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EIGENVALUE PROBLEMS FOR KIRCHHOFF-TYPE EQUATIONS IN VARIABLE EXPONENT SOBOLEV SPACES

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Abstract. In this article, we consider an eigenvalue problem for the Kirchhoff-type equation containing $p(\cdot)$-Laplacian and the mean curvature operator with mixed boundary conditions. More precisely, we are concerned with the problem with the Dirichlet condition on a part of the boundary and the Steklov boundary condition on an another part of the boundary. We show that the eigenvalue problem has infinitely many eigenpairs by using the celebrated Ljusternik-Schnirelmann principle in the calculus of variation. Moreover, we derive that in a variable exponent Sobolev space, there are two cases where the infimum of all eigenvalues is equal to zero and is positive.

1. Introduction

In this article, we consider the following eigenvalue problem with mixed boundary conditions

$$
\begin{gathered}
-M\Big(\int_\Omega A(x,\nabla u(x))\,dx\Big)\operatorname{div}[\mathbf{a}(x,\nabla u(x))]=0 \quad \text{in } \Omega,\\
u(x)=0 \quad \text{on } \Gamma_1,\\
M\Big(\int_\Omega A(x,\nabla u(x))\,dx\Big)\mathbf{n}(x)\cdot\mathbf{a}(x,\nabla u(x))=\lambda g(x,u(x)) \quad \text{on } \Gamma_2.
\end{gathered}
\tag{1.1}
$$

Here Ω is a bounded domain of $\mathbb{R}^N$ $(N\ge 2)$ with a Lipschitz-continuous ($C^{0,1}$ for short) boundary Γ satisfying that

$$
\Gamma_1 \text{ and } \Gamma_2 \text{ are disjoint non-empty open subsets of } \Gamma \text{ such that } \overline{\Gamma_1}\cup\overline{\Gamma_2}=\Gamma, \tag{1.2}
$$

and the vector field $\mathbf{n}$ denotes the unit, outer, normal vector to Γ. Furthermore, $\mathbf{a}(x,\boldsymbol{\xi})$ is a Carathéodory function on $\Omega\times\mathbb{R}^N$ satisfying some structure conditions associated with an anisotropic exponent function $p(x)$ and $A(x,\boldsymbol{\xi})$ is a function satisfying $\nabla_{\boldsymbol{\xi}}A(x,\boldsymbol{\xi})=\mathbf{a}(x,\boldsymbol{\xi})$. Here we say that $\mathbf{a}(x,\boldsymbol{\xi})$ is a Carathéodory function on $\Omega\times\mathbb{R}^N$, if for a.e. $x\in\Omega$, the map $\mathbb{R}^N\ni\boldsymbol{\xi}\mapsto\mathbf{a}(x,\boldsymbol{\xi})$ is continuous and for every $\boldsymbol{\xi}\in\mathbb{R}^N$, the map $\Omega\ni x\mapsto\mathbf{a}(x,\boldsymbol{\xi})$ is measurable on Ω. The operator $u\mapsto\operatorname{div}[\mathbf{a}(x,\nabla u(x))]$ is more general than the $p(\cdot)$-Laplacian $\Delta_{p(x)}u(x):=\operatorname{div}[|\nabla u(x)|^{p(x)-2}\nabla u(x)]$ and the mean curvature operator $\operatorname{div}[(1+|\nabla u(x)|^2)^{(p(x)-2)/2}\nabla u(x)]$. This generality brings about difficulties and requires

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some conditions. The function $M = M(s)$ defined in $[0,\infty)$ satisfies the following condition

(A1) $M : [0,\infty) \to [0,\infty)$ is continuous and monotone non-decreasing, and there exist $0 < m_0 \le m_1 < \infty$ and $k \ge l \ge 1$ such that
$$m_0 s^{l-1} \le M(s) \le m_1(1 + s^{k-1}) \quad \text{for } s \ge 0. \tag{1.3}$$

We impose the mixed boundary conditions, that is, the Dirichlet condition on Γ_1 and the Steklov condition on Γ_2. The given data $g : \Gamma_2 \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function of special type and λ is a real number.

The study of differential equations with $p(\cdot)$-growth conditions is a very interesting topic recently. Studying such problem stimulated its application in mathematical physics, in particular, in elastic mechanics (Zhikov [36]), in electrorheological fluids (Diening [12], Halsey [21], Mihăilescu and Rădulescu [28], Růžička [31]). As recent works, we can find some interesting related articles. See Alves et al. [2], Alves and Tavares [3].

However, in even the case $M \equiv 1$, as we only find a few papers associate with the problem with the mixed boundary condition in variable exponent Sobolev space as in (1.1) (for example, Aramaki [5, 6]), we are convinced of the reason for existence of this article.

When $p(x) \equiv p$ (a constant), there are many articles for the p-Laplacian. For example, see Lê [24], Anane [4], Friedlander [20]. For the p-Laplacian Dirichlet eigenvalue problem:
$$\begin{gathered} -\Delta_p u(x) = \lambda |u(x)|^{p-2} u(x) \quad \text{in } \Omega, \\ u(x) = 0 \quad \text{on } \Gamma, \end{gathered}$$
we can see that the following properties hold.

(1) There exists a nondecreasing sequence of positive eigenvalues $\{\lambda_n\}$ tending to ∞ as $n \to \infty$.
(2) The first eigenvalue λ_1 is simple and only eigenfunctions associated with λ_1 do not change sign.
(3) The set of eigenvalues is closed.
(4) The first eigenvalue λ_1 is isolated.

On the contrary, recently many authors study the $p(\cdot)$-Laplacian. In particular, Fan [15] has studied the eigenvalue problem for the $p(\cdot)$-Laplacian with zero Neumann boundary condition in a bounded domain, and Fan et al. [19] has studied the eigenvalue problem for the $p(\cdot)$-Laplacian Dirichlet problem. Mihăilescu and Rădulescu [29] have studied nonhomogeneous quasilinear eigenvalue problem with variable exponent. In Deng [11], the author treats only the $p(\cdot)$-Laplacian in the case $\Gamma_1 = \emptyset$, that is,
$$\begin{gathered} -\Delta_{p(x)} u(x) + |u(x)|^{p(x)-2} u(x) = 0 \quad \text{in } \Omega, \\ |\boldsymbol{\nabla} u(x)|^{p(x)-2} \frac{\partial u(x)}{\partial \mathbf{n}} = \lambda |u(x)|^{p(x)-2} u(x) \quad \text{on } \Gamma. \end{gathered} \tag{1.4}$$

As the author takes the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ as the base space, the second term in the left-hand side of the first equation of (1.4) takes the essential role. However, if we assume that $\Gamma_1 \neq \emptyset$, we can delete such a term according to the Poincaré type inequality due to Ciarlet and Dinca [10].

For physical motivation to the problem (1.1), we consider the case where $\Gamma = \Gamma_1$ and $p(x) = 2$. Then the equation

$$M(\|\nabla u\|^2_{\mathbf{L}^2(\Omega)})\Delta u(x) = f(x, u(x)) \tag{1.5}$$

is the Kirchhoff equation which arises in nonlinear vibration, namely

$$\begin{gathered} u_{tt} - M(\|\nabla u\|^2_{\mathbf{L}^2(\Omega)})\Delta u = f(x,u) \quad \text{in } \Omega \times (0,T), \\ u = 0 \quad \text{on } \Gamma \times (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega. \end{gathered} \tag{1.6}$$

Equation (1.5) is the stationary counterpart of (1.6). Such a hyperbolic equation is a general version of the Kirchhoff equation

$$\rho u_{tt} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \big|\frac{\partial u}{\partial x}\big|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0$$

presented by Kirchhoff [22]. This equation extends the classical d'Alembert wave equation by considering the effect of the changes in the length of the strings during the vibrations, where L, h, E, ρ and ρ_0 are constants. In Afrouzu and Mirzapour [1], the authors studied the $p(\cdot)$-Kirchhoff type eigenvalue problem

$$\begin{gathered} -M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u(x)|^{p(x)}\,dx\Big)\Delta_{p(x)}u(x) = \lambda|u(x)|^{q(x)-2}u(x) \quad \text{in } \Omega, \\ u(x) = 0 \quad \text{on } \Gamma. \end{gathered} \tag{1.7}$$

They derived the existence of a nontrivial weak solution under some conditions on the functions $M, p(\cdot), q(\cdot)$ and a real number λ. Mendéz [27] considered the problem

$$\begin{aligned} &-M\Big(\int_\Omega |\nabla u(x)|^{p(x)}\,dx\Big)\operatorname{div}[p(x)|\nabla u(x)|^{p(x)-2}\nabla u(x)] \\ &= \lambda p(x)|u(x)|^{p(x)-2}u(x) \quad \text{in } \Omega, \\ &u(x) = 0 \quad \text{on } \Gamma. \end{aligned} \tag{1.8}$$

The author showed that for any $r > 0$, there exists a eigenpair $(u, \lambda) \in W_0^{1,p(\cdot)}(\Omega) \times \mathbb{R}$ of (1.8) satisfying $\widehat{M}\big(\int_\Omega |\nabla u(x)|^{p(x)}\,dx\big) = r$, where $\widehat{M}(t) = \int_0^t M(s)\,ds$.

In this article, we extend these results to a class of operators containing $p(\cdot)$-Laplacian and the mean curvature operator. The purpose of this article is to solve eigenvalue problem (1.1). According to some assumptions on the given function g, we use the Ljusternik-Schnirelmann principle in the constrained variational method. See Ljusternik and Schnirelmann [25] and Szulkin [32]. We will deal with the mixed boundary value eigenvalue problem (1.1) for a class of operators involving the $p(\cdot)$-Laplacian and the mean curvature operator which seems to be a new topic. We will show that there exist infinitely many eigenvalues $\{\lambda_{(n,\alpha)}\}$ tending to ∞ as $n \to \infty$ for any fixed $\alpha > 0$. Moreover, we will derive that under some condition, the infimum λ_* of the set of all eigenvalues of (1.1) is equal to zero, so there does not exist a principal eigenvalue and the set of eigenvalues is not closed. We also show that under some condition on the function g and variable exponent function p in (1.1), there is a case where λ_* is positive.

This article is organized as follows. In Section 2, we recall some results on variable exponent Lebesgue-Sobolev spaces. In Section 3, we give the setting of problem (1.1) rigorously and a main theorem (Theorems 3.20) on the eigenvalue problem (1.1) in which we show the existence of infinitely many eigenpairs of (1.1).

In Section 4, we present some sufficient conditions for the cases $\lambda_* = 0$ and $\lambda_* > 0$, respectively.

2. Preliminaries

Throughout this article, Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with a $C^{0,1}$-boundary Γ and Ω is locally on the same side of Γ. Moreover, we assume that Γ satisfies (1.2).

We only consider real vector spaces of real valued functions over $\mathbb{R}$. For any space B, we denote B^N by the boldface character $\mathbf{B}$. Hereafter, we use this character to denote vectors and vector-valued functions, and we denote the standard inner product of vectors $\mathbf{a} = (a_1, \dots, a_N)$ and $\mathbf{b} = (b_1, \dots, b_N)$ in $\mathbb{R}^N$ by $\mathbf{a}\cdot\mathbf{b} = \sum_{i=1}^N a_i b_i$ and $|\mathbf{a}| = (\mathbf{a}\cdot\mathbf{a})^{1/2}$. Furthermore, we denote the dual space of B by B^* and the duality bracket by $\langle\cdot,\cdot\rangle_{B^*,B}$.

