1}{2}\int_{\tau_{\varepsilon,x,T}}^T [\mu_{0,\tau}^*(X_0(s), y, s) - \lambda(X_0(s), y, s)]^2 ds \mid X_0(T) = x\right], \end{aligned} \tag{2.57}$$

where the function $f_\tau : [\tau, T] \rightarrow \mathbb{R}$ is a continuous function depending on τ , and $C(\tau)$ is a constant also depending on τ . We choose now τ to be a minimizer of

the action (2.28). Since $\tau_{\varepsilon,x,T} > \tau$ with probability 1, it follows that the last expectation on the RHS of (2.57) is bounded above by $q_0(x, y, T)$.

To bound the first term on the RHS of (2.57) we note from the optional sampling theorem that $E[Z(s \vee \tau_{\varepsilon,x,T})] = 0$. Hence $E[Z(s); \tau_{\varepsilon,x,T} < s] = -E[Z(\tau_{\varepsilon,x,T}); \tau_{\varepsilon,x,T} > s]$. Letting $M = \sup_{\tau < s \leq T} X_{\text{class}}(s)$, we see from (2.43) that $0 < Z(\tau_{\varepsilon,x,T}) < M/\sqrt{\varepsilon}$. Hence we have that

$$\sqrt{\varepsilon} E \left[\int_{\tau_{\varepsilon,x,T}}^T f_{\tau}(s) Z(s) ds \right] \leq C_1 E[\tau_{\varepsilon,x,T} - \tau], \tag{2.58}$$

for some constant C_1 independent of ε . For $0 < \lambda < T - \tau$ we let $\gamma(\lambda) = \inf_{\tau+\lambda < s < T} X_{\text{class}}(s)$, whence $\gamma : (0, T - \tau] \rightarrow \mathbb{R}$ is a positive decreasing function. We have from (2.53) and reflection symmetry that

$$\begin{aligned} P(\tau_{\varepsilon,x,T} - \tau > \lambda) &\leq P\left(\sup_{\tau+\lambda < s < T} Z(s) > \gamma(\lambda)/\sqrt{\varepsilon}\right) \\ &\leq \frac{C_2 \sqrt{\varepsilon}}{\sqrt{\lambda} \gamma(\lambda)} \exp\left[-\frac{c_2 \lambda \gamma(\lambda)^2}{\varepsilon}\right]. \end{aligned} \tag{2.59}$$

We see from (2.59) that $\lim_{\varepsilon \rightarrow 0} P(\tau_{\varepsilon,x,T} - \tau > \lambda) = 0$ for all $\lambda > 0$, whence $\lim_{\varepsilon \rightarrow 0} E[\tau_{\varepsilon,x,T} - \tau] = 0$ by dominated convergence.

We see from (2.47) that the second term on the RHS of (2.57) diverges if we replace $\tau_{\varepsilon,x,T}$ by τ . Therefore it is again necessary to estimate the distribution of the variable $\tau_{\varepsilon,x,T} - \tau > 0$ as $\varepsilon \rightarrow 0$. With M as in the previous paragraph, and using (2.46), (2.47) we have from (2.43) that

$$\begin{aligned} P(\tau_{\varepsilon,x,T} - \tau < \lambda) &\leq P\left(\sup_{\tau+\lambda < s < T} Z(s) < M/\sqrt{\varepsilon}\right) \\ &= P(|Z(\tau + \lambda)| < M/\sqrt{\varepsilon}) \\ &\leq C_2(\lambda/\varepsilon)^{1/2}, \quad 0 < \lambda < T - \tau, \end{aligned} \tag{2.60}$$

where $c_2, C_2 > 0$ are constants. We write now

$$E \left[\int_{\tau_{\varepsilon,x,T}}^T Z(s)^2 ds \right] \leq E \left[\int_{\tau+\varepsilon}^T Z(s)^2 ds \right] + \sum_{n=0}^{\infty} a_n, \tag{2.61}$$

where

$$a_n = E \left[\int_{\tau+2^{-(n+1)}\varepsilon}^{\tau+2^{-n}\varepsilon} Z(s)^2 ds; \tau_{\varepsilon,x,T} - \tau < 2^{-n}\varepsilon \right], \quad n = 0, 1, \dots \tag{2.62}$$

It follows from (2.47) that the first term on the RHS of (2.61) is bounded by $C_3 |\log \varepsilon|$ for some constant C_3 . From the Schwarz inequality we have that

$$a_n \leq 2^{-(n+1)/2} \sqrt{\varepsilon} P(\tau_{\varepsilon,x,T} - \tau < 2^{-n}\varepsilon)^{1/2} E \left[\int_{\tau+2^{-(n+1)}\varepsilon}^{\tau+2^{-n}\varepsilon} Z(s)^4 ds \right]^{1/2}. \tag{2.63}$$

It follows from (2.47), (2.60), (2.63) that $a_n \leq C_4 P(\tau_{\varepsilon,x,T} - \tau < 2^{-n}\varepsilon)^{1/2} \leq C_5 2^{-n/2}$ for some constants C_4, C_5 . We have shown that the second term on the RHS of (2.57) converges to 0 as $\varepsilon \rightarrow 0$. \square

Remark 2.3. One can obtain a rate of convergence $\varepsilon \log \varepsilon$ in Lemma 2.2 as $\varepsilon \rightarrow 0$ by making a further assumption that the classical trajectory $X_0(\cdot)$ has the property $X'_0(\tau) > 0$. In that case $\lim_{\lambda \rightarrow 0} \gamma(\lambda) \geq c_1$ for some constant $c_1 > 0$, whence (2.59) implies that $E[\tau_{\varepsilon,x,T} - \tau] \leq C_2 \varepsilon$ for some constant C_2 .

To obtain a lower bound for q_ε by q_0 plus a constant which vanishes as $\varepsilon \rightarrow 0$ we need to show that the variational formula (2.26) yields a lower bound when μ_ε is chosen to be the optimal controller μ_ε^* given by (2.27). We have from propositions 3.1 and 3.2 that the function $(x, T) \rightarrow \mu_\varepsilon^*(x, y, T)$ is C^1 on the domain $x, T > 0$ and $\mu_\varepsilon^*(x, y, T) \geq \lambda(x, y, T)$ for $x, y, T > 0$. Hence the SDE (2.25) with $\mu_\varepsilon = \mu_\varepsilon^*$ may be solved backwards in time. Letting $X_\varepsilon^*(s)$, $s \leq T$, be the solution with terminal condition $X_\varepsilon^*(T) = x$, then $X_\varepsilon^*(s) \leq X_\varepsilon(s)$, $0 < s \leq T$, where $X_\varepsilon(\cdot)$ is given by (2.16).

Lemma 2.4. *For $x, T > 0$ and paths $X_\varepsilon^*(s)$, $s < T$, with $X_\varepsilon^*(T) = x$ we define $\tau_{\varepsilon, x, T}^* = \inf\{s > 0 : X_\varepsilon^*(s') > 0, s \leq s' \leq T\}$. Then $\tau_{\varepsilon, x, T}^* > 0$ with probability 1 and*

$$E\left[\left(\frac{1}{\tau_{\varepsilon, x, T}^*}\right)^{1/2-\nu}\right] < \infty \quad \text{for all } \nu \text{ with } 0 < \nu \leq 1/2, \quad (2.64)$$

$$q_\varepsilon(x, y, T) \geq E\left[\frac{1}{2} \int_{\tau_{\varepsilon, x, T}^*}^T [\mu_\varepsilon^*(X_\varepsilon^*(s), y, s) - \lambda(X_\varepsilon^*(s), y, s)]^2 ds \mid X_\varepsilon^*(T) = x\right]. \quad (2.65)$$

Proof. We consider the stochastic integral $s \rightarrow M(s)$, $0 < s \leq T$, defined by (2.49), (2.50) with $\mu_\varepsilon = \mu_\varepsilon^*$. For $K > 0$ and $0 < x < K$ let $\tau_{\varepsilon, x, T, K}^*$ be the first exit time of $X_\varepsilon^*(s)$, $s < T$, with $X_\varepsilon^*(T) = x$ from the interval $(0, K)$. Since q_ε is non-negative, we have from (2.27) and the optional sampling theorem that

$$\begin{aligned} q_\varepsilon(x, y, T) &\geq E\left[\frac{1}{2} \int_{s \vee \tau_{\varepsilon, x, T, K}^*}^T [\mu_\varepsilon^*(X_\varepsilon^*(s), y, s) - \lambda(X_\varepsilon^*(s), y, s)]^2 ds \mid X_\varepsilon^*(T) = x\right] \end{aligned} \quad (2.66)$$

for $0 < s \leq T$. Since $X_\varepsilon^*(\cdot) \leq X_\varepsilon(\cdot)$, it follows that $s \vee \tau_{\varepsilon, x, T, K}^* \rightarrow s \vee \tau_{\varepsilon, x, T}^*$ with probability 1 as $K \rightarrow \infty$, whence the inequality (2.66) holds with $\tau_{\varepsilon, x, T}^*$ in place of $\tau_{\varepsilon, x, T, K}^*$.

Letting $X_\varepsilon(\cdot)$ be the solution to (2.25) with terminal condition $X_\varepsilon(T) = x > 0$, we have from (2.15), (2.25) that

$$\begin{aligned} &\frac{m_{1,A}(s)}{\sigma_A^2(s)} [\mu_\varepsilon(X_\varepsilon(s), y, s) - \lambda(X_\varepsilon(s), y, s)] ds \\ &= d\left[\frac{m_{1,A}(s)X_\varepsilon(s)}{\sigma_A^2(s)}\right] - \sqrt{\varepsilon} \frac{m_{1,A}(s)}{\sigma_A^2(s)} dB(s) - d\left[\frac{m_{1,A}(s)^2 \sigma_A^2(s, T)}{\sigma_A^2(T) \sigma_A^2(s)}\right] y \\ &\quad - d\left[\frac{m_{1,A}(T)m_{2,A}(s, T)}{\sigma_A^2(T)} - \frac{m_{1,A}(s)m_{2,A}(s) \sigma_A^2(s, T)}{\sigma_A^2(T) \sigma_A^2(s)}\right], \quad 0 < s \leq T. \end{aligned} \quad (2.67)$$

Let δ satisfy $0 < \delta \leq T$ and τ_δ be the stopping time $\tau_\delta = \delta \vee \tau_{\varepsilon,x,T}^*$. On integrating (2.67) with $\mu_\varepsilon = \mu_\varepsilon^*$ over the interval $\tau_\delta < s < T$, we obtain the identity

$$\begin{aligned} & \int_{\tau_\delta}^T \frac{m_{1,A}(s)}{\sigma_A^2(s)} [\mu_\varepsilon^*(X_\varepsilon^*(s), y, s) - \lambda(X_\varepsilon^*(s), y, s)] ds \\ &= \frac{m_{1,A}(T)x}{\sigma_A^2(T)} - \frac{m_{1,A}(\tau_\delta)X_\varepsilon^*(\tau_\delta)}{\sigma_A^2(\tau_\delta)} - \sqrt{\varepsilon}Z(\tau_\delta) \\ &+ \frac{m_{1,A}(\tau_\delta)^2\sigma_A^2(\tau_\delta, T)}{\sigma_A^2(T)\sigma_A^2(\tau_\delta)}y + \frac{m_{1,A}(T)m_{2,A}(\tau_\delta, T)}{\sigma_A^2(T)} \\ &- \frac{m_{1,A}(\tau_\delta)m_{2,A}(\tau_\delta)\sigma_A^2(\tau_\delta, T)}{\sigma_A^2(T)\sigma_A^2(\tau_\delta)}, \end{aligned} \quad (2.68)$$

where the martingale $Z(\cdot)$ is defined in (2.16). From the Schwarz inequality we see that the LHS of (2.68) is bounded above by

$$\frac{\alpha y}{2} \int_{\tau_\delta}^T \frac{m_{1,A}(s)^2}{\sigma_A^4(s)} ds + \frac{1}{2\alpha y} \int_{\tau_\delta}^T [\mu_\varepsilon^*(X_\varepsilon^*(s), y, s) - \lambda(X_\varepsilon^*(s), y, s)]^2 ds, \quad (2.69)$$

for any $\alpha > 0$. Choosing α sufficiently small, we conclude from (2.68) that

$$\begin{aligned} \frac{y}{\tau_{\varepsilon,x,T}^*} &\leq C_1 [1 + \sqrt{\varepsilon} |Z(\tau_\delta)|] \\ &+ \frac{C_1}{2\alpha y} \int_{\tau_{\varepsilon,x,T}^*}^T [\mu_\varepsilon^*(X_\varepsilon^*(s), y, s) - \lambda(X_\varepsilon^*(s), y, s)]^2 ds \end{aligned} \quad (2.70)$$

if $\delta \leq \tau_{\varepsilon,x,T}^* \leq T/2$, for some constants C_1, α depending only on T . Multiplying (2.70) by $(\tau_{\varepsilon,x,T}^*)^{1/2+\nu}$ and taking the expected value, we conclude from the limit of (2.66) as $K \rightarrow \infty$ that

$$\begin{aligned} & yE \left[\left(\frac{1}{\tau_{\varepsilon,x,T}^*} \right)^{1/2-\nu}; \delta < \tau_{\varepsilon,x,T}^* < T/2 \right] \\ & \leq C_1 \left[T^{1/2+\nu} + \sqrt{\varepsilon}E \left[\tau_\delta^{1/2+\nu} |Z(\tau_\delta)| \right] \right] + \frac{C_1 T^{1/2+\nu} q_\varepsilon(x, y, T)}{\alpha y}. \end{aligned} \quad (2.71)$$

Let τ be a stopping time for the martingale $s \rightarrow Z(s)$, $0 < s < T$, of (2.16). Then for $\nu > 0$ there is a constant C_ν , depending on ν , such that

$$E[\tau^{1/2+\nu}; |Z(\tau)| > a] \leq \frac{C_\nu}{a^{1+2\nu}} \quad \text{for } a > 0. \quad (2.72)$$

To show (2.72), let $\tau_a = \sup\{s : 0 < s < T, |Z(s)| \geq a\}$. Then any stopping time τ for $Z(\cdot)$ has the property

$$\{\tau > s, |Z(\tau)| > a\} \subset \{\tau_a > s\}, \quad (2.73)$$

since $\{\tau_a > s\}$ is the largest Borel set in $\mathcal{F}_{s,T}$, the σ -field generated by $B(s')$, $s < s' < T$, on which $\sup_{s < s' < T} |Z(s')| > a$. Hence

$$\begin{aligned} E[\tau^{1/2+\nu}; |Z(\tau)| > a] &= \left(\frac{1}{2} + \nu\right) \int_0^T s^{\nu-1/2} P(\tau > s, |Z(\tau)| > a) ds \\ &\leq \left(\frac{1}{2} + \nu\right) \int_0^T s^{\nu-1/2} P(\tau_a > s) ds \\ &= E[\tau_a^{1/2+\nu}]. \end{aligned} \quad (2.74)$$

The reflection principle applies to $Z(\cdot)$ and also $\text{Var}[Z(s)]$ satisfies an inequality (2.47) with $\tau = 0$. We have therefore from (2.53) that

$$P(\tau_a > n/a^2) = P\left(\sup_{n/a^2 < s < T} |Z(s)| \geq a\right) \leq \frac{2C_2}{\sqrt{n}} e^{-c_2 n}, \quad n = 1, 2, \dots \quad (2.75)$$

The inequality (2.72) follows now from (2.74) and (2.75).

We easily see from (2.72) that $E[\tau_\delta^{1/2+\nu} |Z(\tau_\delta)|]$ is bounded by a constant, uniformly in δ as $\delta \rightarrow 0$. Thus for any $a > 0$,

$$\begin{aligned} E[\tau_\delta^{1/2+\nu} |Z(\tau_\delta)|] &\leq aE[\tau_\delta^{1/2+\nu}] + \sum_{n=0}^\infty 2^{n+1} aE[\tau_\delta^{1/2+\nu}; |Z(\tau_\delta)| > 2^n a] \\ &\leq aT^{1/2+\nu} + \frac{C_{1,\nu}}{a^{2\nu}}, \end{aligned} \quad (2.76)$$

where $C_{1,\nu}$ depends on $\nu > 0$, but not on δ . Choosing a to minimize the RHS of (2.76) we conclude that $E[\tau_\delta^{1/2+\nu} |Z(\tau_\delta)|] \leq C_{2,\nu} T^\nu$ for some constant depending only on ν . Hence the RHS of (2.71) is bounded by a constant independent of δ . Letting $\delta \rightarrow 0$ we conclude that $\tau_{\varepsilon,x,T}^* > 0$ with probability 1 and (2.64) holds.

The inequality (2.65) follows from (2.66) and the monotone convergence theorem by letting $K \rightarrow \infty$ first and then $s \rightarrow 0$. \square

Lemma 2.5. *For x, y, T positive one has $\liminf_{\varepsilon \rightarrow 0} [q_\varepsilon(x, y, T) - q_0(x, y, T)] = 0$.*

Proof. Let $X_\varepsilon^*(s)$, $s \leq T$, be a solution to (2.25) with $\mu_\varepsilon = \mu_\varepsilon^*$ and terminal condition $X_\varepsilon^*(T) = x$. We associate with $X_\varepsilon^*(\cdot)$ the differentiable path $X_{\varepsilon,c}^*(\cdot)$ defined by

$$\frac{dX_{\varepsilon,c}^*(s)}{ds} = \lambda(X_{\varepsilon,c}^*(s), y, s) + [\mu_\varepsilon^*(X_\varepsilon^*(s), y, s) - \lambda(X_\varepsilon^*(s), y, s)], \quad s < T, \quad (2.77)$$

with terminal condition $X_{\varepsilon,c}^*(T) = x$. From (2.15), (2.25), and (2.77) we see that

$$d\{X_\varepsilon^*(s) - X_{\varepsilon,c}^*(s)\} = \left[A(s) + \frac{1}{\sigma_A^2(s)}\right] \{X_\varepsilon^*(s) - X_{\varepsilon,c}^*(s)\} + \sqrt{\varepsilon} dB(s), \quad s < T, \quad (2.78)$$

with zero terminal condition at $s = T$. Comparing (2.78) to (2.14), we conclude from (2.16) that

$$X_{\varepsilon,c}^*(s) = X_\varepsilon^*(s) + \sqrt{\varepsilon} \frac{\sigma_A^2(s)}{m_{1,A}(s)} Z(s), \quad s < T, \quad (2.79)$$

with $Z(\cdot)$ as given in (2.16). We define a classical action which generalizes (2.28). Thus for $x, y, T > 0$ and $z \in \mathbb{R}$ we define $q_0(x, y, z, T)$ just as in (2.28) but with terminal condition $x(\tau) = z$ instead of $x(\tau) = 0$, and without the positivity constraint $x(\cdot) > 0$. It follows from (2.77), (2.79) and (2.65) of Lemma 2.4 that

$$q_\varepsilon(x, y, T) \geq E[q_0(x, y, \sqrt{\varepsilon} Z_\varepsilon, T)], \quad Z_\varepsilon = \frac{\sigma_A^2(\tau_{\varepsilon,x,T}^*)}{m_{1,A}(\tau_{\varepsilon,x,T}^*)} Z(\tau_{\varepsilon,x,T}^*), \quad (2.80)$$

where $Z(\cdot)$ is as in (2.16).

We have already observed that a minimizing τ in (2.28) satisfies $0 < \tau < T$. It follows from this there exist constants $C, \delta > 0$, depending on x, y, T , such that if $|z| < \delta$ then $|q_0(x, y, z, T) - q_0(x, y, T)| \leq C|z|$. Hence from (2.80) we have that

$$q_\varepsilon(x, y, T) \geq q_0(x, y, T) [1 - P(|Z_\varepsilon| > \delta/\sqrt{\varepsilon})] - C\sqrt{\varepsilon} E[|Z_\varepsilon|]. \quad (2.81)$$

From (2.3), (2.4) we see that $|Z_\varepsilon| \leq C_1 \tau_{\varepsilon,x,T}^* |Z(\tau_{\varepsilon,x,T}^*)|$ for some constant C_1 . Hence from the proof of Lemma 2.4 we have that $E[|Z_\varepsilon|] \leq C_2$ for some constant C_2 independent of ε as $\varepsilon \rightarrow 0$. We conclude from (2.81) and the Chebyshev inequality that $q_\varepsilon(x, y, T) \geq q_0(x, y, T) - C_3 \sqrt{\varepsilon}$ for some constant C_3 . \square

We summarize the main result of this section.

Theorem 2.6. *Assume $A(\cdot)$ is continuous and the function q_ε is defined by (2.18), (2.21). Then For x, y, T positive one has $\lim_{\varepsilon \rightarrow 0} q_\varepsilon(x, y, T) = q_0(x, y, T)$, where the function q_0 is defined by (2.28).*

3. REGULARITY AND BOUNDS ON THE FUNCTION q_ε

We first prove a regularity result for the function $(x, y, t, T) \rightarrow G_{\varepsilon,D}(x, y, t, T)$, which will imply the regularity results for the function $(x, y, T) \rightarrow q_\varepsilon(x, y, T)$ we shall need.

Proposition 3.1. *Let $A : [0, \infty) \rightarrow \mathbb{R}$ be a continuous function and $G_{\varepsilon,D}(x, y, t, T)$, $x, y, > 0, 0 \leq t < T < \infty$, be the Dirichlet Green's function for the PDE (1.1) with drift (1.8). Then the derivatives*

$$\frac{\partial^n}{\partial x^n} \frac{\partial^m}{\partial y^m} G_{\varepsilon,D}(x, y, t, T) \quad \text{with } 0 \leq n, m \leq 2, n + m \leq 3, \tag{3.1}$$

$$\frac{\partial^k}{\partial t^k} \frac{\partial^l}{\partial T^l} \frac{\partial^n}{\partial x^n} \frac{\partial^m}{\partial y^m} G_{\varepsilon,D}(x, y, t, T) \quad \text{with } 0 \leq k + l, k + m, l + n \leq 1, \tag{3.2}$$

exist and are continuous in the region $\mathcal{D} = \{(x, y, t, T) : x, y \geq 0, 0 \leq t < T < \infty\}$. Let $G(x, t)$ be the Gaussian distribution with mean 0 and variance t ,

$$G(x, t) = \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{x^2}{2t}\right], \quad x \in \mathbb{R}, t > 0. \tag{3.3}$$

For any $L_0, T_0 > 0$ define \mathcal{D}_{L_0, T_0} to be the region $\mathcal{D}_{L_0, T_0} = \{(x, y, t, T) : 0 \leq x, y, \leq L_0, 0 \leq t < T \leq T_0, T - t \leq L_0^2\}$. Then there is a constant $C(L_0, T_0, \varepsilon)$ such that if m, n satisfy the conditions of (3.1), then

$$\left| \frac{\partial^n}{\partial x^n} \frac{\partial^m}{\partial y^m} G_{\varepsilon,D}(x, y, t, T) \right| \leq \frac{C(L_0, T_0, \varepsilon)}{(T - t)^{(n+m)/2}} G(x - y, 2\varepsilon(T - t)), \tag{3.4}$$

for $(x, y, t, T) \in \mathcal{D}_{L_0, T_0}$. Similarly if k, l, m, n satisfies the conditions of (3.2), then

$$\left| \frac{\partial^k}{\partial t^k} \frac{\partial^l}{\partial T^l} \frac{\partial^n}{\partial x^n} \frac{\partial^m}{\partial y^m} G_{\varepsilon,D}(x, y, t, T) \right| \leq \frac{C(L_0, T_0, \varepsilon)}{(T - t)^{k+l+(n+m)/2}} G(x - y, 2\varepsilon(T - t)), \tag{3.5}$$

for $(x, y, t, T) \in \mathcal{D}_{L_0, T_0}$.

For each $y > 0$ the function $(x, T) \rightarrow \partial^2 G_{\varepsilon,D}(x, y, 0, T) / \partial x^2$, with domain $\{(x, T) : x, T > 0\}$ is continuous up to the boundary $x = 0$, and is also continuously differentiable in T , twice continuously differentiable in x .

Proof. Since the drift $b(\cdot, \cdot)$ is continuous and satisfies for each $T_0 > 0$ the bound $\sup\{|\partial b(y, t) / \partial y| : y \geq 0, 0 \leq t \leq T_0\} < \infty$ we may apply the perturbation argument of [7, Lemma3.4]. From this we see that the derivatives (3.1) with $n \leq 1, m \leq 2$ are continuous and satisfy the inequality (3.4). In making this conclusion we are using the backwards in time PDE (1.1). Since the adjoint PDE (1.4) is similar to (1.1) except run forwards in time, we conclude that (3.4) also holds with $n \leq 2, m \leq 1$. The continuity of the derivatives (3.2) and bounds (3.5)

follow from the fact that $G_{\varepsilon,D}$ is a solution to the PDEs (1.1), (1.4). Hence the derivatives in (3.2) can be expressed as a sum of the derivatives in (3.1).

Letting $v_\varepsilon(x, T) = \exp[2 \int_0^T A(s) ds] \partial^2 G_{\varepsilon,D}(x, y, 0, T) / \partial x^2$, we see by differentiating twice the PDE (1.4) that v_ε is also a solution to (1.4). It is also continuous up to the boundary $x = 0$. For any $L_0 > 0$ we consider v_ε to be a solution to (1.4) in the region $\{(x, T) : 0 < x < L_0, T > 0\}$. Let $G_{\varepsilon,D,L_0}(x, y, t, T)$ be the corresponding Dirichlet Green's function. The function G_{ε,D,L_0} has the same differentiability properties as $G_{\varepsilon,D}$ given in (3.1)-(3.5). We also have the integral representation

$$\begin{aligned} v_\varepsilon(x, T) &= \int_0^{L_0} G_{\varepsilon,D,L_0}(x, y, t, T) v_\varepsilon(y, t) dy + \varepsilon \int_t^T \frac{\partial G_{\varepsilon,D,L_0}(x, 0, s, T)}{\partial y} v_\varepsilon(0, s) ds \\ &\quad - \varepsilon \int_t^T \frac{\partial G_{\varepsilon,D,L_0}(x, L_0, s, T)}{\partial y} v_\varepsilon(L_0, s) ds, \quad 0 < x < L_0, 0 \leq t < T. \end{aligned} \tag{3.6}$$

The differentiability properties of the function v_ε follow from (3.6) and the differentiability properties of G_{ε,D,L_0} by differentiating under the integral and using the estimates (3.4), (3.5). Note that while the function $(x, T) \rightarrow v_\varepsilon(x, T)$ itself is continuous up to the boundary $x = 0$, our proof does not establish that the derivatives are continuous up to the boundary. \square

In the case $A(\cdot) \equiv 0$ the process $X_\varepsilon(\cdot)$ of (2.16) is the standard Brownian Bridge (BB) process from x at time T to y at time 0. In that case

$$X_\varepsilon(s) = \frac{sx + (T - s)y}{T} - \sqrt{\varepsilon} s \int_s^T \frac{dB(s')}{s'}, \quad 0 < s < T, \tag{3.7}$$

and from (1.10) we have that

$$P\left(\inf_{0 \leq s \leq T} X_\varepsilon(s) < 0 \mid X_\varepsilon(T) = x\right) = \exp\left[-\frac{2xy}{\varepsilon T}\right]. \tag{3.8}$$

We can obtain a linear upper bound on the function $x \rightarrow q_\varepsilon(x, y, T)$ in the case of non-trivial $A(\cdot)$ by comparing $X_\varepsilon(\cdot)$ to the BB process.

Proposition 3.2. *Let $A : [0, \infty) \rightarrow \mathbb{R}$ be continuous and q_ε be defined by (2.18), (2.21). Then for any $y, T > 0$ the function $x \rightarrow q_\varepsilon(x, y, T)$, $x \geq 0$, is continuous increasing with $q_\varepsilon(0, y, T) = 0$. For any $T_0 > 0$ there exists a constant $C_A(T_0)$, depending only on T_0 and $\sup_{0 \leq t \leq T_0} |A(t)|$, such that q_ε satisfies the inequality*

$$\left|q_\varepsilon(x, y, T) - \frac{2m_{1,A}(T)xy}{\sigma_A^2(T)}\right| \leq C_A(T_0)Ty, \quad x, y > 0, 0 < T \leq T_0. \tag{3.9}$$

Proof. The monotonicity of the function $x \rightarrow q_\varepsilon(x, y, T)$ follows from (2.17), (2.21). Since $X_\varepsilon(s)$, $s \leq T$, is the solution to (2.14), it follows from the non-intersection of paths property that the function $x \rightarrow v_\varepsilon(x, y, T)$ is decreasing.

