Electronic Journal of Differential Equations, Vol. 2025 (2025), No. 16, pp. 1–17.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2025.16

L^2 SOLUTIONS FOR CUBIC NLS EQUATION WITH HIGHER ORDER FRACTIONAL ELLIPTIC/HYPERBOLIC OPERATORS ON $\mathbb{R}\times\mathbb{T}$ AND $\mathbb{R}^2$

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Abstract. In this work, we consider the Cauchy problem for the cubic Schrödinger equation posed on cylinder $\mathbb{R}\times\mathbb{T}$ with fractional derivatives $(-\partial_y^2)^\alpha$, $\alpha>0$, in the periodic direction. The spatial operator includes elliptic and hyperbolic regimes. We prove L^2 global well-posedness results when $\alpha\ge 1$ by proving a L^4-L^2 Strichartz inequality for the linear equation, following the ideas in [19], where it was considered the elliptical case with $\alpha=1$. Further, these results remain valid on the Euclidean environment $\mathbb{R}^2$, so well-posedness in L^2 are also achieved in this case. Our proof in the elliptic (hyperbolic) case does not work in the small directional dispersion case $0<\alpha<1$ $(0<\alpha\le 1)$, respectively.

1. Introduction

We consider the initial value problem (IVP) associated with the cubic nonlinear Schrödinger equation (3-NLS) on cylinder $\mathbb{R}\times\mathbb{T}$ with higher fractional derivatives in the periodic direction. More precisely, we will study the IVP

$$\begin{gathered} i\partial_t u+\mathcal{L}_\alpha^\pm u=\varepsilon|u|^2u,\quad (x,y)\in\mathbb{R}\times\mathbb{T},\ t\in\mathbb{R},\\ u(0;x,y)=\phi(x,y),\quad (x,y)\in\mathbb{R}\times\mathbb{T}, \end{gathered}\tag{1.1}$$

where $\varepsilon\in\mathbb{R}^*$, $u=u(t;x,y)$ is a complex-valued function on $\mathbb{R}\times\mathbb{R}\times\mathbb{T}$, with $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$, and $\mathcal{L}_\alpha^\pm$ denotes the pseudo-differential operator acting in the space variables, given by

$$\mathcal{L}_\alpha^\pm:=\partial_x^2\mp(-\partial_y^2)^\alpha,\quad \text{with } \alpha>0,\tag{1.2}$$

defined in Fourier variables by

$$\widehat{\mathcal{L}_\alpha^\pm f}(\xi,n)=-(\xi^2\pm|n|^{2\alpha})\widehat{f}^{x,y}(\xi,n),\tag{1.3}$$

where $\widehat{f}^{x,y}$ is the Fourier transform of f on the cylinder $\mathbb{R}\times\mathbb{T}$, that is,

$$\mathcal{F}[f](\xi,n)=\widehat{f}^{x,y}(\xi,n):=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^{2\pi}f(x,y)e^{-i(x\xi+ny)}\,dx\,dy.\tag{1.4}$$

2020 *Mathematics Subject Classification.* 35Q55, 35Q35, 35Q60.
Key words and phrases. Elliptic/hyperbolic cubic nonlinear Schrödinger equation; Cauchy problem; well-poseddness.

Submitted August 7, 2024. Published February 21, 2025.

So, the linear propagator $U_\alpha^\pm$ of (1.1) is given by

$$U_\alpha^\pm(t)\phi = e^{it\,\mathcal{L}_\alpha^\pm}\phi = \mathcal{F}^{-1}\big(e^{-it(\xi^2 \pm |n|^{2\alpha})}\mathcal{F}[\phi]\big). \tag{1.5}$$

The cases $\varepsilon < 0$ and $\varepsilon > 0$ are known as the *focusing* and *defocusing* regimes, respectively. Also, note that in the case $\alpha = 1$,

$$\mathcal{L}_1^\pm = \begin{cases} \partial_x^2 + \partial_y^2 =: \Delta & \text{(Laplacian operator) in the case (+)}, \\ \partial_x^2 - \partial_y^2 =: \Box & \text{(wave operator) in the case (-)}. \end{cases}$$

When $\alpha = 1$, the elliptic case appears as a model in several physical problems (see for example the references [14, 16, 21]). On the other hand, in the hyperbolic case it describes, for instance, the gravity waves on liquid surface and ion-cyclotron waves in plasma (see [7, 13, 18]). Both cases (elliptic/hyperbolic) of the IVP (1.1) with $\alpha > 0$, posed on the Euclidean domain $\mathbb{R}^2$, are covered in the study carried out in [8] about the existence of analytic solutions. In the case $0 < \alpha < 1$, the elliptic operators $\mathcal{L}_\alpha^+$ (fractional directional Laplacian) in (1.2) appear in the context of parabolic equations (see [3, 4] and references therein) as toy models to describe local diffusion occurring only in the x direction, while non-local diffusion occurs in the y-direction. Moreover, the defocusing elliptic case of the IVP (1.1) with $\alpha = 1/2$ (called Half-wave-Schrödinger equation), defined on the cylinder $\mathbb{R}\times\mathbb{T}$, was considered in [22] to establish a modified scattering theory between small solutions to this model and small solutions to the cubic Szegő equation. In the same work, the author infers the existence of unbounded global solutions of (1.1) in the anisotropic space $L_x^2 H_y^s(\mathbb{R}\times\mathbb{T})$ for every $s > 1/2$. In [1], the authors considered the Half-wave-Schrödinger equation with a more general nonlinearity ($|u|^{p-1}u$), posed on the Euclidean domain $\mathbb{R}^2$, and they showed local well-posedness ($1 < p \le 5$) in anisotropic space $L_x^2 H_y^s(\mathbb{R}^2)$ for $s > 1/2$. Furthermore, they presented some results concerning the existence and orbital stability of solitary waves. Recently, in [9], the elliptic case of IVP (1.1), posed on $\mathbb{R}^2$, is considered when $0 < \alpha < 1$ and with nonlinearity $|u|^{p-2}u,\ 2 < p < 2\frac{(1+\alpha)}{1-\alpha}$. More precisely, conditions for the existence of blow-up solutions are investigated.

For every $\alpha > 0$, the IVP (1.1) (elliptic/hyperbolic) formally enjoys the mass conservation law:

$$M[u](t) = \int_{\mathbb{R}}\int_{\mathbb{T}} |u(t;x,y)|^2\,dx\,dy = M[u](0), \tag{1.6}$$

for all $t \in \mathbb{R}$. Therefore, it would be interesting to establish a well-posedness theory in $L^2(\mathbb{R}\times\mathbb{T})$ similarly to the case $\alpha = 1$ (Laplacian and wave operators). The main goal of this paper is to answer this question.

1.1. Case $\alpha = 1$.

(some previous well-posedness results in isotropic Sobolev spaces) In the case of classical Laplacian/wave operator ($\alpha = 1$), local and global well-posedness for time evolution of the flow associated to the IVP (1.1), with initial data ϕ belonging to the classical Sobolev spaces $H^s(\mathcal{D})$ on plane domains $\mathcal{D} = \mathbb{R}^2,\ \mathbb{T}^2$ or $\mathbb{R}\times\mathbb{T}$, has been considered for many authors. For instance, we have the following results:

• $\mathcal{D} = \mathbb{R}^2$: Well-posedness for in $L^2(\mathbb{R}^2)$ is a consequence of the time decay estimates coming from dispersion and the Strichartz inequalities, which have the same form for the elliptic or hyperbolic linear Schrödinger equations. For the elliptic operator

$\mathcal{L}_1^+$ see [6] and for the hyperbolic case $\mathcal{L}_1^-$ we refer to [10], where the authors deduced Strichartz inequalities for operators $e^{it\mathcal{L}}$, where

$$\mathcal{L} = \sum_{1\le i,j\le n} a_{ij}\partial_{x_i}\partial_{x_j}, \quad a_{ij} \in \mathbb{R},$$

with non-degenerate quadratic form $A = \big(a_{ij}\big)$. In particular, $L^2(\mathbb{R}^2)$ is the Sobolev critical regularity ($s = 0$) for well-posedeness, which is suggested by the scale invariance of the model, that is, if u is a solution of (1.1) with $\alpha = 1$ and initial data $\phi(x, y)$, then

$$u_\lambda(t; , x, y) = u(\lambda^2 t; \lambda x, \lambda y), \ \lambda > 0$$

is the respective solution of (1.1) with initial data $\lambda\phi(\lambda x, \lambda y)$.

