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NULL-CONTROLLABILITY FOR 1-D DEGENERATE QUASILINEAR PARABOLIC EQUATIONS

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Abstract. In this article, we prove local null-controllability for one-dimensional degenerate quasilinear parabolic equations. We apply a well-known local inversion argument used by Fursikov and Imanuvilov. The strategy is to use Carleman estimates, previously obtained for weak and strong degenerate parabolic problems.

1. Introduction

In this article, we investigate the controllability of the quasilinear degenerate parabolic system

$$
\begin{gathered}
u_t - \ell(au)\big(a(x)u_x\big)_x + f(t,x,u) = h\chi_\omega, \quad (t,x)\in Q,\\
u(t,1)=0, \quad \text{in } (0,T),\\
\begin{cases}
u(t,0)=0, & \text{(Weak)},\ t\in(0,T),\\
\text{or} & \\
(au_x)(t,0)=0 & \text{(Strong)},\ t\in(0,T),
\end{cases}\\
u(0,x)=u_0(x), \quad x\in(0,1),
\end{gathered}
\tag{1.1}
$$

where $T>0$ is given, $Q:=(0,T)\times(0,1)$, $\omega=(\alpha,\beta)\subset(0,1)$, $u_0\in L^2(0,1)$ and $h\in L^2(Q_\omega)$ is a control that acts on the system through $Q_\omega:=(0,T)\times\omega$. During this section, we will specify some conditions on the functions $a:[0,1]\to\mathbb{R}$, $\ell:\mathbb{R}\to\mathbb{R}$ and $f:[0,T]\times[0,1]\times\mathbb{R}\to\mathbb{R}$, under which the discussion will be developed.

Assumption 1.1. Let $a\in C([0,1])\cap C^1((0,1])$ be a nondecreasing function satisfying $a(0)=0$ and $a>0$ on $(0,1]$. Additionally, suppose that there exists $K\in\mathbb{R}$ such that

$$
xa'(x)\le Ka(x), \quad \forall x\in[0,1], \tag{1.2}
$$

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where $K \in [0,1)$, for the *weakly degenerate case* (WDC), and $K \in [1,2)$, for the *strongly degenerate case* (SDC). Only for the (SDC), we also assume that

$$\begin{aligned} &\exists\theta \in (1,K] \text{ such that } \theta a \le xa' \text{ near zero, if } K>1;\\ &\exists\theta \in (0,1) \text{ such that } \theta a \le xa' \text{ near zero, if } K=1. \end{aligned} \tag{1.3}$$

Next, we provide some examples and comments about assumption 1.1.

(a) For $\gamma \in (0,1)$ and $\alpha \ge 0$, putting $\beta = \arctan(\alpha)$, the function $a_1(x) = x^\gamma \cos(\beta x)$ satisfies (1.2) for the (WDC). On the other hand, if $\gamma \in (1,2)$, then a_1 becomes an example for the (SDC);

(b) For each $p \in (0,1)$, the function $a_2(x) = x^p + x$ satisfies (1.2) for the (WDC). Analogously, if $p \in (1,2)$, then $a_3(x) = x^p + x$ satisfies (1.2) for the (SDC).

Since our main results are associated with (1.1), we should make some comments about the controllability of one-dimensional degenerate or quasilinear problems. Many applied phenomena are closely related to degenerate parabolic equations, calling a notorious attention to their mathematical point of view. Motivated by the properties already known for the uniformly parabolic case, a complete qualitative investigation for degenerate operators is also expected (see a well-posedness result in [7], for instance). This brief comment certainly includes control theory, where much more development is still desired. In one dimension, it seems to us that [12] and [13] are the two first articles dealing with the controllability of degenerate parabolic equations, which clearly inspired much relevant work since then (see [3, 6, 9, 14, 15, 22, 24, 29, 33] and the references therein). On the other hand, to our best knowledge, there are not many controllability results involving quasilinear equations, where the second-order differential operator is associated with a nonlinearity which depends on the state (see [28], for instance). So that, in this paper, the main intention is a investigation about the controllability of one-dimensional degenerate quasilinear equations. To be more precise, we will prove a local null-controllability result for (1.1), at any time, with controls acting on a small subinterval $\omega \subset (0,1)$. In other words, given any time $T > 0$ and a sufficiently small initial data u_0, there exists a state-control pair (u_h, h) for (1.1), such that $u_h(T, \cdot) = 0$ in $[0,1]$. The proof will be based on [30], where a meticulous local inversion argument is developed, using Lyusternik's Theorem. This goal passes by a certain linearization of (1.1), for which a global null-controllability result and some additional estimates will also be obtained.

In the current literature, it is undeniable the strength of the Carleman estimates method, because it provides a refined technique that makes the one-dimensional degenerate controllability field well-understood (see [1, 8, 10, 11, 32] and the references aforementioned). In [28], the local null-controllability result, proved for nondegenerate quasilinear equations, also follows Carleman's approach. To summarize, up to this moment, the controllability of quasilinear equations, where the diffusion term depends nonlinearly on the state, has not been widely investigated, even for the nondegenerate case. It is exactly the motivation for the current research, where we would like to contribute providing a controllability study for the degenerate quasilinear problem (1.1).

To complement the state of the art associated with degenerate problems, we also mention [4], where the boundary null controllability of the degenerate heat equation was obtained as the limit of internal controllability. To be more precise, taking $\omega = \omega_\varepsilon := (1-\varepsilon, 1)$, for each $\varepsilon \in (0,1)$, is built a family of state-control pairs $\{(u_\varepsilon, h_\varepsilon); \varepsilon \in (0,1)\}$ solving (2.2), with $c \equiv 0$ and $g \equiv 0$, with the following property: $(u_\varepsilon, h_\varepsilon) \to (u, g)$ in a suitable functional space, as $\varepsilon \to 0$, where (u, g) solves the boundary null controllability problem for the degenerate heat equation. However, this kind of question keeps not understood if $\omega = (0, \varepsilon)$, as explained in [4, Section 5]. On the other hand, following this direction, some effort has been made to prove an analogous fact for the degenerate wave equation (see [5]). Naturally, it would also be very interesting to analyze the asymptotic behavior of families of state-control pairs corresponding to nonlinear evolution PDEs (degenerate and nondegenerate cases. The discussion above is completely related to the relevance of degenerate operators in Partial Differential Equations, including the Control Theory setting. Specifically talking about this theme, in the presence of nonlinearities, we should mention [20, 21, 19], where the authors have obtained the local null-controllability for a class of degenerate parabolic problems with nonlocal terms, dealing with theoretical and numerical aspects. We emphasize that this work relies on those Carleman estimates achieved in [20] and [19] for the (WDC) and the (SDC), respectively.

Next, we present some important functional spaces, introduced in [1], which are closely related to the initial data of (1.1) and its linearization. Other than that, it also has to do with the statement of our main result.

Definition 1.2 (Weighted Sobolev spaces)**.** Let us consider a real function $a = a(x)$, as in (1.1).

(a) For the (WDC), we set $H^1_a := \big\{u \in L^2(0,1)$ such that u is absolutely continuous in $[0,1]$, $\sqrt{a}u_x \in L^2(0,1)$, and $u(1) = u(0) = 0\big\}$, equipped with the natural norm

$$\|u\|_{H^1_a} := \Big(\|u\|^2_{L^2(0,1)} + \|\sqrt{a}u_x\|^2_{L^2(0,1)}\Big)^{1/2}.$$

(b) For the (SDC), we set $H^1_a := \big\{u \in L^2(0,1)$ such that u is absolutely continuous in $(0,1]$, $\sqrt{a}u_x \in L^2(0,1)$, and $u(1) = 0\big\}$, with the same norm taken for the (WDC);

(c) In both situations, the (WDC) and the (SDC),

$$H^2_a := \{u \in H^1_a : au_x \in H^1(0,1)\}$$

with the norm

$$\|u\|_{H^2_a} := (\|u\|^2_{H^1_a} + \|(au_x)_x\|^2_{L^2(0,1)})^{1/2}.$$

Now, let us state the properties of the functions $\ell : \mathbb{R} \to \mathbb{R}$ and $f : [0,T] \times [0,1] \times \mathbb{R} \to \mathbb{R}$, both mentioned in (1.1).

Assumption 1.3. Let $\ell : \mathbb{R} \to \mathbb{R}$ be a C^1 function with bounded derivative and suppose that $\ell(0) = 1$. We should observe that our results remain the same if we just suppose that $\ell(0) > 0$.

Assumption 1.4. We assume that $f : [0,T] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is a C^1 function, with bounded derivatives, such that $f(t,x,0) \equiv 0$ and $c(t,x) = \partial_3 f(t,x,0)$ belongs to $L^\infty(Q)$, where $(t,x) \in [0,T] \times [0,1]$.

To state our main result, we recall the important concept below:

Definition 1.5. System (1.1) is said *locally null-controllable* at a given time $T > 0$ if there exists $\varepsilon > 0$ with the following property: whenever $u_0 \in H_a^1$ and $\|u_0\|_{H_a^1} \leq \varepsilon$, we can find a control function $h \in L^2(Q_\omega)$, associated with a state u, such that

$$u(T,x) = 0, \quad \text{for every } x \in [0,1].$$

Having in mind the considerations above, we state our main result.