To prove (3.9) we make the change of variable $s \leftrightarrow t$ in (2.16) defined by

$$t^2 \frac{ds}{dt} = g(s)^2, \quad g(s) = \frac{\sigma_A^2(s)}{m_{1,A}(s)}, \quad 0 < s < T, s(T) = T. \tag{3.10}$$

Since $\sigma_A(\cdot)$ is strictly positive and $\lim_{s \rightarrow 0} g(s)/s = 1$, the function $s(\cdot)$ is continuous, strictly monotonic and $\lim_{t \rightarrow 0} s(t)/t = 1$. We see that the stochastic integral in

(2.16) becomes

$$\int_s^T \frac{m_{1,A}(s') dB(s')}{\sigma_A^2(s')} = \int_t^T \frac{d\tilde{B}(t')}{t'}, \quad (3.11)$$

where $\tilde{B}(\cdot)$ is a Brownian motion. We define the stochastic process $\tilde{X}_\varepsilon(t)$, $0 < t < T$, by

$$\tilde{X}_\varepsilon(t) = \frac{m_{1,A}(s(t))}{\sigma_A^2(s(t))} t y_{\text{class}}(s(t)) - \sqrt{\varepsilon} t \int_t^T \frac{d\tilde{B}(t')}{t'}, \quad 0 < t \leq T, \quad (3.12)$$

so that the events $\{\inf_{0 < s < T} X_\varepsilon(s) < 0\}$ and $\{\inf_{0 < t < T} \tilde{X}_\varepsilon(t) < 0\}$ are the same. Observe that

$$\begin{aligned} g_{3,A}(s, T) &= \int_s^T \frac{m_{1,A}(s')^2 ds'}{\sigma_A^4(s')} \\ &= \frac{m_{1,A}(s)^2}{\sigma_A^2(s)} - \frac{m_{1,A}(T)^2}{\sigma_A^2(T)} \\ &= \frac{m_{1,A}(s)^2 \sigma_A^2(s, T)}{\sigma_A^2(T) \sigma_A^2(s)}. \end{aligned} \quad (3.13)$$

From (3.10) we see that $g_{3,A}(s(t), T) = (T-t)/tT$, $0 < t < T$. We conclude then from (2.12), (3.10), (3.13) that

$$\frac{m_{1,A}(s(t))}{\sigma_A^2(s(t))} t y_{\text{class}}(s(t)) = t \frac{m_{1,A}(T)}{\sigma_A^2(T)} \left\{ x + \frac{g_{2,A}(s(t), T)}{g_{1,A}(s(t), T)} \right\} + (T-t) \frac{y}{T}, \quad (3.14)$$

for $0 < t < T$, where $g_{1,A}, g_{2,A}$ are defined by (2.39), (2.40).

The expression (2.40) for $g_{2,A}$ can be simplified by observing that

$$\begin{aligned} g_{2,A}(s, T) &= \frac{m_{2,A}(s, T)}{m_{1,A}(T)} \left[\frac{\sigma_A^2(T) - \sigma_A^2(s, T)}{\sigma_A^2(s, T)} \right] - \frac{m_{2,A}(s)}{m_{1,A}(s)} \\ &= \frac{\sigma_A^2(T)}{m_{1,A}(T)} \frac{m_{2,A}(s, T)}{\sigma_A^2(s, T)} - \frac{m_{2,A}(T)}{m_{1,A}(T)}. \end{aligned} \quad (3.15)$$

We have then from (2.39) and (3.15) that

$$\frac{g_{2,A}(s, T)}{g_{1,A}(s, T)} = m_{2,A}(T) - \frac{\sigma_A^2(T) m_{2,A}(s)}{m_{1,A}(s, T) \sigma_A^2(s)}. \quad (3.16)$$

We have from (3.16) that

$$\lim_{s \rightarrow T} \frac{g_{2,A}(s, T)}{g_{1,A}(s, T)} = 0, \quad \frac{d}{ds} \left[\frac{g_{2,A}(s, T)}{g_{1,A}(s, T)} \right] = - \frac{\sigma_A^2(T)}{m_{1,A}(s, T)} \frac{[\sigma_A^2(s) - m_{2,A}(s)]}{\sigma_A^4(s)}. \quad (3.17)$$

It is easy to see from (3.17) that for any $T_0 > 0$,

$$\left| \frac{g_{2,A}(s, T)}{g_{1,A}(s, T)} \right| \leq C_{A,1}(T_0) T(T-s), \quad 0 < s < T \leq T_0, \quad (3.18)$$

where the constant $C_{A,1}(T_0)$ depends only on T_0 and $\sup_{0 \leq t \leq T_0} |A(t)|$.

It follows from (3.12), (3.14), and (3.18) that

$$\begin{aligned} &P\left(\inf_{0 \leq t \leq T} \tilde{X}_\varepsilon(t) < 0 \mid \tilde{X}_\varepsilon(T) = x\right) \\ &\geq P\left(\inf_{0 \leq t \leq T} \left[\frac{tx_1 + (T-t)y}{T} - \sqrt{\varepsilon}t \int_t^T \frac{d\tilde{B}(t')}{t'}\right] < 0\right), \end{aligned} \tag{3.19}$$

$$x_1 = \frac{Tm_{1,A}(T)}{\sigma_A^2(T)} [x + C_{A,1}(T_0)T^2].$$

We conclude from (3.7), (3.8), and (3.19) that

$$q_\varepsilon(x, y, T) \leq \frac{2x_1y}{T} \leq \frac{2m_{1,A}(T)xy}{\sigma_A^2(T)} + C_{A,2}(T_0)Ty, \quad 0 < T \leq T_0, \tag{3.20}$$

for some constant $C_{A,2}(T_0)$ depending only on T_0 and $\sup_{0 \leq t \leq T_0} |A(t)|$. A similar argument yields a lower bound corresponding to (3.20), whence (3.9) follows. \square

For general $A(\cdot)$ one can construct a linear solution to the HJ equation (2.29), which is therefore also a solution to the HJB equation (2.22). To find it we set

$$q_0(x, y, T) = a(y, T) + b(y, T)x. \tag{3.21}$$

Equating the coefficients of x in (2.29), we obtain the ODE

$$\frac{db(y, T)}{dT} = -\left[A(T) + \frac{1}{\sigma_A^2(T)}\right]b(y, T). \tag{3.22}$$

Integrating (3.22), we conclude that

$$b(y, T) = C(y) \frac{m_{1,A}(T)}{\sigma_A^2(T)} \tag{3.23}$$

for a constant $C(y)$ depending on y . We choose the constant $C(y)$ in (3.23) so that our linear solution gives the function $x \rightarrow 2xy/T$, corresponding to (1.10), when $A(\cdot) \equiv 0$. Thus we choose $C(y) = 2y$. Equating the terms independent of x in (2.29), we obtain the ODE

$$\frac{da(y, T)}{dT} = \left[1 - \frac{m_{2,A}(T)}{\sigma_A^2(T)} + \frac{m_{1,A}(T)y}{\sigma_A^2(T)}\right]b(y, T) - \frac{1}{2}b(y, T)^2. \tag{3.24}$$

With the choice of $C(y) = 2y$ in (3.23), this reduces (3.24) to the equation

$$\frac{da(y, T)}{dT} = \left[1 - \frac{m_{2,A}(T)}{\sigma_A^2(T)}\right]b(y, T). \tag{3.25}$$

Integrating (3.25) with initial condition $a(y, 0) = 0$ yields the solution

$$a(y, T) = \frac{2y}{\sigma_A^2(T)} [m_{1,A}(T)m_{2,A}(T) - \sigma_A^2(T)]. \tag{3.26}$$

We conclude from (3.23)-(3.26) that

$$q_{\text{linear}}(x, y, T) = \frac{2y}{\sigma_A^2(T)} [m_{1,A}(T)m_{2,A}(T) - \sigma_A^2(T) + m_{1,A}(T)x] \tag{3.27}$$

is a linear solution to (2.29). We shall show that the linear solution (3.27) tightly bounds the function q_ε defined by (2.18), (2.21) when $A(\cdot)$ is non-negative.

Proposition 3.3. *Assume the function $A(\cdot)$ is continuous non-negative, and let $q_\varepsilon(x, y, T)$ be defined by (2.18), (2.21). Then*

$$\left\{ \frac{\partial q_{\text{linear}}(x', y, T)}{\partial x'} \Big|_{x'=0} \right\} x \leq q_\varepsilon(x, y, T) \leq q_{\text{linear}}(x, y, T) \quad \text{for } x, y, T > 0, \quad (3.28)$$

and

$$q_\varepsilon(x, y, T) \leq -2\lambda(0, y, T)x \quad \text{for } x, y, T > 0. \quad (3.29)$$

Proof. We see using the formula (2.5) for $G_\varepsilon(x, y, 0, T)$ and the fact that the function $(x, t) \rightarrow G_{\varepsilon, D}(x, y, 0, t)$ is a solution to the PDE (1.4) with drift $b(x, t) = A(t)x - 1$, that the function $(x, T) \rightarrow v_\varepsilon(x, y, T)$ defined by (2.18) is a solution to the PDE (2.19). Furthermore, v_ε satisfies the initial and boundary conditions (2.20). Since the function q_{linear} is a solution of (2.22) it follows that $v_{\varepsilon, 1}(x, y, T) = \exp[-q_{\text{linear}}(x, y, T)/\varepsilon]$ is a solution to the PDE (2.19). From the non-negativity of $A(\cdot)$ we also have that $q_{\text{linear}}(0, y, T) \geq 0$ for $T > 0$. In addition, one has for T small that $q_{\text{linear}}(x, y, T) \simeq 2xy/T$. We conclude that $v_{\varepsilon, 1}$ satisfies the initial and boundary conditions

$$v_{\varepsilon, 1}(0, y, T) \leq 1, \quad T > 0, \quad v_{\varepsilon, 1}(x, y, 0) = 0, \quad x > 0. \quad (3.30)$$

Comparing (2.20) and (3.30), we expect that an application of the maximum principle for linear parabolic PDE [16] implies that $v_{\varepsilon, 1}(x, y, T) \leq v_\varepsilon(x, y, T)$ for all $x, T > 0$, whence the upper bound in (3.28).

In the application of the maximum principle we need to take account of the fact that the domain $\{x \in \mathbb{R} : x > 0\}$ is unbounded, and that the drift $\lambda(x, y, T)$ of (2.15) becomes unbounded as $T \rightarrow 0$. To deal with this we apply for any $M, T_0 > 0$, $0 < \delta < T_0$, the maximum principle to a bounded domain $\mathcal{D}_{M, T_0, \delta} = \{(x, T) : 0 < x < M, \delta < T < T_0\}$ on which the drift is continuous and bounded. Then we let $M \rightarrow \infty$, $\delta \rightarrow 0$.

We first consider the case $M \rightarrow \infty$. It is evident from (3.27) that

$$\lim_{M \rightarrow \infty} \sup_{0 < T < T_0} v_{\varepsilon, 1}(M, y, T) = 0.$$

It follows from (3.9) of Proposition 3.2 that also $\lim_{M \rightarrow \infty} \sup_{0 < T < T_0} v_\varepsilon(M, y, T) = 0$. Next we consider the case $\delta \rightarrow 0$. We see from (3.27) that for any $m > 0$ then $\lim_{\delta \rightarrow 0} \sup_{x \geq m} v_{\varepsilon, 1}(x, y, \delta) = 0$. Observe from (2.3), (2.4) that since the function $s \rightarrow A(s)$ is continuous at $s = 0$ then

$$\frac{\sigma_A^2(s)}{m_{1, A}(s)} = s[1 + so(s)], \quad \frac{m_{2, A}(s)}{m_{1, A}(s)^{1/2}} = s[1 + so(s)]. \quad (3.31)$$

It follows from (3.27) and (3.31) that

$$q_{\text{linear}}(x, y, T) = \frac{2xy}{T} + o(T) \quad \text{as } T \rightarrow 0. \quad (3.32)$$

We conclude from (3.32) that

$$\lim_{\delta \rightarrow 0} \sup_{0 < x \leq m} \left\{ v_{\varepsilon, 1}(x, y, \delta) - \exp \left[- \frac{2xy}{\varepsilon \delta} \right] \right\} = 0. \quad (3.33)$$

From Proposition 3.2 we see that $\lim_{\delta \rightarrow 0} \sup_{x \geq m} v_\varepsilon(x, y, \delta) = 0$ and a similar result to (3.33) holds for v_ε . We conclude that

$$\lim_{\delta \rightarrow 0} \sup_{0 < x < \infty} \left\{ v_{\varepsilon, 1}(x, y, \delta) - v_\varepsilon(x, y, \delta) \right\} = 0. \quad (3.34)$$

From Proposition 3.1 it follows that the function $(x, T) \rightarrow u(x, T) = v_{\varepsilon,1}(x, y, T) - v_{\varepsilon}(x, y, T)$ is continuously differentiable in T and twice continuously differentiable in x on the domain $\mathcal{D}_{M,T_0,\delta}$. It is also continuous up to the boundaries $x = 0$, $x = M$, $T = \delta$. It follows then from the maximum principle [16, Theorem 2, Chapter 3] applied to the solution $u(x, T)$ of (2.19) that the maximum of $u(\cdot, \cdot)$ on $\mathcal{D}_{M,T_0,\delta}$ occurs on one of those boundaries. The upper bound in (3.28) follows by letting $\delta \rightarrow 0, M \rightarrow \infty$ and using (2.20), (3.30), (3.34) and the fact that $\lim_{M \rightarrow \infty} \sup_{0 < T < T_0} u(M, T) = 0$.

The lower bound in (3.28) can be established similarly. Thus we define the function $v_{\varepsilon,2}$ by

$$v_{\varepsilon,2}(x, y, T) = \exp \left[- \frac{2m_{1,A}(T)xy}{\varepsilon\sigma_A^2(T)} \right]. \tag{3.35}$$

Let \mathcal{L} be the linear differential operator $\mathcal{L} = -\partial/\partial T + \dots$ such that the PDE (2.19) is $\mathcal{L}v_{\varepsilon} = 0$. Then we have from (3.35) that

$$\mathcal{L}v_{\varepsilon,2}(x, y, T) = - \frac{2m_{1,A}(T)y}{\varepsilon\sigma_A^2(T)} \left[1 - \frac{m_{2,A}(T)}{\sigma_A^2(T)} \right] v_{\varepsilon,2}(x, y, T). \tag{3.36}$$

Since $A(\cdot)$ is non-negative the RHS of (3.36) is less than or equal to 0. Arguing as in the previous paragraphs we also see that

$$\begin{aligned} \lim_{M \rightarrow \infty} \sup_{0 < T < T_0} [v_{\varepsilon}(M, y, T) - v_{\varepsilon,2}(M, y, T)] &= 0, \\ \lim_{\delta \rightarrow 0} \sup_{0 < x < \infty} \{v_{\varepsilon}(x, y, \delta) - v_{\varepsilon,2}(x, y, \delta)\} &= 0. \end{aligned} \tag{3.37}$$

Setting $u(x, T) = v_{\varepsilon}(M, y, T) - v_{\varepsilon,2}(M, y, T)$, we see from (2.19) and (3.36) that $\mathcal{L}u(x, T) \geq 0$ for $(x, T) \in \mathcal{D}_{M,T_0,\delta}$. Applying [16, Theorem 2, Chapter 3] again, we conclude that $u(\cdot, \cdot)$ takes its maximum on the boundaries $x = 0$, $x = M$, $T = \delta$ of $\mathcal{D}_{M,T_0,\delta}$. The lower bound in (3.28) then follows from (3.37) upon letting $\delta \rightarrow 0, M \rightarrow \infty$.

The upper bound (3.29) also follows in a similar way. We define the function

$$\begin{aligned} v_{\varepsilon,3}(x, y, T) &= \exp \left[\frac{2\lambda(0, y, T)x}{\varepsilon} \right] \\ &= \exp \left[- \frac{2x}{\varepsilon} \left\{ 1 - \frac{m_{2,A}(T)}{\sigma_A^2(T)} + \frac{m_{1,A}(T)y}{\sigma_A^2(T)} \right\} \right]. \end{aligned} \tag{3.38}$$

Taking derivatives in (3.38) we see that

$$\mathcal{L}v_{\varepsilon,3}(x, y, T) = 2x\varepsilon^{-1}A(T)v_{\varepsilon,3}(x, y, T). \tag{3.39}$$

Setting $u(x, T) = v_{\varepsilon,3}(x, y, T) - v_{\varepsilon}(x, y, T)$, we have from (2.19) and (3.39) that $\mathcal{L}u(x, T) \geq 0$ for $(x, T) \in \mathcal{D}_{M,T_0,\delta}$. As with (3.37), we have that

$$\lim_{M \rightarrow \infty} \sup_{0 < T < T_0} u(M, T) = 0, \quad \lim_{\delta \rightarrow 0} \sup_{0 < x < \infty} u(x, \delta) = 0. \tag{3.40}$$

Applying [16, Theorem 2 Chapter 3] once more, and taking the limits $\delta \rightarrow 0, M \rightarrow \infty$ using (3.40), then yields the upper bound (3.29). □

Remark 3.4. In the case $A(\cdot) \equiv 0$ the upper and lower bounds in (3.28) are identical, yielding the function $q_{\varepsilon}(x, y, T)$ of (2.24). A similar situation also occurs

when the drift in (2.1) has the form $b(y, s) = -\gamma y$, where γ is constant. In that case the solution to (2.1) is

$$Y_\varepsilon(s) = e^{-\gamma(s-t)}y + \sqrt{\varepsilon} \int_t^s e^{-\gamma(s-s')} dB(s'). \tag{3.41}$$

Hence the random variable $Y_\varepsilon(T)$ conditioned on $Y_\varepsilon(t) = y$ is Gaussian with mean $m(T-t)y$ and variance $\varepsilon\sigma^2(T-t)$, where

$$m(T) = e^{-\gamma T}, \quad \sigma^2(T) = \frac{1}{2\gamma} [1 - e^{-2\gamma T}]. \tag{3.42}$$

The whole line Green's function $G_\varepsilon(x, y, t, T)$ is explicitly given by the formula

$$G_\varepsilon(x, y, t, T) = \frac{1}{\sqrt{2\pi\varepsilon\sigma^2(T-t)}} \exp\left[-\frac{\{x - m(T-t)y\}^2}{2\varepsilon\sigma^2(T-t)}\right]. \tag{3.43}$$

We again define $q_\varepsilon(x, y, T)$ in terms of the Dirichlet Green's function by (2.18), (2.21). Then the function $[x, T] \rightarrow q_\varepsilon(x, y, T)$, $x, T > 0$, is a solution to (2.22) and (2.23) with

$$\lambda(x, y, T) = -\gamma x - \varepsilon \frac{\partial}{\partial x} \log G_\varepsilon(x, y, 0, T) = -\gamma x + \frac{x - m(T)y}{\sigma^2(T)}. \tag{3.44}$$

We may solve (2.22) and (2.23) in the case of (3.42), (3.44) by looking for a solution of the form $q_\varepsilon(x, y, T) = a(T)xy$. Then $a(\cdot)$ is given by the formula

$$a(T) = \frac{2m(T)}{\sigma^2(T)} = \frac{2\gamma}{\sinh \gamma T}. \tag{3.45}$$

The relation with the function \hat{p} of [17, Proposition 20] is

$$\begin{aligned} \hat{p}(T, x, y) &= \exp\left[\frac{\gamma x^2}{\varepsilon}\right] G_{\varepsilon, D}(x, y, 0, T) \\ &= \exp\left[\frac{\gamma x^2}{\varepsilon}\right] G_\varepsilon(x, y, 0, T) [1 - v_\varepsilon(x, y, T)] \quad \text{with } \varepsilon = 1, \end{aligned} \tag{3.46}$$

where G_ε is given by (3.43), and $v_\varepsilon, q_\varepsilon$ are related by (2.21).

Remark 3.5. The upper bound (3.29) suggests that the function $x \rightarrow q_0(x, y, T)$ is concave. To see this consider solutions to the HJ equation (2.29) with the initial and boundary conditions given by (2.23). In view of the boundary condition at $x = 0$ we have that $\partial q_0(0, y, T)/\partial T = 0$ for $T > 0$. It follows then from the PDE (2.29) that

$$\frac{\partial q_0(x, y, T)}{\partial x} \Big|_{x=0} = -2\lambda(0, y, T). \tag{3.47}$$

Letting $\varepsilon \rightarrow 0$ in the inequality (3.29) we see that the graph of the function $x \rightarrow q_0(x, y, T)$ lies below the line through the origin with slope (3.47).

Corollary 3.6. *Assume $A(\cdot)$ is continuous non-negative, and let $q_\varepsilon(x, y, T)$ be defined by (2.18), (2.21). Then the function $x \rightarrow q_\varepsilon(x, y, T)$ is twice continuously differentiable in x for $x \geq 0$ and $\partial^2 q_\varepsilon(x, y, T)/\partial x^2 \leq 0$ at $x = 0$ and $y, T > 0$.*

Proof. The regularity of q_ε follows from Proposition 3.1. Since $q_\varepsilon(0, y, T) = 0$ and the function $x \rightarrow q_\varepsilon(x, y, T)$ is non-negative, we have using the inequality (3.29)

that $0 \leq \partial q_\varepsilon(x, y, T)/\partial x \leq -2\lambda(0, y, T)$ at $x = 0$. Observing that q_ε satisfies the PDE (2.22) and $\partial q_\varepsilon(x, y, T)/\partial T = 0$ at $x = 0$, we also have that

$$\varepsilon \frac{\partial^2 q_\varepsilon(x, y, T)}{\partial x^2} \Big|_{x=0} = \frac{\partial q_\varepsilon(x, y, T)}{\partial x} \Big|_{x=0} \left[2\lambda(0, y, T) + \frac{\partial q_\varepsilon(x, y, T)}{\partial x} \Big|_{x=0} \right]. \tag{3.48}$$

We conclude from (3.48) and our bounds on $\partial q_\varepsilon(x, y, T)/\partial x$ at $x = 0$ that

$$\partial^2 q_\varepsilon(x, y, T)/\partial x^2 \leq 0$$

at $x = 0$. □

4. ESTIMATING SOLUTIONS OF THE HAMILTON-JACOBI PDE

In §2 we already observed that the infinite dimensional variational problem (2.28) may be reduced to a single variable variational problem in the first hitting time parameter τ , $0 < \tau < T$. From (2.33), (2.34) we have that

$$q_0(x, y, T) = \min_{0 < \tau < T} \frac{\gamma(\tau)^2}{2} \int_\tau^T \frac{m_{1,A}(s)^2 ds}{\sigma_A^4(s)}, \tag{4.1}$$

where the function $\gamma(\cdot)$ is defined by (2.37). The RHS of (4.1) can be expressed in terms of the functions $g_{1,A}, g_{2,A}, g_{3,A}$ of (2.39), (2.40), (3.13). Thus we have that

$$q_0(x, y, T) = \min_{0 < \tau < T} \frac{g_{3,A}(\tau, T)}{2} [y + g_{1,A}(\tau, T)x + g_{2,A}(\tau, T)]^2. \tag{4.2}$$

One sees from (2.39) that $\lim_{\tau \rightarrow 0} g_{1,A}(\tau, T) = 0$ and $\lim_{\tau \rightarrow T} g_{1,A}(\tau, T) = \infty$. From (3.15), the function $\tau \rightarrow g_{2,A}(\tau, T)$, $0 < \tau < T$, satisfies $\lim_{\tau \rightarrow 0} g_{2,A}(\tau, T) = 0$ and $\lim_{\tau \rightarrow T} g_{2,A}(\tau, T) = [\sigma_A^2(T) - m_{2,A}(T)]/m_{1,A}(T)$. From (3.13) we see that the function $\tau \rightarrow g_{3,A}(\tau, T)$, $0 < \tau < T$, has the properties $\lim_{\tau \rightarrow 0} g_{3,A}(\tau, T) = \infty$ and $\lim_{\tau \rightarrow T} g_{3,A}(\tau, T) = 0$. It follows then from (4.2) that $q_0(x, 0, T) = q_0(0, y, T) = 0$. In the case of $x \rightarrow 0$ with fixed $y > 0$, the minimizer $\tau(x, y, T)$ in (4.2) satisfies $\tau(x, y, T) \rightarrow T$, with the minimum in (4.2) converging to 0. In the case of $y \rightarrow 0$ with fixed $x > 0$, the minimizer $\tau(x, y, T)$ satisfies $\tau(x, y, T) \rightarrow 0$, with the minimum in (4.2) also converging to 0. For general $x, y > 0$, there may not be a unique minimizer $\tau(x, y, T)$, so one does not expect the function $q_0(x, y, T)$ of (4.2) to be a C^1 solution to the HJ equation (2.29). Note however from (4.2) that the function $x \rightarrow \sqrt{q_0(x, y, T)}$, $x > 0$, is concave for all $y, T > 0$. This is a simple consequence of the fact that the function is the minimum of a set of linear functions. Concavity of the function $x \rightarrow q_0(x, y, T)$ implies concavity of the function $x \rightarrow \sqrt{q_0(x, y, T)}$. We shall prove concavity of $x \rightarrow q_0(x, y, T)$ in the case when $A(\cdot)$ is non-negative.

When $A(\cdot) \equiv 0$ the formula (4.2) becomes

$$\begin{aligned} q_0(x, y, T) &= \min_{0 < \tau < T} \frac{T - \tau}{2\tau T} \left[y + \frac{\tau x}{T - \tau} \right]^2 \\ &= \frac{1}{2T} [2xy + \min_{\alpha > 0} \{ \alpha x^2 + y^2/\alpha \}] = \frac{2xy}{T} \end{aligned} \tag{4.3}$$

with $\tau(x, y, T) = \frac{yT}{x+y}$. In this case the minimization problem (4.3) is convex in α , but one does not expect for general $A(\cdot)$ that (4.2) is a convex minimization problem.

The solution of the variational problem (2.28) with *fixed* τ and without the positivity constraint on $x(\cdot)$ is given by the expression on the RHS of (4.2). In the case when the function $A(\cdot)$ is non-negative this is also the solution to the fixed τ

variational problem *with* the positivity constraint on $x(\cdot)$. We see this by observing that the optimizing trajectory (2.37) for the unconstrained problem is positive. This follows from the fact that the functions $s \rightarrow g_{1,A}(s, T)$ and $s \rightarrow g_{2,A}(s, T)$, $0 < s < T$, are increasing if $A(\cdot)$ is non-negative. The monotonicity of $g_{1,A}$ follows by noting that it may be written as

$$g_{1,A}(s, T) = \frac{1}{m_{1,A}(T)} \left[\frac{\sigma_A^2(T)}{\sigma_A^2(s, T)} - 1 \right]. \tag{4.4}$$

To show monotonicity of $g_{2,A}$ we differentiate (3.15) to obtain the formula

$$\frac{\partial g_{2,A}(s, T)}{\partial s} = \frac{\sigma_A^2(T)}{m_{1,A}(s)\sigma_A^4(s, T)} [m_{1,A}(s, T)m_{2,A}(s, T) - \sigma_A^2(s, T)]. \tag{4.5}$$

It is easy to see that $m_{1,A}(s, T)m_{2,A}(s, T) - \sigma_A^2(s, T) \geq 0$ for $0 < s < T$ if the function $A(\cdot)$ is non-negative.

For fixed $y > 0$ and x large, minimizers $\tau(x, y, T)$ for (4.2) are close to 0. We can use this observation to show that when x is large, $q_0(x, y, T)$ is well approximated by $q_{\text{linear}}(x, y, T)$ of (3.27). To see this first observe from (3.13) that $\lim_{\tau \rightarrow 0} \tau g_3(\tau, T) = 1$. We also have on differentiating (4.4) that

$$\frac{\partial g_{1,A}(s, T)}{\partial s} = \frac{\sigma_A^2(T)m_{1,A}(s, T)^2}{m_{1,A}(T)\sigma_A^4(s, T)}. \tag{4.6}$$

Upon setting $s = 0$ in (4.5), (4.6) we see that

$$\begin{aligned} g_{1,A}(0, T) &= 0, & \frac{\partial g_{1,A}(0, T)}{\partial s} &= \frac{m_{1,A}(T)}{\sigma_A^2(T)}, \\ g_{2,A}(0, T) &= 0, & \frac{\partial g_{2,A}(0, T)}{\partial s} &= \frac{m_{1,A}(T)m_{2,A}(T)}{\sigma_A^2(T)} - 1. \end{aligned} \tag{4.7}$$

Hence if the minimizer $\tau(x, y, T)$ of (4.2) is close to 0, the minimization problem is given to leading order by

$$\begin{aligned} q_0(x, y, T) &\simeq \frac{1}{2} \left\{ \min_{\tau > 0} \left[\frac{y}{\sqrt{\tau}} + \sqrt{\tau} \left\{ \frac{\partial g_{1,A}(0, T)}{\partial s} x + \frac{\partial g_{2,A}(0, T)}{\partial s} \right\} \right] \right\}^2 \\ &= q_{\text{linear}}(x, y, T). \end{aligned} \tag{4.8}$$

The minimizer in (4.8) gives the leading order term in an expansion of $\tau(x, y, T)$, whence

$$\tau(x, y, T) \simeq \frac{2y^2}{q_{\text{linear}}(x, y, T)} \simeq \frac{Ty}{x} \quad \text{for large } x. \tag{4.9}$$

Note from (4.9) that $\tau(x, y, T) = O(1/x)$ as $x \rightarrow \infty$. We make this argument precise in the following.