• $\mathcal{D} = \mathbb{T}^2$: Global well-posedness in $H^s(\mathbb{T}^2)$, $s > 0$, has been proven in the elliptic case U_1^+ (see the works [2, 5]). In [20] the hyperbolic case U_1^- was considered, getting local well-posedness for $s > \frac{1}{2}$ and ill-posedness when $s < 1/2$.

• $\mathcal{D} = \mathbb{R} \times \mathbb{T}$: The elliptical case $\mathcal{L}_1^+$ was treated in [19]. Specifically, the authors obtained global well-posedness for *small data* in $L^2(\mathbb{R} \times \mathbb{T})$ by proving a $L^4 - L^2$ Strichartz inequality for the group $U_1^+(t) = e^{it\Delta}$ with $(x, y) \in \mathbb{R}\times\mathbb{T}$. More precisely, they showed that

$$\|U_1^+(t)\phi\|_{L^4(I\times\mathbb{R}\times\mathbb{T})} \le C_I \|\phi\|_{L^2(\mathbb{R}\times\mathbb{T})}, \tag{1.7}$$

where $I \subset \mathbb{R}$ is an interval containing $t = 0$ and C_I is a positive constant that depends only on $|I|$ (measure of I). By the symmetry of the laplacian operator, the results are also valid in the space $L^2(\mathbb{T} \times \mathbb{R})$. Similar Strichartz estimates were obtained in [11] for the energy critical nonlinear Schrödinger equation in partially periodic domains, for instance: $\mathbb{R}^m \times \mathbb{T}^{4-m}$ with $m = 2, 3$.

In view of the above comments, a natural question is to figure out what happens for the IVP (1.1) posed on domains $\mathbb{R}^2$, $\mathbb{R} \times \mathbb{T}$ or $\mathbb{T} \times \mathbb{R}$, with initial data in L^2, for more general $\alpha > 0$ as already pointed out before the beginning of this section. The notion of criticality given below tells us that we do not expect well-posedness in L^2 for IVP (1.1) for $0 < \alpha < 1$.

1.2. **Notion of criticality in isotropic Sobolev spaces for $\boldsymbol{\alpha > 0}$.** The equation in (1.1) on the domain $\mathbb{R}^2$ has the following scaling symmetric property: if u is a solution to (1.1), u_λ is also a solution to (1.1), where

$$u(t; x, y) \mapsto u_\lambda(t; x, y) := \lambda u(\lambda^2 t; \lambda x, \lambda^{1/\alpha} y), \quad \lambda > 0, \tag{1.8}$$

which establishes a notion of criticality in Sobolev spaces $H^s(\mathbb{R}^2)$ for the IVP (1.1). More precisely,

- the index s is called *critical* if $\|D^s u_\lambda(0; \cdot, \cdot)\|_{L^2(\mathbb{R}^2)} \sim \|D^s u(0; \cdot, \cdot)\|_{L^2(\mathbb{R}^2)}$,
- the index s is called *subcritical* if $\|D^s u_\lambda(0; \cdot, \cdot)\|_{L^2(\mathbb{R}^2)} \to \infty$ as $\lambda \to \infty$,
- the index s is called *supercritical* if $\|D^s u_\lambda(0; \cdot, \cdot)\|_{L^2(\mathbb{R}^2)} \to 0$ as $\lambda \to \infty$.

Computing $\|D^s u_\lambda(0; \cdot, \cdot)\|_{L^2(\mathbb{R}^2)}$ for $\lambda > 0$ we have

$$\|D^s u_\lambda(0; \cdot, \cdot)\|^2_{L^2(\mathbb{R}^2)} = \lambda^{1-1/\alpha} \int_{\mathbb{R}^2} \big(\lambda^2\xi_1^2 + \lambda^{2/\alpha}\xi_2^2\big)^s |\widehat{u}(0; \xi)|^2 d\xi, \tag{1.9}$$

with $\xi = (\xi_1, \xi_2)$. So one obtains

$$\|D^s u_\lambda(0; \cdot, \cdot)\|^2_{L^2(\mathbb{R}^2)} = \begin{cases} \lambda^{1-1/\alpha+2s/\alpha} p_\alpha(\lambda) & \text{if } 0 < \alpha < 1, \\ \lambda^{2s}\|D^s u(0; \cdot, \cdot)\|^2_{L^2(\mathbb{R}^2)} & \text{if } \alpha = 1, \\ \lambda^{1-1/\alpha+2s} q_\alpha(\lambda) & \text{if } \ \alpha > 1, \end{cases} \tag{1.10}$$

where

$$p_\alpha(\lambda) = \int_{\mathbb{R}^2} \big(\lambda^{2(1-\frac{1}{\alpha})}\xi_1^2 + \xi_2^2\big)^s |\widehat{u}(0;\xi)|^2 d\xi$$
$$q_\alpha(\lambda) = \int_{\mathbb{R}^2} \big(\xi_1^2 + \lambda^{2(\frac{1}{\alpha}-1)}\xi_2^2\big)^s |\widehat{u}(0;\xi)|^2 d\xi.$$

Hence, since

$$\lim_{\lambda\to+\infty} p_\alpha(\lambda) = \|D_y^s u(0;\cdot,\cdot)\|^2_{L^2(\mathbb{R}^2)} \text{ and } \lim_{\lambda\to+\infty} q_\alpha(\lambda) = \|D_x^s u(0;\cdot,\cdot)\|^2_{L^2(\mathbb{R}^2)},$$

we conclude that

$$s_c := \begin{cases} \frac{1-\alpha}{2} > 0 & \text{is the critical regularity for } 0<\alpha<1,\\ 0 & \text{is the critical regularity for } \alpha = 1,\\ \frac{1-\alpha}{2\alpha} < 0 & \text{is the critical regularity for } \alpha > 1. \end{cases} \tag{1.11}$$

Hence, it is not expected well-posedness in L^2 for $0<\alpha<1$.

Furthermore, concerning well-posedness in Sobolev spaces to (1.1) we note that some ill-posedness results (below L^2 regularity) for the one-dimensional cubic NLS on the line can be adapted to establish the same results for (1.1) on the cylinder $\mathbb{R}\times\mathbb{T}$. Indeed, if we consider the following Cauchy problem

$$\begin{gathered} i\partial_t u + \mathcal{L}_\alpha^{\pm} u = \varepsilon |u|^2 u, \quad (t;x,y)\in\mathbb{R}\times\mathbb{R}\times\mathbb{T},\\ u(0;x,y) = \varphi(x), \end{gathered} \tag{1.12}$$

with φ depending only on the x-variable and $\varphi\in H^s(\mathbb{R})$, it follows that $\tilde{\varphi}(x,y) := \varphi(x)\in H^s(\mathbb{R}\times\mathbb{T})$ with

$$\|\tilde{\varphi}\|_{H^s(\mathbb{R}\times\mathbb{T})} = \|\varphi\|_{H^s(\mathbb{R})},$$

and solutions of the IVP:

$$\begin{gathered} i\partial_t w + \partial_x^2 w = \varepsilon |w|^2 w, \quad x\in\mathbb{R},\ t\in\mathbb{R},\\ w(x,0) = \varphi(x), \end{gathered} \tag{1.13}$$

are also solutions of (1.12). Assuming the existence of local solutions, the IVP (1.13) is ill-posed in the following situations:

(i) $s\in(-1/2,0)$ and $\varepsilon<0$. In this case, for any $\delta>0$ the uniform continuous of the flow-map

$$\Phi : u_0\in H^s(\mathbb{R}) \mapsto w\in C([0,\delta];H^s(\mathbb{R})) \tag{1.14}$$

fails (see [12]).