Theorem 1.6 (Local Null-Controllability). *Under assumptions 1.1, 1.3 and 1.4, the nonlinear system* (1.1) *is locally null-controllable at any time $T > 0$, in the sense of Definition 1.5, provided that one of the following conditions holds:*

(a) $K \neq 1$;
(b) $K = 1$ *and* $\theta \geq 1/2$.

Conditions (a) and (b) in Theorem 1.6 are both sufficient to assure that $au \in L^\infty(0,1)$, for any $u \in H_a^1$. This will play a very important role in the proofs of Lemma 3.2 and Proposition 3.4. A complete explanation about this will be given in Appendix 5.

The remainder of this paper is organized as follows: in Section 2, we present useful notation and preliminary results. The first part brings some explanation about a local inversion argument, while the second one is concerned with Carleman and observability estimates, valid for both the (WDC) and the (SDC).

In Section 3, we verify some properties of a mapping $\mathcal{H} : E \to F$, set in (2.1), which will allow us to apply Lyusternik's Theorem (stated as Theorem 2.1) to achieve the local null-controllability of (1.1). At this point, the key information comes from the global-null controllability of the linearization of (1.1), given in (2.2), as well as from some additional regularity results.

In Section 4, we prove the main result of this paper (Theorem 1.6), where the local null-controllability of (1.1) is obtained. Additionally, we include some further comments related possible future directions. The last section is Appendix 5, which complements the content studied in Section 3.

2. Preliminary results

In this section, we introduce some notation and useful auxiliary results, which will help us to prove our main result.

2.1. **Notation and results related to the local inversion argument.** The first part of this section is devoted to a brief explanation about the most important strategies for proving our main results. Our approach relies on a version of Lyusternik's Inverse Mapping Theorem (see [2, 30], for instance), whose statement is given below.

Theorem 2.1 (Lyusternik). *Let E and F be two Banach spaces, consider $H \in C^1(E,F)$ and put $\eta_0 = H(0)$. If $H'(0) \in \mathcal{L}(E,F)$ is onto, then there exist $r > 0$ and $\tilde{H} : B_r(\eta_0) \subset F \to E$ such that*

$$H(\tilde{H}(\xi)) = \xi, \quad \forall \xi \in B_r(\eta_0),$$

which means that $\tilde{H}$ is a right inverse of H in $B_r(\eta_0)$. In addition, there exists $K > 0$ such that

$$\|\tilde{H}(\xi)\|_E \leq K\|\xi - \eta_0\|_F, \forall \xi \in B_r(\eta_0).$$

Next, let us indicate how the proof of Theorem 1.6 can be seen as an application of Theorem 2.1. Even though we have not set the desired Hilbert spaces E and F yet, let us put

$$\mathcal{H}(u,h) = (\mathcal{H}_1(u,h), \mathcal{H}_2(u,h)), \tag{2.1}$$

where

$$\mathcal{H}_1(u,h) := u_t - \ell(au)(au_x)_x + f(t,x,u) - h\chi_\omega, \quad \mathcal{H}_2(u,h) := u(0,\cdot).$$

Notice that for $u_0 \in H_a^1$, the first and the second relations in (1.1) are satisfied if, and only if, there exists $(u,h) \in E$ solving

$$\mathcal{H}(u,h) = (0, u_0).$$

From this point, we realize that, among other properties, E and F must be built:

- considering the boundary conditions mentioned in (1.1);
- having some imposition on its elements, assuring that $u(T,\cdot) \equiv 0$. It will be done having in mind some weights which appear in (2.13);
- having in mind that we want $\mathcal{H}'(0,0) \in \mathcal{L}(E,F)$ to be onto. In fact, it is equivalent to say that, given any $(g,u_0) \in F$, the linear system

$$\begin{gathered} u_t - (a(x)u_x)_x + c(t,x)u = h\chi_\omega + g, \quad (t,x) \in Q; \\ u(t,1) = 0, \quad \text{in } (0,T), \\ \begin{cases} u(t,0) = 0, & \text{(Weak)},\ t \in (0,T) \\ \text{or} \\ (au_x)(t,0) = 0, & \text{(Strong)},\ t \in (0,T) \end{cases} \\ u(0,x) = u_0(x), \quad x \in (0,1), \end{gathered} \tag{2.2}$$

  is globally null-controllable at the time $T > 0$, where $h \in L^2(Q_\omega)$ is the control function and a satisfies assumption 1.1. Hence, it seems that E should contain some information involving the well-posedness (and additional regularity) of the linear system (2.2).

The well-posedness of (2.2) was proved in [1], with the following statement.

Proposition 2.2. *For each $g \in L^2(Q)$, $h \in L^2(Q_\omega)$ and $u_0 \in L^2(0,1)$, there exists a unique weak solution $u \in C^0([0,T]; L^2(0,1)) \cap L^2(0,T; H_a^1)$ of (2.2). Moreover, if $u_0 \in H_a^1$, then*

$$u \in \mathcal{U} := H^1(0,T; L^2(0,1)) \cap L^2(0,T; H_a^2) \cap C^0([0,T]; H_a^1),$$

and there exists a constant $C_T > 0$ such that

$$\begin{aligned} &\sup_{t\in[0,T]} (\|u(t)\|_{H_a^1}^2) + \int_0^T \Big(\|u_t\|_{L^2(0,1)}^2 + \|(au_x)_x\|_{L^2(0,1)}^2 \Big) \\ &\le C_T \Big(\|u_0\|_{H_a^1}^2 + \|g\|_{L^2(Q)}^2 + \|h\|_{L^2(Q_\omega)}^2 \Big). \end{aligned} \tag{2.3}$$

Definition 2.3. Let $\delta = \delta(t,x)$ and $f = f(t,x)$ be two real-valued measurable functions defined in Q, where δ is non-negative. We say that f belongs to $L^2(Q;\delta)$ if $\sqrt{\delta} f \in L^2(Q)$. Moreover, the natural norm in $L^2(Q;\delta)$ will be denoted by $\|\cdot\|_\delta$, that is,

$$\|f\|_\delta = \Big(\int_0^T \int_0^1 \delta f^2 \, dx \, dt \Big)^{1/2}$$

for each $f \in L^2(Q;\delta)$.

Let us take $\omega' = (\alpha', \beta') \subset\subset \omega$ and consider $\psi \in C^2([0,1];\mathbb{R})$ satisfying

$$\psi(x) := \begin{cases} \int_0^x \frac{y}{a(y)}dy, & x \in [0, \alpha'); \\ -\int_{\beta'}^x \frac{y}{a(y)}dy, & x \in [\beta', 1]. \end{cases} \tag{2.4}$$

Also, let us set the functions

$$\eta(x) := e^{\lambda(|\psi|_\infty + \psi)}, \quad \eta_r(x) := e^{\lambda(|\psi|_\infty + \psi)} - e^{\lambda r|\psi|_\infty}. \tag{2.5}$$

where $(t, x) \in (0, T) \times [0, 1]$ and $\lambda, r \in (0, +\infty)$. In addition, consider the constants:

$$\hat{\eta} = \min_{x\in[0,1]} \eta(x), \quad \eta^* := \max_{x\in[0,1]} \eta(x), \quad \hat{\eta}_r = \min_{x\in[0,1]} \eta_r(x), \quad \eta_r^* := \max_{x\in[0,1]} \eta_r(x). \tag{2.6}$$

It is important to notice that, if $r \in (3, +\infty)$ is sufficiently large, then $\eta_r(x) < 0$, for any $x \in [0, 1]$, and $3\eta_r^* < 2\hat{\eta}_r$. In this case, putting $\bar{\eta}_r := 3\eta_r^* - 2\hat{\eta}_r$, we can see that $\eta_r(x) \le \bar{\eta}_r < 0$, for any $x \in [0, 1]$. Next, we consider $m \in C^\infty([0, T];\mathbb{R})$ satisfying

$$\begin{gathered} m(t) \ge t^4(T-t)^4, \quad t \in (0, T/2]; \\ m(t) = t^4(T-t)^4, \quad t \in [T/2, T]\,; \\ m(0) > 0, \end{gathered}$$

to define

$$\begin{gathered} \tau(t) := \frac{1}{m(t)}, \quad \zeta(x,t) := \tau(t)\eta(x), \quad \zeta^*(t) := \tau(t)\eta^*, \\ A(t,x) := \tau(t)\eta_r(x), \quad \bar{A}(t) := \tau(t)\bar{\eta}_r, \end{gathered} \tag{2.7}$$

where $(t, x) \in [0, T) \times [0, 1]$ (see Remark 2.4). Finally, we can mention the weight functions

$$\rho_0 = e^{-sA}\zeta^{-5/6}, \quad \hat{\rho} = e^{-s(A+\bar{A})/2}(\zeta^*)^{-11/6}, \quad \rho_* = e^{-s\bar{A}}(\zeta^*)^{-17/6}, \tag{2.8}$$

associated with the desired spaces E and F. We observe that $\hat{\rho}^2 \le \rho_0\rho_*$ and that there exists a constant $C_T > 0$, only depending on T, such that $0 < C_T \le \rho_* \le C\hat{\rho} \le C\rho_0$. Thus,

$$L^2(Q;\rho_0^2) \hookrightarrow L^2(Q;\hat{\rho}^2) \hookrightarrow L^2(Q;\rho_*^2) \hookrightarrow L^2(Q).$$

Remark 2.4. In (2.7), we define $\tau = \tau(t)$ satisfying $\lim_{t\to 0^+} \tau(t) = \tau(0) > 0$. It plays a crucial role in order to guarantee that (1.1) is locally null-controllable at the time $T > 0$, as stated in Theorem 1.6. Precisely, each function given in (2.7) is based on the weights which will appear in (2.13). As a result, since $\rho_0(t) \to +\infty$, as $t \to T^-$, and $\rho_0(0) > 0$ (since $m(0) > 0$), it is possible to conclude that $u(T, x) = 0$ for any for $u \in L^2(Q;\rho_0^2)$. Hence, it seems reasonable to require that, if $(u, h) \in E$, then u must belong to $L^2(Q;\rho_0^2)$.