Proposition 4.1. *Assume the function $A : [0, \infty) \rightarrow \mathbb{R}$ is continuous and non-negative. Then for any $T_0 > 0$ there exists a constant $C_A(T_0)$, depending only on T_0 and $\sup_{0 \leq t \leq T_0} A(t)$, such that the function $(x, T) \rightarrow q_0(x, y, T)$ satisfies the inequalities*

$$-\frac{C_A(T_0)Ty^2}{x} \leq q_0(x, y, T) - q_{\text{linear}}(x, y, T) \leq 0 \tag{4.10}$$

for $x \geq \max\{2y, T^2\}$ and $0 < T < T_0$, and

$$0 \leq q_0(x, y, T) - \frac{2m_{1,A}(T)xy}{\sigma_A^2(T)} \leq C_A(T_0)Tx \quad \text{for } x, y > 0, 0 < T < T_0. \tag{4.11}$$

Proof. All the constants C_1, C_2, \dots , in the following can be chosen to depend only on T_0 and $\sup_{0 \leq t \leq T_0} A(t)$. We first observe from (2.39) and (3.13) that

$$g_{3,A}(s, T)g_{1,A}(s, T) = \frac{m_{1,A}(T)}{\sigma_A^2(T)}, \quad 0 < s < T. \quad (4.12)$$

Next we note from (2.39) and (4.5) that for any $T_0 > 0$, there are constants $C_1, C_2, C_3 > 0$ such that

$$\begin{aligned} \frac{C_1 s}{T-s} \leq g_{1,A}(s, T) \leq \frac{C_2 s}{T-s}, \quad 0 \leq \frac{\partial g_{2,A}(s, T)}{\partial s} \leq C_3 T, \\ 0 \leq g_{2,A}(s, T) \leq C_3 s T, \quad \text{for } 0 < s < T, \quad 0 < T \leq T_0. \end{aligned} \quad (4.13)$$

Evaluating the functional on the RHS of (4.2) at $\tau = Ty/x$ we conclude from (4.12), (4.13) that

$$q_0(x, y, T) \leq \frac{m_{1,A}(T_0)}{2C_1} [1 + 2C_2 + C_3]^2 \frac{xy}{T}, \quad 0 < T \leq T_0, \quad x \geq \max\{2y, T^2\}. \quad (4.14)$$

We also have from (4.12) and (4.13) that

$$\begin{aligned} \frac{g_{3,A}(\tau, T)}{2} [y + g_1(\tau, T)x + g_{2,A}(\tau, T)]^2 &\geq \frac{m_{1,A}(T)}{2\sigma_A^2(T)} g_{1,A}(\tau, T)x^2 \\ &\geq \frac{C_1 \tau x^2}{2m_{1,A}(T_0)T^2}. \end{aligned} \quad (4.15)$$

It follows from (4.14) and (4.15) that there is a constant C_4 such that any minimizing $\tau = \tau(x, y, T)$ in (4.2) satisfies the inequality

$$0 < \tau(x, y, T) \leq \frac{C_4 T y}{x}, \quad 0 < T \leq T_0, \quad x \geq \max\{2y, T^2\}. \quad (4.16)$$

We have from (3.16) that

$$\lim_{\tau \rightarrow 0} \frac{g_{2,A}(\tau, T)}{g_{1,A}(\tau, T)} = m_{2,A}(T) - \frac{\sigma_A^2(T)}{m_{1,A}(T)}. \quad (4.17)$$

We also see from (3.17) that if $A(\cdot)$ is non-negative, the derivative of the function $s \rightarrow g_{2,A}(s, T)/g_{1,A}(s, T)$, $0 < s < T$, is less than or equal to zero and bounded by a constant times T . We conclude therefore from (4.17) there is a constant C_5 such that

$$-C_5 \tau T \leq \frac{g_{2,A}(\tau, T)}{g_{1,A}(\tau, T)} - \left\{ m_{2,A}(T) - \frac{\sigma_A^2(T)}{m_{1,A}(T)} \right\} \leq 0, \quad 0 < \tau < T \leq T_0. \quad (4.18)$$

The inequality (4.10) follows from (4.12), (4.16), and (4.18). Thus from the upper bound in (4.18) we have that

$$q_0(x, y, T) \leq \inf_{0 < \tau < T} \frac{g_{3,A}(\tau, T)}{2} \left[y + \frac{1}{g_{3,A}(\tau, T)} \frac{q_{\text{linear}}(x, y, T)}{2y} \right]^2 = q_{\text{linear}}(x, y, T). \quad (4.19)$$

Similarly from the lower bound in (4.18) we have upon using (4.16) that

$$\begin{aligned}
 q_0(x, y, T) &\geq \inf_{0 < \tau < T} \frac{g_{3,A}(\tau, T)}{2} \left[y \right. \\
 &\quad \left. + \frac{1}{g_{3,A}(\tau, T)} \left\{ \frac{q_{\text{linear}}(x, y, T)}{2y} - \frac{m_{1,A}(T)}{\sigma_A^2(T)} \frac{C_5 C_4 T^2 y}{x} \right\} \right]^2 \quad (4.20) \\
 &= q_{\text{linear}}(x, y, T) - \frac{m_{1,A}(T)}{\sigma_A^2(T)} \frac{2C_5 C_4 T^2 y^2}{x},
 \end{aligned}$$

provided the term inside the curly braces is positive. Observe that the lower bound in (4.10) is trivial if $q_{\text{linear}}(x, y, T) - CTy^2/x \leq 0$, whence we need only consider situations where $q_{\text{linear}}(x, y, T) - CTy^2/x$ is positive. Upon choosing C sufficiently large, we see that the term of (4.20) inside the curly braces is then positive. Hence we obtain the lower bound (4.10) for all $x \geq \max\{2y, T^2\}$.

We can argue similarly to obtain the bound (4.11). Thus since $g_{2,A}(\cdot, \cdot)$ is non-negative, we have that

$$q_0(x, y, T) \geq \inf_{0 < \tau < T} \frac{g_{3,A}(\tau, T)}{2} \left[y + \frac{1}{g_{3,A}(\tau, T)} \frac{m_{1,A}(T)x}{\sigma_A^2(T)} \right]^2 = \frac{2m_{1,A}(T)xy}{\sigma_A^2(T)}. \quad (4.21)$$

From (4.13) we have that $g_{2,A}(s, T) \leq C_3 T^2$, $0 < s < T \leq T_0$, for some constant C_3 . Hence from (4.2) and (4.12) we have that

$$\begin{aligned}
 q_0(x, y, T) &\leq \inf_{0 < \tau < T} \frac{g_{3,A}(\tau, T)}{2} \left[y + C_3 T^2 + \frac{1}{g_{3,A}(\tau, T)} \frac{m_{1,A}(T)x}{\sigma_A^2(T)} \right]^2 \quad (4.22) \\
 &= \frac{2m_{1,A}(T)x(y + C_3 T^2)}{\sigma_A^2(T)}.
 \end{aligned}$$

□

We wish also to understand the behavior of $\partial q_0(x, y, T)/\partial x$ and $\partial^2 q_0(x, y, T)/\partial x^2$ as $x \rightarrow \infty$ and as $T \rightarrow 0$. To do this we will show that the function $[x, T] \rightarrow q_0(x, y, T)$, defined by (2.28) or equivalently (4.2), can be obtained for $[x, T]$ in a certain domain by the method of characteristics applied to solving the HJ equation (2.29) with boundary condition (2.30). To implement the method of characteristics, we first observe from (2.30) that $\partial q_0(x, y, T)/\partial T = 0$ at $x = 0$, whence (2.29) yields the formula

$$u_0(0, y, T) = \frac{\partial q_0(x, y, T)}{\partial x} \Big|_{x=0} = -2\lambda(0, y, T) = \frac{2m_{1,A}(T)}{\sigma_A^2(T)} [y + g_{2,A}(T, T)], \quad (4.23)$$

where from (3.15) we define

$$g_{2,A}(T, T) = \lim_{s \rightarrow T} g_{2,A}(s, T) = [\sigma_A^2(T) - m_{2,A}(T)]/m_{1,A}(T).$$

Next we differentiate the PDE (2.29) with respect to x to obtain the Burgers' equation

$$\begin{aligned}
 \frac{\partial u_0(x, y, T)}{\partial T} + [\lambda(x, y, T) + u_0(x, y, T)] \frac{\partial u_0(x, y, T)}{\partial x} \\
 + \left[A(T) + \frac{1}{\sigma_A^2(T)} \right] u_0(x, y, T) = 0, \quad (4.24)
 \end{aligned}$$

for the function $u_0(x, y, T) = \partial q_0(x, y, T)/\partial x$. We seek to solve (4.24), with the boundary condition $u_0(0, y, T) = \partial q_0(0, y, T)/\partial x$ given by (4.23), by using the

method of characteristics. If $s \rightarrow [x(s), s]$, $s > \tau$, is a characteristic with initial condition $x(\tau) = 0$, then (4.23) and (4.24) yield the ODE initial value problem

$$\begin{aligned} \frac{d}{ds} u_0(x(s), y, s) + \left[A(s) + \frac{1}{\sigma_A^2(s)} \right] u_0(x(s), y, s) &= 0, \quad s > \tau, \\ u_0(x(\tau), y, \tau) &= \frac{2m_{1,A}(\tau)}{\sigma_A^2(\tau)} [y + g_{2,A}(\tau, \tau)]. \end{aligned} \quad (4.25)$$

The solution to (4.25) is

$$u_0(x(s), y, s) = \frac{2m_{1,A}(s)[y + g_{2,A}(\tau, \tau)]}{\sigma_A^2(s)}, \quad s > \tau. \quad (4.26)$$

It follows from (4.24) and (4.26) that the characteristics are solutions to the ODE initial value problem

$$\frac{dx(s)}{ds} = \lambda(x(s), y, s) + \frac{2m_{1,A}(s)[y + g_{2,A}(\tau, \tau)]}{\sigma_A^2(s)}, \quad s > \tau, \quad x(\tau) = 0. \quad (4.27)$$

From (2.15) we see that (4.27) is the same as

$$\frac{dx(s)}{ds} = \left[A(s) + \frac{1}{\sigma_A^2(s)} \right] x(s) + \frac{m_{1,A}(s)}{\sigma_A^2(s)} [y + 2g_{2,A}(\tau, \tau) - g_{2,A}(s, s)] \quad (4.28)$$

for $s > \tau$, $x(\tau) = 0$. The general solution to the ODE (4.28) is

$$x(s) = C \frac{\sigma_A^2(s)}{m_{1,A}(s)} - [y + 2g_{2,A}(\tau, \tau)] m_{1,A}(s) - m_{2,A}(s), \quad (4.29)$$

where C is an arbitrary constant. The constant C is determined for the characteristic by the initial condition $x(\tau) = 0$. In the case $A(\cdot) \equiv 0$ this yields the formula $x(s) = [s/\tau - 1]y$, $s > \tau$, for the characteristic.

We have that

$$\frac{d}{ds} g_{2,A}(s, s) = \frac{A(s)\sigma_A^2(s)}{m_{1,A}(s)}. \quad (4.30)$$

If we assume $A(\cdot)$ non-negative, it follows from (4.30) that the function $s \rightarrow g_{2,A}(s, s)$ is increasing. This implies the characteristics that are solutions to (4.28) may meet, whence one cannot expect the HJ equation (2.29) to have a classical (continuously differentiable) solution. We can make this more precise by considering characteristics $s \rightarrow x(\tau, s)$, $s > \tau > 0$, which are solutions to (4.28) with initial condition $x(\tau, \tau) = 0$. The first variation $D_\tau x(\tau, s) = \partial x(\tau, s)/\partial \tau$, $s > \tau > 0$, is from (4.28), (4.30) the solution to the initial value problem

$$\begin{aligned} \frac{d}{ds} D_\tau x(\tau, s) &= \left[A(s) + \frac{1}{\sigma_A^2(s)} \right] D_\tau x(\tau, s) + \frac{2m_{1,A}(s)}{\sigma_A^2(s)} \frac{A(\tau)\sigma_A^2(\tau)}{m_{1,A}(\tau)}, \quad s > \tau, \\ D_\tau x(\tau, s) \Big|_{s=\tau} &= -\frac{m_{1,A}(\tau)}{\sigma_A^2(\tau)} [y + g_{2,A}(\tau, \tau)]. \end{aligned} \quad (4.31)$$

We note that (4.31) is equivalent to

$$\begin{aligned} \frac{d}{ds} \left[\frac{m_{1,A}(s)}{\sigma_A^2(s)} D_\tau x(\tau, s) \right] &= \frac{2m_{1,A}(s)^2}{\sigma_A^4(s)} \frac{A(\tau)\sigma_A^2(\tau)}{m_{1,A}(\tau)}, \quad s > \tau, \\ D_\tau x(\tau, s) \Big|_{s=\tau} &= -\frac{m_{1,A}(\tau)}{\sigma_A^2(\tau)} [y + g_{2,A}(\tau, \tau)]. \end{aligned} \quad (4.32)$$

Since $D_\tau x(\tau, s) < 0$ at $s = \tau$ and the derivative on the LHS of (4.32) is non-negative, we can have $D_\tau x(\tau, s) = 0$ for some $s > \tau$, from which point the solution to (4.24) cannot be continued by using the method of characteristics.

When the method of characteristics does apply to obtain the solution of (4.24) with boundary data (4.23), we may obtain a formula for $\partial u_0(x, y, T)/\partial x$ along characteristics similarly to how we obtained (4.26) for $u_0(x, y, T)$. To see this first note from (4.23) and (4.30) that

$$\begin{aligned} \frac{\partial u_0(x, y, T)}{\partial T} \Big|_{x=0} &= \frac{d}{dT} u_0(0, y, T) \\ &= -\left[A(T) + \frac{1}{\sigma_A^2(T)} \right] u_0(0, y, T) + 2A(T). \end{aligned} \tag{4.33}$$

Setting $x = 0$ in (4.24) and using (4.33) we conclude that

$$\frac{\partial u_0(x, y, T)}{\partial x} \Big|_{x=0} = -\frac{2A(T)\sigma_A^2(T)}{m_{1,A}(T)[y + g_{2,A}(T, T)]}. \tag{4.34}$$

Differentiating (4.24) with respect to x , we obtain a partial differential equation for $v_0(x, y, T) = \partial u_0(x, y, T)/\partial x$,

$$\begin{aligned} \frac{\partial v_0(x, y, T)}{\partial T} + [\lambda(x, y, T) + u_0(x, y, T)] \frac{\partial v_0(x, y, T)}{\partial x} \\ + v_0(x, y, T)^2 + 2\left[A(T) + \frac{1}{\sigma_A^2(T)} \right] v_0(x, y, T) = 0. \end{aligned} \tag{4.35}$$

From the method of characteristics applied to (4.35), we obtain using (4.34) the ODE initial value problem

$$\begin{aligned} \frac{d}{ds} v_0(x(s), y, s) + v(x(s), y, s)^2 + 2\left[A(s) + \frac{1}{\sigma_A^2(s)} \right] v_0(x(s), y, s) = 0, \quad s > \tau, \\ v_0(x(\tau), y, \tau) = -\frac{2A(\tau)\sigma_A^2(\tau)}{m_{1,A}(\tau)[y + g_{2,A}(\tau, \tau)]}, \end{aligned} \tag{4.36}$$

where $s \rightarrow x(s)$, $s > \tau$, is the characteristic defined by (4.28). It follows from (4.36) that the function $s \rightarrow 1/v_0(x(s), y, s)$ is a solution to a linear differential equation, whence we conclude that $1/v_0(x(s), y, s)$, $s > \tau$, is of the form

$$\frac{1}{v_0(x(s), y, s)} = C \frac{\sigma_A^4(s)}{m_{1,A}(s)^2} - \sigma_A^2(s), \quad s > \tau, \tag{4.37}$$

for some constant C . Choosing C in (4.37) to satisfy the initial condition (4.36), we have then that

$$\begin{aligned} \frac{\partial u_0(x(s), y, s)}{\partial x} = -K(\tau) \frac{m_{1,A}(s)^2}{\sigma_A^4(s)} / \left\{ 1 + K(\tau) \left[\frac{m_{1,A}(s)^2}{\sigma_A^2(s)} - \frac{m_{1,A}(\tau)^2}{\sigma_A^2(\tau)} \right] \right\} \\ \text{for } s > \tau, \text{ where } K(\tau) = \frac{2A(\tau)\sigma_A^6(\tau)}{m_{1,A}(\tau)^3[y + g_{2,A}(\tau, \tau)]}. \end{aligned} \tag{4.38}$$

Since the function $s \rightarrow m_{1,A}(s)/\sigma_A^2(s)$ is decreasing, we see that the formula (4.38) for $\partial u_0(x, y, T)/\partial x$ can blow up to $-\infty$. This is again a consequence of the fact that we cannot in general expect a classical solution to (4.23), (4.24) when $A(\cdot)$ is non-negative.

We assume that $A(\cdot)$ is non-negative. Observe that the condition $x(\tau) = 0$ in (4.28) implies that the constant C in (4.29) is given by the formula

$$\begin{aligned} C &= \frac{[y + 2g_2(\tau, \tau)]m_{1,A}(\tau)^2}{\sigma_A^2(\tau)} + \frac{m_{1,A}(\tau)m_{2,A}(\tau)}{\sigma_A^2(\tau)} \\ &= \frac{[y + g_2(\tau, \tau)]m_{1,A}(\tau)^2}{\sigma_A^2(\tau)} + m_{1,A}(\tau). \end{aligned} \tag{4.39}$$

Substituting (4.39) into (4.29) gives the formula for the characteristic,

$$\begin{aligned} x(s) &= [y + g_2(\tau, \tau)] \left\{ \frac{m_{1,A}(\tau)^2}{\sigma_A^2(\tau)} \frac{\sigma_A^2(s)}{m_{1,A}(s)} - m_{1,A}(s) \right\} \\ &\quad + m_{1,A}(s)m_{1,A}(\tau) \left\{ \frac{\sigma_A^2(s)}{m_{1,A}(s)^2} - \frac{\sigma_A^2(\tau)}{m_{1,A}(\tau)^2} \right\} \\ &\quad - m_{1,A}(s) \left\{ \frac{m_{2,A}(s)}{m_{1,A}(s)} - \frac{m_{2,A}(\tau)}{m_{1,A}(\tau)} \right\}. \end{aligned} \tag{4.40}$$

The first term on the RHS of (4.40) is bounded below by $c_1 [s/\tau - 1] [y + g_{2,A}(\tau, \tau)]$ for $0 < \tau < s \leq T_0$, where constant $c_1 > 0$ depends only on T_0 and $\sup_{0 \leq t \leq T_0} A(t)$. The remaining terms can be expressed as an integral over the interval $[\tau, s]$,

$$m_{1,A}(s) \int_{\tau}^s \left[\frac{m_{1,A}(\tau)}{m_{1,A}(s')} - 1 \right] \frac{ds'}{m_{1,A}(s')} = m_{1,A}(s)f(s). \tag{4.41}$$

The function $s \rightarrow f(s)$, $s \geq \tau$, is decreasing and $f(\tau) = f'(\tau) = 0$. We conclude that the characteristic $s \rightarrow x(\tau, s)$, $s > \tau$, is an increasing function of s for $s > \tau$ such that $s - \tau$ is sufficiently small. However it could decrease for s large, even to 0. We see from (4.40), (4.41) that

$$\begin{aligned} c_1(s - \tau) \left[\frac{[y + g_{2,A}(\tau, \tau)]}{\tau} - C_1(s - \tau) \right] \\ \leq x(\tau, s) \leq C_2(s - \tau) \left[\frac{[y + g_{2,A}(\tau, \tau)]}{\tau} - c_2(s - \tau) \right], \quad 0 < \tau < s \leq T_0, \end{aligned} \tag{4.42}$$

where $c_1, c_2, C_1, C_2 > 0$ depend only on T_0 and $\sup_{0 \leq t \leq T_0} A(t)$. It follows from (4.42) that $x(\tau, s) > 0$ for $0 < \tau < s < \min\{\tau + [y + g_{2,A}(\tau, \tau)]/C_1\tau, T_0\}$.

Next we obtain from the variation equation (4.31) conditions that imply characteristics do not intersect. Thus setting $y(\tau, s) = m_{1,A}(s)D_{\tau}x(\tau, s)/\sigma_A^2(s)$, we have from (4.31), (4.32)

$$\frac{\partial}{\partial s}y(\tau, s) \leq \frac{C_3\tau}{s^2}, \quad \tau < s \leq T_0, \quad y(\tau, \tau) \leq -C_4 \frac{y + g_{2,A}(\tau, \tau)}{\tau^2}, \tag{4.43}$$

for some positive constants C_3 and C_4 depending only on T_0 and $\sup_{0 \leq t \leq T_0} A(t)$. Integrating (4.43) we conclude that

$$y(\tau, s) \leq C_3 \left[1 - \frac{\tau}{s} \right] - C_4 \frac{y + g_{2,A}(\tau, \tau)}{\tau^2} \quad \text{for } \tau < s \leq T_0. \tag{4.44}$$

It follows from (4.44) that $D_{\tau}x(\tau, s) < 0$ for $0 < \tau < s < \min\{\tau + C_4[y + g_{2,A}(\tau, \tau)]/C_3\tau, T_0\}$.

Let $\Lambda_0 > 0$ be a constant such that $\Lambda_0 < 1/C_1$, $\Lambda_0 < C_4/C_3$ and consider the function $T_y(\tau) = \tau + \Lambda_0[y + g_{2,A}(\tau, \tau)]/\tau$, $0 < \tau \leq T_0$. Evidently $T_y(\cdot) \geq \tilde{T}_y(\cdot)$, where \tilde{T}_y is the convex function $\tilde{T}_y(\tau) = \tau + \Lambda_0 y/\tau$. The infimum of \tilde{T}_y is attained at $\tau = \sqrt{\Lambda_0 y}$ and $\inf_{0 < \tau < \infty} \tilde{T}_y(\tau) = 2\sqrt{\Lambda_0 y}$. Since (4.30) implies that $g_{2,A}(\tau, \tau) \leq$

$C\tau^2$, we see that if $2\sqrt{\Lambda_0 y} \leq T_0$ then $\inf_{0 < \tau < T_0} T_y(\tau) \leq 2[1 + C']\sqrt{\Lambda_0 y}$, where $C' \geq 0$ depends only on T_0 and $\sup_{0 \leq t \leq T_0} A(t)$.

We wish to identify the largest domain \mathcal{D}_{y, T_0} contained in $\{[x, T] : x > 0, 0 < T < T_0\}$ such that characteristics do not intersect within the domain. We have already seen that if $2\sqrt{\Lambda_0 y} \geq T_0$ then we may take $\mathcal{D}_{y, T_0} = \{[x, T] : x > 0, 0 < T < T_0\}$, so let us assume that $2\sqrt{\Lambda_0 y} < T_0$. Then \mathcal{D}_{y, T_0} contains $\{[x, T] : x > 0, 0 < T < 2\sqrt{\Lambda_0 y}\}$, so we just need to consider the situation $2\sqrt{\Lambda_0 y} < T < T_0$. Then the equation

$$\tau + \frac{\Lambda[y + g_{2,A}(T, T)]}{\tau} = T \quad (4.45)$$

has two solutions provided that $2\sqrt{\Lambda[y + g_{2,A}(T, T)]} < T$. Since $2\sqrt{\Lambda_0 y} < T < T_0$ we may choose $\Lambda_1 < \Lambda_0$, depending only on T_0 and $\sup_{0 \leq t \leq T_0} A(t)$, such that $4\sqrt{\Lambda_1[y + g_{2,A}(T, T)]} < T$. The larger solution to (4.45) is

$$\tau_{1,\Lambda,y}(T) = \frac{T}{2} + \frac{T}{2} \left\{ 1 - \frac{4\Lambda[y + g_{2,A}(T, T)]}{T^2} \right\}^{1/2}. \quad (4.46)$$

If $\Lambda \leq \Lambda_1$ then $\tau_{1,\Lambda,y}(T)$ satisfy the inequality

$$\frac{\Lambda[y + g_{2,A}(T, T)]}{T} \leq T - \tau_{1,\Lambda,y}(T) \leq \frac{2\Lambda[y + g_{2,A}(T, T)]}{\sqrt{3}T}. \quad (4.47)$$

It follows from (4.30), (4.47) that

$$0 \leq g_{2,A}(T, T) - g_{2,A}(\tau_{1,\Lambda,y}(T), \tau_{1,\Lambda,y}(T)) \leq C\Lambda[y + g_{2,A}(T, T)], \quad (4.48)$$

for some constant C depending only on T_0 and $\sup_{0 \leq t \leq T_0} A(t)$. We conclude from (4.42), (4.48) there is a constant Λ_2 , depending only on T_0 and $\sup_{0 \leq t \leq T_0} A(t)$, such that if

$$2\sqrt{\Lambda_0 y} < T < T_0 \text{ and } 0 < x \leq \frac{\Lambda_2[y + g_{2,A}(T, T)]^2}{T^2} \text{ then } [x, T] \in \mathcal{D}_{y, T_0}. \quad (4.49)$$

We can make a similar argument to show that if $2\sqrt{\Lambda_0 y} < T < T_0$, then $[x, T] \in \mathcal{D}_{y, T_0}$ for x sufficiently large. In that case we define $\tau_{2,\Lambda,y}$ as the smaller solution to the equation $\tau + \Lambda y/\tau = T$. If $\Lambda < \Lambda_0$, then

$$\tau_{2,\Lambda,y}(T) = \frac{T}{2} - \frac{T}{2} \left\{ 1 - \frac{4\Lambda y}{T^2} \right\}^{1/2}. \quad (4.50)$$

If Λ_1 satisfies $4\sqrt{\Lambda_1 y} < T$, then for $\Lambda < \Lambda_1$ we have that

$$\frac{\Lambda y}{T} \leq \tau_{2,\Lambda,y}(T) \leq \frac{2\Lambda y}{\sqrt{3}T}. \quad (4.51)$$

From (4.42) there is a constant Λ_3 , depending only on T_0 and $\sup_{0 \leq t \leq T_0} A(t)$, such that

$$\text{if } 2\sqrt{\Lambda_0 y} < T < T_0 \text{ and } x \geq \Lambda_3 T^2, \text{ then } [x, T] \in \mathcal{D}_{y, T_0}. \quad (4.52)$$

We define the domain

$$\mathcal{U}_{y, T_0} = \{[\tau, s] : 0 < \tau < T_0, \tau < s < \min[T_y(\tau), T_0]\}. \quad (4.53)$$

The mapping $[\tau, s] \rightarrow [x(\tau, s), s]$ is a diffeomorphism from \mathcal{U}_{y, T_0} onto a domain \mathcal{D}_{y, T_0} , which has the properties (4.49), (4.52). The fact that the mapping is onto follows from the intermediate value theorem since we see from (4.42) that $\lim_{\tau \rightarrow 0} x(\tau, s) = \infty$ for all $0 < s < T_0$. It is one-one since $D_\tau x(\tau, s) < 0$ for $[\tau, s] \in \mathcal{U}_{y, T_0}$.

Proposition 4.2. *Assume the function $A : [0, \infty) \rightarrow \mathbb{R}$ is continuous and non-negative. For $[x, T] \in \mathcal{D}_{y, T_0}$ let $[\tau, T] \in \mathcal{U}_{y, T_0}$ be such that $x(\tau, T) = x$, and define $q_0(x, y, T)$ by*

$$q_0(x, y, T) = \frac{g_{3,A}(\tau, T)}{2} [y + g_{1,A}(\tau, T)x + g_{2,A}(\tau, T)]^2. \tag{4.54}$$

Then the function $[x, T] \rightarrow q_0(x, y, T)$ is a C^1 solution of the HJ equation (2.29) on \mathcal{D}_{y, T_0} and satisfies the boundary condition $\lim_{x \rightarrow 0} q_0(x, y, T) = 0$, $0 < T < T_0$. Furthermore, the function $x \rightarrow q_0(x, y, T)$ is C^2 on \mathcal{D}_{y, T_0} and the derivatives $\partial q_0(x, y, T)/\partial x$, $\partial^2 q_0(x, y, T)/\partial x^2$ are given respectively by the formulas (4.26) and (4.38).