(ii) $s\le -1/2$ and $\varepsilon\neq 0$. In this case, for any $\delta>0$ the flow-map (1.14) is discontinuous everywhere in $H^s(\mathbb{R})$ (norm inflation arguments, see [15]).

Hence, the statements (i) and (ii) imply similar ill-posedness results for negative Sobolev regularity ($s<0$) to the IVP (1.12) for all $\alpha>0$.

In view of the previous discussion we study in this work well-posedness for IVP (1.12) with initial data in $L^2(\mathbb{R}\times\mathbb{T})$ and we get global well-posedness under smallness assumption on data. This result remains valid in $L^2(\mathbb{R}^2)$, but unfortunately cannot be adapted to the case $L^2(\mathbb{T}\times\mathbb{R})$.

1.3. **Main results.** Consider the IVP (1.1), with $\alpha \geq 1$ and initial data $\phi \in L^2(\mathbb{R}\times\mathbb{T})$. In this work we show that it is possible to get a Strichartz estimate similar to (1.7) in the case of the group $U_\alpha^{\pm}$ defined in (1.5) for $\alpha \geq 1$ in the elliptic case (+) and for $\alpha > 1$ in the hyperbolic case (-). More precisely, we will prove the main result.

Theorem 1.1 (Strichartz estimate on $I\times\mathbb{R}\times\mathbb{T}$)**.** *Let $\alpha \geq 1$ and $I \subset \mathbb{R}_t$ an interval containing $t = 0$. Then, there exists a positive constant $C_{\alpha,I}$, depending only on α and the measure of I, such that*

$$\big\|U_\alpha^{\pm}(t)\phi\big\|_{L^4(I\times\mathbb{R}\times\mathbb{T})} \leq C_{\alpha,I}\|\phi\|_{L^2(\mathbb{R}\times\mathbb{T})}, \tag{1.15}$$

for each $\phi \in L^2(\mathbb{R}\times\mathbb{T})$ with $\alpha \geq 1$ for the case (+), and $\alpha > 1$ for the case (-). Moreover, there exists a positive constant $\widetilde{C}_{\alpha,I}$, depending only on α and the measure of I, such that

$$\Big\|\int_0^t U_\alpha^{\pm}(t-t')f(t';\cdot)dt'\Big\|_{L^4(I\times\mathbb{R}\times\mathbb{T})} \leq \widetilde{C}_{\alpha,I}\|f\|_{L^{4/3}(I\times\mathbb{R}\times\mathbb{T})}, \tag{1.16}$$

for any $f \in L^{4/3}(I\times\mathbb{R}\times\mathbb{T})$.

Remark 1.2. We highlight that, in comparison with the case $U_1^+(t)$ covered in [19], the study in the case $\alpha \neq 1$ causes extra technical difficulty in the manipulation of symbol $\tau + \xi^2 - |n|^{2\alpha}$, since one cannot make use of the good algebraic structure of quadratic polynomial in two variables corresponding to the symbol when $\alpha = 1$. Indeed, to estimate the measure of the set $\mathcal{G}_K$ in the proof of crucial Lemma 2.1 we had to introduce two appropriate auxiliary sets, which is not necessary in the case of $U_1^+(t)$.

As in [19], in the context of Cauchy problem for the cubic elliptic NLS, Theorem 1.1 combined with Picard iteration scheme applied to the integral equation

$$u(t) = U_\alpha^{\pm}(t)\phi - i\varepsilon\int_0^t U_\alpha^{\pm}(t-t')|u(t')|^2u(t')dt' \tag{1.17}$$

and the mass conservation law (1.6) imply the following result.

Theorem 1.3 (Well-posedness in L^2)**.** *The Cauchy problem* (1.1) *is globally well-posed for sufficiently small data ϕ in $L^2(\mathbb{R}\times\mathbb{T})$ with $\alpha \geq 1$ for the case (+) and $\alpha > 1$ for the case (-).*

1.4. **Comments.** Finally, we point out some facts.

(I) From the notion of criticality given in (1.11) it is natural to expect well-posedness results for the IVP (1.1) for $s > \frac{1-\alpha}{2} > 0$ when $0 < \alpha < 1$. In fact, our proof of Theorem 1.1 does not work in this setting. See remarks 2.2 and 3.2.

(II) The proof of Theorem 1.1 also fails for the group U_1^-, where the critical regularity suggested by the scaling is L^2. Thus, the hyperbolic case with $\alpha = 1$ remains an interesting open problem on the cylinder domain. At this point it is important to remember that in the hyperbolic case with $\alpha = 1$ on $\mathbb{R}\times\mathbb{R}$, the critical regularity for well-posedness is L^2, but on $\mathbb{T}\times\mathbb{T}$ it was flagged in [20] that the optimal regularity must be $s = \frac{1}{2}$.

(III) The proof of Theorem 1.1 can also be performed in $\mathbb{R}^2$. Indeed, similar to the case U_1^+ treated in [19], the Bourgain's method to obtain Strichartz inequalities on cylinder for $U_\alpha^{\pm}$ also provides a proof of the Strichartz estimate with data on $\mathbb{R}^2$

without using the time decay estimates coming from dispersion. Also, for subcritical nonlinearity $|u|^{p-1}u\,(1<p<3)$ instead the critical case $|u|^2u$ $(p=3)$ globall well-posedness for any data in $L^2(\mathbb{R}\times\mathbb{T})$ can be achieved in the same way as done in [19] in the case U_1^+.

(IV) Our approach cannot automatically adapt to the cylinder $\mathbb{T}\times\mathbb{R}$ since our strategy is based on the quadratic structure of the operator symbol with respect to the continuous propagation.

1.5. **Notation and an elementary inequality.** Throughout the paper we will use the following notation:

- $|J|$ denotes the Lebesgue measure of a set $J\subset\mathbb{R}$,
- $\mathfrak{m}(\cdot)$ denotes the product measure of the one-dimensional Lebesgue and counting measure,
- $0<A(v)\lesssim B(v)$ means that there exists a positive constant c such that $A(v)\le cB(v)$, for any v varying on a certain set,
- $\lfloor x\rfloor$ denote the integer part of x.

Furthermore, $X^{b,s}_{\alpha\pm}(\mathbb{R}\times\mathbb{R}\times\mathbb{T})$ will denote the Bourgain space associated to the group $U_\alpha^{\pm}$, equipped with the norm

$$\begin{aligned}\|f\|^2_{X^{b,s}_{\alpha\pm}} &:= \|U_\alpha^{\pm}(-t)f\|^2_{H^b_tH^s(\mathbb{R}\times\mathbb{T})}\\ &= \int_\tau\int_\xi\sum_{n\in\mathbb{Z}}\langle|\xi|+|n|\rangle^{2s}\langle\tau+q_\pm(\xi,n)\rangle^{2b}\big|\widehat{f}^{t,x,y}(\tau;\xi,n)\big|^2\,d\tau\,d\xi,\end{aligned} \tag{1.18}$$

where $\langle\cdot\rangle=1+|\cdot|$,

$$q_\pm(\xi,n):=\xi^2\pm|n|^{2\alpha} \tag{1.19}$$

and $\widehat{f}^{t,x,y}(\tau;\xi,n)$ denotes the Fourier transform

$$\widehat{f}^{t,x,y}(\tau;\xi,n)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{2\pi}f(t;x,y)e^{-i(t\tau+x\xi+ny)}dt\,dx\,dy$$

with $(\tau;\xi,n)\in\mathbb{R}\times\mathbb{R}\times\mathbb{T}$. The next inequality will be useful to establish some future estimates.

Lemma 1.4. *Let λ be a positive number such that $0<\lambda<1$. Then, it holds that*

$$(a+b)^\lambda\le a^\lambda+b^\lambda, \tag{1.20}$$

for all $a,b\ge0$.