Now, we are ready to define E and F. Let us consider

$$\mathcal{U} := H^1(0, T; L^2(0, 1)) \cap L^2(0, T; H_a^2) \cap C^0([0, T]; H_a^1),$$

as in Proposition 2.2, and put $\mathcal{L}u := u_t - (au_x)_x$ for each $u \in \mathcal{U}$. Under all these notations, we set, for the (WDC), the Hilbert spaces

$$E := \big\{(u, h) \in \mathcal{U} \times L^2(Q_\omega;\rho_*^2) : \rho_0 u, \rho_0(\mathcal{L}u - h\chi_\omega) \in L^2((0, T) \times (0, 1))\big\}, \tag{2.9}$$

and

$$F := L^2(Q;\rho_0^2) \times H_a^1, \tag{2.10}$$

equipped with the norms

$$\|(u,h)\|_E := \Big(\|u\|^2_{\rho_0^2} + \|h\chi_\omega\|^2_{\rho_*^2} + \|\mathcal{L}u - h\chi_\omega\|^2_{\rho_0^2} + \|u(0,\cdot)\|^2_{H^1_a}\Big)^{1/2},$$

and

$$\|(g,v)\|_F := \Big(\|g\|^2_{\rho_0^2} + \|v\|^2_{H^1_a}\Big)^{1/2},$$

respectively. We observe that, for the (SDP), the definition of E must also contain the condition $au_x(t,0) \equiv 0$, a.e. in $[0,T]$, while the definition of F remains the same.

Since we have already defined the weight functions, and identified the Hilbert spaces E and F, as well as the mapping $\mathcal{H} : E \to F$, given in (2.1), we are supposed to verify that $\mathcal{H}$ satisfies the hypotheses of Theorem 2.1 (Section 3). To do that, we need to establish a Carleman estimate that will guarantee those hypotheses.

2.2. **Carleman inequality.** In this second part of Section 2, we present a key Carleman estimate, closely related to those properties of $\mathcal{H} : E \to F$ that we expect to prove. We start taking into consideration the adjoint system associated with (2.2), given by

$$\begin{gathered} -v_t - (a(x)v_x)_x + c(t,x)v = F, \quad (t,x) \in Q, \\ \begin{cases} v(t,0) = 0, & t \in (0,T) \\ \text{or} \\ (av_x)(t,0) = 0, & t \in (0,T) \end{cases} \\ v(T,x) = v_T(x), \quad x \in (0,1), \end{gathered} \tag{2.11}$$

where $F \in L^2(Q)$ and $v_T \in L^2(0,1)$. Now, we consider the functions and the constants given in (2.5) and (2.6), and define

$$\sigma(x,t) := \frac{\eta(x)}{[t(T-t)]^4}, \quad \text{and} \quad \varphi(x,t) := \frac{\eta_r(x)}{[t(T-t)]^4},$$

where $(t,x) \in (0,T) \times [0,1]$. Our desired Carleman and observability inequalities, mentioned above, will be achieved as consequences of the next lemma, whose proof can be found in [20], for the (WDC), and in [19], for the (SDC).

Lemma 2.5. *There exist $C > 0$ and $\lambda_0, s_0 > 0$ such that every solution v of (2.11) satisfies, for all $s \geq s_0$ and $\lambda \geq \lambda_0$, the estimate*

$$\begin{aligned} &\int_0^T \int_0^1 e^{2s\varphi}\big((s\lambda)\sigma a v_x^2 + (s\lambda)^{5/3}\sigma^{5/3}v^2\big) \\ &\leq C\Big(\int_0^T \int_0^1 e^{2s\varphi}|F|^2 \; + (\lambda s)^{17/3}\int_0^T \int_\omega e^{2s\varphi}\sigma^{17/3}v^2\Big), \end{aligned} \tag{2.12}$$

where the constants C, λ_0 and s_0 only depend on ω, a, $\|c\|_{L^\infty(Q)}$ and T.

The functions in (2.8) were not directly based on the weights which appear in (2.12), because

$$\lim_{t\to 0^+} \frac{1}{[t(T-t)]^4} = +\infty.$$

Instead of that, we have taken $\tau = \tau(t)$ in order to build ρ_0, $\hat{\rho}$ and ρ_* (recall Remark 2.4).

Next, we will state a new version of (2.12), involving the weights given in (2.8), whose proof can be done following the same steps of [19, Prop. 3.6].

Proposition 2.6 (Carleman Inequality)**.** *There exist $C > 0$ and $\lambda_0, s_0 > 0$ such that every solution v of* (2.11) *satisfies, for all $s \geq s_0$ and $\lambda \geq \lambda_0$, the estimate*

$$\begin{aligned}&\int_0^T \int_0^1 e^{2sA}\big[s\lambda\zeta a|v_x|^2 + (s\lambda)^{5/3}\zeta^{5/3}|v|^2\big]\\ &\leq C\Big(\int_0^T \int_0^1 e^{2sA}|F|^2 + (s\lambda)^{17/3}\int_0^T\int_\omega e^{2sA}(\zeta^*)^{17/3}|v|^2\Big),\end{aligned} \tag{2.13}$$

where the constants C, λ_0 and *s_0 only depend on ω, a, $\|c\|_{L^\infty(Q)}$ and T.*

We would like to complete this section making some brief comments about Lemma 2.5 and Proposition 2.6.

(a) For the (WDC), Lemma 2.5 and Proposition 2.6 are both detailed in [20]. In that paper, the discussion is organized under the presentation of several technical lemmas, whose proofs rely on energy estimates, as well as on the crucial Hardy-Poincaré inequality proved in [1];

(b) For the (SDC), we follow the same strategy used for the (WDC). However, considering $K \in [1, 2)$, as in assumption 1.1, the case $K = 1$ deserves a special attention (precisely, see [19, Lemma 3.2]);

(c) The definition of ρ_0 in (2.8) is inspired by the integral

$$(s\lambda)^{5/3}\int_0^T\int_0^1 e^{2sA}\zeta^{5/3}|v|^2,$$

which appears in (2.13), according to a standard argument. Likewise, $\hat{\rho}$ and ρ_* are set having in mind the definition of ρ_0, however, there are technical reasons to consider $\bar{A} = \bar{A}(t)$ and $\zeta^* = \zeta^*(t)$ in their expressions;

(d) It is well-known that (2.13) implies the observability inequality

$$\|v(0)\|^2_{L^2(0,1)} \leq C\int_0^T\int_\omega e^{2sA}(s\lambda)^{17/3}(\zeta^*)^{17/3}|v|^2, \tag{2.14}$$

valid for any solution v of (2.11), with $F \equiv 0$. In fact, this inequality holds if $\lambda > 0$ and $s > 0$ are sufficiently large.

3. Properties of the mapping $\mathcal{H}$

This section we prove the properties defined in (2.1), which required to apply Lyusternik's Theorem, namely: $\mathcal{H}'(0, 0)$ must be onto and $\mathcal{H}$ must belong to $C^1(E, F)$. These tasks will be done in the next two subsections. However, before presenting them, let us state a global null-controllability result for the linearized system (2.2), as well as some additional regularity of this system that, as we have already pointed out in Section 2, will be necessary to check the required hypotheses over $\mathcal{H}$. Its proof will be given at the end of this section.

Proposition 3.1. *If $T > 0$ and $(u_0, g) \in H^1_a \times L^2(Q; \rho_0^2)$, then there exists a state-control pair $(u, h) \in L^2(Q; \rho_0^2) \times L^2(Q_\omega; \rho_*^2)$ such that the null-controllability of* (2.2)*, at time $T > 0$, holds.*

Furthermore, we have

$$\sqrt{a}u_x \in L^2(Q; \hat{\rho}^2) : u_t, (au_x)_x \in L^2(Q; \hat{\rho}^2),\ \hat{\rho}u, \rho_*\sqrt{a}u_x \in L^\infty(0, T; L^2(0, 1)),$$

and there exists $C > 0$ such that

$$
\begin{aligned}
&\sup_{[0,T]} \|\hat{\rho} u(t,\cdot)\|^2_{L^2(0,1)} + \sup_{[0,T]} \|\rho_* \sqrt{a} u_x(t,\cdot)\|^2_{L^2(0,1)} \\
&+ \|\sqrt{a} u_x\|^2_{\hat{\rho}^2} + \|u_t\|^2_{\rho_*^2} + \|(a u_x)_x\|^2_{\rho_*^2} \\
&\le C(\|u\|^2_{\rho_0^2} + \|h\chi_\omega\|^2_{\rho_*^2} + \|g\|^2_{\rho_0^2} + \|u_0\|^2_{H^1_a}).
\end{aligned} \tag{3.1}
$$

3.1. **Surjectiveness of $\mathcal{H}'(\mathbf{0},\mathbf{0})$.** Before we establish the properties of $\mathcal{H}$, let us verify that $\mathcal{H}$ is well defined. To check that, it will be essential to know that $au \in L^\infty(0,1)$, for each $u \in H^1_a$. It is always true for $K \neq 1$, where $K \in [0,2)$ is mentioned in Hypothesis 1.1. For the (WDC), it comes from the continuous embedding $H^1_a \hookrightarrow L^\infty(0,1)$. For the (SDC) it comes from the fact that $a \in W^{1,\infty}(0,1)$. For the case $K = 1$, we could just prove it for $\theta \ge 1/2$, where θ is given in (1.3). The case $0 < \theta < 1/2$ remains open. A detailed proof of these facts will be presented in Appendix 5 (see Propositions 5.2 and 5.1).