Also $\tau = \tau(x, y, T)$ in (4.54) is the unique minimizer in the variational problems (2.28), (4.2) for $[x, T]$ with $x > 0$ and $0 < T < T_0$, in the following regions: (a) all $x > 0$ if $2\sqrt{\Lambda_0 y} \geq T$, otherwise (b) $0 < x \leq \Lambda y[y + g_{2,A}(T, T)]/T^2$, (c) $x \geq T^2/\Lambda$, where $\Lambda > 0$ is chosen sufficiently small depending only on T_0 and $\sup_{0 \leq t \leq T_0} A(t)$. Therefore if $[x, T]$ is in one of the regions (a), (b), (c) the functions (4.2) and (4.54) are identical.

Proof. All constants in the following can be chosen to depend only on T_0 and $\sup_{0 \leq t \leq T_0} A(t)$. To show regularity of the function $[x, T] \rightarrow q_0(x, y, T)$ we first differentiate (4.54) with respect to x . The resulting formula for $\partial q_0(x, y, T)/\partial x$ involves $g_{j,A}(\tau, T)$, $j = 1, 2, 3$, and their first derivatives with respect to τ . It also involves $\partial \tau(x, y, T)/\partial x = [D_\tau x(\tau, T)]^{-1}$, which we see from (4.31), (4.32) is a continuous function of $[\tau, T]$ and hence of $[x, T]$. We conclude that the function $x \rightarrow q_0(x, y, T)$ is differentiable and the function $[x, T] \rightarrow \partial q_0(x, y, T)/\partial x$ continuous. We can make a similar argument to see that the function $T \rightarrow q_0(x, y, T)$ is differentiable and the function $[x, T] \rightarrow \partial q_0(x, y, T)/\partial T$ continuous. In that case we need to show the continuity of the function $[x, T] \rightarrow \partial \tau(x, y, T)/\partial T$, which is given by the formula

$$\frac{\partial \tau(x, y, T)}{\partial T} = -\frac{D_T x(\tau, T)}{D_\tau x(\tau, T)}. \tag{4.55}$$

Evidently $D_T x(\tau, T)$ is given by the RHS of (4.28) with $s = T$, $x(s) = x$ and hence is a continuous function of $[x, T]$. We conclude that the function $[x, T] \rightarrow q_0(x, y, T)$ is C^1 on \mathcal{D}_{y, T_0} .

To show that the function (4.54) is a solution to the HJ equation (2.29), we proceed by the standard method [10], writing (2.29) as

$$\frac{\partial q_0(x, y, T)}{\partial T} + H\left(x, y, \frac{\partial q_0}{\partial x}, T\right) = 0, \tag{4.56}$$

where the Hamiltonian is

$$H(x, y, p, T) = \lambda(x, y, T)p + \frac{1}{2}p^2. \tag{4.57}$$

The corresponding Hamiltonian equations of motion are

$$\frac{dx}{ds} = \frac{\partial H(x, y, p, s)}{\partial p}, \quad \frac{dp}{ds} = -\frac{\partial H(x, y, p, s)}{\partial x}. \tag{4.58}$$

We solve (4.58) with initial conditions

$$x(\tau) = 0, \quad H(0, y, p(\tau), \tau) = 0. \tag{4.59}$$

Note that the initial condition (4.59) for $p(\cdot)$ is the same as in (4.25). If we solve the second equation in (4.58) with initial condition (4.59) we obtain

$$p(\tau, s) = \frac{2m_{1,A}(s)}{\sigma_A^2(s)} [y + g_2(\tau, \tau)], \quad s > \tau, \quad (4.60)$$

corresponding to (4.26). Taking $p(\tau, s)$ to be given by (4.60), the first equation in (4.58) becomes identical to the characteristic equation (4.27). We define the function $w : \mathcal{U}_{y, T_0} \rightarrow \mathbb{R}$ by

$$\frac{\partial}{\partial s} w(\tau, s) = -H(x(\tau, s), p(\tau, s), s) + p(\tau, s) \frac{\partial H(x(\tau, s), p(\tau, s), s)}{\partial p} \quad (4.61)$$

for $s > \tau$ and initial condition $w(\tau, \tau) = 0$. Then by standard theory the function $q_0(\cdot, y, \cdot)$ defined on \mathcal{D}_{y, T_0} by $q_0(x(\tau, s), y, s) = w(\tau, s)$ is a solution to (4.56) and $p(\tau, s) = \partial q_0(x(\tau, s), y, s) / \partial x$.

We see that

$$w(\tau, s) = \frac{g_{3,A}(\tau, s)}{2} [y + g_{1,A}(\tau, s)x(\tau, s) + g_{2,A}(\tau, s)]^2, \quad (4.62)$$

by verifying that the RHS of (4.62) is a solution to the differential equation (4.61), where $x(\tau, s)$ and $p(\tau, s)$ are given by (4.29), (4.39), (4.60) and with initial condition $w(\tau, \tau) = 0$. To obtain the formula (4.38) for $\partial^2 q_0 / \partial x^2$ we observe that

$$\frac{\partial}{\partial \tau} \frac{\partial q_0(x(\tau, s), y, s)}{\partial x} = \frac{\partial^2 q_0(x(\tau, s), y, s)}{\partial x^2} \frac{\partial x(\tau, s)}{\partial \tau} = \frac{\partial p(\tau, s)}{\partial \tau}, \quad (4.63)$$

and use the formulas (4.40), (4.60). Note that we may choose Λ_0 sufficiently small, depending only on T_0 , such that the denominator in the formula (4.38) is positive if $[\tau, s] \in \mathcal{U}_{y, T_0}$.

Finally we consider for which $[x, T] \in \mathcal{D}_{y, T_0}$ that $\tau = \tau(x, y, T)$ is the minimizer for (4.2). We apply the standard verification theorem to paths $[x(s), s]$ and $\tau < s \leq T$, with $x(\tau) = 0$ and $x(T) = x$, which lie in \mathcal{D}_{y, T_0} . Using the fact that q_0 is a C^1 solution of (4.56), (4.57) on \mathcal{D}_{y, T_0} and $q(0, y, \tau) = 0$, we have that

$$\begin{aligned} & q_0(x, y, T) \\ &= \int_{\tau}^T \frac{d}{ds} q_0(x(s), y, s) ds \\ &= \int_{\tau}^T \frac{\partial q_0(x(s), y, s)}{\partial x} \left[\frac{dx(s)}{ds} - \lambda(x(s), y, s) \right] - \frac{1}{2} \int_{\tau}^T \left[\frac{\partial q_0(x(s), y, s)}{\partial x} \right]^2 ds \\ &\leq \frac{1}{2} \int_{\tau}^T \left[\frac{dx(s)}{ds} - \lambda(x(s), y, s) \right]^2 ds. \end{aligned} \quad (4.64)$$

If $\tau = \tau(x, y, T)$ and $x(s) = x(\tau, s)$, $\tau < s < T$, then from (4.26) and (4.27) we obtain equality in (4.64). Let $F_0(x, y, \tau, T)$ be the function on the RHS of (4.2). We wish to find $[x, T]$ such that q_0 , defined by (4.54), satisfies $q_0(x, y, T) = \inf_{0 < \tau < T} F_0(x, y, \tau, T)$. We shall identify a subset $S_{x, y, T}$ of $(0, T)$ such that $F_0(x, y, \tau, T) > F_0(x, y, \tau(x, y, T), T) = q_0(x, y, T)$ for $\tau \in S_{x, y, T}$. Hence it is necessary only to consider $\tau \in (0, T) - S_{x, y, T}$. We have already observed that the variational problem (2.28) with fixed τ is quadratic and has the unique solution

(2.37) given by $\Gamma(\tau, s, T, x) = a(\tau, s, T)x + b(\tau, s, T)$, $\tau < s < T$, where

$$\begin{aligned} \sigma_A^2(T)a(\tau, s, T) &= m_{1,A}(s)\sigma_A^2(s, T) [g_{1,A}(s, T) - g_{1,A}(\tau, T)] , \\ \sigma_A^2(T)b(\tau, s, T) &= m_{1,A}(s)\sigma_A^2(s, T) [g_{2,A}(s, T) - g_{2,A}(\tau, T)] . \end{aligned} \tag{4.65}$$

In view of the verification result (4.64), if we show that the path $s \rightarrow \Gamma(\tau, s, T, x)$, $\tau < s < T$, lies in \mathcal{D}_{y, T_0} when $\tau \in (0, T) - S_{x, y, T}$, then it follows that $\tau = \tau(x, y, T)$ is the unique minimizer for (4.2). If $T < 2\sqrt{\Lambda_0 y}$ then $[x, T] \in \mathcal{D}_{y, T_0}$ for all $x > 0$. Since the functions $\tau \rightarrow g_{1,A}(\tau, T), g_{2,A}(\tau, T)$, $0 < \tau < T$, are increasing when $A(\cdot)$ is non-negative, we see from (4.65) that the path $s \rightarrow \Gamma(\tau, s, T, x)$, $\tau < s < T$, lies in \mathcal{D}_{y, T_0} when $0 < \tau < T$. Hence $\tau = \tau(x, y, T)$ is the unique minimizer for (4.2) when $x > 0$ and $0 < T < 2\sqrt{\Lambda_0 y}$.

For $x, y \geq 0$ let

$$\tilde{q}_0(x, y, T) = \min_{0 < \tau < T} \frac{g_{3,A}(\tau, T)}{2} [y + g_{1,A}(\tau, T)x]^2 . \tag{4.66}$$

We have from (4.2) that

$$\tilde{q}_0(x, y, T) \leq \inf_{0 < \tau < T} F_0(x, y, \tau, T) \leq \tilde{q}_0(x, y + g_{2,A}(T, T), T) . \tag{4.67}$$

Using identity (4.12) we see that the minimizing τ for the RHS of (4.66) is

$$g_{3,A}(\tau, T) = \frac{m_{1,A}(T)x}{\sigma_A^2(T)y} . \tag{4.68}$$

Substituting (4.68) into the RHS of (4.66) yields the formula

$$\tilde{q}_0(x, y, T) = \frac{2m_{1,A}(T)xy}{\sigma_A^2(T)} . \tag{4.69}$$

It is intuitively clear from (4.67) that the function q_0 defined by (4.54) on \mathcal{D}_{y, T_0} should satisfy the inequality

$$q_0(x, y, T) \leq \tilde{q}_0(x, y + g_{2,A}(T, T), T) , \quad [x, T] \in \mathcal{D}_{y, T_0} . \tag{4.70}$$

To see this observe that the function $v(\cdot, \cdot)$ defined by $v(\tau, s) = \tilde{q}_0(x(\tau, s), y + g_{2,A}(s, s), s) - w(\tau, s)$, $0 < \tau < s < T_0$, with $w(\cdot, \cdot)$ as in (4.62) satisfies the differential equation

$$\frac{\partial v(\tau, s)}{\partial s} = 2A(s)x(\tau, s) - \frac{2m_{1,A}(s)^2}{\sigma_A^4(s)} [g_{2,A}(s, s) - g_{2,A}(\tau, \tau)]^2 . \tag{4.71}$$

In deriving (4.71) we use the fact that $w(\tau, s) = q_0(x(\tau, s), y, s)$ and that the function $[x, T] \rightarrow q_0(x, y, T)$ is a solution to the HJ equation (2.29). Integrating (4.71) over an interval $[\tau, T]$ with $0 < \tau < T \leq T_0$, we obtain using (4.30), (4.42) the inequality,

$$v(\tau, T) \geq c_1 \frac{y + g_{2,A}(\tau, \tau)}{\tau} \int_{\tau}^T (s - \tau)A(s) ds - 2 \int_{\tau}^T \left[\int_{\tau}^s A(s') ds' \right]^2 ds , \tag{4.72}$$

provided $0 < s - \tau \leq [y + g_{2,A}(\tau, \tau)]/2C_1\tau$. Using the inequality

$$\left[\int_{\tau}^s A(s') ds' \right]^2 \leq 2 \left[\sup_{\tau < s' < s} A(s') \right] \int_{\tau}^s (s' - \tau)A(s') ds' , \tag{4.73}$$

we see that the RHS of (4.72) is non-negative for $0 < T - \tau \leq [y + g_{2,A}(\tau, \tau)]/C_2\tau$, provided C_2 is chosen sufficiently large.

Consider now the function $f(\cdot)$ defined by

$$f(\lambda) = \frac{\lambda}{2}[y_1 + x_1/\lambda]^2, \quad \text{where } x_1, y_1 > 0. \quad (4.74)$$

Evidently the minimizing $\lambda = \lambda_{\min}$ and minimizer are

$$\lambda_{\min} = x_1/y_1, \quad f(\lambda_{\min}) = 2x_1y_1. \quad (4.75)$$

We have furthermore that

$$f(\lambda_{\min}/8) = f(8\lambda_{\min}) \geq 5x_1y_1. \quad (4.76)$$

We consider $[x, T] \in \mathcal{D}_{y, T_0}$ which satisfies (4.49). Let us assume now that $g_{2,A}(T, T) \leq y$. Then we have from (4.70) that

$$q_0(x, y, T) \leq 2\tilde{q}_0(x, y, T). \quad (4.77)$$

Suppose $\tau \in (0, T)$ lies outside the region

$$\frac{m_{1,A}(T)x}{8\sigma_A^2(T)y} \leq g_{3,A}(\tau, T) \leq \frac{8m_{1,A}(T)x}{\sigma_A^2(T)y}. \quad (4.78)$$

Then from (4.68), (4.75), upon setting $y_1 = y$ and $x_1 = m_{1,A}(T)x/\sigma_A^2(T)$ in (4.76), we obtain the inequality

$$F_0(x, y, \tau, T) \geq 5\tilde{q}_0(x, y, T)/2. \quad (4.79)$$

It follows from (4.77), (4.79) that $\tau \in S_{x,y,T}$ if τ does not satisfy (4.78). Observe from (4.49) that $x \leq 4\Lambda_2 y^2/T^2$, which implies that $Tx/y \leq 4\Lambda_2 y/T \leq \Lambda_2 T/\Lambda_0$. Using (4.12), (4.13), (4.78) it follows on choosing Λ_2 sufficiently small, that $\tau \in (0, T) - S_{x,y,T}$ satisfies the inequalities

$$\frac{T}{2} < \tau < T \quad \text{and} \quad C_1 \frac{Tx}{y} \leq T - \tau \leq C_2 \frac{Tx}{y}, \quad (4.80)$$

for some constants $C_1, C_2 > 0$. We have from (4.5), (4.6), (4.65) there are constants $c_3, C_3 > 0$, depending only on T_0 , such that

$$c_3(s - \tau) \frac{x}{T - \tau} \leq \Gamma(\tau, s, T, x) \leq C_3(s - \tau) \left\{ \frac{x}{T - \tau} + T - s \right\}, \quad (4.81)$$

for $0 < \tau < s < T$. Hence we have from (4.80), (4.81) that if $\tau \in (0, T) - S_{x,y,T}$ then

$$\begin{aligned} \Gamma(\tau, s, T, x) &\leq C_3 \left\{ x + \frac{1}{4}(T - \tau)^2 \right\} \\ &\leq C_3 x \left\{ 1 + \frac{C_2^2 T^2 x}{4y^2} \right\} \\ &\leq C_3 x \{ 1 + C_2^2 \Lambda_2 \}. \end{aligned} \quad (4.82)$$

We conclude there exists $\Lambda > 0$, depending only on T_0 , such that if $x \leq \Lambda y^2/T^2$ and $\tau \in (0, T) - S_{x,y,T}$ then the path $s \rightarrow \Gamma(\tau, s, T, x)$, $\tau < s < T$, lies in \mathcal{D}_{y, T_0} . It follows that $\tau = \tau(x, y, T)$ is the unique minimizer for the function $\tau \rightarrow F_0(x, y, \tau, T)$, $0 < \tau < T$, when $x \leq \Lambda y^2/T^2$.

Next we consider the case $g_{2,A}(T, T) > y$ and consider $[x, T] \in \mathcal{D}_{y, T_0}$ which satisfies (4.49). We may estimate $\tau(x, y, T)$ from (4.42). Thus we have from (4.42),

(4.48) that

$$\frac{Tx}{2C_2[y + g_{2,A}(T, T)]} \leq T - \tau(x, y, T) \leq \frac{2Tx}{c_1[y + g_{2,A}(T, T)]} \tag{4.83}$$

$$\text{if } x \leq \frac{\Lambda[y + g_{2,A}(T, T)]^2}{T^2},$$

provided $\Lambda \leq \Lambda_2$ is chosen sufficiently small. From (4.12), (4.48), (4.70) we see that (4.83) implies

$$\begin{aligned} \tilde{q}_0(x, \{y + g_{2,A}(T, T)\}/3, T) &\leq F_0(x, y, \tau(x, y, T), T) \\ &\leq \tilde{q}_0(x, y + g_{2,A}(T, T), T), \end{aligned} \tag{4.84}$$

if Λ is sufficiently small. We have already observed that if q_0 is defined by (4.2) then $\lim_{y \rightarrow 0} q_0(x, y, T) = 0$. It follows from the lower bound (4.84) that

$$\inf_{0 < \tau < T} F_0(x, y, \tau, T) \neq F_0(x, y, \tau(x, y, T), T)$$

if y is sufficiently small.

We show for sufficiently small $\Lambda > 0$ that

$$\text{if } x \leq \frac{\Lambda y[y + g_{2,A}(T, T)]}{T^2}, \text{ then} \tag{4.85}$$

$$\inf_{0 < \tau < T} F_0(x, y, \tau, T) = F_0(x, y, \tau(x, y, T), T).$$

To see this observe from (4.5) that

$$\begin{aligned} \frac{c_3 T}{(T - s)^2} \int_s^T (T - s')A(s') ds' &\leq \frac{\partial g_{2,A}(s, T)}{\partial s} \\ &\leq \frac{C_3 T}{(T - s)^2} \int_s^T (T - s')A(s') ds', \end{aligned} \tag{4.86}$$

where c_3, C_3 are constants. Similarly we have from (4.30) that

$$c_4 \int_0^T sA(s) ds \leq g_{2,A}(T, T) \leq C_4 \int_0^T sA(s) ds. \tag{4.87}$$

We define $\delta, 0 < \delta < 1$, by

$$\int_0^{(1-\delta)T} sA(s) ds = \frac{1}{2} \int_0^T sA(s) ds. \tag{4.88}$$

It follows from (4.87), (4.88) that δ satisfies the inequality

$$\delta \geq \frac{g_{2,A}(T, T)}{2\|A\|_\infty C_4 T^2}. \tag{4.89}$$

Integrating (4.86) over the interval $0 < s < \tau$ we have from the lower bound the inequality,

$$g_{2,A}(\tau, T) \geq c_3 \left[\int_0^\tau sA(s) ds + \frac{\tau}{T - \tau} \int_\tau^T (T - s)A(s) ds \right]. \tag{4.90}$$

We assume first that $\delta \geq 1/2$, whence (4.87), (4.90) imply that $g_{2,A}(\tau, T) \geq c_3 g_{2,A}(T, T)/2C_4$ if $\tau \geq T/2$. We have then from (4.12), (4.13) that

$$F_0(x, y, \tau, T) \geq \frac{g_{3,A}(\tau, T)g_{2,A}(\tau, T)y}{2} \geq \frac{c_5(T - \tau)g_{2,A}(T, T)y}{T^2} \tag{4.91}$$

if $\tau \geq T/2$, for some constant $c_5 > 0$. If $0 < \tau < T/2$ then we again see from (4.87), (4.90) that $g_{2,A}(\tau, T) \geq c_3\tau g_{2,A}(T, T)/2C_4(T - \tau)$. In this case we obtain the inequality

$$F_0(x, y, \tau, T) \geq \frac{c_6 g_{2,A}(T, T)y}{T} \quad \text{if } \tau < T/2, \tag{4.92}$$

where $c_6 > 0$ is constant. It is easy to see from (4.69), the upper bound (4.84) and (4.92) that if $0 < \tau < T/2$ then $F_0(x, y, \tau, T) > F_0(x, y, \tau(x, y, T), T)$ provided x satisfies (4.85) with $\Lambda > 0$ sufficiently small. Similarly from (4.91) we conclude that $F_0(x, y, \tau, T) > F_0(x, y, \tau(x, y, T), T)$ if $T - \tau \geq C_7Tx/y$, for some constant C_7 . Hence if we show that the paths $s \rightarrow \Gamma(\tau, s, T, x)$, $\tau < s < T$, lie in \mathcal{D}_{y, T_0} if $T - \tau < C_7Tx/y$ then it follows that $\inf_{0 < \tau < T} F_0(x, y, \tau, T) = F_0(x, y, \tau(x, y, T), T)$. We see from (4.49) (4.81), (4.82) that this is the case provided x satisfies (4.85) with $\Lambda > 0$ sufficiently small.

Next we assume that $\delta < 1/2$, whence we have from (4.88) and (4.90) that

$$g_{2,A}(\tau, T) \geq c_3 \frac{\tau\delta}{2(T - \tau)} \int_0^T sA(s) ds \quad \text{if } T - \tau \geq \delta T. \tag{4.93}$$

Then using (4.12), (4.13) again together with (4.87), (4.89) we conclude from (4.93) the inequality

$$F_0(x, y, \tau, T) \geq c_8 \frac{g_{2,A}(T, T)^2 y}{T^3} \quad \text{if } T - \tau \geq \delta T, \tag{4.94}$$

where $c_8 > 0$ is constant. It follows from (4.69), the upper bound (4.84) and (4.94) that if $T - \tau \geq \delta T$ then $F_0(x, y, \tau, T) > F_0(x, y, \tau(x, y, T), T)$ provided x satisfies (4.85) with $\Lambda > 0$ sufficiently small. If $T - \tau < \delta T$ then $g_{2,A}(\tau, T) \geq c_3g_{2,A}(T, T)/2C_4$, whence the inequality (4.91) holds provided $T - \tau < \delta T$. We may argue now as in the previous paragraph to conclude that $\inf_{0 < \tau < T} F_0(x, y, \tau, T) = F_0(x, y, \tau(x, y, T), T)$ if x satisfies (4.85) with $\Lambda > 0$ sufficiently small.

Finally we show that for $2\sqrt{\Lambda_0 y} < T$ and $\Lambda > 0$ sufficiently small, if $x \geq T^2/\Lambda$ then $\inf_{0 < \tau < T} F_0(x, y, \tau, T) = F_0(x, y, \tau(x, y, T), T)$. Observe that in this case $x \geq \max\{2y, T^2\}$, whence (4.16) gives a bound on any minimizing τ for the function $\tau \rightarrow F_0(x, y, \tau, T)$, $0 < \tau < T$. Since $x \geq T^2/\Lambda$, this implies $0 < \tau \leq C_4\Lambda y/T$. Hence if we show that all paths $s \rightarrow \Gamma(\tau, s, T, x)$, $\tau < s < T$, lie in \mathcal{D}_{y, T_0} if $0 < \tau \leq C_4\Lambda y/T$ it follows that $\inf_{0 < \tau < T} F_0(x, y, \tau, T) = F_0(x, y, \tau(x, y, T), T)$. We see from (4.42) there exists $\Lambda_3 > 0$ such that that if $0 < \tau \leq \Lambda_3 y/T = \tau^*$ then the characteristic $s \rightarrow x(\tau, s)$, $\tau < s \leq T$, lies in \mathcal{D}_{y, T_0} . Furthermore $x(\tau^*, s) \leq C_5T(s - \tau)$, where C_5 is constant. Since $x \geq T^2/\Lambda$ the lower bound (4.81) implies that $\Gamma(\tau, s, T, x) \geq c_3T(s - \tau)/\Lambda, \tau < s < T$. We conclude that if $\Lambda < c_3/C_5$ then the path $s \rightarrow \Gamma(\tau, s, T, x)$, $\tau < s < T$, lies in \mathcal{D}_{y, T_0} . \square

Remark 4.3. The interval (b) in the statement of Proposition 4.2 is more or less an optimal interval for which the method of characteristics yields the minimizer in (4.2). One can see this by choosing $A(\cdot)$ to have support in a small neighborhood of T . Thus for any δ , $0 < \delta < 1/2$, we set the function $A(\cdot) = A_\delta(\cdot)$, where $A_\delta(s) = A(T)[1 - (T - s)/\delta T]$ for $0 \leq T - s \leq \delta T$ and $A_\delta(s) = 0$ for $T - s > \delta T$. Then the ratio $g_{2,A}(\tau, T)/g_{2,A}(T, T) \simeq 1/N$ if $T - \tau \simeq N\delta T$.

Corollary 4.4. *Assume the function $A : [0, \infty) \rightarrow \mathbb{R}$ is continuous and non-negative. Then for any $T_0 > 0$, there exist constants $C_1, C_2 > 0$, depending only on*

T_0 and $\sup_{0 \leq t \leq T_0} A(t)$, such that the function $(x, T) \rightarrow q_0(x, y, T)$ defined by (4.54) satisfies the inequalities

$$0 \leq \frac{\partial q_0(x, y, T)}{\partial x} - \frac{2m_{1,A}(T)y}{\sigma_A^2(T)} \leq \frac{C_1Ty^2}{x^2}, \quad -\frac{C_1Ty^2}{x^3} \leq \frac{\partial^2 q_0(x, y, T)}{\partial x^2} \leq 0, \quad (4.95)$$

for $[x, T] \in \mathcal{D}_{y, T_0}$, $x \geq C_2 \max\{2y, T^2\}$, $0 < T < T_0$. In addition for each $T_0 > 0$ there is a constant $C > 0$ such that

$$0 \leq \frac{\partial q_0(x, y, T)}{\partial x} - \frac{2m_{1,A}(T)y}{\sigma_A^2(T)} \leq CT, \quad -\frac{CT}{y} \leq \frac{\partial^2 q_0(x, y, T)}{\partial x^2} \leq 0, \quad (4.96)$$

for $[x, T] \in \mathcal{D}_{y, T_0}$, $x, y > 0$, $0 < T < T_0$.

Proof. Since $[x, T] \in \mathcal{D}_{y, T_0}$ we may use the formulas (4.26), (4.38) to show (4.95), (4.96). For (4.95) we use the inequality (4.16) for $\tau(x, y, T)$. The first inequality follows from (4.26) and the fact that $0 \leq g_{2,A}(\tau, \tau) \leq C\tau^2$, where C is constant. To obtain the second inequality we observe that the function $K(\tau)$ of (4.38) satisfies an inequality $K(\tau) \leq C_3\tau^3/y$, where C_3 is constant. Hence for $\tau = \tau(x, y, T)$ we have using (4.16) that

$$\begin{aligned} \frac{K(\tau)m_{1,A}(\tau)^2}{\sigma_A^2(\tau)} &\leq \frac{C_3m_{1,A}(T_0)^2\tau^2}{y} \\ &\leq \frac{C_3m_{1,A}(T_0)^2C_4^2T^2y}{x^2} \\ &\leq \frac{C_3m_{1,A}(T_0)^2C_4^2}{2C_2^2} \end{aligned} \quad (4.97)$$

for $x \geq C_2 \max\{2y, T^2\}$, $0 < T \leq T_0$. We choose C_2 large enough so that the final expression on the RHS of (4.97) is less than $1/2$. We have then from (4.38) the lower bound

$$\frac{\partial^2 q(x, y, T)}{\partial x^2} \geq -\frac{2C_3m_{1,A}(T_0)^2\tau^3}{T^2y}, \quad (4.98)$$

whence the second inequality of (4.95) follows on using the bound (4.16) for $\tau = \tau(x, y, T)$ in (4.98). To prove (4.96) we again use the formulas (4.26), (4.38), observing that $\tau(x, y, T) < T$. \square

5. UNIFORM BOUNDS ON q_ε AND ITS DERIVATIVES

In this section our goal is to show that the bounds on q_0 and its first two space derivatives obtained in Proposition 4.1 and Corollary 4.4 can be extended to q_ε with $\varepsilon > 0$. First we prove results for $\partial q_\varepsilon(x, y, T)/\partial x$ analogous to the bounds on $q_\varepsilon(x, y, T)$ obtained in Proposition 3.3. In fact the lower bound in (5.1) implies the lower bound in (3.28), and the upper bound in (5.1) implies (3.29).