We recall some well-known results on variable exponent Lebesgue and Sobolev spaces. See Fan and Zhang [17], Kováčik and Rácosník [23], Diening et al. [13] and references therein for more details. We consider some new properties on variable exponent Lebesgue space. We define $C(\overline{\Omega}) = \{p \text{ is a continuous function on } \overline{\Omega}\}$, and for any $p \in C(\overline{\Omega})$, put

$$p^+ = p^+(\Omega) = \sup_{x\in\Omega} p(x) = \max_{x\in\overline{\Omega}} p(x), \quad p^- = p^-(\Omega) = \inf_{x\in\Omega} p(x) = \min_{x\in\overline{\Omega}} p(x).$$

For any $p \in C(\overline{\Omega})$ with $p^- \geq 1$ and for any measurable function u on Ω, a modular (for this notation, see [13, Definition 2.1.1]) $\rho_{p(\cdot)} = \rho_{p(\cdot),\Omega}$ is defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\,dx.$$

The variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\Omega) = \{u; u : \Omega \to \mathbb{R} \text{ is a measurable function satisfying } \rho_{p(\cdot)}(u) < \infty\}$$

equipped with the (Luxemburg) norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\big\{\tau > 0; \rho_{p(\cdot)}\big(\frac{u}{\tau}\big) \leq 1\big\}.$$

Then $L^{p(\cdot)}(\Omega)$ is a Banach space. We also define the Sobolev space

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega); |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

where ∇ is a gradient operator, that is, $\nabla u = (\partial_1 u, \dots, \partial_N u)$, $\partial_i = \partial/\partial x_i$, endowed with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{\mathbf{L}^{p(\cdot)}(\Omega)},$$

and $\|\nabla u\|_{\mathbf{L}^{p(\cdot)}(\Omega)} = \||\nabla u|\|_{L^{p(\cdot)}(\Omega)}$.

The following three propositions are well known (see [19], Fan and Zhao [18], Zhao et al. [35]).

Proposition 2.1. *Let $p \in C(\overline{\Omega})$ with $p^- \geq 1$, and let $u, u_n \in L^{p(\cdot)}(\Omega)$ $(n = 1, 2, \dots)$. Then we have the following properties.*

(i) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1 (= 1, > 1) \Leftrightarrow \rho_{p(\cdot)}(u) < 1 (= 1, > 1)$.

(ii) $\|u\|_{L^{p(\cdot)}(\Omega)} > 1 \Rightarrow \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} \leq \rho_{p(\cdot)}(u) \leq \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}$.

(iii) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1 \Rightarrow \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \leq \rho_{p(\cdot)}(u) \leq \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}$.

(iv) $\lim_{n\to\infty} \|u_n - u\|_{L^{p(\cdot)}(\Omega)} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{p(\cdot)}(u_n - u) = 0$.

(v) $\|u_n\|_{L^{p(\cdot)}(\Omega)} \to \infty$ *as* $n \to \infty \Leftrightarrow \rho_{p(\cdot)}(u_n) \to \infty$ *as* $n \to \infty$.

The following proposition is a generalized Hölder inequality.

Proposition 2.2. *Let* $p \in C_+(\overline{\Omega})$, *where* $C_+(\overline{\Omega}) := \{p \in C(\overline{\Omega}); p^- > 1\}$. *For any* $u \in L^{p(\cdot)}(\Omega)$ *and* $v \in L^{p'(\cdot)}(\Omega)$, *we have*

$$\int_\Omega |u(x)v(x)|dx \le \Big(\frac{1}{p^-} + \frac{1}{(p')^-}\Big)\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)} \le 2\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)}.$$

Here and from now on, for any $p \in C_+(\overline{\Omega})$, $p'(\cdot)$ *denotes the conjugate exponent of* $p(\cdot)$, *that is,* $p'(x) = p(x)/(p(x)-1)$ *for* $x \in \overline{\Omega}$.

For $p \in C_+(\overline{\Omega})$, we define, for $x \in \overline{\Omega}$,

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \ge N. \end{cases}$$

Proposition 2.3. *Let* Ω *be a bounded domain of* $\mathbb{R}^N$ *with* $C^{0,1}$*-boundary and let* $p \in C_+(\overline{\Omega})$. *Then we have the following properties.*

(i) *The spaces* $L^{p(\cdot)}(\Omega)$ *and* $W^{1,p(\cdot)}(\Omega)$ *are separable, reflexive and uniformly convex Banach spaces.*
(ii) *If* $q(\cdot) \in C(\overline{\Omega})$ *with* $q^- \ge 1$ *satisfies* $q(x) \le p(x)$ *for all* $x \in \overline{\Omega}$, *then* $W^{1,p(\cdot)}(\Omega) \hookrightarrow W^{1,q(\cdot)}(\Omega)$, *where* $\hookrightarrow$ *means that the embedding map is continuous.*
(iii) *If* $q(x) \in C(\overline{\Omega})$ *with* $q^- \ge 1$ *satisfies that* $q(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$, *then the embedding map* $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ *is compact.*

Next we consider the trace (cf. Fan [16]). Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$-boundary Γ and $p \in C(\overline{\Omega})$ with $p^- \ge 1$. Since $W^{1,p(\cdot)}(\Omega) \subset W^{1,1}(\Omega)$, the trace $u\big|_\Gamma$ to Γ of any function u in $W^{1,p(\cdot)}(\Omega)$ is well defined as a function in $L^1(\Gamma)$. We define

$$\begin{aligned}\mathrm{Tr}(W^{1,p(\cdot)}(\Omega)) &= (\mathrm{Tr}\, W^{1,p(\cdot)})(\Gamma) \\ &= \{f; f \text{ is the trace to } \Gamma \text{ of a function } F \in W^{1,p(\cdot)}(\Omega)\}\end{aligned}$$

equipped with the norm

$$\|f\|_{(\mathrm{Tr}\, W^{1,p(\cdot)})(\Gamma)} = \inf\{\|F\|_{W^{1,p(\cdot)}(\Omega)}; F \in W^{1,p(\cdot)}(\Omega) \text{ satisfying } F\big|_\Gamma = f\}$$

for $f \in (\mathrm{Tr}\, W^{1,p(\cdot)})(\Gamma)$, where the infimum can be achieved. Then we can see that $(\mathrm{Tr}\, W^{1,p(\cdot)})(\Gamma)$ is a Banach space. In the later, we also write $F\big|_\Gamma = g$ by $F = g$ on Γ. Moreover, for $i = 1, 2$, we denote

$$(\mathrm{Tr}\, W^{1,p(\cdot)})(\Gamma_i) = \{f\big|_{\Gamma_i}; f \in (\mathrm{Tr}\, W^{1,p(\cdot)})(\Gamma)\}$$

equipped with the norm

$$\|g\|_{(\mathrm{Tr}\, W^{1,p(\cdot)})(\Gamma_i)} = \inf\{\|f\|_{(\mathrm{Tr}\, W^{1,p(\cdot)})(\Gamma)}; f \in (\mathrm{Tr}\, W^{1,p(\cdot)})(\Gamma) \text{ satisfying } f\big|_{\Gamma_i} = g\},$$

where the infimum can also be achieved, so for any $g \in (\mathrm{Tr}\, W^{1,p(\cdot)})(\Gamma_i)$, there exists $F \in W^{1,p(\cdot)}(\Omega)$ such that $F\big|_{\Gamma_i} = g$ and $\|F\|_{W^{1,p(\cdot)}(\Omega)} = \|g\|_{(\mathrm{Tr}\, W^{1,p(\cdot)})(\Gamma_i)}$.

Let $q \in C_+(\Gamma) := \{q \in C(\Gamma); q^- > 1\}$ and denote the surface measure on Γ induced from the Lebesgue measure dx on Ω by $d\sigma_x$. We define

$$L^{q(\cdot)}(\Gamma) = \big\{u : \Gamma \to \mathbb{R} \text{ is a measurable function with respect to } d\sigma_x$$

$$\text{satisfying } \int_\Gamma |u(x)|^{q(x)}\,d\sigma_x < \infty\big\}$$

and the norm is defined by

$$\|u\|_{L^{q(\cdot)}(\Gamma)} = \inf\big\{\tau > 0; \int_\Gamma \big|\frac{u(x)}{\tau}\big|^{q(x)}\,d\sigma_x \le 1\big\},$$

and we also define a modular on $L^{q(\cdot)}(\Gamma)$ by

$$\rho_{q(\cdot),\Gamma}(u) = \int_\Gamma |u(x)|^{q(x)}\,d\sigma_x.$$

Similarly as Proposition 2.1, we have the following proposition.

Proposition 2.4. *Let $q \in C(\Gamma)$ with $q^- \ge 1$, and let $u, u_n \in L^{q(\cdot)}(\Gamma)$. Then we have the following properties.*

- (i) $\|u\|_{L^{q(\cdot)}(\Gamma)} < 1 (= 1, > 1) \Leftrightarrow \rho_{q(\cdot),\Gamma}(u) < 1 (= 1, > 1)$.
- (ii) $\|u\|_{L^{q(\cdot)}(\Gamma)} > 1 \Rightarrow \|u\|^{q^-}_{L^{q(\cdot)}(\Gamma)} \le \rho_{q(\cdot),\Gamma}(u) \le \|u\|^{q^+}_{L^{q(\cdot)}(\Gamma)}$.
- (iii) $\|u\|_{L^{q(\cdot)}(\Gamma)} < 1 \Rightarrow \|u\|^{q^+}_{L^{q(\cdot)}(\Gamma)} \le \rho_{q(\cdot),\Gamma}(u) \le \|u\|^{q^-}_{L^{q(\cdot)}(\Gamma)}$.
- (iv) $\|u_n\|_{L^{q(\cdot)}(\Gamma)} \to 0 \Leftrightarrow \rho_{q(\cdot),\Gamma}(u_n) \to 0$.
- (v) $\|u_n\|_{L^{q(\cdot)}(\Gamma)} \to \infty \Leftrightarrow \rho_{q(\cdot),\Gamma}(u_n) \to \infty$.

The Hölder inequality also holds for functions on Γ.

Proposition 2.5. *Let $q \in C(\Gamma)$ with $q^- > 1$. Then the following inequality holds.*

$$\int_\Gamma |f(x)g(x)|d\sigma_x \le 2\|f\|_{L^{q(\cdot)}(\Gamma)}\|g\|_{L^{q'(\cdot)}(\Gamma)} \quad \text{for all } f \in L^{q(\cdot)}(\Gamma),\ g \in L^{q'(\cdot)}(\Gamma).$$

Proposition 2.6. *Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$-boundary Γ and let $p \in C_+(\overline{\Omega})$. If $f \in (\operatorname{Tr} W^{1,p(\cdot)})(\Gamma)$, then $f \in L^{p(\cdot)}(\Gamma)$ and there exists a constant $C > 0$ such that*

$$\|f\|_{L^{p(\cdot)}(\Gamma)} \le C\|f\|_{(\operatorname{Tr} W^{1,p(\cdot)})(\Gamma)}.$$

In particular, If $f \in (\operatorname{Tr} W^{1,p(\cdot)})(\Gamma)$, then $f \in L^{p(\cdot)}(\Gamma_i)$ and

$$\|f\|_{L^{p(\cdot)}(\Gamma_i)} \le C\|f\|_{(\operatorname{Tr} W^{1,p(\cdot)})(\Gamma)}$$

for $i = 1, 2$.

For $p \in C_+(\overline{\Omega})$, we define, for $x \in \overline{\Omega}$,

$$p^\partial(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \ge N. \end{cases}$$

The next proposition follows from Yao [33, Proposition 2.6].