Lemma 3.2. *The mapping $\mathcal{H} : E \to F$, given in* (2.1)*, is well defined, recalling that the spaces E and F are defined in* (2.9) *and* (2.10)*, respectively.*

Proof. For each $(u,h) \in E$, let us check that $\mathcal{H}(u,h) \in F$. Clearly, $\mathcal{H}_2(u,h) = u(\cdot,0) \in H^1_a$. Also, recalling assumptions 1.1, 1.3 and 1.4, we have

$$
\begin{aligned}
&\int_0^T \int_0^1 \rho_0^2 |\mathcal{H}_1(u,v,h)|^2 \\
&= \int_0^T \int_0^1 \rho_0^2 \left| u_t - \ell(au)(a u_x)_x + f(t,x,u) - h\chi_\omega \right|^2 \\
&\le 3\int_0^T \int_0^1 \rho_0^2 |\mathcal{L}u - h\chi_\omega|^2 + 3\int_0^T \int_0^1 \rho_0^2 \left|\ell(au) - \ell(0)\right|^2 |(a u_x)_x|^2 \\
&\quad + 3\int_0^T \int_0^1 \rho_0^2 |f(t,x,u) - f(t,x,0)|^2 \\
&\le 3\|(u,h)\|^2_E + C\int_0^T \int_0^1 \rho_0^2 |au|^2 |(a u_x)_x|^2 + C\int_0^T \int_0^1 \rho_0^2 |u|^2 \\
&\le C\|(u,h)\|^2_E + C\int_0^T \int_0^1 \rho_0^2 |au|^2 |(a u_x)_x|^2 .
\end{aligned}
$$

At this point, we must estimate $\mathcal{I} := \int_0^T \int_0^1 \rho_0^2 |au|^2 |(a u_x)_x|^2$. We start recalling that $A = \tau(t)\eta_r(x) \ge \tau(t)\hat{\eta}_r$ and $au \in L^\infty(0,1)$ to obtain

$$
\begin{aligned}
\mathcal{I} &= \int_0^T \int_0^1 e^{-2sA} \zeta^{-5/3} |au|^2 |(a u_x)_x|^2 \\
&\le \int_0^T \int_0^1 e^{-2s\tau\hat{\eta}_r} \eta^{-5/3} \tau^{-5/3} |au|^2 |(a u_x)_x|^2 \\
&\le C\int_0^T e^{-2s\tau\hat{\eta}_r} \tau^{-5/3} \int_0^1 |au|^2 |(a u_x)_x|^2 \\
&\le C\int_0^T e^{-2s\tau\hat{\eta}_r} \tau^{-5/3} (\|u\|^2_{L^2(0,1)} + \|\sqrt{a} u_x\|^2_{L^2(0,1)}) \int_0^1 |(a u_x)_x|^2 \\
&= C\Big(\int_0^T e^{-2s\tau\hat{\eta}_r} \tau^{-5/3} \|u\|^2_{L^2(0,1)} \|(a u_x)_x\|^2_{L^2(0,1)}
\end{aligned}
$$

$$+\int_0^T e^{-2s\tau\hat{\eta}_r}\tau^{-5/3}\|\sqrt{a}u_x\|^2_{L^2(0,1)}\|(au_x)_x\|^2_{L^2(0,1)}\Big) =: \mathcal{I}_1+\mathcal{I}_2\,. \tag{3.2}$$

Recall that $\bar{\eta}_r = 3\eta_r^* - 2\hat{\eta}_r < 0$ which implies $\kappa := \hat{\eta}_r - 2\bar{\eta}_r > 0$ and, consequently,

$$\begin{aligned}&e^{-2s\tau\hat{\eta}_r}\tau^{-5/3}\rho_*^{-4}\\ &= e^{-2s\tau\hat{\eta}_r}\tau^{-5/3}e^{4s\bar{A}}(\zeta^*)^{34/3}\\ &= e^{-2s\tau(\hat{\eta}_r-2\bar{\eta}_r)}\tau^{29/3} = e^{-2s\kappa\tau}\tau^{29/3}\le C.\end{aligned} \tag{3.3}$$

Since $\rho_* \le C\hat{\rho}$, from inequality (3.1), we have

$$\begin{aligned}\mathcal{I}_1 &= \int_0^T (e^{-2s\tau\hat{\eta}_r\tau}\tau^{-5/3}\rho_*^{-4})(\rho_*^2\|u\|^2_{L^2(0,1)})(\rho_*^2\|(au_x)_x\|^2_{L^2(0,1)})\\ &\le C\sup_{t\in[0,T]}\|\hat{\rho}u(t,\cdot)\|^2_{L^2(0,1)}\|(au_x)_x\|^2_{\rho_*^2}\le C\|(u,h)\|_E^4\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\mathcal{I}_2 &= \int_0^T (e^{-2s\tau\hat{\eta}_r\tau}\tau^{-5/3}\rho_*^{-4})(\rho_*^2\|\sqrt{a}u_x\|^2_{L^2(0,1)})(\rho_*^2\|(au_x)_x\|^2_{L^2(0,1)})\\ &\le C\sup_{t\in[0,T]}\|\rho_*\sqrt{a}u_x(t,\cdot)\|^2_{L^2(0,1)}\|(au_x)_x\|^2_{\rho_*^2}\\ &\le C\|(u,h)\|_E^4.\end{aligned} \tag{3.5}$$

As a conclusion, $\mathcal{H}_1(u,v)\in L^2(Q,\rho_0^2)$ and the proof is complete. □

Proposition 3.3. *$\mathcal{H}'(0,0)\in\mathcal{L}(E;F)$ is onto.*

Proof. Take $(g,u_0)\in F$. By Propositions 3.1 and 2.2, there exists $(u,h)\in E$ that solves (2.2). In other words,

$$\begin{aligned}\mathcal{H}'(0,0)(u,h) &= (\mathcal{H}_1'(0,0)(u,h),\mathcal{H}_2'(0,0)(u,h))\\ &= (u_t-(a(x)u_x)_x+c(t,x)u-h\chi_w, u(\cdot,0))\\ &= (g,u_0).\end{aligned}$$

This completes the proof. □

3.2. $\mathcal{H}$ **is continuously differentiable.** In this subsection, we will prove that $\mathcal{H}\in C^1(E,F)$. The proof will rely on the additional regularity described in (3.1).

Proposition 3.4. *The mapping $\mathcal{H}$ is continuously differentiable.*

Proof. It is clear that $\mathcal{H}_2\in C^1(E,F)$. So that, the proof is focused on checking that $\mathcal{H}_1$ has a continuous Gateaux derivative on E.

For $(u,h),(\bar{u},\bar{h})\in E$ and $\lambda>0$, set

$$b := \ell(au)a(x),\quad b_\lambda := \ell(a(u+\lambda\bar{u}))a(x),\quad f := f(t,x,u),$$
$$f_\lambda := f(t,x,u+\lambda\bar{u}),\quad f_3 := D_3f(t,x,u).$$

Claim 1: Given $(u,h)\in E$, the linear mapping $L: E\to L^2(Q;\rho_0^2)$, defined by

$$L(\bar{u},\bar{h}) := \bar{u}_t - \ell'(au)a\bar{u}(au_x)_x + \ell(au)(a\bar{u}_x)_x + f_3\bar{u} - \bar{h}\chi_\omega,$$

is the Gateaux derivative of $\mathcal{H}_1$ at $(u,h)\in E$.

Indeed, for each $(\bar{u},\bar{h})\in E$, we have

$$\Big\|\frac{1}{\lambda}(\mathcal{H}_1(u+\lambda\bar{u},h+\lambda\bar{h})-\mathcal{H}_1(u,h)) - L(\bar{u},\bar{h})\Big\|$$

$$
\begin{aligned}
&= \Big\| \bar{u}_t - \Big[\frac{\ell(a(u+\lambda\bar{u}))(a(u+\lambda\bar{u})_x)_x - \ell(au)(au_x)_x}{\lambda}\Big] \\
&\quad + \frac{1}{\lambda}(f_\lambda - f) - \bar{h}\chi_\omega - L(\bar{u},\bar{h})\Big\| \\
&\le \Big\| \Big[\frac{\ell(a(u+\lambda\bar{u})) - \ell(au)}{\lambda} - \ell'(au)a\bar{u}\Big](au_x)_x\Big\| \\
&\quad + \|[\ell(a(u+\lambda\bar{u})) + \ell(au)](a\bar{u}_x)_x\|_{\rho_0^2} \\
&\quad + \|\frac{1}{\lambda}(f_\lambda - f) - f_3\bar{u}\|_{\rho_0^2} \\
&=: B_1 + B_2 + B_3.
\end{aligned}
$$

We will see that $B_i \to 0$, as $\lambda \to 0$, for any $i = 1,2,3$.