For the sake of completeness we give a proof for this inequality.

Proof. The cases $a=0$ or $b=0$ are obvious, so we consider a,b strictly positive. Since $\lambda\in(0,1)$ we have

$$\frac{a}{a+b}<\Big(\frac{a}{a+b}\Big)^\lambda\quad\text{and}\quad\frac{b}{a+b}<\Big(\frac{b}{a+b}\Big)^\lambda,$$

which imply

$$1<\Big(\frac{a}{a+b}\Big)^\lambda+\Big(\frac{b}{a+b}\Big)^\lambda, \tag{1.21}$$

for all $a,b>0$. The inequality (1.20) is an immediate consequence of (1.21). $\square$

2. Bilinear estimate in the hyperbolic case

In this section, we present the proof of the key bilinear estimate, which allows to get the L^4 Strichartz estimate for U_α^-, with $\alpha > 1$. To derive this bilinear estimate we need, as in the elliptic case treated in [19], to obtain uniform estimates of the measures for certain special sets (see [19, Lemma 2.1]). In our context the corresponding sets are unbounded and to estimate their measures is necessary to deal with series estimation. In particular, the convergence of such series occurs in the case $\alpha > 1$ and fails in the case $0 < \alpha \leq 1$.

We start by showing a similar result to that present in [19, Lemma 2.1], which in our context reads as follows.

Lemma 2.1. *Let $\alpha > 1$, $\xi_0 \in \mathbb{R}$, $n_0 \in \mathbb{Z}$ and $C > 0$. Then for all $K \geq 1$, the set*

$$\mathcal{G}_K := \big\{(\xi, n) \in \mathbb{R} \times \mathbb{Z} : C \leq \big|(\xi - \xi_0)^2 - \frac{1}{2}\big(|n|^{2\alpha} + |n - n_0|^{2\alpha}\big)\big| \leq C + K\big\}$$

satisfies the estimate

$$\sup_{(\xi_0, n_0, C) \in \Lambda} \mathfrak{m}(\mathcal{G}_K) \lesssim_\alpha K, \tag{2.1}$$

where $\Lambda := \mathbb{R} \times \mathbb{Z} \times \mathbb{R}^+$.

Proof. By translation invariance it suffices to consider the case $\xi_0 = 0$. First we write

$$\mathcal{G}_K = \mathcal{G}_K^+ \,\dot\cup\, \mathcal{G}_K^-, \tag{2.2}$$

where

$$\mathcal{G}_K^+ = \mathcal{G}_K \cap \big\{(\xi, n) \in \mathbb{R} \times \mathbb{Z} : \xi^2 > \frac{1}{2}\big(|n|^{2\alpha} + |n - n_0|^{2\alpha}\big)\big\}, \tag{2.3}$$

$$\mathcal{G}_K^- = \mathcal{G}_K \cap \big\{(\xi, n) \in \mathbb{R} \times \mathbb{Z} : \xi^2 \leq \frac{1}{2}\big(|n|^{2\alpha} + |n - n_0|^{2\alpha}\big)\big\}. \tag{2.4}$$

Next we estimate the measures of the sets $\mathcal{G}_K^+$ and $\mathcal{G}_K^-$.

Estimate of $\mathfrak{m}(\mathcal{G}_K^+)$. Using (2.3) we write

$$\mathcal{G}_K^+ = \big\{(\xi, n) \in \mathbb{R} \times \mathbb{Z} : C \leq \xi^2 - \frac{1}{2}\big(|n|^{2\alpha} + |n - n_0|^{2\alpha}\big) \leq C + K\big\}. \tag{2.5}$$

Let l such that $l \geq 1$ and define

$$h(l) := \mathfrak{m}\Big(\big\{(\xi, n) \in \mathbb{R} \times \mathbb{Z} : \xi^2 - \frac{1}{2}\big(|n|^{2\alpha} + |n - n_0|^{2\alpha}\big) \leq l\big\}\Big); \tag{2.6}$$

then

$$\mathfrak{m}\big(\mathcal{G}_K^+\big) = h(C + K) - h(C). \tag{2.7}$$

Note that for a fixed $n \in \mathbb{Z}$ we have

$$\big|\big\{\xi \in \mathbb{R} : \xi^2 - \frac{1}{2}\big(|n|^{2\alpha} + |n - n_0|^{2\alpha}\big) \leq l\big\}\big| = 2\sqrt{\frac{1}{2}\big(|n|^{2\alpha} + |n - n_0|^{2\alpha}\big) + l};$$

therefore,

$$\begin{aligned} h(l) &= 2\sum_{n=0}^{\infty} \sqrt{\frac{1}{2}\big(|n|^{2\alpha} + |n - n_0|^{2\alpha}\big) + l} \\ &\quad + 2\sum_{n=1}^{\infty} \sqrt{\frac{1}{2}\big(|n|^{2\alpha} + |n + n_0|^{2\alpha}\big) + l}. \end{aligned} \tag{2.8}$$

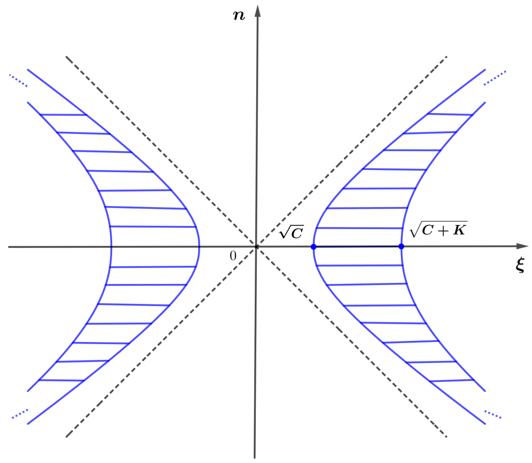

FIGURE 1. The dashed region represents the set $\mathcal{G}_K^+$ with $\alpha = 1.3$ and $n_0 = 0$.

Then, from(2.7) and (2.8) it follows that

$$\mathfrak{m}\big(\mathcal{G}_K^+\big) = S_1^+ + S_2^+, \tag{2.9}$$

where

$$S_1^+ = 2\sum_{n=0}^{\infty}\sqrt{\frac{1}{2}\big(|n|^{2\alpha} + |n-n_0|^{2\alpha}\big) + C + K} - 2\sum_{n=0}^{\infty}\sqrt{\frac{1}{2}\big(|n|^{2\alpha} + |n-n_0|^{2\alpha}\big) + C}$$

$$S_2^+ = 2\sum_{n=1}^{\infty}\sqrt{\frac{1}{2}\big(|n|^{2\alpha} + |n+n_0|^{2\alpha}\big) + C + K} - 2\sum_{n=1}^{\infty}\sqrt{\frac{1}{2}\big(|n|^{2\alpha} + |n+n_0|^{2\alpha}\big) + C}.$$

For S_1^+ we have

$$S_1^+ = 2\sum_{n=0}^{\infty}\frac{K}{\sqrt{\frac{1}{2}\big(|n|^{2\alpha} + |n-n_0|^{2\alpha}\big) + C + K} + \sqrt{\frac{1}{2}\big(|n|^{2\alpha} + |n-n_0|^{2\alpha}\big) + C}},$$

then

$$\begin{aligned} S_1^+ &\le 2\sqrt{2}\sum_{n=0}^{\infty}\frac{K}{\sqrt{n^{2\alpha} + C + K}} \\ &\le 2\sqrt{2}\Big(\frac{K}{\sqrt{C+K}} + K\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}\Big) \lesssim_{\alpha} K, \end{aligned} \tag{2.10}$$

where in the last estimate it has been used that $\alpha > 1$ and $C + K \ge 1$. In a similar way one obtains

$$S_2^+ \lesssim_{\alpha} K.$$

Therefore, from (2.9) we have

$$\mathfrak{m}\big(\mathcal{G}_K^+\big) \lesssim_{\alpha} K. \tag{2.11}$$

Estimate of $\mathfrak{m}(\mathcal{G}_K^-)$. This case is more delicate. We write

$$\mathcal{G}_K^- = \big\{(\xi,n) \in \mathbb{R}\times\mathbb{Z} : -(C+K) \le \xi^2 - \frac{1}{2}\big(|n|^{2\alpha} + |n-n_0|^{2\alpha}\big) \le -C\big\} \tag{2.12}$$

and observe that

$$\mathcal{G}_K^- \subset \mathcal{G}_{1,K}^- \cup \mathcal{G}_{2,K}^-, \tag{2.13}$$

where

$$\begin{aligned} \mathcal{G}_{1,K}^- &= \big\{(\xi,n) \in \mathbb{R}\times\mathbb{Z} : |n-n_0|^{2\alpha} - (C+K) \le \xi^2 \le |n|^{2\alpha} - C\big\}, \\ \mathcal{G}_{2,K}^- &= \big\{(\xi,n) \in \mathbb{R}\times\mathbb{Z} : |n|^{2\alpha} - (C+K) \le \xi^2 \le |n-n_0|^{2\alpha} - C\big\}. \end{aligned}$$