Proposition 5.1. *Assume the function $A : [0, \infty) \rightarrow \mathbb{R}$ is continuous non-negative, and let $q_\varepsilon(x, y, T)$ be defined by (2.18) and (2.21). Then*

$$\frac{2m_{1,A}(T)y}{\sigma_A^2(T)} \leq \frac{\partial q_\varepsilon(x, y, T)}{\partial x} \leq 2\left[1 - \frac{m_{2,A}(T)}{\sigma_A^2(T)}\right] + \frac{2m_{1,A}(T)y}{\sigma_A^2(T)}, \quad x, y, T > 0. \quad (5.1)$$

Proof. Letting $u_\varepsilon(x, y, T) = \partial q_\varepsilon(x, y, T)/\partial x$, we note from Proposition 3.1 that the function $(x, T) \rightarrow u_\varepsilon(x, y, T)$ is continuous in the domain $\{(x, T) : x, T > 0\}$, with continuous derivatives $\partial u_\varepsilon(x, y, T)/\partial x$, $\partial u_\varepsilon(x, y, T)/\partial T$, $\partial^2 u_\varepsilon(x, y, T)/\partial x^2$. Furthermore $(x, T) \rightarrow u_\varepsilon(x, y, T)$ is continuous up to the boundary $\{(x, T) : x = 0, T >$

0}. Differentiating (2.22) with respect to x , we see using (2.15) that $u_\varepsilon(x, y, T)$ is a solution to the diffusive Burger's PDE

$$\begin{aligned} & \frac{\partial u_\varepsilon(x, y, T)}{\partial T} + [\lambda(x, y, T) + u_\varepsilon(x, y, T)] \frac{\partial u_\varepsilon(x, y, T)}{\partial x} \\ & + \left[A(T) + \frac{1}{\sigma_A^2(T)} \right] u_\varepsilon(x, y, T) \\ & = \frac{\varepsilon}{2} \frac{\partial^2 u_\varepsilon(x, y, T)}{\partial x^2}, \quad x, y, T > 0. \end{aligned} \quad (5.2)$$

It follows from (5.2) that

$$\begin{aligned} & \frac{\partial v_\varepsilon(x, y, T)}{\partial T} + [\lambda(x, y, T) + u_\varepsilon(x, y, T)] \frac{\partial v_\varepsilon(x, y, T)}{\partial x} = \frac{\varepsilon}{2} \frac{\partial^2 v_\varepsilon(x, y, T)}{\partial x^2}, \\ & \text{where } v_\varepsilon(x, y, T) = \frac{\sigma_A^2(T) u_\varepsilon(x, y, T)}{m_{1,A}(T)}. \end{aligned} \quad (5.3)$$

From Proposition 3.2 it follows that the function $(x, T) \rightarrow v_\varepsilon(x, y, T)$ is non-negative. Since $q_\varepsilon(0, y, T) = 0$ we also have from the lower bound (3.28) of Proposition 3.3 that $v_\varepsilon(0, y, T) \geq 2y$, $y, T > 0$.

Using Ito's lemma and the martingale optional sampling theorem, we have that

$$v_\varepsilon(x, y, T) = E \left[v_\varepsilon(X_\varepsilon^*(s \vee \tau_{\varepsilon,x,T,K}^*), y, s \vee \tau_{\varepsilon,x,T,K}^*) \mid X_\varepsilon^*(T) = x \right] \quad (5.4)$$

for $0 < s < T$, where the stopping time $\tau_{\varepsilon,x,T,K}^*$ is defined in the proof of Lemma 2.4. In view of the non-negativity of v_ε and lower bound on $v_\varepsilon(0, y, \cdot)$, we conclude from (5.4) that

$$\begin{aligned} v_\varepsilon(x, y, T) & \geq E \left[v_\varepsilon(0, y, \tau_{\varepsilon,x,T}^*); \tau_{\varepsilon,x,T}^* = \tau_{\varepsilon,x,T,K}^* > s \right] \\ & \geq 2yP(\tau_{\varepsilon,x,T}^* = \tau_{\varepsilon,x,T,K}^* > s), \quad 0 < s < T. \end{aligned} \quad (5.5)$$

Observe that

$$\{\tau_{\varepsilon,x,T}^* > s\} = \{\tau_{\varepsilon,x,T}^* = \tau_{\varepsilon,x,T,K}^* > s\} \cup \{\tau_{\varepsilon,x,T,K}^* > s, X_\varepsilon^*(\tau_{\varepsilon,x,T,K}^*) = K\}. \quad (5.6)$$

As in the proof of Lemma 2.4 we use the inequality $X_\varepsilon^*(\cdot) \leq X_\varepsilon(\cdot)$, where $X_\varepsilon(\cdot)$ is given by (2.16), to show that $\limsup_{K \rightarrow \infty} P(\tau_{\varepsilon,x,T,K}^* > s, X_\varepsilon^*(\tau_{\varepsilon,x,T,K}^*) = K) \leq \limsup_{K \rightarrow \infty} P(\sup_{s < s' < T} X_\varepsilon(s') > K) = 0$. Letting $K \rightarrow \infty$ in (5.5) we then have from (5.6) that $v_\varepsilon(x, y, T) \geq 2yP(\tau_{\varepsilon,x,T}^* > s)$ for $0 < s < T$. Since Lemma 2.4 implies that $\lim_{s \rightarrow 0} P(\tau_{\varepsilon,x,T}^* > s) = 1$ we conclude that $v_\varepsilon(x, y, T) \geq 2y$, whence the lower bound in (5.1).

To obtain the upper bound in (5.1) we define for $h > 0$ a function $q_{\varepsilon,h}$ by

$$q_{\varepsilon,h}(x, y, T) = q_\varepsilon \left(x + \frac{\sigma_A^2(T)}{m_{1,A}(T)} h, y, T \right) \quad x, T > 0. \quad (5.7)$$

The function $(x, T) \rightarrow q_{\varepsilon,h}(x, y, T)$ is also a solution to (2.22), whence the function $v_{\varepsilon,h}(x, y, T) = [q_{\varepsilon,h}(x, y, T) - q_\varepsilon(x, y, T)]/h$ is a solution to the PDE

$$\begin{aligned} & \frac{\partial v_{\varepsilon,h}(x, y, T)}{\partial T} + \lambda(x, y, T) \frac{\partial v_{\varepsilon,h}(x, y, T)}{\partial x} \\ & + \frac{1}{2} \left[\frac{\partial q_{\varepsilon,h}(x, y, T)}{\partial x} + \frac{\partial q_\varepsilon(x, y, T)}{\partial x} \right] \frac{\partial v_{\varepsilon,h}(x, y, T)}{\partial x} \\ & = \frac{\varepsilon}{2} \frac{\partial^2 v_{\varepsilon,h}(x, y, T)}{\partial x^2}, \quad x, T > 0. \end{aligned} \quad (5.8)$$

Let $X_{\varepsilon,h}^*(\cdot)$ denote solutions to the backwards in time SDE (2.25) with μ_ε given by

$$\mu_\varepsilon(x, y, T) = \lambda(x, y, T) + \frac{1}{2} \left[\frac{\partial q_{\varepsilon,h}(x, y, T)}{\partial x} + \frac{\partial q_\varepsilon(x, y, T)}{\partial x} \right]. \tag{5.9}$$

Letting $\tau_{\varepsilon,h,x,T,K}^*$ be the first exit time for $X_{\varepsilon,h}^*(s)$, $s < T$, with terminal condition $X_{\varepsilon,h}^*(T) = x$ from the interval $(0, K)$ we have again from Ito's lemma and the martingale optional sampling theorem the identity

$$v_{\varepsilon,h}(x, y, T) = E \left[v_{\varepsilon,h}(X_{\varepsilon,h}^*(s \vee \tau_{\varepsilon,h,x,T,K}^*), y, s \vee \tau_{\varepsilon,h,x,T,K}^*) \mid X_{\varepsilon,h}^*(T) = x \right] \tag{5.10}$$

for $0 < s < T$. We can simplify the expression (5.10) by taking $K \rightarrow \infty$. Observe that

$$\begin{aligned} & E \left[v_{\varepsilon,h}(K, y, \tau_{\varepsilon,h,x,T,K}^*); \tau_{\varepsilon,h,x,T,K}^* > s, X_{\varepsilon,h}^*(\tau_{\varepsilon,h,x,T,K}^*) = K \mid X_{\varepsilon,h}^*(T) = x \right] \\ & \leq \sup_{s < s' < T} |v_{\varepsilon,h}(K, y, s')| P \left(\sup_{s < s' < T} X_{\varepsilon,h}^*(s') > K \mid X_{\varepsilon,h}^*(T) = x \right). \end{aligned} \tag{5.11}$$

Since the drift μ_ε of (5.9) satisfies $\mu_\varepsilon(x, y, T) \geq \lambda(x, y, T)$, we see that the probability on the RHS of (5.11) is bounded by $P(\sup_{s < s' < T} X_\varepsilon(s') > K \mid X_\varepsilon(T) = x)$, where $X_\varepsilon(\cdot)$ is given by (2.16). We may bound this latter probability by using the reflection principle (2.46), whence the probability converges to zero as $K \rightarrow \infty$ like $\exp[-cK^2]$ for some constant $c > 0$. From Proposition 3.3 we see that $\sup_{s < s' < T} |v_{\varepsilon,h}(K, y, s')|$ is bounded linearly in K as $K \rightarrow \infty$. We conclude that the RHS of (5.11) converges to 0 as $K \rightarrow \infty$. Letting $K \rightarrow \infty$ in (5.10) we have then that

$$\begin{aligned} v_{\varepsilon,h}(x, y, T) &= E \left[v_{\varepsilon,h}(0, y, \tau_{\varepsilon,h,x,T}^*); \tau_{\varepsilon,h,x,T}^* > s \mid X_{\varepsilon,h}^*(T) = x \right] \\ &+ E \left[v_{\varepsilon,h}(X_{\varepsilon,h}^*(s), y, s); \tau_{\varepsilon,h,x,T}^* < s \mid X_{\varepsilon,h}^*(T) = x \right], \end{aligned} \tag{5.12}$$

for $0 < s < T$, where $\tau_{\varepsilon,h,x,T}^*$ is the first hitting time at 0 for the diffusion $X_{\varepsilon,h}^*(s)$, $s < T$, with terminal condition $X_{\varepsilon,h}^*(T) = x$.

It is easy to bound from above the first expectation on the RHS of (5.12) by using Proposition 3.3. Thus we have from the upper bound (3.29) the inequality $v_{\varepsilon,h}(0, y, \tau) \leq 2g_{2,A}(\tau, \tau) + 2y, \tau > 0$, where $g_{2,A}(\tau, \tau) = [\sigma_A^2(\tau) - m_{2,A}(\tau)]/m_{1,A}(\tau)$ has derivative (4.30). Since $A(\cdot)$ is non-negative, whence the function $\tau \rightarrow g_{2,A}(\tau, \tau)$ is increasing, we see that the first expectation on the RHS of (5.12) is bounded above by $2g_{2,A}(T, T) + 2y$ for all s satisfying $0 < s < T$. We shall show that the limit of the second expectation on the RHS of (5.12) converges to 0 as $s \rightarrow 0$. The upper bound in (5.1) follow then by first letting $s \rightarrow 0$ in (5.12) and then $h \rightarrow 0$.

Using the bound (3.9) of Proposition 3.2, we see that the second expectation on the RHS of (5.12) is bounded in absolute value by

$$\frac{C(T)y}{sh} E \left[X_{\varepsilon,h}^*(s) + sh + s^2; \tau_{\varepsilon,h,x,T}^* < s \mid X_{\varepsilon,h}^*(T) = x \right], \quad 0 < s < T, \tag{5.13}$$

where the constant $C(T)$ depends only on T . To estimate the expression (5.13) we use the lower bound (5.1) which has been already proven. Let $X_{\varepsilon,\text{linear}}(\cdot)$ denote solutions to the SDE (2.25) with μ_ε given by

$$\mu_\varepsilon(x, y, T) = \lambda(x, y, T) + \frac{\partial q_{\text{linear}}(x, y, T)}{\partial x} = \lambda(x, -y, T). \tag{5.14}$$

If $X_{\varepsilon,h}^*(T) = X_{\varepsilon,\text{linear}}(T) = x$ then $X_{\varepsilon,h}^*(s) \leq X_{\varepsilon,\text{linear}}(s)$ for all $\tau_{\varepsilon,h,x,T}^* < s < T$. Letting $\tau_{\varepsilon,\text{linear},x,T}$ be the first hitting time at 0 for $X_{\varepsilon,\text{linear}}(s)$, $s < T$, with

$X_{\varepsilon,\text{linear}}(T) = x$, we see that $\tau_{\varepsilon,h,x,T}^* \geq \tau_{\varepsilon,\text{linear},x,T}$. Hence we have that

$$\begin{aligned} E [X_{\varepsilon,h}^*(s); \tau_{\varepsilon,h,x,T}^* < s \mid X_{\varepsilon,h}^*(T) = x] \\ \leq E [X_{\varepsilon,\text{linear}}(s); \tau_{\varepsilon,\text{linear},x,T} < s \mid X_{\varepsilon,\text{linear}}(T) = x]. \end{aligned} \tag{5.15}$$

Since the drift (5.14) is linear the SDE (2.25) can be explicitly solved in this case and the solution is obtained by replacing y by $-y$ in the formula (2.16). Thus we have that

$$X_{\varepsilon}(s) = x_{\text{class}}(s) - \sqrt{\varepsilon} \frac{\sigma_A^2(s)}{m_{1,A}(s)} Z(s), \quad s < T, \tag{5.16}$$

where $Z(\cdot)$ is defined in (2.16), and from (2.12) we have that

$$\begin{aligned} \sigma_A^2(T)x_{\text{class}}(s) &= xm_{1,A}(s,T)\sigma_A^2(s) - ym_{1,A}(s)\sigma_A^2(s,T) \\ &\quad + m_{1,A}(s,T)m_{2,A}(s,T)\sigma_A^2(s) - m_{2,A}(s)\sigma_A^2(s,T). \end{aligned} \tag{5.17}$$

We see from (5.17) that $\lim_{s \rightarrow 0} x_{\text{class}}(s) = -y$, whence there exists s_0 with $0 < s_0 < T$ such that $x_{\text{class}}(s) \leq -y/2$ for $0 < s \leq s_0$. It follows then from (5.16) that

$$P(\tau_{\varepsilon,\text{linear},x,T} < s) \leq P\left(\inf_{s < s' < s_0} s'Z(s') > cy/\sqrt{\varepsilon}\right), \quad 0 < s \leq s_0, \tag{5.18}$$

for some constant $c > 0$ depending only on s_0 . The variables $sZ(s)$, $s < T$, are Gaussian with zero mean and variance bounded above by $C(T)s$ for some constant $C(T)$ depending only on T . Hence the probability on the RHS of (5.18) is bounded above by $\exp[-c/s]$ for some constant $c > 0$. Observing also that $\sup_{0 < s < T} E [X_{\varepsilon,\text{linear}}(s)^2 \mid X_{\varepsilon,\text{linear}}(T) = x] < \infty$, we conclude from (5.18) and the Schwarz inequality applied to the RHS of (5.15) that the expression (5.13) converges to 0 as $s \rightarrow 0$. \square

Proposition 5.2. *Assume the function $A : [0, \infty) \rightarrow \mathbb{R}$ is continuous and non-negative. Then for any $T_0 > 0$, there exists a constant $C > 0$, depending only on T_0 and $\sup_{0 \leq t \leq T_0} A(t)$, such that the function $(x, T) \rightarrow q_{\varepsilon}(x, y, T)$ satisfies the inequality*

$$-\frac{CTy^2}{x} \leq q_{\varepsilon}(x, y, T) - q_{\text{linear}}(x, y, T) \tag{5.19}$$

for $xy \geq \varepsilon T$, $x \geq \max\{2y, T^2\}$, $0 < T < T_0$, and the inequality

$$q_{\varepsilon}(x, y, T) - \frac{2m_{1,A}(T)xy}{\sigma_A^2(T)} \leq CTx \quad \text{for } x > 0, 0 < T < T_0. \tag{5.20}$$

Proof. All constants in the following can be chosen to depend only on T_0 and $\sup_{0 \leq t \leq T_0} A(t)$. We consider the stochastic integral $s \rightarrow M(s)$ defined similarly to (2.49) but with q_{linear} in place of q_{ε} and μ_{ε} in (2.25) given by $\mu_{\varepsilon} = \mu_{\varepsilon}^*$ of (2.27). Then similarly to (2.51) we obtain the inequality

$$\begin{aligned} q_{\text{linear}}(x, y, T) \\ \leq E[q_{\text{linear}}(X_{\varepsilon}^*(s \vee \tau_{\varepsilon,x,T,K}^*), y, s \vee \tau_{\varepsilon,x,T,K}^*)] \\ + E\left[\frac{1}{2} \int_{s \vee \tau_{\varepsilon,x,T,K}^*}^T [\mu_{\varepsilon}^*(X_{\varepsilon}^*(s), y, s) - \lambda(X_{\varepsilon}^*(s), y, s)]^2 ds \mid X_{\varepsilon}^*(T) = x\right] \end{aligned} \tag{5.21}$$

for $0 < s \leq T$, where $X_{\varepsilon}^*(\cdot)$ is the solution to the SDE (2.25) with $\mu_{\varepsilon} = \mu_{\varepsilon}^*$. The stopping time $\tau_{\varepsilon,x,T,K}^*$ in (5.21) is the first exit time of $X_{\varepsilon}^*(s)$, $s < T$, with $X_{\varepsilon}^*(T) = x$ from the interval $(0, K)$. As in the proof of Lemma 2.4 we use the

inequality $X_\varepsilon^*(\cdot) \leq X_\varepsilon(\cdot)$, where $X_\varepsilon(\cdot)$ is given by (2.16). Letting $K \rightarrow \infty$ in (5.21) we have then using (2.65) of Lemma 2.4 the inequality

$$q_{\text{linear}}(x, y, T) \leq E[q_{\text{linear}}(X_\varepsilon^*(s \vee \tau_{\varepsilon,x,T}^*), y, s \vee \tau_{\varepsilon,x,T}^*)] + q_\varepsilon(x, y, T) \tag{5.22}$$

for $0 < s < T$, with $\tau_{\varepsilon,x,T}^*$ the stopping time defined in Lemma 2.4. Using the inequality $q_{\text{linear}}(x, y, \tau) \leq Cy[x/\tau + \tau]$, $0 < \tau \leq T_0$, where C is constant, we have from (5.22) the inequality

$$q_{\text{linear}}(x, y, T) - q_\varepsilon(x, y, T) \leq CyE[\tau_{\varepsilon,x,T}^*] + \frac{Cy}{s} E[X_\varepsilon^*(s) + s^2; \tau_{\varepsilon,x,T}^* < s], \tag{5.23}$$

for $0 < s < T$. Just as in the proof of Proposition 5.1 we use the lower bound (5.1) to show that the second expectation on the RHS of (5.23) converges to 0 as $s \rightarrow 0$.

To bound $E[\tau_{\varepsilon,x,T}^*]$ we use the identity (2.68), observing since $A(\cdot)$ is non-negative that the sum of the last two terms on the RHS are non-negative. We have then upon applying the Schwarz inequality to the RHS of (2.68) that for any $T_0 > 0$ one has

$$\begin{aligned} & \frac{C_1}{\sqrt{\tau_{\varepsilon,x,T}^*}} \left\{ \frac{1}{2} \int_{\tau_{\varepsilon,x,T}^*}^T [\mu_\varepsilon^*(X_\varepsilon^*(s), y, s) - \lambda(X_\varepsilon^*(s), y, s)]^2 ds \right\}^{1/2} \\ & \geq \frac{c_1 x}{T} - \sqrt{\varepsilon} Z(\tau_{\varepsilon,x,T}^*) \quad \text{for } 0 < T \leq T_0, \end{aligned} \tag{5.24}$$

where $C_1, c_1 > 0$ are constants. It follows from (5.24) that if $|Z(\tau_{\varepsilon,x,T}^*)| \leq c_1 x / 2\sqrt{\varepsilon} T$ then

$$\tau_{\varepsilon,x,T}^* \leq \left(\frac{2TC_1}{c_1 x} \right)^2 \frac{1}{2} \int_{\tau_{\varepsilon,x,T}^*}^T [\mu_\varepsilon^*(X_\varepsilon^*(s), y, s) - \lambda(X_\varepsilon^*(s), y, s)]^2 ds. \tag{5.25}$$

We conclude from (2.65) of Lemma 2.4 and (5.25) that

$$E[\tau_{\varepsilon,x,T}^*; |Z(\tau_{\varepsilon,x,T}^*)| \leq c_1 x / 2\sqrt{\varepsilon} T] \leq \left(\frac{2TC_1}{c_1 x} \right)^2 q_\varepsilon(x, y, T). \tag{5.26}$$

From (2.74), (2.75) we have that

$$E[\tau_{\varepsilon,x,T}^*; |Z(\tau_{\varepsilon,x,T}^*)| > a] \leq E[\tau_a] \leq \frac{C_2}{a^2}, \quad a > 0, \quad 0 < T \leq T_0, \tag{5.27}$$

where C_2 is constant. It follows from (5.27) that

$$E[\tau_{\varepsilon,x,T}^*; |Z(\tau_{\varepsilon,x,T}^*)| > c_1 x / 2\sqrt{\varepsilon} T] \leq \frac{4C_2 \varepsilon T^2}{c_1^2 x^2}, \quad 0 < T \leq T_0. \tag{5.28}$$

We conclude from (5.26), (5.28) and Proposition 3.2 that

$$E[\tau_{\varepsilon,x,T}^*] \leq \frac{C_3 T y}{x} \quad \text{if } x \geq 2y, \quad x \geq T^2, \quad xy > \varepsilon T, \quad 0 < T \leq T_0, \tag{5.29}$$

where C_3 is constant. The lower bound (5.19) follows from (5.23), (5.29) on letting $s \rightarrow 0$ in (5.23).

To prove (5.20), we first observe that if $x \geq y$ the inequality follows from the inequality $q_\varepsilon(x, y, T) \leq q_{\text{linear}}(x, y, T)$. Hence we may assume $0 < x < y$. We consider the stochastic integral $s \rightarrow M(s)$ defined similarly to (2.49) but with

μ_ε in (2.25) given by (5.14). Arguing as in the proof of Proposition 5.1 we have analogously to (2.51) the inequality

$$q_\varepsilon(x, y, T) \leq E \left[\frac{1}{2} \int_{\tau_{\varepsilon, \text{linear}, x, T}}^T [\mu_\varepsilon(X_\varepsilon(s), y, s) - \lambda(X_\varepsilon(s), y, s)]^2 ds \mid X_\varepsilon(T) = x \right]. \tag{5.30}$$

Next we consider the stochastic integral $s \rightarrow M(s)$ defined similarly to (2.49) but with q_{linear} in place of q_ε and μ_ε in (2.25) again given by (5.14). Then using Ito's formula and the martingale optional sampling theorem we have the identity

$$q_{\text{linear}}(x, y, T) = E [q_{\text{linear}}(0, y, \tau_{\varepsilon, \text{linear}, x, T})] + E \left[\frac{1}{2} \int_{\tau_{\varepsilon, \text{linear}, x, T}}^T [\mu_\varepsilon(X_\varepsilon(s), y, s) - \lambda(X_\varepsilon(s), y, s)]^2 ds \mid X_\varepsilon(T) = x \right]. \tag{5.31}$$

Observe next that

$$q_{\text{linear}}(x, y, T) - E [q_{\text{linear}}(0, y, \tau_{\varepsilon, \text{linear}, x, T})] \leq \frac{2m_{1,A}(T)xy}{\sigma_A^2(T)} + C_4 y E [T - \tau_{\varepsilon, \text{linear}, x, T}], \quad 0 < T \leq T_0, \tag{5.32}$$

for some constant C_4 . The inequality (5.20) follows from (5.30)-(5.32) if we can show that

$$E [T - \tau_{\varepsilon, \text{linear}, x, T}] \leq \frac{C_5 T x}{y}, \quad 0 < x < y, \quad 0 < T \leq T_0, \tag{5.33}$$

for a constant C_5 .

We show that (5.33) holds if $y \geq C_6 T^2$ for some constant C_6 . We choose C_6 such that the drift μ_ε defined by (5.14) satisfies the inequality $\mu_\varepsilon(x, y, s) \geq y/C_5 s$, $0 < s \leq T_0$ for some constant $C_5 > 0$. This follows from (2.15) since $\sigma_A^2(s) - m_{2,A}(s) \leq C_7 s^2$, $0 < s \leq T_0$, where C_7 is constant. Then the LHS of (5.33) is bounded above by $u_\varepsilon(x) = E [T - \tau_{\varepsilon, \text{linear}, x, T}^*]$, where $\tau_{\varepsilon, \text{linear}, x, T}^*$ is the first exit time from $(0, \infty)$ for the diffusion $X_\varepsilon(s)$, $s < T$, which is the solution to (2.25) with terminal condition $X_\varepsilon(T) = x$ and drift $\mu_\varepsilon(x, y, T) = y/C_5 T$. Now $u_\varepsilon(\cdot)$ is the solution to the boundary value problem,

$$-\frac{\varepsilon}{2} \frac{d^2 u_\varepsilon(x)}{dx^2} + \frac{y}{C_5 T} \frac{du_\varepsilon(x)}{dx} = 1, \quad x > 0, \quad u_\varepsilon(0) = 0. \tag{5.34}$$

The solution to (5.34) is the linear function $u_\varepsilon(x) = C_5 T x / y$, whence we obtain the upper bound (5.33).

To finish the proof of (5.20) we need to deal with the case $0 < x < y < C_6 T^2$. In this case (5.20) reduces to the inequality $q_\varepsilon(x, y, T) \leq C_8 T x$, $0 < T \leq T_0$, where C_8 is constant. We consider again the stochastic integral $s \rightarrow M(s)$ defined similarly to (2.49) but with μ_ε in (2.25) given by $\mu_\varepsilon(x, y, s) = \lambda(x, y, s) + C_9 T$, $0 < s < T \leq T_0$, where the constant $C_9 > 0$ is chosen sufficiently large. Let $\tau_{\varepsilon, x, T}$ be the first exit time for the diffusion $X_\varepsilon(s)$, $s < T$, of (2.25) with $X_\varepsilon(T) = x$. Then a similar inequality to (5.30) holds, whence we have

$$q_\varepsilon(x, y, T) \leq \frac{C_9^2 T^2}{2} E [T - \tau_{\varepsilon, x, T}], \quad 0 < x \leq y \leq C_6 T^2, \quad 0 < T \leq T_0. \tag{5.35}$$

We also see as before that $E[T - \tau_{\varepsilon,x,T}] \leq C_{10}x/T$, $0 < T \leq T_0$, where C_{10} is constant. The result follows. \square

Remark 5.3. The inequality (5.20) also follows from the upper bound in (5.1) by integration over the interval $[0, x]$. In our proof in Proposition 5.2 we use the fact that the optimizing τ in (4.2) is close to T as $x \rightarrow 0$.

We have already shown in Proposition 5.1 that all of the bounds on the derivative $\partial q_\varepsilon(x, y, T)/\partial x$ with $\varepsilon = 0$ in Corollary 4.4, with the exception of the upper bound (4.95), extend to $\varepsilon > 0$. Next we extend the upper bound (4.95) to $\varepsilon > 0$.

Proposition 5.4. *Assume the function $A : [0, \infty) \rightarrow \mathbb{R}$ is continuous and non-negative. Then for any $T_0 > 0$ there exists a constant $C > 0$, depending only on T_0 and $\sup_{0 \leq t \leq T_0} A(t)$, such that the function $(x, T) \rightarrow q_\varepsilon(x, y, T)$ satisfies the inequality*

$$\frac{\partial q_\varepsilon(x, y, T)}{\partial x} - \frac{2m_{1,A}(T)y}{\sigma_A^2(T)} \leq \frac{CTy^2}{x^2}, \tag{5.36}$$

for $xy \geq \varepsilon T$, $x \geq \max\{2y, CT^2\}$, and $0 < T \leq T_0$.