Firstly, from assumption 1.4, for each $(t,x) \in (0,1)\times(0,T)$, we apply mean value theorem to obtain $u_\lambda^* = u_\lambda^*(t,x) \in \mathbb{R}$ such that

$$
B_3^2 \le \int_0^T \int_0^1 \rho_0^2 \,|(D_3 f(t,x,u_\lambda^*) - D_3 f(t,x,u))\bar{u}|^2 \to 0,
$$

as $\lambda \to 0$, where this convergence comes from Lebesgue's theorem.

Secondly, applying assumption 1.3 and the mean value theorem again, there exists $s_\lambda = s_\lambda(t,x) \in \mathbb{R}$ such that

$$
\begin{aligned}
B_1^2 &= \int_0^T \int_0^1 \rho_0^2 \Big|\frac{\ell(a(u+\lambda\bar{u})) - \ell(au)}{\lambda} - \ell'(au)a\bar{u}\Big|^2 |(au_x)_x|^2 \\
&= \int_0^T \int_0^1 \rho_0^2 |\ell'(s_\lambda) - \ell'(au)|^2 |a\bar{u}|^2 |(au_x)_x|^2 \to 0,
\end{aligned}
$$

as $\lambda \to 0$. Since we can argue as in (3.2), (3.4) and (3.5) we obtain

$$
\begin{aligned}
&\int_0^T \int_0^1 \rho_0^2 |\ell'(s_\lambda) - \ell'(au)|^2 |a\bar{u}|^2 |(au_x)_x|^2 \le \int_0^T \int_0^1 \rho_0^2 |a\bar{u}|^2 |(au_x)_x|^2 \\
&\le C\Big(\sup_{t\in[0,T]} \|\hat{\rho}\bar{u}(t,\cdot)\|_{L^2(0,1)}^2 \|(au_x)_x\|_{\rho_*^2}^2 \\
&\quad + \sup_{t\in[0,T]} \|\rho_*\sqrt{a}\bar{u}_x(t,\cdot)\|_{L^2(0,1)}^2 \|(au_x)_x\|_{\rho_*^2}^2 \Big) \\
&\quad \le C\|(u,h)\|_E^2 \|(\bar{u},\bar{h})\|_E^2.
\end{aligned}
$$

Analogously, there exists $u_\lambda = u_\lambda(t,x) \in \mathbb{R}$ such that

$$
\begin{aligned}
B_2^2 &= \int_0^T \int_0^1 \rho_0^2 |\ell(a(u+\lambda\bar{u})) + \ell(au)|^2 |(au_x)_x|^2 \\
&= \int_0^T \int_0^1 \rho_0^2 |\ell'(u_\lambda)|^2 |a\lambda\bar{u}|^2 |(a\bar{u}_x)_x|^2 \\
&\le C\lambda^2 \int_0^T \int_0^1 \rho_0^2 |a\bar{u}|^2 |(a\bar{u}_x)_x|^2 \\
&\le C\lambda^2 \|(\bar{u},\bar{h})\|_E^2 \to 0,
\end{aligned}
$$

as $\lambda \to 0$. Thus, the Claim 1 is concluded.

Claim 2: The Gateaux derivative $\mathcal{H}_1' : E \to \mathcal{L}(E; L^2(Q;\rho_0^2))$ is continuous. Take $(u,h) \in E$ and let $((u^n,h^n))_{n=1}^\infty$ be a sequence such that $\|(u^n,h^n) - (u,h)\|_E \to 0$.

We will prove that $\|\mathcal{H}_1'(u^n,h^n)-\mathcal{H}_1'(u,h)\|_{\mathcal{L}(E;L^2(Q;\rho_0^2))}\to 0$. In fact, we consider $(\bar u,\bar h)$ on the unit sphere of E. Since

$$\mathcal{H}_1'(u,h)(\bar u,\bar h)=\bar u_t-\ell'(au)a\bar u(au_x)_x+\ell(au)(a\bar u_x)_x+f_3\bar u-\bar h\chi_\omega$$

and

$$\mathcal{H}_1'(u^n,h^n)(\bar u,\bar h)=\bar u_t-\ell'(au^n)a\bar u(au_x^n)_x+\ell(au^n)(a\bar u_x)_x+D_3f(t,x,u^n)\bar u-\bar h\chi_\omega,$$

we obtain

$$\begin{aligned}
&\|(H_1'(u^n,h^n)-H_1'(u,h))(\bar u,\bar h)\|^2_{\rho_0^2}\\
&\le C\int_0^T\int_0^1\rho_0^2|\ell'(au^n)|^2|a\bar u|^2|[a(u-u^n)_x]_x|^2\\
&\quad+C\int_0^T\int_0^1\rho_0^2|\ell'(au^n)-\ell'(au)|^2|a\bar u|^2|(au_x)_x|^2\\
&\quad+C\int_0^T\int_0^1\rho_0^2|\ell(au^n)-\ell(au)|^2|(a\bar u_x)_x|^2\\
&\quad+C\int_0^T\int_0^1\rho_0^2|\bar u|^2|D_3f(t,x,u^n)-D_3f(t,x,u)|^2\\
&=: C(J_1+J_2+J_3+J_4).
\end{aligned}$$

Once again, arguing as in (3.2), (3.4) and (3.5), we obtain

$$\begin{aligned}
J_1&\le\int_0^T\int_0^1\rho_0^2|a\bar u|^2|(a(u-u^n)_x)_x|^2\\
&\le C\Big(\sup_{t\in[0,T]}\|\hat\rho\bar u(t,\cdot)\|^2_{L^2(0,1)}\|[a(u-u^n)_x]_x\|^2_{\rho_*^2}\\
&\quad+\sup_{t\in[0,T]}\|\rho_*\sqrt{a}\bar u_x(t,\cdot)\|^2_{L^2(0,1)}\|[a(u-u^n)_x]_x\|^2_{\rho_*^2}\Big)\\
&\le C\|(\bar u,\bar h)\|_E^2\,\|(u^n,h^n)-(u,h)\|_E^2.
\end{aligned}$$

Next, applying relation (3.3), we have

$$\begin{aligned}
J_2&\le\int_0^T(e^{-2s\tau\hat\eta_r}\tau^{-5/3}\rho_*^{-4})(\rho_*^2\|a\bar u\|^2_\infty)\int_0^1\eta^{-5/3}\rho_*^2|\ell'(au^n)-\ell'(au)|^2|(au_x)_x|^2\\
&\le C\Big[\int_0^1\rho_*^2(\|\bar u\|^2_{L^2(0,1)}+\|\sqrt{a}\bar u_x\|^2_{L^2(0,1)})\int_0^1\rho_*^2|\ell'(au^n)-\ell'(au)|^2|(au_x)_x|^2\Big]\\
&\le C\Big[\sup_{t\in[0,T]}\|\hat\rho\bar u(t,\cdot)\|^2_{L^2(0,1)}+\sup_{t\in[0,T]}\|\rho_*\sqrt{a}\bar u_x(t,\cdot)\|^2_{L^2(0,1)}\Big]\\
&\quad\times\int_0^T\int_0^1\rho_*^2|\ell'(au^n)-\ell'(au)|^2|(au_x)_x|^2\\
&\le C\|(\bar u,\bar h)\|_E^2\int_0^T\int_0^1\rho_*^2|\ell'(au^n)-\ell'(au)|^2|(au_x)_x|^2\to 0,
\end{aligned}$$

as $n\to+\infty$, where the convergence is a consequence of Lebesgue's theorem. In a very similar way,

$$J_3\le\int_0^T(e^{-2s\tau\hat\eta_r}\tau^{-5/3}\rho_*^{-4})(\rho_*^2\|a(u^n-u)\|^2_\infty)\int_0^1\eta^{-5/3}\rho_*^2|(a\bar u_x)_x|^2$$

$$
\begin{aligned}
&\le C\Big[\sup_{t\in[0,T]}\|\hat\rho(u^n-u)(t,\cdot)\|^2_{L^2(0,1)}+\sup_{t\in[0,T]}\|\rho_*\sqrt{a}(u^n-u)_x(t,\cdot)\|^2_{L^2(0,1)}\Big]\\
&\quad\times\int_0^T\int_0^1\rho_*^2|(a\bar u_x)_x|^2\\
&\le C\|(u^n,h^n)-(u,h)\|^2_E\|(\bar u,\bar h)\|^2_E.
\end{aligned}
$$

At last, applying Hypothesis 1.3 and (3.1), we obtain

$$
\begin{aligned}
J_4&=\Big(\int_0^T\int_0^1\rho_0^2|\bar u|^2|D_3f(t,x,u^n)-D_3f(t,x,u)|^2\Big)\\
&\le\sup_{(t,x)\in Q}|D_3f(t,x,u^n)-D_3f(t,x,u)|^2\int_0^T\int_0^1\rho_0^2|\bar u|^2\\
&\le\|(u^n,h^n)-(u,h)\|^2_E\|(\bar u,\bar h)\|^2_E,
\end{aligned}
$$

where we have also used the continuous embedding $C([0,T];H^1_a)\hookrightarrow C(Q)$. Therefore,

$$
\mathcal{H}_1'(u^n,h^n)\to\mathcal{H}_1'(u,h)
$$

in $\mathcal{L}(E;L^2(Q;\rho_0^2))$, which means that $\mathcal{H}_1':E\to\mathcal{L}(E;L^2(Q;\rho_0^2))$ is a continuous mapping, as stated in Claim 2. This completes the proof. □