Indeed, $\mathcal{G}_{1,K}^-$ contains the points of $\mathcal{G}_K^-$ with $|n-n_0| \le |n|$ and $\mathcal{G}_{2,K}^-$ those that satisfy $|n-n_0| > |n|$.

Similar arguments to those used to estimate $\mathcal{G}_K^+$ give us

$$\mathfrak{m}\big(\mathcal{G}_{1,K}^-\big) = 4\Big(\sum_{n=\lfloor C^{\frac{1}{2\alpha}}\rfloor+1}^{\infty} \sqrt{n^{2\alpha}-C} - \sum_{n-n_0=\lfloor (C+K)^{\frac{1}{2\alpha}}\rfloor+1}^{\infty} \sqrt{(n-n_0)^{2\alpha}-C-K}\Big),$$

which implies

$$\begin{aligned} \mathfrak{m}\big(\mathcal{G}_{1,K}^-\big) &= 4\Big(\sum_{n=\lfloor C^{\frac{1}{2\alpha}}\rfloor+1}^{\infty} \sqrt{n^{2\alpha}-C} - \sum_{n=\lfloor (C+K)^{\frac{1}{2\alpha}}\rfloor+1}^{\infty} \sqrt{n^{2\alpha}-C-K}\Big) \\ &:= 4(S_1^- + S_2^-), \end{aligned} \tag{2.14}$$

where

$$S_1^- = \sum_{n=\lfloor C^{\frac{1}{2\alpha}}\rfloor+1}^{\lfloor (C+K)^{\frac{1}{2\alpha}}\rfloor} \sqrt{n^{2\alpha}-C},$$

$$S_2^- = \sum_{n=\lfloor (C+K)^{\frac{1}{2\alpha}}\rfloor+1}^{\infty} \big(\sqrt{n^{2\alpha}-C} - \sqrt{n^{2\alpha}-C-K}\big).$$

To estimate S_1^- we observe that

$$\begin{aligned} S_1^- &\le \int_{C^{\frac{1}{2\alpha}}}^{\lfloor (C+K)^{\frac{1}{2\alpha}}\rfloor} \sqrt{z^{2\alpha}-C}\,dz + \sqrt{\lfloor (C+K)^{\frac{1}{2\alpha}}\rfloor^{2\alpha} - C} \\ &\le \underbrace{\int_{C^{\frac{1}{2\alpha}}}^{(C+K)^{\frac{1}{2\alpha}}} \sqrt{z^{2\alpha}-C}\,dz}_{\mathcal{J}_1} + \sqrt{K}. \end{aligned} \tag{2.15}$$

To estimate the integral $\mathcal{J}_1$ we use that $z \mapsto \sqrt{z^{2\alpha}-C}$ is an increasing function to obtain

$$\mathcal{J}_1 \le \sqrt{K}\big((C+K)^{\frac{1}{2\alpha}} - C^{\frac{1}{2\alpha}}\big).$$

Then, applying Lemma 1.4 (remember that $\alpha > 1$) one obtains

$$\mathcal{J}_1 \le \sqrt{K}K^{\frac{1}{2\alpha}} \le K,$$

where in the last estimate it has been used that $K \ge 1$ and $\alpha > 1$. Therefore, from (2.15) and the estimate for $\mathcal{J}_1$ we conclude that $S_1^- \lesssim K$.

Now we estimate S_2^- as follows

$$\begin{aligned} S_2^- &= \sum_{n=\lfloor (C+K)^{\frac{1}{2\alpha}}\rfloor+1}^{\infty} \frac{K}{\sqrt{n^{2\alpha}-C}+\sqrt{n^{2\alpha}-C-K}} \\ &\leq \sum_{n=\lfloor (C+K)^{\frac{1}{2\alpha}}\rfloor+1}^{\infty} \frac{K}{\sqrt{n^{2\alpha}-C}} \\ &\leq \frac{K}{\sqrt{\left(\lfloor (C+K)^{\frac{1}{2\alpha}}\rfloor+1\right)^{2\alpha}-C}} + \underbrace{\int_{(C+K)^{\frac{1}{2\alpha}}}^{\infty} \frac{K}{\sqrt{z^{2\alpha}-C}}dz}_{\mathcal{J}_2} \\ &\leq \sqrt{K}+\mathcal{J}_2. \end{aligned} \tag{2.16}$$

Now we proceed to estimate the integral $\mathcal{J}_2$. Considering the nonlinear change of variables $\rho^{2\alpha}=z^{2\alpha}-C$, and using that $\alpha>1$, $C>0$ and $K\geq 1$, we have

$$\begin{aligned} \mathcal{J}_2 &= K\int_{K^{\frac{1}{2\alpha}}}^{\infty} \frac{\rho^{2\alpha-1}}{\rho^{\alpha}\left(\rho^{2\alpha}+C\right)^{1-\frac{1}{2\alpha}}}d\rho \\ &\leq K\int_{K^{\frac{1}{2\alpha}}}^{\infty} \rho^{-\alpha}d\rho \\ &= \frac{1}{\alpha-1}KK^{\frac{1-\alpha}{2\alpha}} \lesssim_{\alpha} K. \end{aligned} \tag{2.17}$$

Then, inserting (2.17) in (2.16) one obtains $S_2^- \lesssim_\alpha K$.

Therefore, from (2.14) and the estimates obtained for S_1^- and S_2^- we have

$$\mathfrak{m}\big(\mathcal{G}_{1,K}^-\big) \lesssim_\alpha K.$$

By the same way we have $\mathfrak{m}\big(\mathcal{G}_{2,K}^-\big) \lesssim_\alpha K$. So,

$$\mathfrak{m}\big(\mathcal{G}_{K}^-\big) \leq \mathfrak{m}\big(\mathcal{G}_{1,K}^-\big)+\mathfrak{m}\big(\mathcal{G}_{2,K}^-\big) \lesssim_\alpha K,$$

which joint with (2.11) give us (2.1). This completes the proof of Lemma 2.1. □

Remark 2.2. For the case $0<\alpha\leq 1$ the result stated in Lemma 2.1 fails. Indeed, the series S_1^+ in (2.10) diverges for all $0<\alpha\leq 1$.