Proof. We use the identity (5.4). The upper bound (5.1) implies that for any $T_0 > 0$ there is a constant C_1 such that $v_\varepsilon(x, y, \tau) \leq 2y + C_1\tau^2$, $0 < \tau \leq T_0$. Hence we may let $K \rightarrow \infty$ and $s \rightarrow 0$ in (5.4) to obtain the identity

$$v_\varepsilon(x, y, T) = E[v_\varepsilon(0, y, \tau_{\varepsilon,x,T}^* | X_\varepsilon^*(T) = x)], \quad 0 < s < T. \tag{5.37}$$

Inequality (5.36) follows then from (5.37) if we can show that

$$E[(\tau_{\varepsilon,x,T}^*)^2] \leq \frac{C_2T^2y^2}{x^2}, \quad x \geq \max\{2y, C_2T^2\}, \quad 0 < T \leq T_0, \tag{5.38}$$

for a constant C_2 . To prove (5.38) we use the upper bound (5.1). Analogously to (5.14) we consider solutions $X_{\varepsilon,\text{linear}}^*(\cdot)$ to the SDE (2.25) with μ_ε given by

$$\mu_\varepsilon(x, y, T) = \lambda(x, -y, T) + 2\left[1 - \frac{m_{2,A}(T)}{\sigma_A^2(T)}\right]. \tag{5.39}$$

Letting $\tau_{\varepsilon,\text{linear},x,T}^*$ be the first hitting time at 0 for $X_{\varepsilon,\text{linear}}^*(s)$, $s < T$, with $X_{\varepsilon,\text{linear}}^*(T) = x$, we see that $\tau_{\varepsilon,x,T}^* \leq \tau_{\varepsilon,\text{linear},x,T}^*$. To prove (5.38) it will be sufficient therefore to estimate $E[(\tau_{\varepsilon,\text{linear},x,T}^*)^2]$.

Since the drift (5.39) is linear the SDE (2.25) can be again explicitly solved, and the solution is

$$X_\varepsilon(s) = x_{\text{class}}^*(s) - \sqrt{\varepsilon} \frac{\sigma_A^2(s)}{m_{1,A}(s)} Z(s), \quad s < T, \tag{5.40}$$

where $Z(\cdot)$ is defined in (2.16), and $x_{\text{class}}^*(\cdot)$ is obtained from (5.17) by switching the signs of the terms which do not involved x or y . Thus we have that

$$\begin{aligned} \sigma_A^2(T)x_{\text{class}}^*(s) &= xm_{1,A}(s, T)\sigma_A^2(s) - ym_{1,A}(s)\sigma_A^2(s, T) \\ &\quad - m_{1,A}(s, T)m_{2,A}(s, T)\sigma_A^2(s) + m_{2,A}(s)\sigma_A^2(s, T). \end{aligned} \tag{5.41}$$

We have from (2.40), (4.5) and (5.41) that

$$x_{\text{class}}^*(s) \geq c_3 \frac{sx}{T} - [1 - \frac{s}{T}] \{C_3y + C_4sT\}, \quad 0 < s < T, \quad 0 < T \leq T_0, \tag{5.42}$$

for some constants $C_3, c_3, C_4 > 0$. We conclude from (5.42) that

$$\tau_{0,\text{linear},x,T}^* \leq T \frac{2C_3y}{c_3x + 2C_3y} \quad \text{if } x \geq \frac{2C_4T^2}{c_3}. \tag{5.43}$$

We may extend the inequality (5.43) to $\varepsilon > 0$ by considering for $n = 1, 2, \dots$, events A_n where $A_n = \{\tau_{\varepsilon,\text{linear},x,T}^* > nTy/x\}$. Assuming $x \geq \max\{2y, 2C_4T^2/c_3\}$, we have from (5.42) the inequality $x_{\text{class}}^*(\tau_{\varepsilon,\text{linear},x,T}^*) \geq c_3\tau_{\varepsilon,\text{linear},x,T}^*(x/4T)$ on the event A_n provided $n \geq 4C_3/c_3$. Hence from (5.40) we have on A_n with $n \geq 4C_3/c_3$ the inequality $Z(\tau_{\varepsilon,\text{linear},x,T}^*) < -c_5x/\sqrt{\varepsilon}T$, where $c_5 > 0$ depends only on T_0 . From (2.75) we have that

$$P\left(\sup_{s < s' < T} |Z(s')| > a\right) \leq \frac{C_6}{(a^2s)^4}, \quad a > 0, \quad 0 < s < T \leq T_0, \tag{5.44}$$

where $C_6 > 0$ depends only on T_0 . Choosing integers n_0, n_1 such that $n_0 \geq 4C_3/c_3$ and $n_1 \geq x/y$, we have from (5.44) that

$$\begin{aligned} E[(\tau_{\varepsilon,\text{linear},x,T}^*)^2] &\leq \frac{T^2y^2}{x^2} \left[n_0^2 + \sum_{n_0 \leq n \leq n_1} (n+1)^2 P(A_n) \right] \\ &\leq \frac{T^2y^2}{x^2} \left[n_0^2 + \frac{C_6}{c_5^8} \left(\frac{\varepsilon T}{xy}\right)^4 \sum_{n_0 \leq n \leq n_1} \frac{(n+1)^2}{n^4} \right]. \end{aligned} \tag{5.45}$$

The inequality (5.38) follows from (5.45) provided $xy \geq \varepsilon T$. □

Finally we show that the function $x \rightarrow q_\varepsilon(x, y, T)$ is concave, thereby extending the upper bound on $\partial^2 q_\varepsilon(x, y, T)/\partial x^2$ with $\varepsilon = 0$ in Corollary 4.4 to $\varepsilon > 0$. Because of the singularity in the drift $[x, T] \rightarrow \lambda(x, y, T)$, $x, y, T > 0$, of (2.15) as $T \rightarrow 0$, we use an approximation method.

Lemma 5.5. *Assume the function $A : [0, \infty) \rightarrow \mathbb{R}$ is continuous and non-negative. Then for any $\delta > 0$ there is a unique classical solution $[x, T] \rightarrow q_{\varepsilon,\delta}(x, y, T)$, $x > 0$, $T > \delta$, to the PDE (2.22) with boundary and initial conditions*

$$q_{\varepsilon,\delta}(0, y, T) = 0, \quad T > \delta, \quad q_{\varepsilon,\delta}(x, y, \delta) = \frac{2m_{1,A}(\delta)xy}{\sigma_A^2(\delta)}, \quad x > 0. \tag{5.46}$$

Furthermore, the function $[x, T] \rightarrow q_{\varepsilon,\delta}(x, y, T)$ satisfies the inequalities

$$\frac{2m_{1,A}(T)xy}{\sigma_A^2(T)} \leq q_{\varepsilon,\delta}(x, y, T) \leq -2\lambda(0, y, T)x, \quad x > 0, \quad T > \delta, \tag{5.47}$$

$$\frac{2m_{1,A}(T)y}{\sigma_A^2(T)} \leq \frac{\partial q_{\varepsilon,\delta}(x, y, T)}{\partial x} \leq 2\left[1 - \frac{m_{2,A}(T)}{\sigma_A^2(T)}\right] + \frac{2m_{1,A}(T)y}{\sigma_A^2(T)} \tag{5.48}$$

for $x > 0$ and $T > \delta$. Let $q_\varepsilon(x, y, T)$ be defined by (2.18) and (2.21). Then

$$\lim_{\delta \rightarrow 0} [q_{\varepsilon,\delta}(x, y, T) - q_\varepsilon(x, y, T)] = 0,$$

and the limit is uniform in all sets $\{[x, T] : x > 0, T_0 < T < T_1\}$ with $0 < T_0 < T_1 < \infty$.

Proof. We proceed as in the proof of Proposition 3.3 by setting $v_{\varepsilon,\delta}(x, y, T) = \exp[-q_{\varepsilon,\delta}(x, y, T)/\varepsilon]$. If $[x, T] \rightarrow v_{\varepsilon,\delta}(x, y, T)$ is a solution to the PDE (2.19) in the region $x > 0, T > \delta$ with boundary and initial conditions

$$v_{\varepsilon,\delta}(0, y, T) = 1, \quad T > \delta, \quad v_{\varepsilon,\delta}(x, y, \delta) = \exp\left[-\frac{2m_{1,A}(\delta)xy}{\sigma_A^2(\delta)\varepsilon}\right], \quad x > 0, \tag{5.49}$$

then $q_{\varepsilon,\delta}(x, y, T) = -\varepsilon \log v_{\varepsilon,\delta}(x, y, T)$ is a solution to (2.22) with boundary and initial conditions (5.46). Since the drift $[x, T] \rightarrow \lambda(x, y, T)$ is linear in x and continuous in T for $T \geq \delta$, standard regularity theory implies that $[x, T] \rightarrow v_{\varepsilon,\delta}(x, y, T)$ is a classical solution to (2.19), (5.49), from whence we conclude that $[x, T] \rightarrow q_{\varepsilon,\delta}(x, y, T)$ is a classical solution to (2.22), (5.46). The proof of (5.47) proceeds as in the proof of Proposition 3.3 by using the maximum principle. With (5.47) established, the proof of (5.48) then follows along the same lines as the proof of Proposition 5.1.

To prove the convergence of $q_{\varepsilon,\delta}$ as $\delta \rightarrow 0$ we define the function $u_{\varepsilon,\delta}(x, y, T) = q_{\varepsilon}(x, y, T) - q_{\varepsilon,\delta}(x, y, T)$. Since both functions $[x, T] \rightarrow q_{\varepsilon}(x, y, T)$ and $[x, T] \rightarrow q_{\varepsilon,\delta}(x, y, T)$ are solutions to (2.22) it follows that the function $[x, T] \rightarrow u_{\varepsilon,\delta}(x, y, T) = q_{\varepsilon}(x, y, T) - q_{\varepsilon,\delta}(x, y, T)$ is a solution to the PDE

$$\begin{aligned} \frac{\partial u_{\varepsilon,\delta}(x, y, T)}{\partial T} &= \frac{\varepsilon}{2} \frac{\partial^2 u_{\varepsilon,\delta}(x, y, T)}{\partial x^2} - \left\{ \lambda(x, y, T) \right. \\ &\quad \left. + \frac{1}{2} \left[\frac{\partial q_{\varepsilon}(x, y, T)}{\partial x} + \frac{\partial q_{\varepsilon,\delta}(x, y, T)}{\partial x} \right] \right\} \frac{\partial u_{\varepsilon,\delta}(x, y, T)}{\partial x}, \end{aligned} \tag{5.50}$$

in the region $\{[x, T] : x > 0, T > \delta\}$. It follows from (5.46) and the upper bound (3.28) of Proposition 3.3 that the boundary and initial conditions satisfy

$$u_{\varepsilon,\delta}(0, y, T) = 0, \quad T > \delta, \quad 0 \leq u_{\varepsilon}(x, y, \delta) \leq C\delta y, \quad x > 0, \tag{5.51}$$

where the constant C may be chosen uniformly in any interval $0 < \delta < \delta_0 < \infty$. From (5.51) and the maximum principle applied to (5.50) we conclude that $0 \leq u_{\varepsilon,\delta}(x, y, T) \leq C\delta y$ for $x > 0, T > \delta$. \square

Lemma 5.6. *Assume the function $A : [0, \infty) \rightarrow \mathbb{R}$ is continuous non-negative, and $q_{\varepsilon,\delta}(x, y, T)$, $x, y > 0, T > \delta$, the function defined in Lemma 5.5. Then for any $T_0 > \delta$ there exist constants $C, M > 0$, depending on $\varepsilon, \delta, y, T_0$ and $\sup_{0 \leq t \leq T_0} A(t)$, such that*

$$\left| \frac{\partial^2 q_{\varepsilon,\delta}(x, y, T)}{\partial x^2} \right| \leq C \quad \text{for } x \geq M, \delta < T \leq T_0. \tag{5.52}$$

Proof. Similarly to the derivation of (5.3) we see that the function

$$v_{\varepsilon,\delta}(x, y, T) = \frac{\sigma^2(T)}{m_{1,A}(T)} \frac{\partial q_{\varepsilon,\delta}(x, y, T)}{\partial x} \tag{5.53}$$

is a solution to the PDE

$$\begin{aligned} \frac{\partial v_{\varepsilon,\delta}(x, y, T)}{\partial T} &+ \left[\lambda(x, y, T) + \frac{m_{1,A}(T)}{\sigma_A^2(T)} v_{\varepsilon,\delta}(x, y, T) \right] \frac{\partial v_{\varepsilon,\delta}(x, y, T)}{\partial x} \\ &= \frac{\varepsilon}{2} \frac{\partial^2 v_{\varepsilon,\delta}(x, y, T)}{\partial x^2} \end{aligned} \tag{5.54}$$

in the region $\{[x, T] : x > 0, T > \delta\}$ with constant initial condition $2y$ on the half line $\{[x, \delta] : x > 0\}$. We make a change of variable to eliminate the linear drift $\lambda(\cdot, \cdot, \cdot)$ from (5.54). To see this consider the PDE

$$\frac{\partial w(x, T)}{\partial T} + [\alpha(T)x + \beta(T)] \frac{\partial w(x, T)}{\partial x} = \frac{\varepsilon}{2} \frac{\partial^2 w(x, T)}{\partial x^2}, \quad x \in \mathbb{R}, T > \delta. \tag{5.55}$$

If we make the transformation $w(x, T) = u(z, t)$ where

$$\begin{aligned} z &= \exp \left[- \int_{\delta}^T \alpha(s) ds \right] x - \int_{\delta}^T \beta(s) \exp \left[- \int_{\delta}^s \alpha(s') ds' \right] ds, \\ t &= \int_{\delta}^T \exp \left[- 2 \int_{\delta}^s \alpha(s') ds' \right] ds. \end{aligned} \tag{5.56}$$

then u is a solution to the heat equation

$$\frac{\partial u(z, t)}{\partial t} = \frac{\varepsilon}{2} \frac{\partial^2 u(z, t)}{\partial z^2}, \quad z \in \mathbb{R}, t > 0. \tag{5.57}$$

Now writing $\lambda(x, y, T) = \alpha(T)x + \beta(T)$ and setting $v_{\varepsilon, \delta}(x, y, T) = u(z, t)$ according to the change of variables (5.56), we see from (5.54) that u is a solution to the Burgers' equation

$$\frac{\partial u(z, t)}{\partial t} + \gamma(t)u(z, t) \frac{\partial u(z, t)}{\partial z} = \frac{\varepsilon}{2} \frac{\partial^2 u(z, t)}{\partial z^2}, \tag{5.58}$$

where

$$\gamma(t) = \exp \left[\int_{\delta}^T \alpha(s) ds \right] \frac{m_{1,A}(T)}{\sigma_A^2(T)}, \quad T \geq \delta. \tag{5.59}$$

For each $z_0 \in \mathbb{R}, t_0 > 0$ we define the domain $\mathcal{D}(z_0, \varepsilon) = \{[z, t] : |z - z_0| < \sqrt{\varepsilon t_0}, 0 < t < t_0\}$. The Dirichlet Green's function for the heat equation (5.57) on the domain $\mathcal{D}(z_0, \varepsilon)$ is simply a space translation and dilation of the Green's function on the domain $\mathcal{D}(0, 1)$. This latter Green's function can be obtained by the method of images. Thus for $t > 0$ let $z \rightarrow G(z, t)$ be the pdf of the Gaussian variable with mean 0 and variance t , so

$$G(z, t) = \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{z^2}{2t}\right], \quad z \in \mathbb{R}. \tag{5.60}$$

Then the Dirichlet Green's function $G_D(z, z', t)$ for $\mathcal{D}(0, 1)$ is given by the series

$$G_D(z, z', t) = \sum_{m=0}^{\infty} p(m)G(z - z_m, t), \tag{5.61}$$

where $z_0 = z'$ and $z_m, m = 1, 2, \dots$, are reflections of z' in the boundaries $z' = \pm\sqrt{t_0}$ with parities $p(m) = \pm 1$. The function

$$u(z, t) = \int_{-\sqrt{t_0}}^{\sqrt{t_0}} G_D(z, z', t)u_0(z') dz', \quad [z, t] \in \mathcal{D}(0, 1) \tag{5.62}$$

is then a solution to (5.57) with $\varepsilon = 1$. It satisfies the initial condition $u(z, 0) = u_0(z), |z| < \sqrt{t_0}$, and boundary condition $u(z, t) = 0, z = \pm\sqrt{t_0}, 0 < t < t_0$.

Letting $t = t_0$ correspond to $T = T_0$ in (5.56), we see from (5.48) of Lemma 5.5 there exist a constant $C_0 > 0$, depending only on T_0 , and a constant M_0 , depending only on δ, T_0 , such that the solution u to the Burgers' equation (5.58) satisfies

$$|u(z, t)| \leq C_0 + 2y \quad \text{for } z \geq M_0, 0 < t \leq t_0. \tag{5.63}$$

We can integrate (5.58) on the domain $\mathcal{D}(z_0, \varepsilon)$ with $z_0 > M_0 + \sqrt{\varepsilon t_0}$ by using the Green's function (5.61). We obtain the integral equation

$$\begin{aligned} u(z, t) &= \int_{-\sqrt{t_0}}^{\sqrt{t_0}} G_D((z - z_0)/\sqrt{\varepsilon}, z', t)u(\sqrt{\varepsilon}z' + z_0, 0) dz' \end{aligned}$$

$$\begin{aligned}
& + 2 \sum_{z'=\pm\sqrt{t_0}} p(z') \int_0^t \frac{\partial G_D((z-z_0)/\sqrt{\varepsilon}, z', (t-s))}{\partial z'} u(\sqrt{\varepsilon}z' + z_0, s) ds \quad (5.64) \\
& + \frac{1}{2\sqrt{\varepsilon}} \int_0^t \int_{-\sqrt{t_0}}^{\sqrt{t_0}} \frac{\partial G_D((z-z_0)/\sqrt{\varepsilon}, z', (t-s))}{\partial z'} \gamma(s) u(\sqrt{\varepsilon}z' + z_0, s)^2 dz' ds
\end{aligned}$$

for $[z, t] \in \mathcal{D}(z_0, \varepsilon)$, where $p(z') = -1$ if $z' = \sqrt{t_0}$ and $p(z') = 1$ if $z' = -\sqrt{t_0}$. We can use the representation (5.64) and the bound (5.63) to obtain a bound on $\partial u(z, t)/\partial z$ at $z = z_0$, $0 < t \leq t_0$, which is independent of z_0 as $z_0 \rightarrow \infty$. Observe from (5.46) that $u(\cdot, 0) \equiv 2y$ is constant. Hence if $u_1(z, t)$ denotes the first term on the RHS of (5.64) we have from (5.60), (5.61) the inequality

$$\left| \frac{\partial u_1(z, t)}{\partial z} \right| \leq \frac{C_1 y}{\sqrt{\varepsilon}}, \quad |z - z_0| < \frac{\sqrt{\varepsilon t_0}}{2}, \quad 0 < t \leq t_0, \quad (5.65)$$

where C_1 depends only on t_0 . Letting $u_2(z, t)$ be the second (boundary) term on the RHS of (5.64), we may use (5.63) to bound $\partial u_2(z, t)/\partial z$ at $z = z_0$. Thus we have that

$$\left| \frac{\partial u_2(z, t)}{\partial z} \right| \leq \frac{C_2(C_0 + 2y)}{\sqrt{\varepsilon}}, \quad |z - z_0| < \frac{\sqrt{\varepsilon t_0}}{2}, \quad 0 < t \leq t_0, \quad (5.66)$$

where C_2 depends only on t_0 .

To bound the derivative of the third term on the RHS of (5.64) we define an operator \mathcal{L} on functions $w : \mathcal{D}(0, 1) \rightarrow \mathbb{R}$ by

$$\mathcal{L}w(z, t) = \int_0^t \int_{-\sqrt{t_0}}^{\sqrt{t_0}} \frac{\partial G_D(z, z', t-s)}{\partial z'} \gamma(s) w(z', s) dz' ds, \quad [z, t] \in \mathcal{D}(0, 1) \quad (5.67)$$

We see from (5.60), (5.61) there is a constant C_3 , depending only on t_0 such that

$$\left| \frac{\partial^2 G_D(z, z', t)}{\partial z \partial z'} \right| \leq \frac{C_3}{t} G(z - z', 2t), \quad [z, t], [z', t] \in \mathcal{D}(0, 1). \quad (5.68)$$

It follows from (5.68) that

$$\begin{aligned}
& \left| \frac{\partial G_D(z_1, z', t)}{\partial z'} - \frac{\partial G_D(z_2, z', t)}{\partial z'} \right| \\
& \leq \frac{C_3 |z_1 - z_2|}{t} \int_0^1 G(\lambda(z_1 - z') + (1 - \lambda)(z_2 - z'), 2t) d\lambda.
\end{aligned} \quad (5.69)$$

Let $U_{z_1, z_2} = \{z' \in \mathbb{R} : |z'| < \sqrt{t_0}, |z_1 - z'| \geq 2|z_1 - z_2|\}$. It is evident that

$$|z_1 - z_2| \leq |\lambda(z_1 - z') + (1 - \lambda)(z_2 - z')| \quad \text{for } 0 < \lambda < 1, z' \in U_{z_1, z_2}. \quad (5.70)$$

Next we write

$$\mathcal{L}w(z_1, t) - \mathcal{L}w(z_2, t) = F_1(z_1, z_2, t) + F_2(z_1, z_2, t), \quad (5.71)$$

where

$$\begin{aligned}
& F_1(z_1, z_2, t) \\
& = \int_0^t \int_{U_{z_1, z_2}} \left[\frac{\partial G_D(z_1, z', t-s)}{\partial z'} - \frac{\partial G_D(z_2, z', t-s)}{\partial z'} \right] \gamma(s) w(z', s) dz' ds.
\end{aligned} \quad (5.72)$$

It follows from (5.69), (5.70) that for all α satisfying $0 < \alpha < 1$ there is a constant C_4 such that

$$|F_1(z_1, z_2, t)| \leq C_4 |z_1 - z_2|^\alpha \|w\|_\infty \int_0^t \frac{ds}{s^{(1+\alpha)/2}} = \frac{2C_4}{1-\alpha} \|w\|_\infty \sqrt{t} \left(\frac{|z_1 - z_2|}{\sqrt{t}}\right)^\alpha. \tag{5.73}$$

To bound $F_2(z_1, z_2, t)$ we use the inequalities

$$\left| \frac{\partial G_D(z, z', t)}{\partial z'} \right| \leq \frac{C_5}{\sqrt{t}} G(z - z', 2t), \quad [z, t], [z', t] \in \mathcal{D}(0, 1), \tag{5.74}$$

$$\int_{|z'| < a} G(z', 2t) dz' \leq C_6 \min \left\{ \frac{a}{\sqrt{t}}, 1 \right\}, \tag{5.75}$$

where C_5, C_6 are constants depending only on t_0 . We have from (5.74), (5.75) that

$$\begin{aligned} & \int_0^t \int_{[-\sqrt{t_0}, \sqrt{t_0}] - U_{z_1, z_2}} \left| \frac{\partial G_D(z_1, z', t-s)}{\partial z'} - \frac{\partial G_D(z_2, z', t-s)}{\partial z'} \right| dz' ds \\ & \leq 2C_5 C_6 \int_0^t \frac{1}{\sqrt{t-s}} \min \left\{ \frac{3|z_1 - z_2|}{\sqrt{t-s}}, 1 \right\} ds \\ & = 6C_5 C_6 |z_1 - z_2| \int_0^{t/9|z_1 - z_2|^2} \min \left\{ \frac{1}{\sqrt{s'}}, 1 \right\} \frac{ds'}{\sqrt{s'}} \\ & \leq \frac{C_7}{1-\alpha} \sqrt{t} \left(\frac{|z_1 - z_2|}{\sqrt{t}}\right)^\alpha \quad \text{if } \frac{|z_1 - z_2|}{\sqrt{t}} \leq 1. \end{aligned} \tag{5.76}$$

We conclude from (5.71), (5.73), (5.76) that

$$|\mathcal{L}w(z_1, t) - \mathcal{L}w(z_2, t)| \leq C_\alpha \|w\|_\infty \sqrt{t} \left(\frac{|z_1 - z_2|}{\sqrt{t}}\right)^\alpha \quad \text{if } \frac{|z_1 - z_2|}{\sqrt{t}} \leq 1, \tag{5.77}$$

where the constant C_α depends only on α, t_0 . It follows from (5.67), (5.74) that $\|\mathcal{L}w\|_\infty \leq C\sqrt{t}\|w\|_\infty$ for some universal constant C . Hence (5.77) holds for all $[z_1, t], [z_2, t] \in \mathcal{D}(0, 1)$.

Next for a continuous function $f : [-\sqrt{t_0}, \sqrt{t_0}] \rightarrow \mathbb{R}$ and α satisfying $0 < \alpha < 1$ define the α Hölder norm of f by

$$\begin{aligned} \|f\|_{0,\alpha} &= \sup\{|f(z)| : z \in [-\sqrt{t_0}, \sqrt{t_0}]\} \\ &+ \sup \left\{ \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^\alpha} : z \in [-\sqrt{t_0}, \sqrt{t_0}] \right\}. \end{aligned} \tag{5.78}$$

We may bound the derivative of $(\partial/\partial z)\mathcal{L}w(z, t)$ in terms of the norms (5.78) for $w(\cdot, s)$, $0 < s < t$. To see this observe from (5.67) that

$$\begin{aligned} & \frac{\mathcal{L}w(z+h, t) - \mathcal{L}w(z, t)}{h} \\ &= \int_0^t \int_{-\sqrt{t_0}}^{\sqrt{t_0}} \int_0^1 \frac{\partial^2 G_D(z + \lambda h, z', t-s)}{\partial z \partial z'} d\lambda \gamma(s) w(z', s) dz' ds \\ &= \int_0^t \int_{-\sqrt{t_0}}^{\sqrt{t_0}} \int_0^1 \frac{\partial^2 G_D(z + \lambda h, z', t-s)}{\partial z \partial z'} \gamma(s) [w(z', s) - w(z + \lambda h, s)] d\lambda dz' ds. \end{aligned} \tag{5.79}$$

We have from (5.68), (5.79) that

$$\left| \frac{\mathcal{L}w(z+h, t) - \mathcal{L}w(z, t)}{h} \right| \leq C \int_0^t \frac{ds}{(t-s)^{1-\alpha/2}} \|w(\cdot, s)\|_{0,\alpha} ds, \tag{5.80}$$

for $0 < \alpha < 1$ $0 < t \leq t_0$, where C_8 is a constant depending on t_0 .

Let $u_3(z, t)$ with $[z, t] \in \mathcal{D}(z_0, \varepsilon)$ be the third term on the RHS of (5.64). Then $u_3(z, t) = \mathcal{L}w_1((z - z_0)/\sqrt{\varepsilon}, t)$ with $[z, t] \in \mathcal{D}(z_0, \varepsilon)$, where w_1 is given by the formula

$$w_1(z, t) = \frac{1}{2\sqrt{\varepsilon}}u^2(\sqrt{\varepsilon}z + z_0, t), \quad [z, t] \in \mathcal{D}(0, 1). \tag{5.81}$$

We have then from (5.63), (5.77) that

$$\|\mathcal{L}w_1(\cdot, t)\|_{0,\alpha} \leq \frac{C_\alpha}{2\sqrt{\varepsilon}}[C_0 + 2y]^2t^{(1-\alpha)/2}, \quad 0 < t < t_0, \quad 0 < \alpha < 1. \tag{5.82}$$

Now defining $w_2; \mathcal{D}(0, 1) \rightarrow \mathbb{R}$ by $w_2((z - z_0)/\sqrt{\varepsilon}, t) = u(z_0 + (z - z_0)/2, t)$, we have from (5.65), (5.66), (5.82) the bound

$$\|w_2(\cdot, t)\|_{0,\alpha} \leq C_9[C_0 + 2y] + \frac{C_\alpha}{2\sqrt{\varepsilon}}[C_0 + 2y]^2t^{(1-\alpha)/2}, \tag{5.83}$$

for $0 < t < t_0$, $0 < \alpha < 1$, and a constant C_9 depending only on t_0 .