3.3. **Proof of Proposition 3.1.**

Proof. Given $T>0$ and $(u_0,g)\in H^1_a\times L^2(Q;\rho_0^2)$, let us consider the problem

$$
\begin{gathered}
u_t-(a(x)u_x)_x+c(t,x)u=h+g,\quad (t,x)\text{ in }Q,\\
u(t,1)=0,\quad t\in(0,T),\\
\begin{cases}u(t,0)=0, & \text{(Weak)},\ t\in(0,T)\\ \text{or}\\ (au_x)(t,0)=0, & \text{(Strong)},\ t\in(0,T)\end{cases}\\
u(0,x)=u_0(x),\quad x\text{in }(0,1),
\end{gathered}
\tag{3.6}
$$

where $h\in L^2(Q)$. Observe that (3.6) is similar to (2.2), where we are replacing $h\chi_\omega$, with support in Q_ω, just by h. Our aim is to define, for each $n\in\mathbb{N}^*$, a functional $J_n:\big[\mathbf{L^2(Q)}\big]^{\mathbf{2}}\to\mathbb{R}$, minimizing each one of them subject to the natural constraint determined by (2.2). It will allow us to obtain a sequence $((u_n,h_n))_{n=1}^\infty$ of solutions to (2.2) converging, in some sense, to

$$
(u,h)\in L^2(Q;\rho_0^2)\times L^2(Q_\omega;\rho_*^2),
$$

which is also a solution to (2.2).

To do so, for each $n\in\mathbb{N}^*$, let us define

$$
A_n(t,x)=\frac{A(T-t)^4}{(T-t)^4+\frac1n},\quad bar A_n(t)=\frac{\bar A(T-t)^4}{(T-t)^4+\frac1n},
$$

where $(t,x)\in[0,T]\times[0,1]$. We also consider

$$
\rho_n=e^{-sA_n},\quad \bar\rho_n=e^{-s\bar A_n},\quad \rho_{0,n}=\rho_n\zeta^{-5/6},\quad \rho_{*,n}=\bar\rho_n\zeta^{*-17/6}m_n,
$$

where

$$
m_n(x)=\begin{cases}1, & x\in\omega,\\ n, & x\notin\omega.\end{cases}
$$

These weight functions are built in such a way that

- $\rho_{0,n}$ and $\rho_{*,n}$ are bounded from below by a positive constant only depending on T;
- $\rho_{0,n}$ and $\rho_{*,n}$ are bounded from above by another positive constant depending on n and T.

For each $n \in \mathbb{N}^*$, we set the functional $J_n : \left[\mathbf{L^2(Q)}\right]^{\mathbf{2}} \to \mathbb{R}$, given by

$$J_n(u,h) = \frac{1}{2}\int_0^T \int_0^1 \rho_{0,n}^2 |u|^2 + \frac{1}{2}\int_0^T \int_0^1 \rho_{*,n}^2 |h|^2,$$

for each $(u,h) \in [L^2(Q)]^2$. Since each J_n is lower semi-continuous, strictly convex and coercive (see [21]), we can apply [23, Proposition 1.2] to obtain a unique (u_n, h_n) satisfying

$$J(u_n,h_n) = \min\{J(u,h); (u,h) \in \mathcal{C}\}$$

where $\mathcal{C} = \{(u,h) \in \left[\mathbf{L^2(Q)}\right]^{\mathbf{2}}; (u,h) \text{ solves (2.2)}\}$. Consequently, by Lagrange's Principle, for each $n \in \mathbb{N}^*$, there exists a function p_n solving the system

$$\begin{gathered}
-p_{n\,t} - (a p_{n\,x})_x + c(t,x) p_n = -\rho_{0,n}^2 u_n, \quad (t,x) \in Q, \\
p_n(t,1) = 0, \quad t \in (0,T), \\
\begin{cases} p_n(t,0) = 0, & \text{(Weak)},\ t \in (0,T) \\ \text{or} \\ (a p_{n\,x})(t,0) = 0, & \text{(Strong)},\ t \in (0,T) \end{cases} \\
p_n(T,x) = 0, \quad x \in (0,1), \\
p_n = \rho_{*,n}^2 h_n, \quad (t,x) \in Q.
\end{gathered} \tag{3.7}$$

By standard arguments, (3.7) can help us to prove that $J_n(u_n,h_n) \le C\sqrt{J_n(u_n,h_n)}$ for all $n \in \mathbb{N}^*$, i.e., $(J_n(u_n,h_n))_{n=1}^{\infty}$ is a numerical bounded sequence.

Since $\rho_{0,n}^2 \ge C_T$ and $\rho_{*,n}^2 \ge C_T m_n$, we deduce that

$$\|u_n\|_{L^2}^2 + \int_0^T \int_\omega |h_n|^2 + n\int_0^T \int_{[0,1]\setminus\omega} |h_n|^2 \le C J_n(u_n,h_n) \le C,$$

whence there exists $(u,h) \in L^2(Q) \times L^2(Q_\omega)$, such that

$$u_n \rightharpoonup u, \text{ in } L^2(Q) \quad \text{and} \quad h_n \rightharpoonup h\chi_\omega \text{ in } L^2(Q),$$

up to subsequences. From this, we have

$$\rho_{0,n} u_n \rightharpoonup \rho_0 u \quad \text{and} \quad \rho_{*,n} h_n \rightharpoonup \rho_* h \chi_\omega \quad \text{in } L^2(Q). \tag{3.8}$$

Consequently, $u \in L^2(Q;\rho_0^2)$ and $h \in L^2(Q_\omega;\rho_*^2)$. Recalling that (u_n,h_n) is a solution of (2.2), for each $n \in \mathbb{N}^*$, a passing to the limit argument implies that (u,h) also solves (2.2).

Thinking about a better presentation, the estimates mentioned in (3.1) will be established in two subsequent lemmas. □

Lemma 3.5. *Under the assumptions of Proposition 3.1, we have that* $\hat\rho u \in L^\infty(0,T;L^2(0,1))$, $\sqrt{a} u_x \in L^2(Q;\hat\rho^2)$, *and there exists* $C > 0$ *such that*

$$\sup_{t\in[0,T]} \|\hat\rho u(t,\cdot)\|_{L^2(0,1)}^2 + \|\sqrt{a} u_x\|_{\hat\rho^2}^2 \le C(\|u\|_{\rho_0^2}^2 + \|h\chi_\omega\|_{\rho_*^2}^2 + \|g\|_{\rho_0^2}^2 + \|u_0\|_{H_a^1}^2).$$

Proof. Multiplying the PDE in (2.2) by $\hat\rho^2 u$, integrating in $[0,1]$ and using the two relations

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \hat\rho^2 u^2 = \int_0^1 \hat\rho^2 u_t u + \int_0^1 \hat\rho\hat\rho_t u^2$$

and

$$\int_0^1 \hat\rho^2 (au_x)_x u = -2\int_0^1 \hat\rho\hat\rho_x a u u_x - \int_0^1 \hat\rho^2 a u_x^2,$$

we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \hat\rho^2 u^2 + \int_0^1 \hat\rho^2 a u_x^2 \\
&= -\int_0^1 \hat\rho^2 c u^2 + \int_0^1 \hat\rho^2 u h\chi_\omega + \int_0^1 \hat\rho^2 g u + \int_0^1 \hat\rho\hat\rho_t u^2 - 2\int_0^1 \hat\rho\hat\rho_x a u u_x \\
&=: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{3.9}$$

Above, we have also used $u(t,0) = u(t,1) \equiv 0$ for (WDP), and $u(t,1) = au_x(t,0) \equiv 0$ for (SDP). Now, since $\rho_* \le C\hat\rho \le C\rho_0$ and $\rho_0\rho_* \ge \hat\rho^2$, we obtain

$$\begin{gathered}
I_1 \le C\int_0^1 \rho_0^2 |u|^2, \\
I_2 \le C\Big(\frac{1}{2}\int_0^1 \rho_*^2 |h\chi_\omega|^2 + \frac{1}{2}\int_0^1 \rho_0^2|u|^2\Big), \\
I_3 \le C\Big(\frac{1}{2}\int_0^1 \rho_0^2 |g|^2 + \frac{1}{2}\int_0^1 \rho_0^2|u|^2\Big).
\end{gathered}$$

Let us estimate I_4. Firstly, rewriting A and $\bar A$ as $A(t,x) = \zeta(t,x)\tilde\eta(x)$, where $\tilde\eta = \eta_r/\eta$, and $\bar A(t,x) = \zeta(t,x)\frac{\bar\eta_r}{\eta}$, we have

$$\begin{aligned}
|\hat\rho_t| &= \Big| - s(\frac{\eta_r + \bar\eta_r}{2\eta})\zeta_t e^{-s(\frac{A+\bar A}{2})}(\zeta^*)^{-11/6} - \frac{11}{6} e^{-s(\frac{A+\bar A}{2})}(\zeta^*)^{-11/6}\zeta_t^*\Big| \\
&\le e^{-sA}\big[s\bar\eta_r(\zeta^*)^{-11/6}|\zeta_t| + \frac{11}{6}(\zeta^*)^{-17/6}|\zeta_t^*|\big].
\end{aligned}$$