Proposition 2.7. *Let $p \in C_+(\overline{\Omega})$. Then if $q \in C_+(\Gamma)$ satisfies $q(x) < p^\partial(x)$ for all $x \in \Gamma$, then the trace mapping $W^{1,p(\cdot)}(\Omega) \to L^{q(\cdot)}(\Gamma)$ is well-defined and compact. In particular, the trace mapping $W^{1,p(\cdot)}(\Omega) \to L^{p(\cdot)}(\Gamma)$ is compact and there exists a constant $C > 0$ such that*

$$\|u\|_{L^{p(\cdot)}(\Gamma)} \le C\|u\|_{W^{1,p(\cdot)}(\Omega)} \quad \text{for } u \in W^{1,p(\cdot)}(\Omega).$$

Now we consider the weighted variable exponent Lebesgue space. Let $p \in C(\overline{\Omega})$ with $p^- \geq 1$ and let $a(x)$ be a measurable function on Ω with $a(x) > 0$ a.e. $x \in \Omega$. We define a modular

$$\rho_{(p(\cdot),a(\cdot))}(u) = \int_\Omega a(x)|u(x)|^{p(x)}\,dx$$

for any measurable function u in Ω. Then the weighted Lebesgue space is defined by

$$L^{p(\cdot)}_{a(\cdot)}(\Omega) = \big\{u \text{ is a measurable function on } \Omega \text{ satisfying } \rho_{(p(\cdot),a(\cdot))}(u) < \infty\big\}$$

equipped with the norm

$$\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} = \inf\big\{\tau > 0; \int_\Omega a(x)\big|\frac{u(x)}{\tau}\big|^{p(x)}\,dx \leq 1\big\}.$$

Then $L^{p(\cdot)}_{a(\cdot)}(\Omega)$ is a Banach space.

We have the following proposition (cf. Fan [14, Proposition 2.5]).

Proposition 2.8. *Let $p \in C(\overline{\Omega})$ with $p^- \geq 1$. For $u, u_n \in L^{p(\cdot)}_{a(\cdot)}(\Omega)$, we have the following.*

(i) *For $u \neq 0$, $\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} = \tau \Leftrightarrow \rho_{(p(\cdot),a(\cdot))}\left(\frac{u}{\tau}\right) = 1$.*

(ii) $\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} < 1 (= 1, > 1) \Leftrightarrow \rho_{(p(\cdot),a(\cdot))}(u) < 1 (= 1, > 1)$.

(iii) $\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} > 1 \Rightarrow \|u\|^{p^-}_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} \leq \rho_{(p(\cdot),a(\cdot))}(u) \leq \|u\|^{p^+}_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}$.

(iv) $\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} < 1 \Rightarrow \|u\|^{p^+}_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} \leq \rho_{(p(\cdot),a(\cdot))}(u) \leq \|u\|^{p^-}_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}$.

(v) $\lim_{n\to\infty} \|u_n - u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{(p(\cdot),a(\cdot))}(u_n - u) = 0$.

(vi) *$\|u_n\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} \to \infty$ as $n \to \infty \Leftrightarrow \rho_{(p(\cdot),a(\cdot))}(u_n) \to \infty$ as $n \to \infty$.*

The author of [14] also derived the following proposition (cf. [14, Theorem 2.1]).

Proposition 2.9. *Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$-boundary and $p \in C_+(\overline{\Omega})$. Moreover, let $a \in L^{\alpha(\cdot)}(\Omega)$ satisfy $a(x) > 0$ a.e. $x \in \Omega$ and $\alpha \in C_+(\overline{\Omega})$. If $q \in C(\overline{\Omega})$ satisfies*

$$1 \leq q(x) < \frac{\alpha(x) - 1}{\alpha(x)} p^*(x) \text{ for all } x \in \overline{\Omega},$$

then the embedding map $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}_{a(\cdot)}(\Omega)$ is compact.

Similarly, let $q \in C(\Gamma)$ with $q^- \geq 1$ and let $b(x)$ be a measurable function with respect to $d\sigma_x$ on Γ with $b(x) > 0$ σ-a.e. $x \in \Gamma$. We define a modular

$$\rho_{(q(\cdot),b(\cdot)),\Gamma}(u) = \int_\Gamma b(x)|u(x)|^{q(x)} d\sigma_x.$$

Then the weighted Lebesgue space on Γ is defined by

$$L^{q(\cdot)}_{b(\cdot)}(\Gamma) = \{u \text{ is a } \sigma\text{-measurable function on } \Gamma \text{ satisfying } \rho_{(q(\cdot),b(\cdot)),\Gamma}(u) < \infty\}$$

equipped with the norm

$$\|u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} = \inf\big\{\tau > 0; \int_\Gamma b(x)\big|\frac{u(x)}{\tau}\big|^{q(x)}\,d\sigma_x \leq 1\big\}.$$

Then $L^{q(\cdot)}_{b(\cdot)}(\Gamma)$ is a Banach space.

Proposition 2.10. *Let $q \in C(\Gamma)$ with $q^- \geq 1$. For $u, u_n \in L^{q(\cdot)}_{b(\cdot)}(\Gamma)$, we have the following.*

(i) $\|u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} < 1 (= 1, > 1) \Leftrightarrow \rho_{(q(\cdot),b(\cdot)),\Gamma}(u) < 1 (= 1, > 1)$.
(ii) $\|u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} > 1 \Rightarrow \|u\|^{q^-}_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} \leq \rho_{(q(\cdot),b(\cdot)),\Gamma}(u) \leq \|u\|^{q^+}_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)}$.
(iii) $\|u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} < 1 \Rightarrow \|u\|^{q^+}_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} \leq \rho_{(q(\cdot),b(\cdot)),\Gamma}(u) \leq \|u\|^{q^-}_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)}$.
(iv) $\lim_{n\to\infty} \|u_n - u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{(q(\cdot),b(\cdot)),\Gamma}(u_n - u) = 0$.
(v) $\|u_n\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} \to \infty$ *as* $n \to \infty \Leftrightarrow \rho_{(q(\cdot),b(\cdot)),\Gamma}(u_n) \to \infty$ *as* $n \to \infty$.

The following proposition plays an important role in the present paper.

Proposition 2.11. *Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$-boundary Γ and let $p \in C_+(\overline{\Omega})$. Assume that $0 < b \in L^{\beta(\cdot)}(\Gamma)$, $\beta \in C_+(\Gamma)$. If $r \in C(\Gamma)$ satisfies*

$$1 \leq r(x) < \frac{\beta(x) - 1}{\beta(x)} p^\partial(x) \text{ for all } x \in \Gamma,$$

then the embedding map $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}_{b(\cdot)}(\Gamma)$ is compact.

Proof. Let $u \in W^{1,p(\cdot)}(\Omega)$. Set $h(x) = \beta'(x) r(x)$. From the hypothesis, we have $h(x) < p^\partial(x)$ for all $x \in \Gamma$. By Proposition 2.7, the embedding map $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{h(\cdot)}(\Gamma)$ is compact. Since $|u(x)|^{r(x)} \in L^{\beta'(\cdot)}(\Gamma)$, it follows from the Hölder inequality (Proposition 2.5) that

$$\int_\Gamma b(x)|u(x)|^{r(x)}\, d\sigma_x \leq 2\|b\|_{L^{\beta(\cdot)}(\Gamma)} \||u|^{r(\cdot)}\|_{L^{\beta'(\cdot)}(\Gamma)} < \infty.$$

Hence $W^{1,p(\cdot)}(\Omega) \subset L^{r(\cdot)}_{b(\cdot)}(\Gamma)$. We show that the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}_{b(\cdot)}(\Gamma)$ is compact. Let $u_n \to 0$ weakly in $W^{1,p(\cdot)}(\Omega)$. Then $u_n \to 0$ strongly in $L^{h(\cdot)}(\Gamma)$. Since

$$\rho_{\beta'(\cdot),\Gamma}(|u_n|^{r(\cdot)}) = \int_\Gamma |u_n(x)|^{r(x)\beta'(x)} d\sigma_x = \int_\Gamma |u_n(x)|^{h(x)}\, d\sigma_x \to 0,$$

we have $\||u_n|^{r(\cdot)}\|_{L^{\beta'(\cdot)}(\Gamma)} \to 0$ from Proposition 2.10 (iv). Therefore,

$$\int_\Gamma b(x)|u_n(x)|^{r(x)}\, d\sigma_x \leq 2\|b\|_{L^{r(\cdot)}(\Gamma)} \||u_n|^{r(\cdot)}\|_{L^{\beta'(\cdot)}(\Gamma)} \to 0.$$

Thus it also follows from Proposition 2.10 (iv) that $\|u_n\|_{L^{r(\cdot)}_{b(\cdot)}(\Gamma)} \to 0$, so $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}_{b(\cdot)}(\Gamma)$ is compact. □

Now we consider the Nemytskii operator.

Proposition 2.12. *Let $q \in C(\overline{\Omega})$ with $q^- \geq 1$ and a be a measurable function with $a(x) > 0$ for a.e. $x \in \Omega$. Assume that*

(1) *A function $F(x,t)$ is a Carathéodory function on $\Omega \times \mathbb{R}$.*
(2) *The growth condition holds: there exist $c \in L^{q_1(\cdot)}(\Omega)$ with $c(x) \geq 0$ a.e. $x \in \Omega$, $q_1 \in C(\overline{\Omega})$ with $q_1^- \geq 1$ and a constant $c_1 > 0$ such that*

$$|F(x,t)| \leq c(x) + c_1 a(x)^{1/q_1(x)} |t|^{q(x)/q_1(x)} \quad \text{for a.e. } x \in \Omega \text{ and all } t \in \mathbb{R}.$$

Then the Nemytskii operator $N_F : L^{q(\cdot)}_{a(\cdot)}(\Omega) \ni u \mapsto F(x, u(x)) \in L^{q_1(\cdot)}(\Omega)$ *is continuous and there exists a constant* $C > 0$ *such that*

$$\rho_{q_1(\cdot)}(N_F(u)) \le C(\rho_{q_1(\cdot)}(c) + \rho_{(q(\cdot),a(\cdot))}(u)) \quad \text{for all } u \in L^{q(\cdot)}_{a(\cdot)}(\Omega).$$

In particular, if $q_1(x) \equiv 1$*, then* $N_F : L^{q(\cdot)}_{a(\cdot)}(\Omega) \to L^1(\Omega)$ *is continuous.*

For a proof of the above proposition, see Aramaki [9, Proposition 7]. The proposition is an extension of [6, Proposition 2.12]. Similarly we have the following proposition.

Proposition 2.13. *Let* $r \in C(\overline{\Gamma_2})$ *with* $r^- \ge 1$ *and* b *be a* σ*-measurable function with* $b(x) > 0$ σ*-a.e.* $x \in \Gamma_2$*. Assume that*

1. *The function* $H(x,t)$ *is a Carathéodory function on* $\Gamma_2 \times \mathbb{R}$*.*
2. *The growth condition holds: there exist* $d \in L^{r_1(\cdot)}(\Gamma_2)$ *with* $d(x) \ge 0$ σ*-a.e.* $x \in \Gamma_2$*,* $r_1 \in C(\overline{\Gamma_2})$ *with* $r_1 \ge 1$*, and a constant* $d_1 > 0$ *such that*

$$|H(x,t)| \le d(x) + d_1 b(x)^{1/r_1(x)} |t|^{r(x)/r_1(x)} \quad \text{for } \sigma\text{-a.e. } x \in \Gamma_2 \text{ and all } t \in \mathbb{R}.$$

Then the Nemytskii operator $N_H : L^{r(\cdot)}_{b(\cdot)}(\Gamma_2) \ni v \mapsto H(x, v(x)) \in L^{r_1(\cdot)}(\Gamma_2)$ *is continuous and there exists a constant* $C > 0$ *such that*

$$\rho_{r_1(\cdot),\Gamma_2}(N_H(v)) \le C(\rho_{r_1(\cdot),\Gamma_2}(d) + \rho_{(r(\cdot),b(\cdot)),\Gamma_2}(v)) \quad \text{for all } v \in L^{r(\cdot)}_{b(\cdot)}(\Gamma_2).$$

In particular, if $r_1(x) \equiv 1$*, then* $N_H : L^{r(\cdot)}_{b(\cdot)}(\Gamma_2) \to L^1(\Gamma_2)$ *is continuous.*

Now we define the space

$$X = \{v \in W^{1,p(\cdot)}(\Omega); v = 0 \text{ on } \Gamma_1\}. \tag{2.1}$$

Then it is clear that X is a closed subspace of $W^{1,p(\cdot)}(\Omega)$, so X is a reflexive and separable Banach space. We can see the following Poincaré-type inequality (cf. [10]).