The inequality (5.83) shows that the solution $u(\cdot, \cdot)$ of (5.58) is Hölder continuous in the domain $\mathcal{D}(z_0, \varepsilon/4)$. We represent $u(z, t)$ again as in (5.64) but with t_0 replaced by $t_0/4$, whence (5.83) implies that $u(\cdot, \cdot)$ is Hölder continuous in the domain $\mathcal{D}(z_0, \varepsilon)$. Letting $w_3(z, t) = w_2(z, t)^2/2\sqrt{\varepsilon}$, then we see that

$$u_3(z, t) = \mathcal{L}w_3((z - z_0)/\sqrt{\varepsilon}, t), \quad \|w_3(\cdot, t)\|_{0,\alpha} \leq \frac{1}{\sqrt{\varepsilon}}\|w_2(\cdot, t)\|_{0,\alpha}^2. \tag{5.84}$$

It follows from (5.80), (5.83), (5.84) that

$$\left| \frac{\partial u_3(z, t)}{\partial z} \right| \leq C \quad \text{for } |z - z_0| < \sqrt{\varepsilon t_0}/2, \quad 0 < t < t_0/2, \tag{5.85}$$

for some constant depending on ε, y, t_0 , but not on z_0 . The inequality (5.52) follows from (5.65), (5.66), and (5.85). □

Theorem 5.7. *Assume the function $A : [0, \infty) \rightarrow \mathbb{R}$ is continuous non-negative, and let $q_\varepsilon(x, y, T)$ be defined by (2.18), (2.21). Then for all $y, T > 0$ the function $x \rightarrow q_\varepsilon(x, y, T)$ is concave.*

Proof. We show that the function $x \rightarrow q_{\varepsilon,\delta}(x, y, T)$ is concave, and then the result follows from Lemma 5.5 by letting $\delta \rightarrow 0$. We define the function $w_{\varepsilon,\delta}$ by

$$w_{\varepsilon,\delta}(x, y, T) = \frac{\sigma_A^4(T)}{m_{1,A}(T)^2} \frac{\partial^2 q_{\varepsilon,\delta}(x, y, T)}{\partial x^2}. \tag{5.86}$$

By differentiating the PDE (5.3) we see from Proposition 3.1 that the function $[x, T] \rightarrow w_{\varepsilon,\delta}(x, y, T)$ is a classical solution of the PDE

$$\begin{aligned} & \frac{\partial w_{\varepsilon,\delta}(x, y, T)}{\partial T} + \left[\lambda(x, y, T) + \frac{\partial q_{\varepsilon,\delta}(x, y, T)}{\partial x} \right] \frac{\partial w_{\varepsilon,\delta}(x, y, T)}{\partial x} \\ & + \frac{m_{1,A}(T)^2}{\sigma_A^4(T)} w_{\varepsilon,\delta}(x, y, T)^2 \\ & = \frac{\varepsilon}{2} \frac{\partial^2 w_{\varepsilon,\delta}(x, y, T)}{\partial x^2}. \end{aligned} \tag{5.87}$$

We proceed as in the proof of Proposition 5.1 using Ito’s lemma and the martingale optional sampling theorem. Thus similarly to (5.4) we have the representation

$$\begin{aligned}
 w_{\varepsilon,\delta}(x, y, T) &= E \left[w_{\varepsilon,\delta}(X_{\varepsilon,\delta}^*(\delta \vee \tau_{\varepsilon,\delta,x,T,K}^*), y, \delta \vee \tau_{\varepsilon,\delta,x,T,K}^*) \mid X_{\varepsilon,\delta}^*(T) = x \right] \\
 &\quad - E \left[\int_{\delta \vee \tau_{\varepsilon,\delta,x,T,K}^*}^T \frac{m_{1,A}(s)^2}{\sigma_A^4(s)} w_{\varepsilon,\delta}(X_{\varepsilon,\delta}^*(s), y, s)^2 ds \right], \tag{5.88}
 \end{aligned}$$

where $X_{\varepsilon,\delta}^*(\cdot)$ is the solution to the SDE (2.25) with drift $\mu_\varepsilon(x, y, T)$ given by the coefficient of $\partial w_{\varepsilon,\delta}(x, y, T)/\partial x$ in (5.87). The stopping time $\tau_{\varepsilon,\delta,x,T,K}^*$ is the first exit time of $X_{\varepsilon,\delta}^*(s)$, $s \leq T$, with $X_{\varepsilon,\delta}^*(T) = x$ from the interval $(0, K)$. From (5.46) we have that $w_{\varepsilon,\delta}(\cdot, y, \delta) \equiv 0$, whence the first term on the RHS of (5.88) can be written as

$$\begin{aligned}
 &E \left[w_{\varepsilon,\delta}(0, y, \tau_{\varepsilon,\delta,x,T,K}^*); \tau_{\varepsilon,\delta,x,T,K}^* > \delta, X_{\varepsilon,\delta}(\tau_{\varepsilon,\delta,x,T,K}^*) = 0 \right] \\
 &+ E \left[w_{\varepsilon,\delta}(K, y, \tau_{\varepsilon,\delta,x,T,K}^*); \tau_{\varepsilon,\delta,x,T,K}^* > \delta, X_{\varepsilon,\delta}(\tau_{\varepsilon,\delta,x,T,K}^*) = K \right]. \tag{5.89}
 \end{aligned}$$

Using Lemma 5.5 and arguing as in Corollary 3.6, we see that $w_{\varepsilon,\delta}(0, y, s) \leq 0$ for $s > \delta, y > 0$. Hence the first term in (5.89) is non-positive. The second term converges to 0 as $K \rightarrow \infty$. To prove this we use Lemma 5.6, which yields a uniform upper bound on $|w_{\varepsilon,\delta}(K, y, s)|$, $\delta < s < T$, as $K \rightarrow \infty$. Then we follow the corresponding argument around (5.11) in the proof of Proposition 5.1. By letting $K \rightarrow \infty$ in (5.88) we conclude that $w_{\varepsilon,\delta}(x, y, T) \leq 0$ for $x \geq 0, y > 0, T > \delta$. \square

6. CONVERGENCE OF THE FUNCTION $\partial q_\varepsilon(x, y, T)/\partial x$ AS $\varepsilon \rightarrow 0$

In this section we assume the function $A(\cdot)$ is non-negative, whence the results of Proposition 4.2 and Corollary 4.4 imply that the function $x \rightarrow q_0(x, y, T)$ of (2.28), (4.2) is C^1 for certain ranges of $[x, T]$. We will show that $\lim_{\varepsilon \rightarrow 0} \partial q_\varepsilon(0, y, T)/\partial x = \partial q_0(0, y, T)/\partial x$. In view of the upper bound (3.29), we only need to prove for small x a lower bound for $q_\varepsilon(x, y, T)$ in terms of $q_0(x, y, T)$ and a correction term which goes to 0 as $\varepsilon \rightarrow 0$. We already obtained such a lower bound in Lemma 2.5. Our starting point was the inequality (2.80), which leads to the inequality (2.81). However the second term on the RHS of (2.81) is not sufficient for our purposes since we need the correction to be bounded by a constant times x as $x \rightarrow 0$. Instead of (2.80) we observe from (2.77)-(2.80) that

$$\begin{aligned}
 &\frac{1}{2} \int_{\tau_{\varepsilon,x,T}^*}^T [\mu_\varepsilon^*(X_\varepsilon^*(s), y, s) - \lambda(X_\varepsilon^*(s), y, s)]^2 ds \\
 &\geq q_0(x, y, T) - q_0(\sqrt{\varepsilon}Z_\varepsilon, y, \tau_{\varepsilon,x,T}^*). \tag{6.1}
 \end{aligned}$$

The function $[x, T] \rightarrow q_0(x, y, T)$ is defined by (2.28) for $x, T > 0$, and for $x < 0, T > 0$ by

$$\begin{aligned}
 q_0(x, y, T) &= - \min \left\{ \frac{1}{2} \int_T^\tau \left[\frac{dx(s)}{ds} - \lambda(x(s), y, s) \right]^2 ds : \tau > T, \right. \\
 &\quad \left. x(T) = x, x(\cdot) < 0, x(\tau) = 0 \right\}. \tag{6.2}
 \end{aligned}$$

Assuming the function $[x, T] \rightarrow q_0(x, y, T)$ defined by (2.28), (6.2) is sufficiently differentiable at $[0, T]$, we can do a Taylor expansion,

$$\begin{aligned}
 & q_0(\sqrt{\varepsilon}Z_\varepsilon, y, \tau_{\varepsilon,x,T}^*) \\
 &= \frac{\partial q_0(0, y, T)}{\partial x} \sqrt{\varepsilon}Z_\varepsilon + \varepsilon Z_\varepsilon^2 \int_0^1 \int_0^1 \lambda d\lambda d\mu \frac{\partial^2 q_0(\lambda\mu\sqrt{\varepsilon}Z_\varepsilon, y, T)}{\partial x^2} \\
 & \quad - [T - \tau_{\varepsilon,x,T}^*] \int_0^1 d\lambda \frac{\partial q_0(\sqrt{\varepsilon}Z_\varepsilon, y, \lambda\tau_{\varepsilon,x,T}^* + (1-\lambda)T)}{\partial t}.
 \end{aligned} \tag{6.3}$$

Then we can estimate the expectation of the correction term in (6.1) by estimating the expectation of each term on the RHS of (6.3).

We may obtain a formula similar to (4.2) for $q_0(x, y, T)$, $x < 0$, defined by (6.2). If the minimization in (6.2) is for fixed $\tau > T$ then there is a unique minimizing trajectory $x(s)$, $T < s < \tau$, given by (2.37), where $\gamma(\tau)$ is chosen so that $x(\tau) = 0$. The functions $s \rightarrow g_{1,A}(s, T)$, $g_{2,A}(s, T)$, which were defined in (2.39), (2.40) for $0 < s < T$ may be extended by the same formulas to $s > T$. Similarly we may extend the function $s \rightarrow g_{3,A}(s, T)$ by using (4.12). Note that the functions $s \rightarrow g_{1,A}(s, T)$, $g_{3,A}(s, T)$, $s > T$, are negative. We have then from (6.2) that

$$q_0(x, y, T) = - \min_{\tau > T} \frac{|g_{3,A}(\tau, T)|}{2} [y + g_{1,A}(\tau, T)x + g_{2,A}(\tau, T)]^2. \tag{6.4}$$

When $A(\cdot) \equiv 0$ the formula (6.4) becomes

$$q_0(x, y, T) = - \min_{\tau > T} \frac{\tau - T}{2\tau T} \left[y + \frac{\tau x}{(T - \tau)} \right]^2 = - \frac{1}{2T} [-2xy + \min_{\alpha > 1} \{\alpha x^2 + y^2/\alpha\}]. \tag{6.5}$$

Hence we have that

$$q_0(x, y, T) = \frac{2xy}{T} \quad \text{if } -y < x < 0, \tag{6.6}$$

$$q_0(x, y, T) = - \frac{(x - y)^2}{2T} \quad \text{if } x < -y. \tag{6.7}$$

For $-y < x < 0$ the minimizing $\tau > T$ in (6.4) is given by $\tau(x, y, T) = yT/(x + y)$. Otherwise the minimum is obtained by letting $\tau \rightarrow \infty$. The function $[x, T] \rightarrow q_0(x, y, T)$ defined by (6.6), (6.7) is a C^1 solution to the HJ equation (2.29) in the region $\{[x, T] : x < 0, T > 0\}$. However the second derivative $\partial^2 q_0(x, y, T)/\partial x^2$ is discontinuous across the boundary $\{[x, T] : x = -y, T > 0\}$. The characteristics which yield the function (6.6) are the same as in the situation $x > 0$ studied in §4, and are given by $x(\tau, s) = (s - \tau)y/\tau$, $s > 0$. Then we have

$$q_0(x, y, T) = \frac{1}{2} \int_\tau^T p(\tau, s)^2 ds = 2 \int_\tau^T \frac{y^2}{s^2} ds = \frac{2xy}{T}, \quad x = x(\tau, T). \tag{6.8}$$

This set of characteristics covers the region $\{[x, T] : x > -y, T > 0\}$ without intersecting, but all characteristics converge to the point $[-y, 0]$. The characteristics which yield the function (6.7) are given by $x(\lambda, s) = \lambda$, $\lambda < -y$, $s > 0$. In that case

$$q_0(x, y, T) = \frac{1}{2} \int_\infty^T p(\lambda, s)^2 ds = \frac{1}{2} \int_\infty^T \frac{(\lambda - y)^2}{s^2} ds = - \frac{(x - y)^2}{2T}, \tag{6.9}$$

where $x = x(\lambda, T)$. This set of characteristics covers the region $\{[x, T] : x < -y, T > 0\}$, also without intersecting.

We shall show that the function $[x, T] \rightarrow q_0(x, y, T)$ defined by (2.28), (6.2) is differentiable for $[x, T]$ in a neighborhood of the initial line $\{[0, T] : T > 0\}$,

by proving it may be obtained via the method of characteristics. To do this we extend the domain \mathcal{D}_{y,T_0} defined just prior to (4.45). It follows from (4.32) that the characteristics $s \rightarrow x(\tau, s)$ do not meet if $0 < s < \tau \leq T_0$. Hence we may extend the domain \mathcal{D}_{y,T_0} of §4 to include the set $\{[x, T] : 0 < T < T_0, x(T_0, T) < x \leq 0\}$, and similarly extend the region \mathcal{U}_{y,T_0} . The inequality (4.42) for the characteristic $s \rightarrow x(\tau, s)$, $\tau \leq s \leq T_0$, continues to hold for $0 < s < \tau$. More precisely, we have from (4.40), (4.41) that

$$\begin{aligned} C_1(s - \tau) & \left[\frac{[y + g_{2,A}(\tau, \tau)]}{\tau} + (\tau - s) \right] \\ & \leq x(\tau, s) \\ & \leq c_1(s - \tau) \left[\frac{[y + g_{2,A}(\tau, \tau)]}{\tau} + c_2(\tau - s) \right], \quad 0 < s < \tau \leq T_0, \end{aligned} \tag{6.10}$$

where C_1, c_1, c_2 depend only on T_0 and $\sup_{0 \leq t \leq T_0} A(t)$. Hence there is a constant $\Lambda_3 > 0$, depending only on T_0 and $\sup_{0 \leq t \leq T_0} A(t)$, such that

$$\begin{aligned} \text{if } 0 < T \leq 3T_0/4 \text{ and } -\Lambda_3[y + g_{2,A}(T_0, T_0)] < x \leq 0, \\ \text{then } [x, T] \in \mathcal{D}_{y,T_0}. \end{aligned} \tag{6.11}$$

We extend the results of Proposition 4.2 to include $[x, T]$ in the region (6.11).

Lemma 6.1. *The results of Proposition 4.2, with $q_0(x, y, T)$ defined by (4.54), continue to hold in the extended region \mathcal{D}_{y,T_0} . Also $\tau = \tau(x, y, T)$ in (4.54) is the unique minimizer in the variational problems (6.2), (6.4) for $[x, T]$ with $x < 0$, $0 < T < T_0/2$, in the following regions:*

- (a) $-\Lambda[y + g_{2,A}(T, T)] < x < 0$ if $4\sqrt{\Lambda_0 y}/3 \geq T$, otherwise
- (b) $-\Lambda[y + g_{2,A}(T, T)]^2/T^2 < x < 0$, where $\Lambda > 0$ is chosen sufficiently small depending only on T_0 and $\sup_{0 \leq t \leq T_0} A(t)$.

Therefore if $[x, T]$ is in one of the regions (a), (b), the functions (6.4) and (4.54) are identical.

Proof. All constants in the following can be chosen to depend only on T_0 and $\sup_{0 \leq t \leq T_0} A(t)$. It is clear from (6.10) that the characteristics $s \rightarrow x(\tau, s)$, $0 < s < \tau$, satisfy $x(\tau, s) < 0$, and from (4.32) that $D_\tau x(\tau, s) < 0$. The differentiability properties of the function $[x, T] \rightarrow q_0(x, y, T)$ and the fact that it is a solution to the HJ equation (2.29) follow as in the proof of Proposition 4.2. We also have from (4.64) that

$$-q_0(x, y, T) \leq \frac{1}{2} \int_T^\tau \left[\frac{dx(s)}{ds} - \lambda(x(s), y, s) \right]^2 ds, \tag{6.12}$$

for any path $s \rightarrow x(s)$, $T < s < \tau < T_0$, in \mathcal{D}_{y,T_0} with $x(T) = x < 0$, $x(\tau) = 0$. Equality holds in (6.12) if $x(\cdot)$ is the characteristic.

Let $F_0(x, y, \tau, T)$ be the function on the RHS of (6.4). We wish to find $[x, T] \in \mathcal{D}_{y,T_0}$ with $x < 0$ such that q_0 , defined by (4.54), satisfying

$$-q_0(x, y, T) = \inf_{\tau > T} F_0(x, y, \tau, T).$$

To do this we first observe from (4.5) that since $A(\cdot)$ is non-negative, the function $s \rightarrow g_{2,A}(s, T)$, $s > 0$, is increasing and hence non-negative. We also have from (2.8) that the function $s \rightarrow \sigma_A^2(s, T)$, $s > T$, is negative and decreasing with $\lim_{s \rightarrow T} \sigma_A^2(s, T) = 0$. Letting $\lim_{s \rightarrow \infty} \sigma_A^2(s, T) = \sigma_A^2(\infty, T)$, one sees in

the case $A(\cdot) \equiv 0$ that $\sigma_A^2(\infty, T) = -\infty$. It is however possible for some non-negative $A(\cdot)$ that $\sigma_A^2(\infty, T) > -\infty$. We have then from (4.4) that the function $s \rightarrow g_{1,A}(s, T)$, $s > T$, is increasing with $\lim_{s \rightarrow T} g_{1,A}(s, T) = -\infty$, and $g_{1,A}(s, T) < -1/m_{1,A}(T)$, $s > T$. It follows that $\lim_{s \rightarrow \infty} g_{1,A}(s, T) = g_{1,A}(\infty, T) \leq -1/m_{1,A}(T)$.

Similarly to (4.66) we consider $[x, T] \in \mathcal{D}_{y, T_0}$ which satisfies (6.11), and define $\tilde{q}_0(x, y, T)$ by

$$-\tilde{q}_0(x, y, T) = \min_{\tau > T} \frac{|g_{3,A}(\tau, T)|}{2} [y + g_{1,A}(\tau, T)x]^2. \tag{6.13}$$

Using the identity (4.12) we see that the minimizing τ for the RHS of (6.13) is given by

$$g_{3,A}(\tau, T) = \frac{m_{1,A}(T)x}{\sigma_A^2(T)y}, \quad \text{if } \frac{y}{g_{1,A}(\infty, T)} < x < 0. \tag{6.14}$$

Substituting (6.14) into the RHS of (6.13) then yields the formula

$$\tilde{q}_0(x, y, T) = \frac{2m_{1,A}(T)xy}{\sigma_A^2(T)}, \tag{6.15}$$

which is the same as (4.69). We should however note that the RHS of (6.15) is negative in this case. Following the argument in the proof of Proposition 4.2, we see that if $\tau > T$ lies outside the region

$$\frac{8m_{1,A}(T)x}{\sigma_A^2(T)[y + g_{2,A}(T, T)]} \leq g_{3,A}(\tau, T) \leq \frac{m_{1,A}(T)x}{8\sigma_A^2(T)[y + g_{2,A}(T, T)]}, \tag{6.16}$$

F_0 satisfies the inequality

$$F_0(x, y, \tau, T) \geq -5\tilde{q}_0(x, y + g_{2,A}(T, T), T)/2. \tag{6.17}$$

In concluding (6.17) we have used the fact that the function $s \rightarrow g_{2,A}(s, T)$, $s > T$, is increasing.

We require $x < 0$ to be sufficiently close to 0 so that if $\tau > T$ with $0 < T \leq T_0/2$ lies in the region (6.16) then $\tau \leq 3T/2 \leq 3T_0/4$. This is the case if $x > -\Lambda_4[y + g_{2,A}(T, T)]$, where $\Lambda_4 > 0$ is constant. Next we determine how small $|x|$ needs to be so that the minimizing paths $s \rightarrow \Gamma(\tau, s, T, x) = a(\tau, s, T)x + b(\tau, s, T)$, $T < s < \tau$, for fixed τ defined by (4.65) lie in \mathcal{D}_{y, T_0} . To see this first note that the functions $s \rightarrow a(\tau, s, T)$, $b(\tau, s, T)$; $T < s < \tau$, are non-negative. Hence if $x < 0$ is sufficiently small one may have $\Gamma(\tau, s, T, x) > 0$ for some $s \in (T, \tau)$. We see from (4.5), (4.6), (4.65) there are constants $C_1, c_1, C_2 > 0$ such that

$$C_1(\tau - s) \frac{x}{\tau - T} \leq \Gamma(\tau, s, T, x) \leq (\tau - s) \left[\frac{c_1 x}{\tau - T} + C_2(s - T) \right], \tag{6.18}$$

for $T < s < \tau < T_0$. Let Λ_0 be the constant defined just after (4.44), whence if $T < 4\sqrt{\Lambda_0 y}/3$ then \mathcal{D}_{y, T_0} contains the domain $\{[x, s] : x > 0, 0 < s < 3T/2\}$. We assume that $\tau > T$ lies in the region (6.16) and that $-\Lambda_4[y + g_{2,A}(T, T)] < x < 0$, whence $\tau \leq 3T_0/4$. Choosing Λ_4 to also satisfy the inequality $C_1\Lambda_4 < \Lambda_3$, we see from (6.11), (6.18) that if $T < 4\sqrt{\Lambda_0 y}/3$ then the path $s \rightarrow \Gamma(\tau, s, T, x)$, $T < s < \tau$, lies in \mathcal{D}_{y, T_0} . In the case $T \geq 4\sqrt{\Lambda_0 y}/3$ we observe that if $\tau > T$ satisfies (6.16) and $-\Lambda_4[y + g_{2,A}(T, T)] < x < 0$ then $\tau - T$ satisfies an inequality

$$\tau - T \leq \frac{C_3 T x}{[y + g_{2,A}(T, T)]}, \quad \text{where } C_3 \text{ is constant.} \tag{6.19}$$

Hence the RHS of (6.18) is negative provided $x < 0$ satisfies the inequality

$$|x| \leq \frac{c_1[y + g_{2,A}(T, T)]^2}{C_2 C_3^2 T^2}. \tag{6.20}$$

We conclude in this case that the path $s \rightarrow \Gamma(\tau, s, T, x)$, $T < s < \tau$, lies in \mathcal{D}_{y, T_0} provided $[x, T]$ satisfies (6.20).

Finally we need to show that if $\tau = \tau(x, y, T)$, then $F_0(x, y, \tau, T) < -5\tilde{q}_0(x, y + g_{2,A}(T, T), T)/2$, which is the same as $q_0(x, y, T) \geq 5\tilde{q}_0(x, y + g_{2,A}(T, T), T)/2$. We first observe from (4.71) and (6.10) that since $x(\tau, s) < 0$, $0 < s < \tau$, we have $\partial v(\tau, s)/\partial s \leq 0$ for $T < s < \tau$ and $v(\tau, \tau) = 0$. We conclude that

$$q_0(x, y, T) \leq \tilde{q}_0(x, y + g_{2,A}(T, T), T), \quad [x, T] \in \mathcal{D}_{y, T_0}, \quad x < 0. \tag{6.21}$$

We assume now that $[x, T]$ lies in the domain $\{[x, T] : 0 < T < T_0/2, -\Lambda[y + g_{2,A}(T, T)] < x < 0\}$, where Λ satisfies $0 < \Lambda \leq \Lambda_3$. It follows from (6.11) that $[x, T] \in \mathcal{D}_{y, T_0}$. Letting $x = x(\tau, T)$ we have from (6.10) that

$$\tau - T \leq \frac{\tau|x|}{c_1[y + g_{2,A}(\tau, \tau)]} \leq \frac{\Lambda\tau}{c_1}. \tag{6.22}$$

Choosing $\Lambda \leq c_1/3$ we see from (6.22) that $\tau - T \leq \Lambda T/[c_1 - \Lambda] \leq T/2$. Observe from (4.30) that

$$1 \leq \frac{y + g_{2,A}(\tau, \tau)}{y + g_{2,A}(T, T)} \leq 1 + C_4\Lambda \tag{6.23}$$

if $T < 4\sqrt{\Lambda_0 y}/3$, where C_4 is constant. It follows then from (4.71), (6.10), (6.23), upon using a lower bound for the integral of the RHS of (4.71) on the interval $T < s < \tau$ similar to the one in (4.72), that

$$q_0(x, y, T) \geq [1 + C_5\Lambda]\tilde{q}_0(x, y + g_{2,A}(T, T), T), \tag{6.24}$$

where C_5 is constant. We assume that $T \geq 4\sqrt{\Lambda_0 y}/3$ and that $[x, T]$ satisfies the inequality

$$0 < T < T_0/2, \quad -\frac{\Lambda[y + g_{2,A}(T, T)]^2}{T^2} < x < 0. \tag{6.25}$$

Then inequality (6.22) continues to hold, whence we see from (4.30) and (6.25) that (6.23) also holds. Similarly to before we see that (6.24) holds in this case also. Now we choose Λ so that $C_5\Lambda \leq 1$. □

Observe from (3.48) that we expect

$$\frac{\partial q_\varepsilon(0, y, T)}{\partial x} - \frac{\partial q_0(0, y, T)}{\partial x} = \varepsilon \frac{\partial^2 q_0(0, y, T)}{\partial x^2} \Big/ \frac{\partial q_0(0, y, T)}{\partial x} + o(\varepsilon) \quad \text{as } \varepsilon \rightarrow 0. \tag{6.26}$$

This evidently suggests that the LHS of (6.26) is $O(\varepsilon)$ as $\varepsilon \rightarrow 0$. Furthermore, we see from (4.23) that $\partial q_0(0, y, T)/\partial x \simeq y/T$, and from (4.34) that $\partial^2 q_0(0, y, T)/\partial x^2 \simeq T/y$ if $y \geq T^2$. Hence we expect the LHS of (6.26) to be bounded by a constant times $\varepsilon T^2/y^2 + o(\varepsilon)$.

Proposition 6.2. *Assume the function $A : [0, \infty) \rightarrow \mathbb{R}$ is continuous non-negative, and let $q_\varepsilon(x, y, T)$ be defined by (2.18), (2.21), and $q_0(x, y, T)$ by (2.28), (4.2). Then for each $T_0 > 0$ there are constants C_1, C_2 , depending only on T_0 and $\sup_{0 \leq t \leq T_0} A(t)$, such that*

$$\left| \frac{\partial q_\varepsilon(0, y, T)}{\partial x} - \frac{\partial q_0(0, y, T)}{\partial x} \right| \leq \frac{C_2\varepsilon}{y} \left[\frac{T^2}{y} + \frac{\varepsilon T}{y^2} \right], \quad 0 < T \leq T_0, \tag{6.27}$$

provided $0 < T \leq T_0, y \geq C_1 T^2, \varepsilon T < y^2$.

Proof. All constants in the following can be chosen to depend only on T_0 and $\sup_{0 \leq t \leq T_0} A(t)$. We define stopping times for the martingale $s \rightarrow Z(s), s < T$, of (2.16). For $\delta > 0$ let τ_δ be given by

$$\tau_\delta = \inf \left\{ s : 0 < s < T, \left| \frac{\sigma_A^2(T)}{m_{1,A}(T)} Z(s') \right| < \delta \text{ for } s < s' < T \right\}. \tag{6.28}$$

It is easy to see that $\tau_\delta > 0$ with probability 1. Since $\tau_{\varepsilon,x,T}^*$, defined in the statement of Lemma 2.4, is also a stopping time for the martingale (2.16), it follows that $\tilde{\tau} = \tilde{\tau}_{\varepsilon,\delta,x,T} = \tau_\delta \vee \tau_{\varepsilon,x,T}^* \vee (T/2)$ is a stopping time. On taking expectations in (6.1) we have from (2.65) of Lemma 2.4 that for any $\delta > 0$,

$$q_\varepsilon(x, y, T) \geq [1 - P(\tau_{\varepsilon,x,T}^* < T/2) - P(\tau_{\varepsilon,x,T}^* < \tau_\delta)] q_0(x, y, T) - E[q_0(\sqrt{\varepsilon}Z_\varepsilon, y, \tau_{\varepsilon,x,T}^*; \tau_{\varepsilon,x,T}^* > \max\{\tau_\delta, T/2\})]. \tag{6.29}$$

Since the function $s \rightarrow \sigma_A^2(s)/m_{1,A}(s), s > 0$, is increasing, we see from (2.80) that $|Z_\varepsilon| \leq \sigma_A^2(T)|Z(\tau_{\varepsilon,x,T}^*)|/m_{1,A}(T)$. Hence if δ is small enough we may use the Taylor expansion (6.3) to estimate the second expectation on the RHS of (6.29). It follows from Proposition 4.2 and Lemma 6.1 that it is sufficient for δ to lie in the interval

$$\sqrt{\varepsilon}\delta \leq \Lambda \min \left\{ \frac{y}{T^2}, 1 \right\} [y + g_{2,A}(T/2, T/2)], \tag{6.30}$$

where Λ depends only on T_0 .