Proposition 2.3 (Hyperbolic bilinear estimate)**.** *Let u_1 and u_2 be two functions in $L^2(\mathbb{R}\times\mathbb{R}\times\mathbb{T})$ with the following support properties:*

$$\operatorname{supp}(\widehat{u_j}) \subseteq \mathcal{H}_{K_j}^-, \quad j=1,2,$$

where

$$\mathcal{H}_{K_j}^- := \big\{(\tau;\xi,n) : \tfrac{1}{2}K_j \leq \big|\tau+\xi^2-|n|^{2\alpha}\big| \leq 2K_j\big\}. \tag{2.18}$$

Then we have the inequality

$$\|u_1u_2\|_{L^2(\mathbb{R}\times\mathbb{R}\times\mathbb{T})} \lesssim (K_1K_2)^{\frac{1}{2}}\|u_1\|_{L^2(\mathbb{R}\times\mathbb{R}\times\mathbb{T})}\|u_2\|_{L^2(\mathbb{R}\times\mathbb{R}\times\mathbb{T})}. \tag{2.19}$$

Proof. Set $(\tau_2;\xi_2,n_2):=(\tau-\tau_1;\xi-\xi_1,n-n_1)$. Using the Cauchy-Schwarz inequality and Plancherel's theorem, we have

$$\begin{aligned}
\|u_1u_2\|^2_{L^2_{txy}} &= \int_\tau\int_\xi\sum_{n\in\mathbb{Z}}\Big|\int_{\tau_1}\int_{\xi_1}\sum_{n_1\in\mathbb{Z}}\mathbf{1}_{\mathcal{A}_{\tau\xi n}}(\tau_1;\xi_1,n_1)\cdot\widehat{u_1}^{t,x,y}(\tau_1;\xi_1,n_1)\\
&\quad\times\widehat{u_2}^{t,x,y}(\tau_2;\xi_2,n_2)d\tau_1 d\xi_1\Big|^2 d\tau\, d\xi\\
&\lesssim \sup_{\tau,\xi,n}\mathfrak{m}(\mathcal{A}_{\tau\xi n})\|u_1\|^2_{L^2_{txy}}\|u_2\|^2_{L^2_{txy}},
\end{aligned}\tag{2.20}$$

with $\mathcal{A}_{\tau,\xi,n}$ defined as follows:

$$(\tau_1;\xi_1,n_1)\in\mathcal{A}_{\tau,\xi,n}\Longleftrightarrow\begin{cases}(\tau_1;\xi_1,n_1)\in\operatorname{supp}(\widehat{u_1}^{t,x,y}) \text{ and}\\ (\tau_2;\xi_2,n_2)\in\operatorname{supp}(\widehat{u_2}^{t,x,y}).\end{cases}$$

Notice that if $(\tau_1;\xi_1,n_1)\in\mathcal{A}_{\tau,\xi,n}$, then

$$\tau_1\in J_1:=\big[a_1-2K_1,\, a_1+2K_1\big] \quad\text{and}\quad \tau_1\in J_2:=\big[a_2-2K_2,\, a_2+2K_2\big],$$

where $a_1=|n_1|^{2\alpha}-\xi_1^2$ and $a_2=\tau+(\xi-\xi_1)^2-|n-n_1|^{2\alpha}$. Hence

$$\tau_1\in J_1\cap J_2,\ \text{with } |J_1\cap J_2|\le 4\min\{K_1,K_2\}.\tag{2.21}$$

On the other hand, for all $(\tau_1;\xi_1,n_1)\in\mathcal{A}_{\tau,\xi,n}$ we can use the triangle inequality to eliminate τ_1 and obtain

$$\begin{aligned}
&\Big|(\xi_1-\frac{\xi}{2})^2-\frac12\big(|n_1|^{2\alpha}+|n-n_1|^{2\alpha}\big)+\frac{\tau}{2}+\frac{\xi^2}{4}\Big|\\
&=\frac12\big|\tau+\xi_1^2-|n_1|^{2\alpha}+(\xi-\xi_1)^2-|n-n_1|^{2\alpha}\big|\\
&\le\frac12\Big(\big|\tau_1+\xi_1^2-|n_1|^{2\alpha}\big|+\big|\tau-\tau_1+(\xi-\xi_1)^2-|n-n_1|^{2\alpha}\big|\Big)\\
&< K_1+K_2.
\end{aligned}\tag{2.22}$$

So,

$$\text{if } (\tau_1;\xi_1,n_1)\in\mathcal{A}_{\tau,\xi,n},\ \text{then } (\xi_1,n_1)\in\mathcal{B}_{\tau,\xi,n},\tag{2.23}$$

where

$$\mathcal{B}_{\tau,\xi,n}:=\Big\{(\xi_1,n_1)\in\mathbb{R}\times\mathbb{Z}:\Big|(\xi_1-\frac{\xi}{2})^2-\frac12\big(|n_1|^{2\alpha}+|n_1-n|^{2\alpha}\big)+\frac{\tau}{2}+\frac{\xi^2}{4}\Big|\le K_1+K_2\Big\}.$$

This is the reason that led us to consider $\mathcal{G}_K$ slightly different from the set that should naturally replace the used in [19, Lemma 2.1].

Now, from the triangle inequality, we note that

$$\mathcal{B}_{\tau,\xi,n}\subset\Big\{(\xi_1,n_1)\in\mathbb{R}\times\mathbb{Z}: C\le\Big|(\xi_1-\frac{\xi}{2})^2-\frac12\big(|n_1|^{2\alpha}+|n_1-n|^{2\alpha}\big)\Big|\le C+2(K_1+K_2)\Big\}$$

with $C=\big|\frac{\tau}{2}+\frac{\xi^2}{4}\big|-(K_1+K_2)$. Hence, if $C>0$ we use Lemma 2.1 (with $\xi_0=\frac{\xi}{2}$ and $n_0=n$) to obtain

$$\mathfrak{m}(\mathcal{B}_{\tau,\xi,n})\lesssim K_1+K_2.$$

Otherwise, if $C<0$ we have

$$\mathfrak{m}(\mathcal{B}_{\tau,\xi,n})\le\mathfrak{m}\Big(\Big\{(\xi_1,n_1):\Big|(\xi_1-\frac{\xi}{2})^2-\frac12\big(|n_1|^{2\alpha}+|n_1-n|^{2\alpha}\big)\Big|\le 2(K_1+K_2)\Big\}\Big)$$

and consequently

$$\mathfrak{m}(\mathcal{B}_{\tau,\xi,n}) \le \lim_{\varepsilon\searrow 0} \mathfrak{m}\Big(\Big\{(\xi_1,n_1) : \varepsilon < \big|(\xi_1-\frac{\xi}{2})^2-\frac{1}{2}\big(|n_1|^{2\alpha}+|n_1-n|^{2\alpha}\big)\big| \le 2(K_1+K_2)\Big\}\Big),$$

so using again Lemma 2.1 it follows that

$$\mathfrak{m}(\mathcal{B}_{\tau,\xi,n}) \lesssim K_1 + K_2. \tag{2.24}$$

Finally, collecting the information in (2.21), (2.23) and (2.24) one obtains

$$\mathfrak{m}(\mathcal{A}_{\tau,\xi,n}) \le |J_1 \cap J_2| \cdot \mathfrak{m}(\mathcal{B}_{\tau,\xi,n}) \lesssim \min\{K_1,K_2\}\max\{K_1,K_2\}$$

and inserting this in (2.20) we obtain

$$\|u_1u_2\|^2_{L^2_{txy}} \lesssim K_1K_2\|u_1\|^2_{L^2_{txy}}\|u_2\|^2_{L^2_{txy}}.$$

Then, the result is proved. □

3. Bilinear estimate in the elliptic case

Now we prove the corresponding bilinear estimate for the elliptic symbol of the operator.

Lemma 3.1. *Let $\alpha \ge 1$, $\xi_0 \in \mathbb{R}$, $n_0 \in \mathbb{Z}$ and $C \ge 1$. Then for all $K \ge 1$, the set*

$$\widetilde{\mathcal{G}}_K := \Big\{(\xi,n) \in \mathbb{R}\times\mathbb{Z} : C \le (\xi-\xi_0)^2 + \frac{1}{2}\big(|n|^{2\alpha} + |n-n_0|^{2\alpha}\big) \le C + K\Big\}$$

satisfies the estimate

$$\sup_{(\xi_0,n_0,C)\in\Lambda} \mathfrak{m}(\widetilde{\mathcal{G}}_K) \lesssim_\alpha K, \tag{3.1}$$

where $\Lambda := \mathbb{R}\times\mathbb{Z}\times\mathbb{R}^+$.