Proposition 2.14. *Let* Ω *be a bounded domain of* $\mathbb{R}^N$ *with a* $C^{0,1}$*-boundary and let* $p \in C_+(\overline{\Omega})$*. Then there exists a constant* $C = C(\Omega, N, p) > 0$ *such that*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \le C\|\nabla u\|_{\mathbf{L}^{p(\cdot)}(\Omega)} \quad \text{for all } u \in X.$$

In particular, $\|\nabla u\|_{\mathbf{L}^{p(\cdot)}(\Omega)}$ *is equivalent to* $\|u\|_{W^{1,p(\cdot)}(\Omega)}$ *for* $u \in X$*.*

For a proof of the above proposition see [5, Lemma 2.5].
Thus we can define the norm on X so that

$$\|v\|_X = \|\nabla v\|_{\mathbf{L}^{p(\cdot)}(\Omega)} \quad \text{for } v \in X, \tag{2.2}$$

which is equivalent to $\|v\|_{W^{1,p(\cdot)}(\Omega)}$ from Proposition 2.14.

3. Assumptions and main theorem

Let $p \in C_+(\overline{\Omega})$ be fixed. Assume that the following:

(A2) $A : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a function satisfying that for a.e. $x \in \Omega$, the function $A(x,\cdot) : \mathbb{R}^N \ni \boldsymbol{\xi} \mapsto A(x,\boldsymbol{\xi})$ is of C^1-class, and for all $\boldsymbol{\xi} \in \mathbb{R}^N$, the function $A(\cdot,\boldsymbol{\xi}) : \Omega \ni x \mapsto A(x,\boldsymbol{\xi})$ is measurable. Moreover, suppose that $A(x,\mathbf{0}) = 0$ and put $\mathbf{a}(x,\boldsymbol{\xi}) = \nabla_{\boldsymbol{\xi}} A(x,\boldsymbol{\xi})$. Then $\mathbf{a}(x,\boldsymbol{\xi})$ is a Carathéodory function.

For items (A3)–(A5), $c, k_0, k_1 > 0$ denote constants, $h_0 \in L^{p'(\cdot)}(\Omega)$ is a non-negative function, and $h_1 \in L^1_{\text{loc}}(\Omega)$ with $h_1(x) \ge 1$ for a.e. $x \in \Omega$.

(A3) $|\mathbf{a}(x,\boldsymbol{\xi})| \le c(h_0(x) + h_1(x)|\boldsymbol{\xi}|^{p(x)-1})$ for all $\boldsymbol{\xi} \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

(A4) A is $p(\cdot)$-uniformly convex, that is,

$$A\left(x, \frac{\boldsymbol{\xi}+\eta}{2}\right) + k_1 h_1(x)|\boldsymbol{\xi}-\eta|^{p(x)} \le \frac{1}{2}A(x,\boldsymbol{\xi}) + \frac{1}{2}A(x,\eta)$$

for all $\boldsymbol{\xi}, \eta \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

(A5) $k_0 h_1(x)|\boldsymbol{\xi}|^{p(x)} \le \mathbf{a}(x,\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \le p(x)A(x,\boldsymbol{\xi})$ for all $\boldsymbol{\xi} \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

(A6) $(\mathbf{a}(x,\boldsymbol{\xi}) - \mathbf{a}(x,\eta)) \cdot (\boldsymbol{\xi}-\eta) > 0$ for all $\boldsymbol{\xi}, \eta \in \mathbb{R}^N$ with $\boldsymbol{\xi} \ne \eta$ and a.e. $x \in \Omega$.

(A7) $A(x,-\boldsymbol{\xi}) = A(x,\boldsymbol{\xi})$ for all $\boldsymbol{\xi} \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

Remark 3.1. (i) The condition (A3) is more general than that of Mashiyev et al. [26] who considered the case $h_1(x) \equiv 1$. In our case, to overcome this we have to consider the space Y defined by (3.2) later as a basic space rather than the space X defined by (2.1).

(ii) (A5) implies that A is $p(\cdot)$-sub-homogeneous, that is,

$$A(x, s\boldsymbol{\xi}) \le A(x,\boldsymbol{\xi})s^{p(x)} \quad \text{for each } \boldsymbol{\xi} \in \mathbb{R}^N, \text{ a.e. } x \in \Omega \text{ and } s \ge 1. \tag{3.1}$$

For a proof, see Aramaki [7, (4.14)].

Example 3.2. Let

(i) $A(x,\boldsymbol{\xi}) = \frac{h(x)}{p(x)}|\boldsymbol{\xi}|^{p(x)}$ with $p^- \ge 2$, $h \in L^1_{\rm loc}(\Omega)$ satisfying $h(x) \ge 1$ a.e. $x \in \Omega$.

(ii) $A(x,\boldsymbol{\xi}) = \frac{h(x)}{p(x)}((1+|\boldsymbol{\xi}|^2)^{p(x)/2} - 1)$ with $p^- \ge 2$, $h \in L^{p'(\cdot)}(\Omega)$ satisfying $h(x) \ge 1$ a.e. $x \in \Omega$.

Then $A(x,\boldsymbol{\xi})$ and $\mathbf{a}(x,\boldsymbol{\xi}) = \boldsymbol{\nabla}_{\boldsymbol{\xi}} A(x,\boldsymbol{\xi})$ of (i) and (ii) satisfy (A2)–(A7).

Remark 3.3. In Example 3.2, when $h(x) \equiv 1$, (i) corresponds to the $p(\cdot)$-Laplacian and (ii) corresponds to the prescribed mean curvature operator for nonparametric surface.

For the function $h_1 \in L^1_{\rm loc}(\Omega)$ with $h_1(x) \ge 1$ for a.e. $x \in \Omega$, we define a modular on X by

$$\widetilde{\rho}_{(p(\cdot),h_1(\cdot))}(v) = \int_\Omega h_1(x)|\boldsymbol{\nabla} v(x)|^{p(x)}\,dx \quad \text{for } v \in X,$$

where the space X is defined by (2.1). We define our basic space

$$Y = Y(\Omega) = \{v \in X; \widetilde{\rho}_{(p(\cdot),h_1(\cdot))}(v) < \infty\} \tag{3.2}$$

equipped with the norm

$$\|v\|_Y = \inf\big\{\tau > 0; \widetilde{\rho}_{(p(\cdot),h_1(\cdot))}\big(\frac{v}{\tau}\big) \le 1\big\}.$$

Proposition 3.4. *The space $(Y, \|\cdot\|_Y)$ is a separable and reflexive Banach space.*

For a proof of the above propositon see Aramaki [8, Proposition 3.4]. We note that $C_0^\infty(\Omega) \subset Y$. Since $h_1(x) \ge 1$ a.e. $x \in \Omega$, it follows that

$$\widetilde{\rho}_{(p(\cdot),h_1(\cdot))}(v) = \rho_{p(\cdot)}(h_1^{1/p(\cdot)}|\boldsymbol{\nabla} v|) \ge \rho_{p(\cdot)}(|\boldsymbol{\nabla} v|) \text{ for } v \in Y$$

and

$$\|v\|_Y = \|h_1^{1/p(\cdot)}\boldsymbol{\nabla} v\|_{\mathbf{L}^{p(\cdot)}(\Omega)} \ge \|\boldsymbol{\nabla} v\|_{\mathbf{L}^{p(\cdot)}(\Omega)} = \|v\|_X \text{ for } v \in Y. \tag{3.3}$$

From (3.3) and Proposition 2.1, we have the following proposition.

Proposition 3.5. *Let $p \in C_+(\overline{\Omega})$ and let $u, u_n \in Y$ $(n = 1, 2, \dots)$. Then the following properties hold:*

- (i) $Y \hookrightarrow X$ *and* $\|u\|_X \le \|u\|_Y$.
- (ii) $\|u\|_Y > 1 (= 1, < 1) \Leftrightarrow \widetilde{\rho}_{(p(\cdot),h_1(\cdot))}(u) > 1 (= 1, < 1)$.
- (iii) $\|u\|_Y > 1 \Rightarrow \|u\|_Y^{p^-} \le \widetilde{\rho}_{(p(\cdot),h_1(\cdot))}(u) \le \|u\|_Y^{p^+}$.
- (iv) $\|u\|_Y < 1 \Rightarrow \|u\|_Y^{p^+} \le \widetilde{\rho}_{(p(\cdot),h_1(\cdot))}(u) \le \|u\|_Y^{p^-}$.
- (v) $\lim_{n\to\infty} \|u_n - u\|_Y = 0 \Leftrightarrow \lim_{n\to\infty} \widetilde{\rho}_{(p(\cdot),h_1(\cdot))}(u_n - u) = 0$.
- (vi) $\|u_n\|_Y \to \infty$ *as* $n \to \infty \Leftrightarrow \widetilde{\rho}_{(p(\cdot),h_1(\cdot))}(u_n) \to \infty$ *as* $n \to \infty$.

We assume that the function g in (1.1) satisfies

- (A8) The function $g(x, t)$ is of the form $g(x, t) = b(x)|t|^{r(x)-2}t$, where b satisfies $0 < b \in L^{\beta(\cdot)}(\Gamma_2)$ with $\beta \in C_+(\overline{\Gamma_2})$, and $r \in C_+(\overline{\Gamma_2})$ satisfies
$$r(x) < \frac{\beta(x) - 1}{\beta(x)} p^{\partial}(x) \quad \text{for all } x \in \overline{\Gamma_2}.$$

If we define
$$G(x, t) = \int_0^t g(x, s)\, ds,$$
then $G(x, t) = \frac{b(x)}{r(x)}|t|^{r(x)}$, so we have
$$r(x)G(x, t) = b(x)|t|^{r(x)} = g(x, t)t > 0 \tag{3.4}$$
for σ-a.e. $x \in \Gamma_2$ and all $0 \ne t \in \mathbb{R}$.

Now we introduce the notion of a weak solution and an eigenfunction for the problem (1.1).

Definition 3.6. (i) We say that a pair $(u, \lambda) \in Y \times \mathbb{R}$ is a weak solution of (1.1), if
$$M\Big(\int_\Omega A(x, \boldsymbol{\nabla} u(x))\, dx\Big) \int_\Omega \mathbf{a}(x, \nabla u(x)) \cdot \nabla v(x)\, dx = \lambda \int_{\Gamma_2} g(x, u(x))v(x) d\sigma_x \tag{3.5}$$
for all $v \in Y$.

(ii) Such a pair $(u, \lambda) \in Y \times \mathbb{R}$ with $u \ne 0$ is called an eigenpair, λ is called an eigenvalue and u is called an associated eigenfunction.

If we define a function associated with the function M by
$$\widehat{M}(t) = \int_0^t M(s)\, ds \quad \text{for } t \ge 0,$$
then we see that $\widehat{M} \in C^1([0, \infty))$ and satisfies
$$\frac{m_0}{l} t^l \le \widehat{M}(t) \le m_1 \left(t + \frac{1}{k} t^k\right) \text{ for } t \ge 0. \tag{3.6}$$
Moreover, since $\widehat{M}'(t) = M(t)$ is monotone non-decreasing and satisfies (1.3), $\widehat{M}(t)$ is convex and strictly monotone increasing on $[0, \infty)$.

We define functionals on Y by
$$\Phi(u) = \int_\Omega A(x, \boldsymbol{\nabla} u(x))\, dx, \quad \Psi(u) = \widehat{M}(\Phi(u)), \quad K(u) = \int_{\Gamma_2} G(x, u(x)) d\sigma_x \tag{3.7}$$
for $u \in Y$. It follows from (A7) and (A8) that Φ, Ψ and K are even functionals, that is, $\Phi(-u) = \Phi(u)$, $\Psi(-u) = \Psi(u)$ and $K(-u) = K(u)$ for all $u \in Y$.