Secondly, we obtain

$$\begin{aligned}
|\hat\rho\hat\rho_t| &\le e^{-2sA}\big[s\bar\eta_r(\zeta^*)^{-11/6}|\zeta_t| + \frac{11}{6}(\zeta^*)^{-17/6}|\zeta_t^*|\big] \\
&\le Ce^{-2sA}(\zeta^{-2}|\zeta_t| + \zeta^{-3}|\zeta_t|)\zeta^{-5/3} \\
&\le C\rho_0^2,
\end{aligned}$$

for all $t \in [0,T]$, following that $I_4 \le C\int_0^1 \rho_0^2|u|^2$. Next, using

$$\begin{aligned}
|\hat\rho_x| &= \big| - s(\frac{A_x + \bar A_x}{2}) e^{-s\left(\frac{A+\bar A}{2}\right)}(\zeta^*)^{-11/6}\big| \\
&\le Ce^{-sA}\zeta e^{-s(\frac{A+\bar A}{2})}(\zeta^*)^{-11/6} \\
&\le e^{-sA}(\zeta)^{-5/6} \\
&= \rho_0,
\end{aligned}$$

we obtain

$$I_5 \le \frac{1}{2}\int_0^1 \hat\rho^2 a u_x^2 + 2\int_0^1 \hat\rho_x^2 a u^2 \le \frac{1}{2}\int_0^1 \hat\rho^2 a u_x^2 + 2\int_0^1 \rho_0^2 u^2.$$

Hence, (3.9) gives us

$$\frac{d}{dt}\int_0^1 \hat\rho^2|u|^2 + \int_0^1 \hat\rho^2 a|u_x|^2 \le C\Big(\int_0^1 \rho_0^2|u|^2 + \int_0^1 \rho_*^2|h\chi_\omega|^2 + \int_0^1 \rho_0^2|g|^2\Big).$$

Integrating in time, we reach the desired estimate. □

Lemma 3.6. *Under the assumptions of Proposition 3.1 we have*

$$\rho_*\sqrt{a}u_x \in L^\infty(0,T;L^2(0,1)); u_t, (au_x)_x \in L^2(Q;\rho_*^2),$$

and there exists $C>0$ such that

$$\begin{aligned}&\sup_{t\in[0,T]}\|\rho_*\sqrt{a}u_x(t,\cdot)\|^2_{L^2(0,1)} + \|u_t\|^2_{\rho_*^2} + \|(au_x)_x\|^2_{\rho_*^2}\\ &\le C(\|u\|^2_{\rho_0^2} + \|h\chi_\omega\|^2_{\rho_*^2} + \|g\|^2_{\rho_0^2} + \|u_0\|^2_{H_a^1}).\end{aligned}$$

Proof. Firstly, let us estimate the first and the second terms on the left side of the desired inequality. Multiplying the PDE in (2.2) by $\rho_*^2 u_t$ and integrating in $[0,1]$, we have

$$\begin{aligned}\int_0^1 \rho_*^2 u_t^2 &= \int_0^1 \rho_*^2 u_t h\chi_\omega + \int_0^1 \rho_*^2 g u_t - \int_0^1 \rho_*^2 c(t,x)uu_t + \int_0^1 \rho_*^2 (au_x)_x u_t\\ &= I_1 + I_2 - I_3 + I_4.\end{aligned} \tag{3.10}$$

Using Young's inequality with ε and $\rho_* \le C\hat\rho \le C\rho_0 \le C\rho$, we obtain

$$I_1 \le \int_0^1 \rho_*^2|h\chi_\omega||u_t| \le \varepsilon\int_0^1 \rho_*^2|u_t|^2 + \frac{1}{4\varepsilon}\int_0^1 \rho_*^2|h\chi_\omega|^2,$$

$$I_2 \le \int_0^1 \rho_*^2|gu_t| \le \varepsilon\int_0^1 \rho_*^2|u_t|^2 + \frac{1}{4\varepsilon}\int_0^1 \rho_*^2|g|^2 \le \varepsilon\int_0^1 \rho_*^2|u_t|^2 + \frac{C}{4\varepsilon}\int_0^1 \rho_0^2|g|^2,$$

$$\begin{aligned}-I_3 &\le \int_0^1 |c(t,x)|\rho_*^2|uu_t|\\ &\le \varepsilon\int_0^1 \rho_*^2|u_t|^2 + \frac{\|c\|_\infty}{4\varepsilon}\int_0^1 \rho_*^2|u|^2\\ &\le \varepsilon\int_0^1 \rho_*^2|u_t|^2 + \frac{C}{4\varepsilon}\int_0^1 \rho_0^2|u|^2\,.\end{aligned}$$

Since $u_t(t,0) = u_t(t,1) \equiv 0$ for the (WDP) and $au_x(t,0) = u_t(t,1) \equiv 0$ for the (SDP), we integrate by parts to obtain

$$\begin{aligned}I_4 &= \rho_*^2 a u_x u_t\big|_{x=0}^{x=1} - \int_0^1 (\rho_*^2 u_{tx} a u_x\\ &= -\frac{1}{2}\frac{d}{dt}\int_0^1 \rho_*^2 a u_x^2 + \frac{1}{2}\int_0^1 (\rho_*^2)_t a u_x^2\\ &= -\frac{1}{2}\frac{d}{dt}\int_0^1 \rho_*^2 a u_x^2 + \frac{1}{2}I_{41}.\end{aligned} \tag{3.11}$$

Hence

$$\int_0^1 \rho_*^2|u_t|^2 + \frac{1}{2}\frac{d}{dt}\int_0^1 \rho_*^2 a|u_x|^2 = I_1 + I_2 - I_3 + \frac{1}{2}I_{41}. \tag{3.12}$$

At this point, we observe that

$$(\rho_*)_t = -s\tau_t\bar\eta_r e^{-s\bar A}(\zeta^*)^{-17/6} - \frac{17}{6}e^{-s\bar A}(\zeta^*)^{-23/6}\tau_t\eta^*$$

and, consequently,

$$\begin{aligned}|\rho_*(\rho_*)_t| &\le Ce^{-2s\bar{A}}[|\tau_t\eta^*|(\zeta^*)^{-17/3} + |\tau_t\eta^*|(\zeta^*)^{-20/3}] \\ &= Ce^{-2s\bar{A}}(\zeta^*)^{-11/3}[(\zeta^*)^{-2} + (\zeta^*)^{-3}]|\zeta^*_t)| \\ &\le C\hat{\rho}^2.\end{aligned}$$

So that

$$I_{41} \le C\int_0^1 \hat{\rho}^2 a u_x^2\,.$$

As a result, taking a sufficiently small $\varepsilon > 0$, we obtain

$$\begin{aligned}&\int_0^1 \rho_*^2 u_t^2 \ + \frac{1}{2}\frac{d}{dt}\int_0^1 \rho_*^2 a u_x^2 \\ &\le C\Big(\int_0^1 \rho_*^2|h\chi_\omega|^2 + \int_0^1 \rho_0^2 g^2 + \int_0^1 \rho_0^2 u^2 + \int_0^1 \hat{\rho}^2 a u_x^2\Big),\end{aligned}$$

which implies

$$\sup_{t\in[0,T]} \|\rho_*\sqrt{a}u_x(t,\cdot)\|^2_{L^2(0,1)} + \|u_t\|^2_{\rho_*^2} \le C(\|u\|^2_{\rho_0^2} + \|h\chi_\omega\|^2_{\rho_*^2} + \|g\|^2_{\rho_0^2} + \|u_0\|^2_{H^1_a}).$$

To estimate $\|(au_x)_x\|^2_{\rho_*^2}$, we proceed analogously, multiplying the PDE in (2.2) by $-\rho_{17}^2(au_x)_x$ and integrating in $[0,1]$. The details can be seen in [20, Lemma 4.3]. □

4. Main result and further comments

Proof of Theorem 1.6. In Section 3, we have proved that $\mathcal{H}: E \to F$ is a continuously differentiable mapping, whose derivative $\mathcal{H}'(0,0) \in \mathcal{L}(E;F)$ is onto (Lemma 3.2, and Propositions 3.3 and 3.4). As a result, Theorem 2.1 can be applied in order to obtain a sufficiently small $\varepsilon > 0$ and a right inverse mapping $\tilde{\mathcal{H}} : B_\varepsilon(\mathbf{0}) \subset F \to E$ of $\mathcal{H}$. Hence, taking $u_0 \in H^1_a$ satisfying $\|u_0\|_{H^1_a} < \varepsilon$, we can see that

$$(u,h) := \tilde{\mathcal{H}}(0,u_0)$$

solves

$$\begin{gathered} u_t - \ell(au)(a(x)u_x)_x + f(t,x,u) = h\chi_\omega, \quad (t,x)\in Q, \\ u(t,1) = 0, \quad \text{in } (0,T), \\ \begin{cases} u(t,0) = 0, & \text{(Weak)},\ t\in(0,T) \\ \text{or} \\ (au_x)(t,0) = 0, & \text{(Strong)},\ t\in(0,T) \end{cases} \\ u(0,x) = u_0, \quad x\in(0,1), \\ u(T,x) = 0, \quad x\in(0,1), \end{gathered} \tag{4.1}$$

where the last condition comes from Remark 2.4. It completes the proof. □

In the context of degenerate equations, there are many important questions which have not been dealt yet, or for which much more investigation should be performed. Among them, we would like to emphasize the following ones: the controllability of linear problems in higher-dimensional spatial domains, using the method of moments; the boundary controllability obtained as the limit of internal controllability, in nonlinear cases