Proof. As in Lemma 2.1 it suffices to consider the case $\xi_0 = 0$. Notice that for all $K \ge 1$,

$$\widetilde{\mathcal{G}}_K \subset \widetilde{\mathcal{G}}_{1,K} \cup \widetilde{\mathcal{G}}_{2,K}, \tag{3.2}$$

where

$$\widetilde{\mathcal{G}}_{1,K} = \big\{(\xi,n) \in \mathbb{R}\times\mathbb{Z} : C - |n|^{2\alpha} \le \xi^2 \le C + K - |n-n_0|^{2\alpha}\big\},$$
$$\widetilde{\mathcal{G}}_{2,K} = \big\{(\xi,n) \in \mathbb{R}\times\mathbb{Z} : C - |n-n_0|^{2\alpha} \le \xi^2 \le C + K - |n|^{2\alpha}\big\}.$$

Indeed, $\widetilde{\mathcal{G}}_{1,K}$ contains the points of $\widetilde{\mathcal{G}}_K$ with $|n-n_0| \le |n|$ and $\widetilde{\mathcal{G}}_{2,K}$ those that satisfy $|n-n_0| > |n|$.

Similar analysis as performed in the hyperbolic case shows that

$$\begin{aligned}\mathfrak{m}(\widetilde{\mathcal{G}}_{1,K}) &= 2\sum_{|n-n_0|=0}^{\lfloor (C+K)^{\frac{1}{2\alpha}}\rfloor} \sqrt{C+K-|n-n_0|^{2\alpha}} - 2\sum_{|n|=0}^{\lfloor C^{\frac{1}{2\alpha}}\rfloor} \sqrt{C-|n|^{2\alpha}} \\ &= 2\sum_{|n|=0}^{\lfloor (C+K)^{\frac{1}{2\alpha}}\rfloor} \sqrt{C+K-|n|^{2\alpha}} - 2\sum_{|n|=0}^{\lfloor C^{\frac{1}{2\alpha}}\rfloor} \sqrt{C-|n|^{2\alpha}} \\ &:= \widetilde{S}_1 + \widetilde{S}_2,\end{aligned} \tag{3.3}$$

where

$$\widetilde{S}_1 = 2\sum_{|n|=0}^{\lfloor C^{\frac{1}{2\alpha}}\rfloor} \Big(\sqrt{C+K-|n|^{2\alpha}} - \sqrt{C-|n|^{2\alpha}}\Big),$$

$$\widetilde{S}_2 = 2 \sum_{|n|=\lfloor C^{\frac{1}{2\alpha}} \rfloor+1}^{\lfloor (C+K)^{\frac{1}{2\alpha}} \rfloor} \sqrt{C+K-|n|^{2\alpha}}.$$

To estimate $\widetilde{S}_1$ we use that

$$\begin{aligned}
\widetilde{S}_1 &= 2 \sum_{|n|=0}^{\lfloor C^{\frac{1}{2\alpha}} \rfloor} \frac{K}{\left(\sqrt{C+K-|n|^{2\alpha}} + \sqrt{C-|n|^{2\alpha}}\right)} \\
&\le 2 \sum_{|n|=0}^{\lfloor C^{\frac{1}{2\alpha}} \rfloor -1} \frac{K}{\sqrt{C-|n|^{2\alpha}}} + \frac{2K}{\sqrt{C+K-\lfloor C^{\frac{1}{2\alpha}} \rfloor^{2\alpha}}} \\
&\le 4K \int_0^{C^{\frac{1}{2\alpha}}} \frac{dz}{\sqrt{C-z^{2\alpha}}} + 2\sqrt{K}.
\end{aligned} \tag{3.4}$$

Making now the change of variables $z = C^{\frac{1}{2\alpha}} \rho^{\frac{1}{\alpha}}$ with $\alpha \ge 1$, from (3.4) we have

$$\begin{aligned}
\widetilde{S}_1 &\le \frac{4K}{\alpha C^{\frac{\alpha-1}{2\alpha}}} \int_0^1 \frac{d\rho}{\rho^{1-\frac{1}{\alpha}} \sqrt{1-\rho^2}} + 2\sqrt{K} \\
&\lesssim \frac{4K}{\alpha C^{\frac{\alpha-1}{2\alpha}}} \Big(\int_0^1 \frac{d\rho}{\rho^{1-\frac{1}{\alpha}}} + \int_0^1 \frac{d\rho}{\sqrt{1-\rho^2}} \Big) + 2\sqrt{K} \lesssim_\alpha K,
\end{aligned} \tag{3.5}$$

where it has been used that $C \ge 1$ and $\alpha \ge 1$.

Now we proceed to estimate $\widetilde{S}_2$. First note that

$$\begin{aligned}
\widetilde{S}_2 &= 4 \sum_{n=\lfloor C^{\frac{1}{2\alpha}} \rfloor+1}^{\lfloor (C+K)^{\frac{1}{2\alpha}} \rfloor} \sqrt{C+K-n^{2\alpha}} \\
&\le 4\sqrt{K} + \int_{\lfloor C^{\frac{1}{2\alpha}} \rfloor+1}^{\lfloor (C+K)^{\frac{1}{2\alpha}} \rfloor} \sqrt{C+K-z^{2\alpha}} dz \\
&\le 4\sqrt{K} + \int_{C^{\frac{1}{2\alpha}}}^{(C+K)^{\frac{1}{2\alpha}}} \sqrt{C+K-z^{2\alpha}} dz \\
&\le 4\sqrt{K} + \sqrt{K}\big((C+K)^{\frac{1}{2\alpha}} - C^{\frac{1}{2\alpha}}\big).
\end{aligned} \tag{3.6}$$

On the other hand, Lemma 1.20 gives us that

$$(C+K)^{\frac{1}{2\alpha}} - C^{\frac{1}{2\alpha}} \le K^{\frac{1}{2\alpha}} \tag{3.7}$$

for all $C \ge 1$ and $\alpha \ge 1$. Combining (3.6) and (3.7) we have

$$\widetilde{S}_2 \lesssim K,$$

then the proof is complete. □

Remark 3.2. For the case $0 < \alpha < 1$ the result stated in Lemma 3.1 fails. Indeed, it is not difficult to see that the $\widetilde{S}_1$ satisfies (3.5) in the following way

$$\widetilde{S}_1 \sim \Big(\frac{1}{C^{\frac{\alpha-1}{2\alpha}}} + 1 \Big) K,$$

where $\frac{1}{C^{\frac{\alpha-1}{2\alpha}}} \to +\infty$ as $C \to +\infty$ whenever $0 < \alpha < 1$.

In the same way as in the hyperbolic case, Lemma 3.1 implies the following result.

Proposition 3.3 (Elliptic bilinear estimate)**.** *Let u_1 and u_2 be two functions in $L^2(\mathbb{R}\times\mathbb{R}\times\mathbb{T})$ with the following support properties:*

$$\mathrm{supp}(\widehat{u_j}) \subseteq \mathcal{H}^+_{K_j}, \quad j=1,2,$$

where

$$\mathcal{H}^+_{K_j} := \big\{(\tau;\xi,n) : \tfrac{1}{2}K_j \le \big|\tau+\xi^2+|n|^{2\alpha}\big| \le 2K_j\big\}. \tag{3.8}$$

Then we have the inequality

$$\|u_1u_2\|_{L^2(\mathbb{R}\times\mathbb{R}\times\mathbb{T})} \lesssim (K_1K_2)^{1/2}\ \|u_1\|_{L^2(\mathbb{R}\times\mathbb{R}\times\mathbb{T})}\ \|u_2\|_{L^2(\mathbb{R}\times\mathbb{R}\times\mathbb{T})}. \tag{3.9}$$

4. Strichartz inequality on $\mathbb{R}\times\mathbb{T}$

This section is devoted to the proof of Theorem 1.1, which follows the same lines as previous proofs of related results in [19] and we reproduce a sketch of it for the sake of completeness.

Proof of Theorem 1.1 (1.15)*.* The first step is the proof of the following L^4 Strichartz estimate in the Bourgain space $X^{b,0}_{\alpha\pm}$.