Lemma 3.7. (i) *We have*

$$\frac{k_0}{p^+}\widetilde{\rho}_{(p(\cdot),h_1(\cdot))}(u) \le \Phi(u) \le c(2\|h_0\|_{L^{p'(\cdot)}(\Omega)}\|\nabla u\|_{\mathbf{L}^{p(\cdot)}(\Omega)} + \widetilde{\rho}_{(p(\cdot),h_1(\cdot))}(u))$$

for $u \in Y$, where c and k_0 are the constants in (A.1) *and* (A5).

(ii) *We have*

$$\Phi\Big(\frac{u+v}{2}\Big) + k_1\widetilde{\rho}_{(p(\cdot),h_1(\cdot))}(u-v) \le \frac{1}{2}\Phi(u) + \frac{1}{2}\Phi(v)$$

for all $u, v \in Y$, where k_1 is the constant in (A4). *In particular, Φ is convex, that is, $\Phi((1-\tau)u + \tau v) \le (1-\tau)\Phi(u) + \tau\Phi(v)$ for all $u, v \in Y$ and $\tau \in [0,1]$.*

Proof. (i) easily follows from (A5) and the Hölder inequality (Proposition 2.2). (ii) easily follows from (A4) and the continuity of $A(x, \boldsymbol{\xi})$ with respect to $\boldsymbol{\xi}$. □

The functional Ψ defined by (3.7) is a continuous modular on a real Banach space Y in the sense of [13, Definition 2.1.11], that is, Ψ has the following properties:

(a) $\Psi(0) = 0$. This easily follows from $A(x, \mathbf{0}) = 0$ and the definition of $\widehat{M}$.
(b) $\Psi(-u) = \Psi(u)$ for every $u \in Y$. This follows from (A7).
(c) Ψ is convex. Indeed, since $\widehat{M}$ is convex and strictly monotone increasing and Φ is convex, for any $u, v \in Y$ and $\tau \in [0,1]$ we have

$$\begin{aligned}\Psi((1-\tau)u + \tau v) &= \widehat{M}(\Phi((1-\tau)u + \tau v)) \\ &\le \widehat{M}((1-\tau)\Phi(u) + \tau\Phi(v)) \\ &\le (1-\tau)\Psi(u) + \tau\Psi(v).\end{aligned}$$

(d) The function $[0,\infty) \ni \lambda \mapsto \Psi(\lambda u)$ is continuous for every $u \in Y$. Indeed, let $[0,\infty) \ni \lambda_n \to \lambda_0$ as $n \to \infty$. Here we can assume that $0 \le \lambda_n \le \lambda_0 + 1$ for large $n \in \mathbb{N}$. From (A.0) and (A5), we have

$$|A(x, \lambda_n \nabla u(x))| \le c(\lambda_0 + 1)h_0(x)|\nabla u(x)| + c(\lambda_0+1)^{p^+}h_1(x)|\nabla u(x)|^{p(x)}.$$

Since $h_0 \in L^{p'(\cdot)}(\Omega)$ and $|\nabla u(\cdot)| \in L^{p(\cdot)}(\Omega)$ and $u \in Y$, the right-hand side in the above inequality is an integrable function independent of n. Clearly, we see that $A(x, \lambda_n \nabla u(x)) \to A(x, \lambda_0 \nabla u(x))$ as $n \to \infty$ for a.e. $x \in \Omega$. By the Lebesgue dominated convergent theorem, we see that $\Phi(\lambda_n u) \to \Phi(\lambda_0 u)$ as $n \to \infty$, so $\Psi(\lambda_n u) \to \Psi(\lambda_0 u)$.
(e) $\Psi(u) = 0$ implies $u = 0$. Indeed, if $\Psi(u) = 0$, then $\Phi(u) = 0$. Hence it follows from (A5) and the Poincaré-type inequality (Proposition 2.14) that $u = 0$. .

Thus we can define a modular space

$$Y_\Psi = \{u \in Y; \lim_{\tau\to 0}\Psi(\tau u) = 0\} = \{u \in Y; \Psi(\tau u) < \infty \text{ for some } \tau > 0\}$$

with the Luxemburg norm

$$\|u\|_\Psi = \inf\big\{\tau > 0; \Psi\big(\frac{u}{\tau}\big) \le 1\big\} \quad \text{for } u \in Y_\Phi.$$

Then $(Y_\Psi, \|\cdot\|_\Psi)$ is a normed linear space over $\mathbb{R}$ from [13, Theorem 2.1.7]. Clearly we see that $Y_\Psi = Y$, and the norms $\|\cdot\|_\Psi$ and $\|\cdot\|_Y$ are equivalent (cf. [8, Lemma 4.3]).

From now on, we denote $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$ for any real numbers a and b. Since

$$\Phi(u) \geq \frac{k_0}{p^+} \int_\Omega h_1(x) |\nabla u(x)|^{p(x)} \, dx \geq \frac{k_0}{p^+} (\|u\|_Y^{p^+} \wedge \|u\|_Y^{p^-}),$$

it follows from (3.6) that

$$\Psi(u) = \widehat{M}(\Phi(u)) \geq \frac{m_0}{l} \Big(\frac{k_0}{p^+} (\|u\|_Y^{p^+} \wedge \|u\|_Y^{p^-}) \Big)^l, \tag{3.8}$$

Lemma 3.8. *If $u_n \to u$ weakly in Y and $\Psi(u_n) \to \Psi(u)$ as $n \to \infty$, then we have $\Psi\left(\frac{u_n - u}{2}\right) \to 0$ as $n \to \infty$. In particular, $u_n \to u$ strongly in Y as $n \to \infty$.*

Proof. Let $u_n \to u$ weakly in Y and $\Psi(u_n) \to \Psi(u)$ as $n \to \infty$. Then, if we use [13, Lemma 2.4.17] (cf. Aramaki [9, Lemma 20]), then we can show that $\Psi\left(\frac{u_n - u}{2}\right) \to 0$ as $n \to \infty$, so $u_n \to u$ strongly in Y using (3.8). □

First we list the properties of Ψ.

Proposition 3.9. (i) *Ψ is coercive, that is, $\Psi(u) \to \infty$ as $\|u\|_Y \to \infty$.*

(ii) *Ψ is sequentially weakly lower-semicontinuous on Y.*

(iii) *$\Psi \in C^1(Y, \mathbb{R})$ and the Fréchet derivative Ψ' of Ψ satisfies*

$$\langle \Psi'(u), v \rangle_{Y^*, Y} = M(\Phi(u)) \int_\Omega \mathbf{a}(x, \nabla u(x)) \cdot \nabla v(x) \, dx \quad \text{for } u, v \in Y. \tag{3.9}$$

(iv) *$\Psi \in \mathcal{W}_Y$, that is, if $u_n \to u$ weakly in Y and $\liminf_{n\to\infty} \Psi(u_n) \leq \Psi(u)$, then the sequence $\{u_n\}$ has a strongly convergent subsequence.*

(v) *Ψ is bounded on every bounded subset of Y.*

Proof. (i) follows from (3.8). (ii) follows from Aramaki [7, Proposition 4.4] and the fact that $\widehat{M}$ is monotone increasing and continuous. (iii) follows from [7, Proposition 4.1] and $\widehat{M} \in C^1([0, \infty))$.

(iv) Let $u_n \to u$ weakly in Y and $\liminf_{n\to\infty} \Psi(u_n) \leq \Psi(u)$. Since Ψ is sequentially weakly lower semi-continuous, $\Psi(u) \leq \liminf_{n\to\infty} \Psi(u_n)$, so that $\liminf_{n\to\infty} \Psi(u_n) = \Psi(u)$. Hence there exists a subsequence $\{u_{n'}\}$ of $\{u_n\}$ such that $\lim_{n'\to\infty} \Psi(u_{n'}) = \Psi(u)$. By Lemma 3.8, we see that $u_{n'} \to u$ strongly in Y.

(v) follows from Lemma 3.7 (i) and (3.6). □

Next we derive the properties of Ψ'.

Proposition 3.10. (i) *Ψ' is strictly monotone in Y, that is,*

$$\langle \Psi'(u) - \Psi'(v), u - v \rangle_{Y^*, Y} > 0 \quad \text{for all } u, v \in Y \text{ with } u \neq v.$$

Moreover, Ψ' is bounded on every bounded subset of Y and coercive in the sense that

$$\lim_{\|u\|_Y \to \infty} \frac{\langle \Psi'(u), u \rangle_{Y^*, Y}}{\|u\|_Y} = \infty.$$

(ii) *Ψ' is of (S_+)-type, that is, if $u_n \to u$ weakly in Y and*

$$\limsup_{n\to\infty} \langle \Psi'(u_n), u_n - u \rangle_{Y^*. Y} \leq 0,$$

then $u_n \to u$ strongly in Y.

(iii) *The mapping $\Psi' : Y \to Y^*$ is a homeomorphism.*

Proof. (i) In general, when a functional $f : Y \to \mathbb{R}$ is of C^1-class, f is strictly convex if and only if $f' : Y \to Y^*$ is strictly monotone (cf. Zeidler [34, Proposition 25.10]), that is,

$$\langle f'(u) - f'(v), u - v\rangle_{Y^*,Y} > 0 \quad \text{for all } u, v \in Y \text{ with } u \neq v.$$

From (A6),

$$\langle \Phi'(u)-\Phi'(v), u-v\rangle_{Y^*,Y} = \int_\Omega (\mathbf{a}(x, \nabla u(x)-\mathbf{a}(x, \nabla v(x)))\cdot(\nabla u(x)-\nabla v(x))\,dx > 0$$

for all $u, v \in Y$ with $u \neq v$, so Φ' is strictly monotone in Y, so Φ is strictly convex. The function $\widehat{M}$ is strictly monotone increasing and convex. Hence for $u, v \in Y$ with $u \neq v$ and $\tau \in (0, 1)$, since $\Phi((1-\tau)u + \tau v) < (1-\tau)\Phi(u) + \tau\Phi(v)$, we have

$$\widehat{M}(\Phi(1-\tau)u + \tau v)) < \widehat{M}((1-\tau)\Phi(u) + \tau\Phi(v)) \le (1-\tau)\widehat{M}(\Phi(u)) + \tau\widehat{M}(\Phi(v)),$$

so $\Psi((1-\tau)u + \tau v) < (1-\tau)\Psi(u) + \tau\Psi(v)$. Thus Ψ is strictly convex, so $\Psi'(\cdot) = M(\Phi(\cdot))\Phi'(\cdot)$ is strictly monotone in Y.