We show that

$$\lim_{x \rightarrow 0} P(\tau_{\varepsilon,x,T}^* < T/2) = 0, \quad \lim_{x \rightarrow 0} P(\tau_{\varepsilon,x,T}^* < \tau_\delta) = 0. \tag{6.31}$$

To prove the first limit in (6.31) we use the inequality $\partial q_\varepsilon(x, y, T)/\partial x \geq 0$, whence it follows that $X_\varepsilon^*(s) \leq X_\varepsilon(s), 0 < s < T$, where $X_\varepsilon(\cdot)$ is defined by (2.16). This was already observed just prior to Lemma 2.4. We have from (2.12) that one can choose a constant ν with $0 < \nu < 1/2$, such that $y_{\text{class}}(s) < 2x$ if $0 < T - s < T\nu x/(y + T^2)$, provided $x < y + T^2, 0 < T \leq T_0$. The first limit in (6.31) follows if we can show that

$$\lim_{x \rightarrow 0} P\left(\sqrt{\varepsilon} \inf_{T - T\nu x/(y + T^2) < s < T} Z(s) > -\frac{2m_{1,A}(T/2)x}{\sigma_A^2(T/2)}\right) = 0, \tag{6.32}$$

since $P(\tau_{\varepsilon,x,T}^* < T - T\nu x/(y + T^2))$ is smaller than the probability in (6.32). The limit in (6.32) follows from the reflection principle. Similarly to (2.75) we have that the probability in (6.32) is bounded in terms of a probability for the standard normal variable Y by

$$P(|Y| < 2x/\sqrt{\varepsilon}\sigma(x)), \text{ where } \sigma(x)^2 \geq \frac{c\nu Tx}{(y + T^2)}, \tag{6.33}$$

with $c > 0$ a constant. Since the probability (6.33) is bounded by a constant times \sqrt{x} the limit (6.32) follows. To prove the second limit in (6.31) we argue similarly, using the inequality $P(\tau_{\varepsilon,x,T}^* < \tau_\delta) \leq P(\tau_{\varepsilon,x,T}^* < T - T\nu x/(y + T^2)) + P(\tau_\delta > T - T\nu x/(y + T^2))$. Using the reflection principle again we see that

$$\lim_{x \rightarrow 0} P(\tau_\delta > T - T\nu x/(y + T^2)) = 0.$$

We estimate the contribution of the first term in the Taylor expansion (6.3) to the expectation in (6.29). To do this we use the inequality

$$\left| Z_\varepsilon - \frac{\sigma_A^2(T)}{m_{1,A}(T)} Z(\tau_{\varepsilon,x,T}^*) \right| \leq C_1 [T - \tau_{\varepsilon,x,T}^*] |Z(\tau_{\varepsilon,x,T}^*)|, \tag{6.34}$$

where C_1 is a constant. Using that

$$s \rightarrow Z(s)^2 - \int_s^T \frac{m_{1,A}(s')^2}{\sigma_A^4(s')} ds', \quad 0 < s < T, \tag{6.35}$$

is a martingale, we have from (6.35) and the optional stopping theorem that

$$\begin{aligned} E [Z(\tau_{\varepsilon,x,T}^*)^2; \tau_{\varepsilon,x,T}^* > T/2] &\leq \frac{C_2}{T^2} \{E [T - \tau_{\varepsilon,x,T}^*] + TP (\tau_{\varepsilon,x,T}^* < T/2)\} \\ &\leq \frac{3C_2}{T^2} E [T - \tau_{\varepsilon,x,T}^*], \end{aligned} \tag{6.36}$$

where C_2 is constant. To bound the RHS of (6.36) we use the lower bound (5.1) of Proposition 5.1. Recalling the definition of $\tau_{\varepsilon,\text{linear},x,T}$ after (5.14), we have that $E[T - \tau_{\varepsilon,x,T}^*] \leq E[T - \tau_{\varepsilon,\text{linear},x,T}]$. We obtain then from (5.33) an upper bound for $E [T - \tau_{\varepsilon,x,T}^*]$, provided $y \geq C_3 T^2$ where C_3 is constant. We may also obtain an inequality $E [(T - \tau_{\varepsilon,x,T}^*)^2] \leq v_\varepsilon(x)$, where $v_\varepsilon(\cdot)$ is the solution to a boundary value problem. Thus

$$-\frac{\varepsilon}{2} \frac{d^2 v_\varepsilon(x)}{dx^2} + \frac{y}{C_5 T} \frac{dv_\varepsilon(x)}{dx} = 2u_\varepsilon(x), \quad x > 0, \quad v_\varepsilon(0) = 0, \tag{6.37}$$

where $u_\varepsilon(\cdot)$ is the solution to (5.34). Evidently we have that

$$v_\varepsilon(x) = \left(\frac{C_5 T}{y}\right)^3 x \left[\varepsilon + \frac{xy}{C_5 T}\right]. \tag{6.38}$$

It follows from (6.34), upon using the Schwarz inequality and (6.36)-(6.38), that

$$\sqrt{\varepsilon} \limsup_{x \rightarrow 0} \frac{1}{x} E \left[\left| Z_\varepsilon - \frac{\sigma_A^2(T)}{m_{1,A}(T)} Z(\tau_{\varepsilon,x,T}^*) \right|; \tau_{\varepsilon,x,T}^* > T/2 \right] \leq C_4 \varepsilon \frac{T}{y^2}, \tag{6.39}$$

where C_4 is a constant.

Applying the optional sampling theorem to the martingale $s \rightarrow Z(s)$, $0 < s < T$, we have that

$$\begin{aligned} &\frac{\sigma_A^2(T)}{m_{1,A}(T)} |E [Z(\tau_{\varepsilon,x,T}^*); \tau_{\varepsilon,x,T}^* > \max\{T/2, \tau_\delta\}]| \\ &\leq \frac{\sigma_A^2(T)}{m_{1,A}(T)} E[|Z(T/2)|; \tau_{\varepsilon,x,T}^* < T/2] + \delta P (\tau_{\varepsilon,x,T}^* < \tau_\delta). \end{aligned} \tag{6.40}$$

To bound the first term on the RHS of (6.40) we observe that

$$\begin{aligned} &\sqrt{\varepsilon} \frac{\sigma_A^2(T)}{m_{1,A}(T)} E[|Z(T/2)|; \tau_{\varepsilon,x,T}^* < T/2] \\ &\leq C_5 E [|X_\varepsilon(T/2) - x_{\text{class}}(T/2)|; \tau_{\varepsilon,\text{linear},x,T} < T/2], \end{aligned} \tag{6.41}$$

where $X_\varepsilon(\cdot)$ is given by (5.16) and C_5 is a constant. From (5.17) and the Chebyshev inequality we have, using the inequality $y \geq C_3 T^2$, that

$$|x_{\text{class}}(T/2)| |P (\tau_{\varepsilon,\text{linear},x,T} < T/2)| \leq \frac{C_6 y v_\varepsilon(x)}{T^2}, \tag{6.42}$$

where C_6 is a constant. Letting $X_\varepsilon^*(s)$, $s < T$, be the diffusion with drift $\mu_\varepsilon(x, y, T) = y/C_5T$ defined just after (5.33), we have that

$$E [X_\varepsilon(T/2); \tau_{\varepsilon, \text{linear}, x, T} < T/2] \leq E [X_\varepsilon^*(T/2); \tau_{\varepsilon, \text{linear}, x, T}^* < T/2]. \tag{6.43}$$

Let $[x, t] \rightarrow u_\varepsilon(x, t)$, $x > 0$, $t > 0$, be the solution to the PDE

$$\frac{\partial u_\varepsilon(x, t)}{\partial t} = -\mu \frac{\partial u_\varepsilon(x, t)}{\partial x} + \frac{\varepsilon}{2} \frac{\partial^2 u_\varepsilon(x, t)}{\partial x^2}, \quad x > 0, t > 0, \tag{6.44}$$

with boundary and initial conditions

$$u_\varepsilon(0, t) = 0, t > 0, \quad u_\varepsilon(x, 0) = x, x > 0. \tag{6.45}$$

Then one has

$$u_\varepsilon(x, T - t) = E [X_\varepsilon^*(t); \tau_{\varepsilon, \text{linear}, x, T}^* < t], \quad t < T, \quad \text{when } \mu = \frac{y}{C_5T}. \tag{6.46}$$

The solution to (6.44), (6.45) is

$$u_\varepsilon(x, t) = \int_0^\infty G_{\varepsilon, D}(x, x', t)x' dx', \tag{6.47}$$

where the Green's function $G_{\varepsilon, D}$ has the formula

$$G_{\varepsilon, D}(x, x', t) = \frac{1}{\sqrt{2\pi\varepsilon t}} \exp\left[-\frac{(x - x' - \mu t)^2}{2\varepsilon t}\right] \left\{1 - \exp\left[-\frac{2xx'}{\varepsilon t}\right]\right\}. \tag{6.48}$$

Using the inequality $1 - e^{-z} \leq z$, $z \geq 0$, we have from (6.47), (6.48) that

$$\begin{aligned} \limsup_{x \rightarrow 0} \frac{1}{x} u_\varepsilon(x, t) &\leq \frac{2}{\sqrt{2\pi}(\varepsilon t)^{3/2}} \exp\left[-\frac{\mu^2 t}{2\varepsilon}\right] \int_0^\infty e^{-\mu x' / \varepsilon x'^2} dx' \\ &= \frac{4}{\sqrt{2\pi}} \left(\frac{\varepsilon}{\mu^2 t}\right)^{3/2} \exp\left[-\frac{\mu^2 t}{2\varepsilon}\right]. \end{aligned} \tag{6.49}$$

We conclude from (6.38) and (6.42)-(6.49), that the first term on the RHS of (6.40) is bounded as

$$\sqrt{\varepsilon} \limsup_{x \rightarrow 0} \frac{1}{x} \frac{\sigma_A^2(T)}{m_{1,A}(T)} E[|Z(T/2)|; \tau_{\varepsilon, x, T}^* < T/2] \leq \frac{C_7 \varepsilon T}{y^2} \quad \text{if } \varepsilon T \leq y^2, \tag{6.50}$$

where C_7 is a constant.

To find a bound for the second term on the RHS of (6.40) we use the inequality $P(\tau_{\varepsilon, x, T}^* < \tau_\delta) \leq P(\tau_{\varepsilon, x, T}^* < \nu T) + P(\nu T < \tau_{\varepsilon, x, T}^* < \tau_\delta)$, where $1/2 < \nu < 1$ and ν is a suitably chosen constant. We may then bound $\sqrt{\varepsilon} \delta P(\tau_{\varepsilon, x, T}^* < \nu T)$ from the inequalities obtained in the previous paragraph. Assuming $y \geq T^2$, it follows from (6.30) that we may take $\sqrt{\varepsilon} \delta = \Lambda y$, whence $\limsup_{x \rightarrow 0} x^{-1} \sqrt{\varepsilon} \delta P(\tau_{\varepsilon, x, T}^* < \nu T) \leq C_8 \varepsilon T / y^2$ for some constant C_8 . We estimate the second probability as

$$\begin{aligned} &P(\nu T < \tau_{\varepsilon, x, T}^* < \tau_\delta) \\ &\leq P(\tau_\delta > \nu T, \tau_{\varepsilon, \text{linear}, x, T} < \tau_\delta) \\ &\leq P\left(\sup_{\tau_{\varepsilon, \text{linear}, x, T} \vee (\nu T) < s < T} |X_\varepsilon(s) - x_{\text{class}}(s)| > c_8 \sqrt{\varepsilon} \delta\right), \end{aligned} \tag{6.51}$$

where $X_\varepsilon(\cdot)$, $x_{\text{class}}(\cdot)$ are given in (5.16), (5.17), and $c_8 > 0$ is a constant. We choose ν so that $-c_8 \Lambda y / 2 \leq x_{\text{class}}(s) \leq x$ for $\nu T < s < T$. Hence if $\nu T < s < T$, $0 < x < c_8 \Lambda y$ and $X_\varepsilon(s) - x_{\text{class}}(s) < -c_8 \Lambda y$ then $X_\varepsilon(s) < 0$. Since $X_\varepsilon(s) > 0$ for $\tau_{\varepsilon, \text{linear}, x, T} < s < T$ it follows that if s satisfies $\tau_{\varepsilon, \text{linear}, x, T} \vee (\nu T) < s < T$ and $|X_\varepsilon(s) - x_{\text{class}}(s)| > c_8 \Lambda y$, then $X_\varepsilon(s) > c_8 \Lambda y / 2$. We see therefore, using the

inequality $X_\varepsilon^*(\cdot) \geq X_\varepsilon(\cdot)$, that the probability on the RHS of (6.51) is bounded above by

$$w_\varepsilon(x) = P\left(\sup_{\tau_{\varepsilon, \text{linear}, x, T}^* < s < T} X_\varepsilon^*(s) > c_8 \Lambda y / 2\right). \tag{6.52}$$

The function $w_\varepsilon : [0, c_8 \Lambda y / 2] \rightarrow [0, 1]$ is the solution to the boundary value problem,

$$\begin{aligned} -\frac{\varepsilon}{2} \frac{d^2 w_\varepsilon(x)}{dx^2} + \frac{y}{C_5 T} \frac{dw_\varepsilon(x)}{dx} &= 0, \\ 0 < x < c_8 \Lambda y / 2, \quad w_\varepsilon(0) &= 0, \quad w_\varepsilon(c_8 \Lambda y / 2) = 1, \end{aligned} \tag{6.53}$$

which has solution

$$w_\varepsilon(x) = \left\{ \exp\left[\frac{2xy}{C_5 \varepsilon T}\right] - 1 \right\} / \left\{ \exp\left[\frac{c_8 \Lambda y^2}{C_5 \varepsilon T}\right] - 1 \right\}. \tag{6.54}$$

Taking limits in (6.54) we see that

$$\Lambda y \lim_{x \rightarrow 0} \frac{1}{x} w_\varepsilon(x) = \frac{2\Lambda y^2}{C_5 \varepsilon T} / \left\{ \exp\left[\frac{c_8 \Lambda y^2}{C_5 \varepsilon T}\right] - 1 \right\} \leq \frac{C_9 \varepsilon T}{y^2}, \tag{6.55}$$

for some constant C_9 , provided $\varepsilon T < y^2$.

We conclude now a bound on the contribution of the first term in the Taylor expansion (6.3) to the RHS of (6.29). Using the formula (4.23) and the inequality $y \geq C_3 T^2$, we have that

$$\limsup_{x \rightarrow 0} \frac{1}{x} \frac{\partial q_0(0, y, T)}{\partial x} \sqrt{\varepsilon} |E[Z_\varepsilon : \tau_\delta \vee (T/2) < \tau_{\varepsilon, x, T}^*]| \leq \frac{C_{10} \varepsilon}{y}, \tag{6.56}$$

where $\delta = \Lambda y$ and C_{10} is a constant. To bound the contribution of the second term in the Taylor expansion (6.3) we use (4.38) to obtain the inequality $|\partial^2 q_0(x, y, T) / \partial x^2| \leq C_{11} T / y$, provided $|x| < \Lambda y$ and $y \geq C_3 T^2$. We have then from (5.33), (6.36) that

$$\begin{aligned} &\limsup_{x \rightarrow 0} \frac{1}{x} \left| E \left[Z_\varepsilon^2 \int_0^1 \int_0^1 \lambda d\lambda d\mu \frac{\partial^2 q_0(\lambda \mu \sqrt{\varepsilon} Z_\varepsilon, y, T)}{\partial x^2}; \tau_\delta \vee (T/2) < \tau_{\varepsilon, x, T}^* \right] \right| \\ &\leq \frac{C_{11} \varepsilon T}{y} \limsup_{x \rightarrow 0} \frac{1}{x} E \left[Z_\varepsilon^2; \tau_\delta \vee (T/2) < \tau_{\varepsilon, x, T}^* \right] \\ &\leq \frac{C_{12} \varepsilon T^2}{y^2}, \end{aligned} \tag{6.57}$$

where C_{12} is a constant. To bound the contribution of the final term in the Taylor expansion (6.3) we use the fact that $\partial q_0(0, y, t) / \partial t = 0$, whence

$$\frac{\partial q_0(x, y, t)}{\partial t} = x \int_0^1 \frac{\partial^2 q_0(\mu x, y, t)}{\partial t \partial x'} d\mu, \tag{6.58}$$

where we have from (2.29) that

$$\begin{aligned} \frac{\partial^2 q_0(x', y, t)}{\partial t \partial x'} &= - \left[A(t) + \frac{1}{\sigma_A^2(t)} \right] \frac{\partial q_0(x', y, t)}{\partial x'} \\ &\quad - \left[\lambda(x', y, t) + \frac{\partial q_0(x', y, t)}{\partial x'} \right] \frac{\partial^2 q_0(x', y, t)}{\partial x'^2}. \end{aligned} \tag{6.59}$$

It follows from (4.26) and (4.38) there are constants C_{11}, C_{12} such that

$$\left| \left[A(t) + \frac{1}{\sigma_A^2(t)} \right] \frac{\partial q_0(x', y, t)}{\partial x'} \right| \leq \frac{C_{11} y}{T^2}, \tag{6.60}$$

$$\left| \left[\lambda(x', y, t) + \frac{\partial q_0(x', y, t)}{\partial x'} \right] \frac{\partial^2 q_0(x', y, t)}{\partial x'^2} \right| \leq C_{12}, \tag{6.61}$$

provided $0 < x' < \Lambda y$, $T/2 < t < T$, and $y \geq C_3 T^2$. Hence we may estimate the expectation of the final term in the Taylor expansion by combining our estimate on the expectation of the RHS of (6.34) with (6.58)-(6.61). We obtain an inequality

$$\begin{aligned} & \limsup_{x \rightarrow 0} \frac{1}{x} \left| E \left[(T - \tau_{\varepsilon, x, T}^*) \int_0^1 d\lambda \frac{\partial q_0(\sqrt{\varepsilon} Z_\varepsilon, y, \lambda \tau_{\varepsilon, x, T}^* + (1 - \lambda)T)}{\partial t}; \right. \right. \\ & \quad \left. \left. \tau_\delta \vee (T/2) < \tau_{\varepsilon, x, T}^* \right] \right| \\ & \leq \frac{C_{13}\varepsilon}{y} \quad \text{for } \delta = \Lambda y, y \geq C_3 T^2. \end{aligned} \tag{6.62}$$

It follows then from (6.56), (6.57), and (6.62) that the LHS of (6.27) is bounded by $C_{14}\varepsilon/y$ provided $y \geq C_3 T^2$ and $\varepsilon T \leq y^2$.

This bound on the LHS of (6.27) may be improved by noting a cancellation in the Taylor expansion (6.3). To see this we use the identity

$$\frac{\sigma_A^2(T)}{m_{1,A}(T)} - \frac{\sigma_A^2(\tau)}{m_{1,A}(\tau)} = \int_\tau^T \left[A(t) + \frac{1}{\sigma_A^2(t)} \right] \frac{\sigma_A^2(t)}{m_{1,A}(t)} dt. \tag{6.63}$$

Hence using (6.58), (6.59), and (6.63) we have that the sum of the first and third terms in the Taylor expansion (6.3) may be written as

$$\begin{aligned} & \frac{\partial q_0(0, y, T)}{\partial x} \sqrt{\varepsilon} Z_\varepsilon - \int_\tau^T dt \frac{\partial q_0(\sqrt{\varepsilon} Z_\varepsilon, y, t)}{\partial t} \\ & = \frac{\partial q_0(0, y, T)}{\partial x} \sqrt{\varepsilon} \frac{\sigma_A^2(T)}{m_{1,A}(T)} Z(\tau) - \sqrt{\varepsilon} \int_\tau^T dt \left[A(t) + \frac{1}{\sigma_A^2(t)} \right] \\ & \quad \times \left\{ \frac{\sigma_A^2(t)}{m_{1,A}(t)} \frac{\partial q_0(0, y, T)}{\partial x} - \frac{\sigma_A^2(\tau)}{m_{1,A}(\tau)} \int_0^1 d\mu \frac{\partial q_0(\mu \sqrt{\varepsilon} Z_\varepsilon, y, t)}{\partial x} \right\} Z(\tau) \\ & \quad + \sqrt{\varepsilon} Z_\varepsilon \int_\tau^T dt \int_0^1 d\mu \left[\lambda(\mu \sqrt{\varepsilon} Z_\varepsilon, y, t) \right. \\ & \quad \left. + \frac{\partial q_0(\mu \sqrt{\varepsilon} Z_\varepsilon, y, t)}{\partial x'} \right] \frac{\partial^2 q_0(\mu \sqrt{\varepsilon} Z_\varepsilon, y, t)}{\partial x'^2}, \end{aligned} \tag{6.64}$$

where $\tau = \tau_{\varepsilon, x, T}^*$.

To bound the expectation of the first term on the RHS of (6.64) we observe that

$$\begin{aligned} & \limsup_{x \rightarrow 0} \frac{1}{x} \sqrt{\varepsilon} \frac{\sigma_A^2(T)}{m_{1,A}(T)} \left| E[Z(\tau_{\varepsilon, x, T}^*) : \tau_\delta \vee (T/2) < \tau_{\varepsilon, x, T}^*] \right| \\ & \leq \limsup_{x \rightarrow 0} \frac{1}{x} \left[u_\varepsilon(x, T/2) + C_{15}y \{ w_\varepsilon(x) + P(\tau_{\varepsilon, \text{linear}, x, T}^* < \nu T) \} \right], \end{aligned} \tag{6.65}$$

where we assume $y \geq C_3 T^2, \varepsilon T \leq y^2, \sqrt{\varepsilon} \delta = \Lambda y$. Then from (6.49) and (6.55) we see that $\limsup_{x \rightarrow 0} x^{-1} [u_\varepsilon(x, T/2) + y w_\varepsilon(x)] \leq C_{16}(\varepsilon T/y^2)^2$. Instead of using the bound $P(\tau_{\varepsilon, \text{linear}, x, T}^* < \nu T) \leq v_\varepsilon(x)/(1 - \nu)^2 T^2$ as in (6.42), we use the identity $P(\tau_{\varepsilon, \text{linear}, x, T}^* < \nu T) = u_\varepsilon(x, (1 - \nu)T)$, where $[x, t] \rightarrow u_\varepsilon(x, t)$, $x, t > 0$, is the solution to the PDE (6.44) with boundary and initial conditions $u_\varepsilon(0, t) = 0$,

$u_\varepsilon(x,0) = 1$. Similarly to (6.49) we have now that

$$\begin{aligned} \limsup_{x \rightarrow 0} \frac{1}{x} u_\varepsilon(x,t) &\leq \frac{2}{\sqrt{2\pi}(\varepsilon t)^{3/2}} \exp\left[-\frac{\mu^2 t}{2\varepsilon}\right] \int_0^\infty e^{-\mu x'/\varepsilon} x' dx' \\ &= \frac{2}{\mu t \sqrt{2\pi}} \left(\frac{\varepsilon}{\mu^2 t}\right)^{1/2} \exp\left[-\frac{\mu^2 t}{2\varepsilon}\right]. \end{aligned} \tag{6.66}$$

It follows from (6.66) that $y \limsup_{x \rightarrow 0} x^{-1} P(\tau_{\varepsilon,\text{linear},x,T}^* < \nu T) \leq C_{17}(\varepsilon T/y^2)^2$. We conclude from (6.65) that the \limsup as $x \rightarrow 0$ of x^{-1} times the expectation of the first term on the RHS of (6.64) is bounded by $C_{18}\varepsilon^2 T/y^3$.

Using (6.61) and arguing as in the previous paragraph, we see from (6.36), (6.38) and the Schwarz inequality that the \limsup as $x \rightarrow 0$ of x^{-1} times the expectation of the third term on the RHS of (6.64) is bounded by $C_{19}\varepsilon T^2/y^2$. The expectation of the second term on the RHS of (6.64) is bounded by

$$\begin{aligned} &\frac{C_{20}\sqrt{\varepsilon}}{T} \frac{\partial q_0(0,y,T)}{\partial x} E[(T - \tau_{\varepsilon,x,T}^*)^2 | Z(\tau_{\varepsilon,x,T}^*); \tau_\delta \vee (T/2) < \tau_{\varepsilon,x,T}^*] \\ &+ C_{21}\varepsilon T \sup_{0 < x' < \sqrt{\varepsilon}\delta, T/2 < t < T} \left| \frac{\partial^2 q_0(x',y,t)}{\partial x'^2} \right| \\ &\times E[(T - \tau_{\varepsilon,x,T}^*) Z(\tau_{\varepsilon,x,T}^*)^2; \tau_\delta \vee (T/2) < \tau_{\varepsilon,x,T}^*]. \end{aligned} \tag{6.67}$$

We bound the first expectation in (6.67) by using the Schwarz inequality and the inequality $E[(T - \tau_{\varepsilon,x,T}^*)^4] \leq E[(T - \tau_{\varepsilon,\text{linear},x,T}^*)^4]$. This latter expectation can be estimated by considering the function

$$u_{\varepsilon,\alpha}(x) = E[\exp\{-\alpha(T - \tau_{\varepsilon,\text{linear},x,T}^*)\}] , \quad x, \alpha > 0. \tag{6.68}$$

Then $u_{\varepsilon,\alpha}(\cdot)$ is the solution to a boundary value problem

$$\frac{\varepsilon}{2} \frac{d^2 u_{\varepsilon,\alpha}(x)}{dx^2} - \mu \frac{du_{\varepsilon,\alpha}(x)}{dx} = \alpha u_{\varepsilon,\alpha}(x), \quad x > 0, \quad u_{\varepsilon,\alpha}(0) = 1, \tag{6.69}$$

with $\mu = y/C_5 T$. Evidently we have that

$$u_{\varepsilon,\alpha}(x) = \exp\left[-\frac{x}{\varepsilon} \left\{ \sqrt{\mu^2 + 2\varepsilon\alpha} - \mu \right\}\right]. \tag{6.70}$$

From (6.70) we conclude that

$$\begin{aligned} E[(T - \tau_{\varepsilon,\text{linear},x,T}^*)^4] &= \left(\frac{\partial}{\partial \alpha}\right)^4 u_{\varepsilon,\alpha}(x) \Big|_{\alpha=0} \\ &= \frac{15\varepsilon^3 x}{\mu^7} + \frac{15\varepsilon^2 x^2}{\mu^6} + \frac{6\varepsilon x^3}{\mu^5} + \frac{x^4}{\mu^4}. \end{aligned} \tag{6.71}$$

It follows from (6.36), (6.71) that the \limsup of x^{-1} times the first expectation in (6.67) as $x \rightarrow 0$ is bounded by $C_{22}\varepsilon^2 T/y^3$.

We also use the Schwarz inequality to bound the second expectation in (6.67) by using the fact that for $\alpha \in \mathbb{R}$ the function

$$s \rightarrow \exp\left[\alpha Z(s) - \frac{\alpha^2}{2} \int_s^T \frac{m_{1,A}(s')^2}{\sigma_A^4(s')} ds'\right], \quad 0 < s < T, \tag{6.72}$$

is also a martingale. On differentiating (6.72) twice with respect to α and setting $\alpha = 0$ we see that (6.35) is a martingale. On differentiating four times we have that

$$s \rightarrow Z(s)^4 - 6Z(s)^2 \int_s^T \frac{m_{1,A}(s')^2}{\sigma_A^4(s')} ds' + 3 \left(\int_s^T \frac{m_{1,A}(s')^2}{\sigma_A^4(s')} ds' \right)^2, \quad (6.73)$$

for $0 < s < T$, is a martingale. Hence

$$\begin{aligned} & E \left[Z(\tau_{\varepsilon,x,T}^*)^4 ; \tau_{\varepsilon,x,T}^* > T/2 \right] \\ & \leq C_{23} \left\{ \frac{1}{T^4} E[(T - \tau_{\varepsilon,x,T}^*)^2] + \frac{1}{T} E[Z(T/2)^2 ; \tau_{\varepsilon,x,T}^* < T/2] \right\}. \end{aligned} \quad (6.74)$$

We have already seen that the first expectation on the RHS of (6.74) is bounded by a constant times $v_\varepsilon(x)$ of (6.38). To bound the second expectation we proceed in a similar way to how we bounded the expectation in (6.41). We have that

$$\begin{aligned} & \varepsilon \frac{\sigma_A^4(T)}{m_{1,A}(T)^2} E[Z(T/2)^2 ; \tau_{\varepsilon,x,T}^* < T/2] \\ & \leq C_{24} E \left[[X_\varepsilon(T/2) - x_{\text{class}}(T/2)]^2 ; \tau_{\varepsilon,\text{linear},x,T} < T/2 \right], \end{aligned} \quad (6.75)$$

and the expectation on the RHS of (6.75) can be bounded using solutions to the PDE (6.44). Thus we have that the lim sup of x^{-1} times the RHS of (6.75) as $x \rightarrow 0$ is bounded by $C_{24}\varepsilon^2 T^2/y^3$, whence the lim sup of x^{-1} times the RHS of (6.74) as $x \rightarrow 0$ is bounded by $C_{25}\varepsilon/Ty^3$. We conclude that the lim sup of x^{-1} times the second expectation in (6.67) as $x \rightarrow 0$ is bounded by $C_{26}\varepsilon^2 T^3/y^4 \leq C_{27}\varepsilon^2 T/y^3$. \square

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