Specifically talking about some numerical perspective regarding this current paper, we include some comments pointing out possible future works. In order to solve the proposed controllability problem numerically, some different approaches can be combined to perform an approximate and reliable analysis of the system. A practical and well-established strategy in the literature involves the use of the finite element method (FEM). This methodology makes possible a complete discretization of the problem into a finite-dimensional space, allowing numerical approximations through well-defined iterative processes. So that, a natural future study could be the comparison between two approaches: the primal method and the dual one. The primal method is more straightforward, focusing directly on the finite element formulation by discretizing the spatial and temporal domains, updating the solution iteratively in an approximate solution space V_h, with $\dim(V_h) < \infty$. On the other hand, the dual method introduces some pre-programming complexity, since it is incorporated dual variables into the weak formulation of the problem. This process reformulates the original problem as a constrained optimization problem within a variational framework, where iterative algorithms are employed to solve both primal and dual variables, simultaneously. The advantage and disadvantage of each on of this methods are properly discussed in [27]. At this point, we should say that some initial numerical insights into the class of problems proposed here can be found in [19], where iterative algorithms, adapted for nonlinear parabolic problems, are presented. Besides, in [19], an effective approach for numerical iterations is considered, by adjusting quasi-Newton method for null controllability problems. The whole numerical analysis of null-controllability problems is completely associated with well-chosen weight functions, such those defined in Section 2. We think that numerical simulations for the null-controllability of degenerate quasilinear equations could be based on [17, 18, 19, 28, 30] and [31].

5. Appendix: Essential boundedness of au

This appendix shows that, for each $u \in H^1_a$, we have $au \in L^\infty(0,1)$. This fact is essential in to prove that the mapping $\mathcal{H} : E \to F$, set in (2.1), is well defined and continuously differentiable. For the whole discussion, let us consider $K \in [0,2)$ mentioned in assumption 1.1 and $\theta \in \mathbb{R}$ given in (1.3).

Proposition 5.1. *Given $u \in H^1_a$, we have $au \in L^\infty(0,1)$ and $C_a > 0$, only depending on the function a, such that*

$$\|au\|_{L^\infty(0,1)} \le C_a \|u\|_{H^1_a},$$

provided that one of the following conditions holds:

(a) $K \neq 1$;

(b) $K = 1$ *and* $\theta \ge 1/2$.

The proof of this proposition will be a consequence of the four next lemmas.

Lemma 5.2. *The continuous embedding $H^1_a \hookrightarrow L^\infty(0,1)$ holds for the (WDC). In particular, $au \in L^\infty(0,1)$, for any $u \in H^1_a$.*

Proof. In fact, given $u = u(x) \in H^1_a$, we can take

$$|u(x)| \le \Big| \int_x^1 u_x \Big| \le \Big(\int_x^1 \frac{1}{a} \Big)^{1/2} \Big(\int_x^1 a u_x^2 \Big)^{1/2} \le \|\frac{1}{a}\|^2_{L^1} \|u\|_{H^1_a},$$

for each $x \in (0,1]$. Hence, there exists $C_a > 0$, only depending on the function a, such that $\|u\|_{L^\infty} \le C_a \|u\|_{H^1_a}$. □

Lemma 5.3. *If $a \in W^{1,\infty}(0,1)$, then $au \in L^\infty(0,1)$, for each $u \in H^1_a$.*

Proof. Indeed, given $y \in (0,1]$, since $a \in C^1([y,1])$ and u is absolutely continuous in $[y,1]$, we have

$$\begin{aligned}
|au(y)| &\le \int_y^1 |a'u|\,dx + \int_y^1 a|u_x|\,dx \\
&\le \int_0^1 |a'u|\,dx + \int_0^1 a|u_x|\,dx \\
&\le \|a'\|L^\infty(0,1)\|u\|_{L^2(0,1)} + \int_0^1 \sqrt{a}\sqrt{a}|u_x|\,dx \\
&\le \|a'\|_{L^\infty(0,1)}\|u\|_{L^2(0,1)} + \|a\|_{L^\infty(0,1)}\|\sqrt{a}u_x\|_{L^2(0,1)} \\
&\le \|a\|_{W^{1\infty}(0,1)}\|u\|_{H^1_a}.
\end{aligned}$$

Since, $y \in (0,1]$ is arbitrary, the desired result follows. □

Lemma 5.4. *If $K > 1$, then $a \in W^{1,\infty}(0,1)$. In particular, $au \in L^\infty(0,1)$.*

Proof. We only need to prove that $a' \in L^\infty(0,1)$. From (1.3), there exists $\varepsilon > 0$ such that $\theta a \le xa'$, $\forall x \in (0,\varepsilon]$. In particular $a' > 0$ and, since $\theta > 1$, the mapping $x \mapsto \frac{a}{x}$ is increasing in $(0,\varepsilon]$. So that, using (1.2), we have

$$0 \le a'(x) \le \frac{Ka(x)}{x} \le \frac{Ka(\varepsilon)}{\varepsilon}, \quad \forall x \in (0,\varepsilon].$$

On the other hand, $a' \in C^0([\varepsilon,1])$, following that $a' \in L^\infty(0,1)$ and

$$\|a'\|_{L^\infty(0,1)} \le \max\Big\{\frac{Ka(\varepsilon)}{\varepsilon}, \max_{x\in[\varepsilon,1]} |a'(x)|\Big\}.$$

□

Lemma 5.5. *If $K = 1$ and $\theta \ge 1/2$, then $au \in L^\infty(0,1)$, for any $u \in H^1_a$.*

Proof. From (1.3), there exists $\varepsilon > 0$ such that $\theta a \le xa'$ for all $x \in (0,\varepsilon]$. This implies that $a' > 0$ and

$$x \mapsto \frac{a}{x^\theta} \text{ is nondecreasing in } (0,\varepsilon]. \tag{5.1}$$

Since $a \in C^0([0,1])$ and $u \in H^1_a \hookrightarrow H^1(\varepsilon,1) \hookrightarrow C^0([\varepsilon,1])$, we have that $au \in L^\infty(\varepsilon,1)$ and

$$|au(y)| \le \|a\|_{L^\infty(0,1)}\|u\|_{H^1_a}, \quad \text{for all } y \in [\varepsilon,1].$$

We just need to prove $au \in L^\infty(0,\varepsilon)$. Indeed, given $y \in (0,\varepsilon]$, we have

$$a^2u^2(y) = a^2u^2(\varepsilon) - \int_y^\varepsilon (a^2u^2(x))_x\,dx,$$

whence,

$$|au(y)|^2 \le \|a\|^2_{L^\infty(0,1)}\|u\|^2_{H^1_a} + 2\int_y^\varepsilon a^2|u||u_x|\,dx + 2\int_y^\varepsilon aa'|u|^2\,dx.$$

Let us estimate each one of the two last integrals. The first of them is easier, since we just use Hölder inequality to obtain

$$\int_y^\varepsilon a^2|u||u_x|\,dx \le \|a\|^{3/2}_{L^\infty(0,1)}\|u\|_{L^2(0,1)}\|\sqrt{a}u_x\|_{L^2(0,1)} \le \|a\|^{3/2}_{L^\infty(0,1)}\|u\|^2_{H^1_a}.$$

To estimate the second integral, let us prove that aa' is bounded in $(0,\varepsilon]$. In fact, using (1.2) and (5.1), we have

$$aa' \le \frac{a^2}{x} = (\frac{a}{x^\theta})^2 x^{2\theta-1} \le \frac{a(\varepsilon)}{\varepsilon}, \quad \forall x \in (0,\varepsilon],$$

since $\theta \ge 1/2$. Hence,

$$\int_y^\varepsilon aa'|u|^2\,dx \le \frac{a(\varepsilon)}{\varepsilon}\int_0^1 |u|^2\,dx = \frac{a(\varepsilon)}{\varepsilon}\|u\|^2_{L^2(0,1)}.$$

Therefore, $au \in L^\infty(0,1)$ and

$$\|au\|_{L^\infty(0,1)} \le C\|u\|_{H^1_a},$$

where

$$C = \Big(\max\big\{\|a\|^2_{L^\infty(0,1)}, 2\|a\|^{3/2}_{L^\infty(0,1)}, \frac{2a(\varepsilon)}{\varepsilon}\big\}\Big)^{1/2} > 0.$$

□

Remark 5.6. The prototype function $\tilde{a}(x) = x^\alpha$, with $\alpha \in [1,2)$, belongs to $W^{1,\infty}(0,1)$, therefore $\tilde{a}u \in L^\infty(0,1)$, for any $u \in H^1_a$. Nevertheless, for any $p \in (0,1)$, consider the function $a(x) = x^p + x$ and note that

- $a' = px^{p-1} + 1 \Rightarrow a \notin W^{1,\infty}(0,1)$;
- $xa' = px^p + x \le x^p + x = a$, that is, a satisfies assumption 1.1, with $K = 1$;
- $aa' = px^{2p-1} + (p+1)x^p + x$ is bounded if, and only if, $p \ge 1/2$.

Therefore, the proof given in Lemma 5.5 does not work for $p < 1/2$.

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