It follows from the Hölder inequality (Proposition 2.2) and Proposition 3.5 (i) that

$$\begin{aligned}
&|\langle \Psi'(u), v\rangle_{Y^*,Y}| \\
&= M(\Phi(u))\Big|\int_\Omega \mathbf{a}(x, \nabla u(x))\cdot\nabla v(x)\,dx\Big| \\
&\le cM(\Phi(u))\int_\Omega (h_0(x)|\nabla v(x)| + h_1(x)|\nabla u(x)|^{p(x)-1}|\nabla v(x)|)\,dx \\
&= cM(\Phi(u))\int_\Omega (h_0(x)|\nabla v(x)| + h_1(x)^{1/p'(x)}|\nabla u(x)|^{p(x)-1}h_1(x)^{1/p(x)}|\nabla v(x)|)\,dx \\
&\le 2cm_1(1 + \Phi(u)^{k-1})(\|h_0\|_{L^{p'(\cdot)}(\Omega)}\|v\|_Y \\
&\quad + \|h_1^{1/p'(\cdot)}|\nabla u|^{p(\cdot)-1}\|_{L^{p'(\cdot)}(\Omega)}\|h_1^{1/p(\cdot)}|\nabla v|\|_{L^{p(\cdot)}(\Omega)} \\
&= 2cm_1(1 + \Phi(u)^{k-1})(\|h_0\|_{L^{p'(\cdot)}(\Omega)} + \|h_1^{1/p'(\cdot)}|\nabla u|^{p(\cdot)-1}\|_{L^{p'(\cdot)}(\Omega)})\|v\|_Y
\end{aligned}$$

for all $v \in Y$. Hence we have

$$\|\Psi'(u)\|_{Y^*} \le 2cm_1(1 + \Phi(u)^{k-1})(\|h_0\|_{L^{p'(\cdot)}(\Omega)} + \|h_1^{1/p'(\cdot)}|\nabla u|^{p(\cdot)-1}\|_{L^{p'(\cdot)}(\Omega)}).$$

Here we note that

$$\begin{gathered}
\Phi(u)^{k-1} \le c^{k-1}(2\|h_0\|_{L^{p'(\cdot)}(\Omega)}\|u\|_Y + \|u\|_Y^{p^+} \vee \|u\|_Y^{p^-})^{k-1}, \\
\rho_{p'(\cdot)}(h_1^{1/p'(\cdot)}|\nabla u|^{p(\cdot)-1}) = \int_\Omega h_1(x)|\nabla u(x)|^{p(x)}\,dx \le \|u\|_Y^{p^+} \vee \|u\|_Y^{p-}.
\end{gathered}$$

If $\|u\| \le M$, then it is clear that there exists a constant $C(A1) > 0$ such that $\|\Psi'(u)\|_{Y^*} \le C(A1)$, so Ψ' is bounded on every bounded subset of Y.

Let $\|u\|_Y > 1$. Then from (A1) and (A5),

$$\begin{aligned}
\langle \Psi'(u), u\rangle_{Y^*,Y} &= M(\Phi(u))\int_\Omega \mathbf{a}(x, \nabla u(x)\cdot\nabla u(x)\,dx \\
&\ge k_0M(\Phi(u))\int_\Omega h_1(x)|\nabla u(x)|^{p(x)}\,dx \\
&\ge \frac{k_0^l}{(p^+)^{l-1}}m_0\|u\|_Y^{(l-1)p^-}\|u\|_Y^{p^-}
\end{aligned}$$

$$= \frac{m_0 k_0^l}{(p^+)^{l-1}} \|u\|_Y^{lp^-}.$$

Since $lp^- > 1$, this implies the coerciveness of Ψ'.

(ii) Let $u_n \to u$ weakly in Y and $\limsup_{n\to\infty} \langle \Psi'(u_n), u_n - u\rangle_{Y^*,Y} \le 0$. Since Ψ' is monotone from (i), $\langle \Psi'(u_n) - \Psi'(u), u_n - u\rangle_{Y^*,Y} \ge 0$. Hence

$$\begin{aligned} 0 &\le \liminf_{n\to\infty} \langle \Psi'(u_n) - \Psi'(u), u_n - u\rangle_{Y^*,Y} \\ &= \liminf_{n\to\infty} \langle \Psi'(u_n), u_n - u\rangle_{Y^*,Y} \\ &\le \limsup_{n\to\infty} \langle \Psi'(u_n), u_n - u\rangle_{Y^*,Y} \le 0. \end{aligned}$$

Therefore, $\lim_{n\to\infty} M(\Phi(u_n))\langle \Phi'(u_n), u_n - u\rangle_{Y^*,Y} = 0$. Since $u_n \to u$ weakly in Y, the sequence $\{\|u_n\|_Y\}$ is bounded. Hence since $M(\Phi(u_n))$ is bounded from Lemma 3.7 (i), we have $\lim_{n\to\infty} M(\Phi(u_n))\langle \Phi'(u), u_n - u\rangle_{Y^*,Y} = 0$. Therefore,

$$\lim_{n\to\infty} M(\Phi(u_n))\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle_{Y^*,Y} = 0.$$

Thereby, since $M(\Phi(u_n)) \ge 0$ and $\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle_{Y^*,Y} \ge 0$, we obtain that $\lim_{n\to\infty} M(\Phi(u_n)) = 0$ or $\lim_{n\to\infty} \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle_{Y^*,Y} = 0$. Indeed, if we put $a_n = M(\Phi(u_n))$ and $b_n = \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle_{Y^*,Y}$, then it suffices to derive that $a_n \ge 0$, $b_n \ge 0$ and $\lim_{n\to\infty} a_n b_n = 0$ implies that $\lim_{n\to\infty} a_n = 0$ or $\lim_{n\to\infty} b_n = 0$. For any subsequence $\{n'\}$ of $\mathbb{N}$, we have $\lim_{n'\to\infty} a_{n'} b_{n'} = 0$. If $\lim_{n'\to\infty} a_{n'}$ does not exist or exists and is equal to a positive number, then there exist $\varepsilon_0 > 0$ and a subsequence $\{a_{n''}\}$ of $\{a_{n'}\}$ such that $a_{n''} \ge \varepsilon_0$ for any $a_{n''}$. Hence we have $a_{n''} b_{n''} \ge \varepsilon_0 b_{n''} \ge 0$. Since $\lim_{n''\to\infty} a_{n''} b_{n''} = 0$, we see that $\lim_{n''\to\infty} b_{n''} = 0$, so according to the convergent principal we have $\lim_{n\to\infty} b_n = 0$. If $\lim_{n'\to\infty} a_{n'} = 0$ for any subsequence $\{a_{n'}\}$, then we clearly have $\lim_{n\to\infty} a_n = 0$.

When $M(\Phi(u_n)) \to 0$ as $n \to \infty$, we have $\Phi(u_n) \to 0 = \Phi(0)$. By Lemma 3.8 with $M \equiv 1$, $u_n \to 0$ strongly in Y (in this case we necessarily have $u = 0$). When

$$\lim_{n\to\infty} \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle_{Y^*,Y} = \lim_{n\to\infty} \langle \Phi'(u_n), u_n - u\rangle_{Y^*,Y} = 0,$$

since Φ' is of (S_+)-type (cf. [9, Proposition 21 (ii)]), we have $u_n \to u$ strongly in Y.

(iii) Since Ψ' is strictly monotone from (i), Ψ' is injective. We show that $\Psi' : Y \to Y^*$ is surjective. Let $w \in Y^*$. Define a functional on Y by

$$\varphi(u) := \Psi(u) - \langle w, u\rangle_{Y^*,Y} \text{ for } u \in Y.$$

From (A1) and Lemma 3.7 (i), for $\|u\|_Y > 1$, we see that

$$\varphi(u) \ge \widehat{M}(\Phi(u)) - \langle w, u\rangle_{Y^*,Y} \ge \Big(\frac{k_0}{p^+}\Big)^l \|u\|_Y^{lp^-} - \|w\|_{Y^*}\|u\|_Y.$$

Since $lp^- > 1$, φ is coercive. Since Ψ is sequentially weakly lower semi-continuous, φ is so. If we put $\gamma = \inf_{u\in Y} \varphi(u) (< \infty)$, then there exists a sequence $\{u_n\} \subset Y$ such that $\gamma = \lim_{n\to\infty} \varphi(u_n)$. Since φ is coercive, the sequence $\{u_n\}$ is bounded. Since Y is a reflexive Banach space, there exist a subsequence $\{u_{n'}\}$ of $\{u_n\}$ and $u_0 \in Y$ such that $u_{n'} \to u_0$ weakly in Y, so $\varphi(u_0) \le \liminf_{n'\to\infty} \varphi(u_{n'}) = \gamma$. This implies that $\gamma > -\infty$ and u_0 is a minimizer of φ, so $\varphi'(u_0) = 0$, i.e., $\Psi'(u_0) = w$. Therefore, Ψ' has an inverse operator $(\Psi')^{-1} : Y^* \to Y$. We show that $(\Psi')^{-1}$ is continuous. Let $f_n \to f$ in Y^* as $n \to \infty$. Then there exist $u_n, u \in Y$ such that $\Psi'(u_n) = f_n$ and $\Psi'(u) = f$. Then $\{u_n\}$ is bounded in Y. Indeed, if $\{u_n\}$ is unbounded, then there exists a subsequence $\{u_{n'}\}$ of $\{u_n\}$ such that $\|u_{n'}\|_Y \to \infty$ as $n' \to \infty$. Hence

$$\langle \Psi'(u_{n'}), u_{n'} \rangle_{Y^*,Y} = \langle f_{n'}, u_{n'} \rangle_{Y^*,Y} \le \|f_{n'}\|_{Y^*} \|u_{n'}\|_Y \le C\|u_{n'}\|_Y$$

for some constant $C > 0$. This contradict the coerciveness of Ψ'.

Since Y is a reflexive Banach space, there exist a subsequence (still denoted by $\{u_{n'}\}$) and $u_0 \in Y$ such that $u_{n'} \to u_0$ weakly in Y. Hence

$$\begin{aligned}\lim_{n'\to\infty} \langle \Psi'(u_{n'}), u_{n'} - u_0 \rangle_{Y^*,Y} &= \lim_{n'\to\infty} \langle \Psi'(u_{n'}) - \Psi'(u), u_{n'} - u_0 \rangle_{Y^*,Y} \\ &= \lim_{n'\to\infty} \langle f_{n'} - f, u_{n'} - u_0 \rangle_{Y^*,Y} = 0.\end{aligned}$$

Since Ψ' is of (S_+)-type, we see that $u_{n'} \to u_0$ strongly in Y. According to the continuity of Ψ', $\Psi'(u_{n'}) = f_{n'} \to f = \Psi'(u_0) = \Psi'(u)$, so we have $u_0 = u$ from the injectiveness of Ψ'. By the convergent principle (cf. [34, Theorem 10.13 (i)]), for full sequence $\{u_n\}$, $u_n \to u$ strongly in Y, that is, $(\Psi')^{-1}(f_n) \to (\Psi')^{-1}(f)$ as $n \to \infty$. □

For the functional K defined by (3.7), we have the following proposition.

Proposition 3.11. *Under hypotheses* (A8)*, we have the following.*

(i) $K \in C^1(Y, \mathbb{R})$ *and*

$$\langle K'(u), v \rangle_{Y^*,Y} = \int_{\Gamma_2} g(x, u(x)) v(x) \, d\sigma_x \quad \text{for } u, v \in Y. \tag{3.10}$$

(ii) K *is sequentially weakly continuous in* Y.

(iii) $K' : Y \to Y^*$ *is weakly-strongly continuous, that is, if* $u_n \to u$ *weakly in* Y *as* $n \to \infty$*, then* $K'(u_n) \to K'(u)$ *strongly in* Y^* *as* $n \to \infty$.