Step 1. Let $\alpha\ge 1$ in the case (+), $\alpha>1$ in the case (-) and $b>1/2$. Then

$$\|u\|_{L^4(\mathbb{R}\times\mathbb{R}\times\mathbb{T})} \lesssim \|u\|_{X^{b,0}_{\alpha\pm}(\mathbb{R}\times\mathbb{R}\times\mathbb{T})} \tag{4.1}$$

for any $u\in X^{b,0}_{\alpha\pm}(\mathbb{R}\times\mathbb{R}\times\mathbb{T})$.

Proof of Step 1. Let u a smooth function and consider the dyadic decomposition

$$u(t;x,y) = \sum_{k=0}^{\infty} u_{2^k}(t;x,y),$$

with $\mathrm{supp}(\widehat{u_{2^k}}^{t,x,y}) \in \mathcal{H}^{\pm}_{2^k}$ defined in (2.18). Using Proposition 2.3, Proposition 3.3 and that $b>1/2$ one obtains

$$\begin{aligned}
\|u\|^2_{L^4(\mathbb{R}\times\mathbb{R}\times\mathbb{T})} &= \|uu\|_{L^2(\mathbb{R}\times\mathbb{R}\times\mathbb{T})}\\
&\le \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \|u_{2^{k_1}}u_{2^{k_2}}\|_{L^2(\mathbb{R}\times\mathbb{R}\times\mathbb{T})}\\
&\lesssim \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} (2^{k_1}2^{k_2})^{1/2}\|\widehat{u_{2^{k_1}}}^{t,x,y}\|_{L^2(\mathbb{R}\times\mathbb{R}\times\mathbb{T})}\|\widehat{u^{t,x,y}_{2^{k_2}}}\|_{L^2(\mathbb{R}\times\mathbb{R}\times\mathbb{T})}.
\end{aligned} \tag{4.2}$$

Since $\mathrm{supp}(\widehat{u_{2^k}}^{t,x,y}) \in \mathcal{H}^{\pm}_{2^k}$ we have $\langle\tau+q_\pm(\xi,n)\rangle \ge |\tau+q_\pm(\xi,n)| \ge 2^{k-1}$. Then, it follows that

$$\begin{aligned}
&\|\widehat{u_{2^{k_j}}}^{t,x,y}\|_{L^2(\mathbb{R}\times\mathbb{R}\times\mathbb{T})}\\
&= \Big(\int_\tau\int_\xi\sum_{n\in\mathbb{Z}} \big|\widehat{u_{2^{k_j}}}^{t,x,y}(\tau;\xi,n)\big|^2 d\tau d\xi\Big)^{1/2}\\
&\le \Big(\int_\tau\int_\xi\sum_{n\in\mathbb{Z}} \frac{\langle\tau+q_\pm(\xi,n)\rangle^{2b}}{2^{(k_j-1)2b}}\big|\widehat{u_{2^{k_j}}}^{t,x,y}(\tau;\xi,n)\big|^2 d\tau d\xi\Big)^{1/2}\\
&\lesssim 2^{-bk_j}\|u\|_{X^{b,0}_{\alpha\pm}},
\end{aligned} \tag{4.3}$$

for $j = 1, 2$. So, inserting (4.3) in (4.2) and using that $b > 1/2$, we have

$$\|u\|^2_{L^4(\mathbb{R}\times\mathbb{R}\times\mathbb{T})} \lesssim \sum_{k_1=0}^{\infty} (2^{k_1})^{1/2-b} \sum_{k_2=0}^{\infty} (2^{k_2})^{1/2-b} \|u\|^2_{X^{b,0}_{\alpha\pm}} \lesssim \|u\|^2_{X^{b,0}_{\alpha\pm}}, \tag{4.4}$$

as claimed in (4.1).

Step 2. Let $\delta > 0$, $I = [-\delta, \delta]$ and $b > 1/2$. Then

$$\|U^{\pm}_{\alpha}(t)\phi\|_{L^4(I\times\mathbb{R}\times\mathbb{T})} \lesssim (\delta^{1/2} + \delta^{1/2-b})\|\phi\|_{L^2(\mathbb{R}\times\mathbb{T})}, \tag{4.5}$$

for each $\phi \in L^2(\mathbb{R}\times\mathbb{T})$.

Proof of Step 2. Let $\psi \in C_0^\infty$ be a cut-off function such that $\mathrm{supp}(\psi) \subset (-2, 2)$ and $\psi(t) \equiv 1$ on $[-1, 1]$ and define $\psi_\delta(t) := \psi(\frac{t}{\delta})$. For $b > 0$, one obtains

$$\|\psi_\delta\|_{H^b_t} \lesssim \delta^{1/2}\|\psi\|_{L^2_t} + \delta^{1/2-b}\|\psi\|_{\dot{H}^b_t}.$$

Now, applying (4.1) (here we need $b > 1/2$), we have

$$\begin{aligned}
\|U^{\pm}_{\alpha}(t)\phi\|_{L^4(I\times\mathbb{R}\times\mathbb{T})} &\lesssim \|\psi_\delta(t) U^{\pm}_{\alpha}(t)\phi\|_{L^4(\mathbb{R}\times\mathbb{R}\times\mathbb{T})} \\
&\lesssim \|\psi_\delta(t) U^{\pm}_{\alpha}(t)\phi\|_{X^{b,0}_{\alpha\pm}(\mathbb{R}\times\mathbb{R}\times\mathbb{T})} \\
&= \|\psi_\delta\|_{H^b_t}\|\phi\|_{L^2(\mathbb{R}\times\mathbb{T})} \\
&\lesssim (\delta^{1/2} + \delta^{1/2-b})\|\phi\|_{L^2(\mathbb{R}\times\mathbb{T})},
\end{aligned}$$

so (4.5) is proved. □

Finally, the estimate (4.5) can be extended to an arbitrary interval I in the same way as in [19]. This completes the proof of Theorem 1.1-(1.15).

Proof of Theorem 1.1 (1.16). Consider the linear operator

$$A : L^2(\mathbb{R}\times\mathbb{T}) \longrightarrow L^4(I\times\mathbb{R}\times\mathbb{T}),$$

defined by $A\phi := U^{\pm}_{\alpha}(t)\phi$. Due to the estimate (1.15) we observe that A is bounded. Let us denote by A^* the adjoint operator of A, then

$$A^*(f) = \int_I U^{\pm}_{\alpha}(-t') f(t';\cdot)dt',$$

which is bounded from $L^{4/3}(I\times\mathbb{R}\times\mathbb{T})$ to $L^2(\mathbb{R}\times\mathbb{T})$, i.e.,

$$\Big\| \int_I U^{\pm}_{\alpha}(-t') f(t';\cdot)dt' \Big\|_{L^2(\mathbb{R}\times\mathbb{T})} \le C^*_I \|f\|_{L^{4/3}(I\times\mathbb{R}\times\mathbb{T})}$$

for some constant C^*_I, depending only on the measure of I. Therefore,

$$AA^* = \int_I U^{\pm}_{\alpha}(t - t') f(t';\cdot)dt',$$

is bounded from $L^{4/3}(I\times\mathbb{R}\times\mathbb{T})$ to $L^4(I\times\mathbb{R}\times\mathbb{T})$, satisfying

$$\Big\| \int_I U^{\pm}_{\alpha}(t - t') f(t';\cdot)dt' \Big\|_{L^4(I\times\mathbb{R}\times\mathbb{T})} \le C_I C^*_I \|f\|_{L^{4/3}(I\times\mathbb{R}\times\mathbb{T})}.$$

Finally, to complete the proof of (1.16) is used the arguments in [17, Lemma 3.1]. □

Acknowledgments. This research was partially supported by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior-Brasil (CAPES)-Finance Code 001. A. J. Corcho first author wassupported by CNPq/Brazil grant no. 307616/2020-7 and "Beatriz Galindo" research position at the University of Córdoba/Spain. Finally, we wish to thank the referees for their comments and suggestions that actually improved this article..

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