Proof. (i) and (ii) follows from Aramaki [7, Proposition 4.2, Proposition 4.4]. So we only verify (iii). Let $u_n \to u$ weakly in Y. Then

$$\langle K'(u_n) - K'(u), v \rangle_{Y^*,Y} = \int_{\Gamma_2} (g(x, u_n(x)) - g(x, u(x))) v(x) d\sigma_x \quad \text{for } v \in Y.$$

From Proposition 2.11 and (A8), the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}_{b(\cdot)}(\Gamma_2)$ is compact. Since $Y \hookrightarrow X \hookrightarrow W^{1,p(\cdot)}(\Omega)$, there exists a constant $C > 0$ such that

$$\|v\|_{L^{r(\cdot)}_{b(\cdot)}(\Gamma_2)} \le C\|v\|_Y \quad \text{for all } v \in Y.$$

By the Hölder inequality (Proposition 2.5), for any $v \in Y$, we have

$$\begin{aligned}&|\langle K'(u_n) - K'(u), v \rangle_{Y^*,Y}| \\ &\le \int_{\Gamma_2} b(x)^{-1/r(x)} |g(x, u_n(x)) - g(x, u(x))| b(x)^{1/r(x)} |v(x)| d\sigma_x \\ &\le 2\|b(\cdot)^{-1/r(\cdot)} |g(\cdot, u_n(\cdot)) - g(\cdot, u(\cdot))|\|_{L^{r'(\cdot)}(\Gamma_2)} \|b(\cdot)^{1/r(\cdot)} |v(\cdot)|\|_{L^{r(\cdot)}(\Gamma_2)}.\end{aligned}$$

Since

$$\|b(\cdot)^{1/r(\cdot)} v(\cdot)\|_{L^{r(\cdot)}(\Gamma_2)} = \|v\|_{L^{r(\cdot)}_{b(\cdot)}(\Gamma_2)} \le C\|v\|_Y,$$

we have

$$\|K'(u_n) - K'(u)\|_{Y^*} \le 2C\|b(\cdot)^{-1/r(\cdot)} |g(\cdot, u_n(\cdot)) - g(\cdot, u(\cdot))|\|_{L^{r'(\cdot)}(\Gamma_2)}.$$

We want to show that $\|K'(u_n) - K'(u)\|_{Y^*} \to 0$ as $n \to \infty$. By Proposition 2.4 (iv), it suffices to show that

$$\rho_{r'(\cdot),\Gamma_2}\Big(b(\cdot)^{-1/r(\cdot)}g(\cdot,u_n(\cdot)) - b(\cdot)^{-1/r(\cdot)}g(\cdot,u(\cdot))\Big) \to 0 \quad \text{as } n \to \infty. \tag{3.11}$$

We can see that

$$\begin{aligned}
&\rho_{r'(\cdot),\Gamma_2}\Big(b(\cdot)^{-1/r(\cdot)}g(\cdot,u_n(\cdot)) - b(\cdot)^{-1/r(\cdot)}g(\cdot,u(\cdot))\Big)\\
&= \int_{\Gamma_2} b(x)^{-r'(x)/r(x)}|g(x,u_n(x)) - g(x,u(x))|^{r'(x)}d\sigma_x.
\end{aligned}$$

Since $u_n \to u$ weakly in Y and the embedding map $Y \hookrightarrow L^{r(\cdot)}_{b(\cdot)}(\Gamma_2)$ is compact, we can see that $u_n \to u$ strongly in $L^{r(\cdot)}_{b(\cdot)}(\Gamma_2)$. From [6, Theorem A.1], there exist a subsequence $\{u_{n'}\}$ of $\{u_n\}$ and $f \in L^{r(\cdot)}(\Gamma_2)$ such that $b(x)^{1/r(x)}u_{n'}(x) \to b(x)^{1/r(x)}u(x)$ σ-a.e. $x \in \Gamma_2$ and $|b(x)^{1/r(x)}u_{n'}(x)| \le f(x)$ for σ-a.e. $x \in \Gamma_2$. Since $b(x) > 0$ σ-a.e. $x \in \Gamma_2$, $u_{n'}(x) \to u(x)$ σ-a.e. $x \in \Gamma_2$, so we see that $g(x,u_{n'}(x)) \to g(x,u(x))$ σ-a.e. $x \in \Gamma_2$. From (A8), we have

$$\begin{aligned}
&b(x)^{-r'(x)/r(x)}|g(x,u_{n'}(x) - g(x,u(x))|^{r'(x)}\\
&\le b(x)^{-r'(x)/r(x)}(b(x)|u_{n'}(x)|^{r(x)-1} + b(x)|u(x)|^{r(x)-1})^{r'(x)}\\
&\le b(x)^{r'(x)-r'(x)/r(x)}(|u_{n'}(x)|^{r(x)} + |u(x)|^{r(x)})\\
&\le b(x)(|u_{n'}(x)|^{r(x)} + |u(x)|^{r(x)})\\
&\le 2f(x)^{r(x)}.
\end{aligned}$$

The last term is an integrable function in Ω independent of n'. Thus by the Lebesgue dominated convergence theorem, we have

$$\rho_{r'(\cdot),\Gamma_2}\Big(b(\cdot)^{-1/r(\cdot)}g(\cdot,u_{n'}(\cdot)) - b(\cdot)^{-1/r(\cdot)}g(\cdot,u(\cdot))\Big) \to 0 \quad \text{as } n' \to \infty.$$

From the convergent principle [34, Proposition 10.13], we see that (3.11) holds, so $\|K'(u_n) - K'(u)\|_{Y^*} \to 0$ as $n \to \infty$. □

Remark 3.12. From (3.9), (3.10) and Definition 3.6, we can see that $(u,\lambda) \in Y \times \mathbb{R}$ is a weak solution of (1.1) if and only if

$$\Psi'(u) = \lambda K'(u). \tag{3.12}$$

In particular, we have $\langle \Psi'(u), u\rangle_{Y^*,Y} = \lambda\langle K'(u), u\rangle_{Y^*,Y}$. If (u,λ) is an eigenpair of (1.1), then from (A5), (A1) and (A8)it follows that

$$\begin{aligned}
&\langle \Psi'(u), u\rangle_{Y^*,Y}\\
&= M(\Phi(u))\int_\Omega \mathbf{a}(x,\nabla u(x))\cdot\nabla u(x)\,dx\\
&\ge m_0\Big(\int_\Omega A(x,\nabla u(x))\,dx\Big)^{l-1}\int_\Omega \mathbf{a}(x,\nabla u(x))\cdot\nabla u(x)\,dx\\
&\ge m_0\Big(\int_\Omega \frac{1}{p(x)}\mathbf{a}(x,\nabla u(x))\cdot\nabla u(x)\,dx\Big)^{l-1}\int_\Omega \mathbf{a}(x,\nabla u(x))\cdot\nabla u(x)\,dx\\
&\ge \frac{m_0}{(p^+)^{l-1}}\Big(\int_\Omega \mathbf{a}(x,\nabla u(x))\cdot\nabla u(x)\,dx\Big)^l
\end{aligned}$$

$$\geq \frac{m_0 k_0^l}{(p^+)^{l-1}}\Big(\int_\Omega h_1(x)|\nabla u(x)|^{p(x)}\,dx\Big)^l$$
$$\geq \frac{m_0 k_0^l}{(p^+)^{l-1}}(\|u\|_Y^{p^+}\wedge \|u\|_Y^{p^-})^l > 0$$

and from (3.12) and (3.4),

$$\langle K'(u), u\rangle_{Y^*,Y} = \int_{\Gamma_2} g(x,u(x))u(x)d\sigma_x > 0,$$

so we have

$$\lambda = \frac{\langle \Psi'(u), u\rangle_{Y^*,Y}}{\langle K'(u), u\rangle_{Y^*,Y}} > 0. \tag{3.13}$$

This means that any eigenvalue of problem (1.1) is positive.

To solve the eigenvalue problem (3.12), we apply the constrained variational method. We take Ψ as an objective functional and K as a constraint functional. For any fixed $\alpha > 0$, put

$$M_\alpha = \{u \in Y; K(u) = \alpha\}. \tag{3.14}$$

If $u \in M_\alpha$, then from (A8),

$$\begin{aligned}\langle K'(u), u\rangle_{Y^*,Y} &= \int_{\Gamma_2} g(x,u(x))u(x)d\sigma_x \\ &\geq r^- \int_{\Gamma_2} G(x,u(x))d\sigma_x \\ &= r^- K(u) = r^- \quad \alpha > 0,\end{aligned} \tag{3.15}$$

so $K'(u) \neq 0$. Hence M_α is a C^1-submanifold of Y with codimension one. Moreover, M_α is weakly closed subset of Y. Indeed, let $u_j \in M_\alpha$ and $u_j \to u$ weakly in Y as $j \to \infty$. Since K is sequentially weakly continuous from Proposition 3.11 (ii), $\alpha = K(u_j) \to K(u)$, so $u \in M_\alpha$.

It is well known that when $u \in M_\alpha$, a pair $(u,\lambda) \in Y \times \mathbb{R}$ solves (3.12) if and only if u is a critical point of Ψ with respect to M_α, that is,

$$\langle \Psi'(u), h\rangle_{Y^*,Y} = 0 \quad \text{for all } h \in T_u M_\alpha,$$

(see for example [34, Proposition 43.21]). Here $T_u M_\alpha$ is the tangent space of M_α at $u \in M_\alpha$ and we can see that

$$T_u M_\alpha = \mathrm{Ker}(K'(u)) = \{v \in Y; \langle K'(u), v\rangle_{Y^*,Y} = 0\}.$$

Let $P : Y \to T_u M_\alpha$ be the natural projection. Note that the bounded linear map $K'(u) : Y \to \mathbb{R}$ is surjective. We denote the restriction of Ψ to M_α by $\widetilde{\Psi} = \Psi\big|_{M_\alpha}$ and the derivative $d\widetilde{\Psi}(u) \in Y^*$ of $\widetilde{\Psi}$ at $u \in M_\alpha$ can be defined by $\langle d\widetilde{\Psi}(u), v\rangle_{Y^*,Y} = \langle \Psi'(u), Pv\rangle_{Y^*,Y}$ for $v \in Y$.

For $u \in M_\alpha$, put $w = (\Psi')^{-1}(K'(u))$. Then since we have (3.15), we see that $K'(u) \neq 0$. From (A7), the functional Ψ is even, so Ψ' is odd and so $\Psi'(0) = 0$. Since $(\Psi')^{-1}$ is injective, we have $w \neq 0$. From strict monotonicity of Ψ' (Proposition 3.11 (i)),

$$\langle K'(u), w\rangle_{Y^*,Y} = \langle K'(u), (\Psi')^{-1}(K'(u))\rangle_{Y^*,Y} = \langle \Psi'(w), w\rangle_{Y^*,Y} > 0. \tag{3.16}$$

Hence since $w = (\Psi')^{-1}(K'(u)) \notin T_u M_\alpha$, we can see that

$$Y = T_u M_\alpha \oplus \{\beta(\Psi')^{-1}(K'(u)); \beta \in \mathbb{R}\}.$$

For every $v \in Y$, there exists a unique $\beta \in \mathbb{R}$ such that $v = Pv + \beta(\Psi')^{-1}(K'(u))$. Since $Pv \in T_u M_\alpha = \mathrm{Ker}(K'(u))$, we have

$$\langle K'(u), v\rangle_{Y^*,Y} = \beta\langle K'(u), (\Psi')^{-1}(K'(u))\rangle_{Y^*,Y}.$$

Thus from (3.14), we can write

$$\beta = \frac{\langle K'(u), v\rangle_{Y^*,Y}}{\langle K'(u), (\Psi')^{-1}(K'(u))\rangle_{Y^*,Y}}.$$

Now we have

$$\begin{aligned}
&\langle d\widetilde{\Psi}(u), v\rangle_{Y^*,Y} \\
&= \langle \Psi'(u), Pv\rangle_{Y^*,Y} \\
&= \langle \Psi'(u), v\rangle_{Y^*,Y} - \Big\langle \Psi'(u), \frac{\langle K'(u), v\rangle_{Y^*,Y}}{\langle K'(u), (\Psi')^{-1}(K'(u))\rangle_{Y^*,Y}}(\Psi')^{-1}(K'(u))\Big\rangle_{Y^*,Y} \\
&= \Big\langle \Psi'(u) - \frac{\langle \Psi'(u), (\Psi')^{-1}(K'(u))\rangle_{Y^*,Y}}{\langle K'(u), (\Psi')^{-1}(K'(u))\rangle_{Y^*,Y}} K'(u), v\Big\rangle_{Y^*,Y} \quad \text{for all } v \in Y.
\end{aligned}$$

Thus we have

$$d\widetilde{\Psi}(u) = \Psi'(u) - \lambda(u)K'(u),$$

where

$$\lambda(u) = \frac{\langle \Psi'(u), (\Psi')^{-1}(K'(u))\rangle_{Y^*,Y}}{\langle K'(u), (\Psi')^{-1}(K'(u))\rangle_{Y^*,Y}}.$$

Proposition 3.13. *For each $\alpha > 0$, the functional $\widetilde{\Psi} : M_\alpha \to \mathbb{R}$ satisfies $(\mathrm{PS})_c$-condition for any $c \in \mathbb{R}$, that is, if any sequence $\{u_n\} \subset M_\alpha$ such that $\widetilde{\Psi}(u_n) \to c$ and $\|d\widetilde{\Psi}(u_n)\|_{Y^*} \to 0$ as $n \to \infty$, then $\{u_n\}$ contains a convergent subsequence.*

Proof. Let $\{u_n\} \subset M_\alpha$ satisfy that $\widetilde{\Psi}(u_n) \to c$ and $d\widetilde{\Psi}(u_n) \to 0$ in Y^* as $n \to \infty$. Then since from (3.6) and (A5),

$$\begin{aligned}
\widetilde{\Psi}(u_n) &= \widehat{M}(\Phi(u_n)) \\
&\geq \frac{m_0}{l}\Big(\frac{k_0}{p^+}\int_\Omega h_1(x)|\nabla u_n(x)|^{p(x)}\,dx\Big)^l \\
&\geq \frac{m_0}{l}\Big(\frac{k_0}{p^+}\|u_n\|_Y^{p^+} \wedge \|u_n\|_Y^{p^-}\Big)^l,
\end{aligned}$$

$\{u_n\}$ is bounded in Y. Since Y is a reflexive Banach space from Proposition 3.4, there exist a subsequence $\{u_{n'}\}$ of $\{u_n\}$ and $u_0 \in Y$ such that $u_{n'} \to u_0$ weakly in Y. By Proposition 3.11 (ii) and (iii), $K'(u_{n'}) \to K'(u_0)$ in Y^* and $K(u_{n'}) \to K(u_0)$ as $n \to \infty$. Thereby, $u_0 \in M_\alpha$. Put $w_{n'} = (\Psi')^{-1}(K'(u_{n'}))$. Since $K'(u_{n'}) \to K'(u_0) \neq 0$ in Y^* from (3.15), we see that $w_{n'} \to w_0 \neq 0$ in Y, where $w_0 = (\Psi')^{-1}(K'(u_0))$. Thus

$$\begin{aligned}
&\langle K'(u_{n'}), (\Psi')^{-1}(K'(u_{n'}))\rangle_{Y^*,Y} \\
&= \langle \Psi'(w_{n'}), w_{n'}\rangle_{Y^*,Y} \to \langle \Psi'(w_0), w_0\rangle_{Y^*,Y} > 0.
\end{aligned} \tag{3.17}$$

On the other hand,

$$|\langle \Psi'(u_{n'}), (\Psi')^{-1}(K'(u_{n'}))\rangle_{Y^*,Y}| = |\langle \Psi'(u_{n'}), w_{n'}\rangle_{Y^*,Y}| \leq \|\Psi'(u_{n'})\|_{Y^*}\|w_{n'}\|_Y.$$

Since $u_{n'} \to u_0$ weakly in Y, we see that $\{u_{n'}\}$ is bounded in Y, so by Proposition 3.10 (i), $\|\Psi'(u_{n'})\|_{Y^*}$ is bounded. Hence, there exists a constant $c_2 > 0$ such that

$$|\langle \Psi'(u_{n'}), (\Psi')^{-1}(K'(u_{n'}))\rangle_{Y^*,Y}| \le c_2. \tag{3.18}$$

From (3.17) and (3.18), $\{\lambda(u_{n'})\}$ is bounded in $\mathbb{R}$. Passing to a subsequence, we may assume that $\lambda(u_{n'}) \to \lambda_0$ for some $\lambda_0 \in \mathbb{R}$. Since $d\widetilde{\Psi}(u_{n'}) \to 0$ in Y^*, we see that $\Phi'(u_{n'}) - \lambda(u_{n'})K'(u_{n'}) \to 0$ as $n' \to \infty$. Hence, since $K'(u_{n'}) \to K'(u_0)$ in Y^*,

$$\Psi'(u_{n'}) = (\Psi'(u_{n'}) - \lambda(u_{n'})K'(u_{n'})) + \lambda(u_{n'})K'(u_{n'}) \to \lambda_0 K'(u_0)$$

in Y^* as $n' \to \infty$. Therefore, we see that $u_{n'} \to (\Psi')^{-1}(\lambda_0 K'(u_0))$ strongly in Y as $n' \to \infty$. □

Here we recall the notion of "genus" which wass introduced in Rabinowitz [30, Chapter 7] or [34, Section 44.3]. Let E be a real Banach space and let $\mathcal{E}$ denote the family of subsets $A \subset E \setminus \{0\}$ such that A is closed in E and symmetric with respect to 0, that is, $x \in A$ implies $-x \in A$. For $\emptyset \ne A \in \mathcal{E}$, define the genus of A to be $n \ge 1$ (denoted by $\gamma(A) = n$) if there is a map $\varphi \in C(A, \mathbb{R}^n \setminus \{\mathbf{0}\})$ with φ odd and n is the smallest integer with this property. When there does not exist a finite such n, set $\gamma(A) = \infty$. Finally set $\gamma(\emptyset) = 0$. The main properties of genus will be listed in the next proposition.

Proposition 3.14. *Let $A, B \in \mathcal{E}$. Then the following properties hold.*

- (i) *If there exits an odd map $f \in C(A, B)$, then $\gamma(A) \le \gamma(B)$.*
- (ii) *If $A \subset B$, then $\gamma(A) \le \gamma(B)$.*
- (iii) $\gamma(A \cup B) \le \gamma(A) + \gamma(B)$.
- (iv) *If A is compact, then $\gamma(A) < \infty$ and there exists $\delta > 0$ such that if we put $N_\delta(A) = \{x \in E; \|x - A\| := \inf\{\|x - y\|; y \in A\} \le \delta\}$, then $N_\delta(A) \in \mathcal{E}$ and $\gamma(N_\delta(A)) = \gamma(A)$.*
- (v) *If Ω is a bounded neighborhood of $\mathbf{0}$ in $\mathbb{R}^n$, and there exists a mapping $h : A \to \partial\Omega$ with h an odd homeomorphism, then $\gamma(A) = n$.*

For a proof of the above proposition, see [30, Lemma 7.5 and Proposition 7.7] or [32, Proposition 2.3]. We note that it can be easily seen that when $A \in \mathcal{E}$, $A \ne \emptyset$ if and only if $\gamma(A) \ge 1$.

We apply the notion with $E = Y$. Let $\Sigma_\alpha = \{H \subset M_\alpha : H$ is compact and symmetric$\}$, $\gamma(H)$ be the genus of $H \in \Sigma_\alpha$, and define

$$c_{(n,\alpha)} = \inf_{H \in \Sigma_\alpha, \gamma(H) \ge n} \sup_{u \in H} \widetilde{\Psi}(u) \quad (n = 1, 2, \dots). \tag{3.19}$$

The following proposition is due to [32, Corollary 4.3].

Proposition 3.15 (Ljusternik-Schnirelmann principle)**.** *Assume that M is a closed symmetric C^1-submanifold of a real Banach space B and $0 \notin M$. Let $f \in C^1(M, \mathbb{R})$ be an even functional and bounded from below. Define*

$$c_j = \inf_{H \in \Gamma_j} \sup_{u \in H} f(u) \quad \textit{for } j = 1, 2, \dots,$$

where

$$\Gamma_j = \{H \subset M : H \textit{ is compact, symmetric and } \gamma(H) \ge j\}.$$

If $\Gamma_k \ne \emptyset$ for some $k \ge 1$ and f satisfies $(\mathrm{PS})_c$-condition for $c := c_m = c_{m+1} = \dots = c_k$ with $1 \le m \le k$, then f has at least $k - m + 1$ distinct pairs of critical points.

Since Y is a separable reflexive Banach space, it is well known that there exist $\{e_n\}_{n=1}^{\infty} \subset Y$ and $\{f_n\}_{n=1}^{\infty} \subset Y^*$ such that $\langle f_n, e_m\rangle_{Y^*,Y} = \delta_{nm}$, where δ_{nm} is the Kronecker delta and

$$Y = \overline{\operatorname{span}\{e_1, e_2, \dots\}} \quad \text{and} \quad Y^* = \overline{\operatorname{span}\{f_1, f_2, \dots\}}.$$

We define the spaces

$$Y_j = \operatorname{span}\{e_j\}, \quad Z_n = \oplus_{j=1}^{n} Y_j, \quad W_n = \overline{\oplus_{j=n}^{\infty} Y_j}.$$

If we apply Proposition 3.15 with $B = Y$, $M = M_\alpha$ and $f = \widetilde{\Psi}$, then we obtain the following lemma. We note that $\widetilde{\Psi}$ is bounded from below on M_α and satisfies $(\mathrm{PS})_c$-condition with respect ot M_α for any $c \in \mathbb{R}$ by Proposition 3.13.

Lemma 3.16. *For any $m \in \mathbb{N}$, we have $\Gamma_m \neq \emptyset$. Thus we see that all $c_{(m,\alpha)}$ defined by* (3.19) *are critical values of $\widetilde{\Psi}$ with respect to M_α and*

$$-\infty < c_{(m,\alpha)} \le c_{(m+1,\alpha)} < \infty \quad \text{for every } m \in \mathbb{N}.$$

Proof. For each fixed $m \in \mathbb{N}$, we claim that

$$c(m) := \inf\{K(u) : u \in Z_m, \|u\|_Y = 1\} > 0. \tag{3.20}$$

Indeed, assume that $c(m) = 0$. Then there exists a sequence $\{u_j\} \subset Z_m$ such that $\|u_j\|_Y = 1$ and

$$0 \le K(u_j) \le \frac{1}{j}. \tag{3.21}$$

Since the sequence $\{u_j\}$ is bounded in Y, there exist a subsequence $\{u_{j'}\}$ of $\{u_j\}$ and $u_0 \in Y$ such that $u_{j'} \to u_0$ weakly in Y as $j' \to \infty$. Since $\langle f_k, u_{j'}\rangle_{Y^*,Y} = 0$ for any $k > m$, we have $\langle f_k, u_0\rangle_{Y^*,Y} = 0$ for all $k > m$, so we see that $u_0 \in Z_m$. Since $\dim Z_m = m < \infty$, $u_{j'} \to u_0$ strongly in Z_m, so in Y. Thereby $\|u_0\|_Y = 1$, so we can see that $K(u_0) > 0$. On the other hand, letting $j' \to \infty$ in (3.21), we see that $K(u_0) = 0$. This is a contradiction.

For $0 \neq u \in Z_m$, since $\|u/\|u\|_Y\|_Y = 1$, it follows from (3.20) and (3.4) that

$$c(m) \le K\Big(\frac{u}{\|u\|_Y}\Big) = \int_{\Gamma_2} \frac{1}{\|u\|_Y^{r(x)}} G(x, u(x))d\sigma_x \le \frac{1}{\|u\|_Y^{r^+} \wedge \|u\|_Y^{r^-}} K(u).$$

Thus we have $K(u) \ge c(m)\|u\|_Y^{r^+} \wedge \|u\|_Y^{r^-}$ for all $u \in Z_m$. Therefore, $Z_m \cap M_\alpha$ is a bounded and closed subset of Z_m, so is compact by $\dim Z_m < \infty$. Since K is an even functional, $Z_m \cap M_\alpha$ is clearly symmetric. Let $G = \{u = u_1e_1 + \dots + u_me_m \in Z_m; K(u) < \alpha\}$. Then G can be identified with an open neighborhood of $\mathbf{0}$ in $\mathbb{R}^m$ by a trivial odd homeomorphism. Since the identity map: $Z_m \cap M_\alpha \to \partial G$ is an odd homeomorphism, using Proposition 3.14 (v), we have $\gamma(Z_m \cap M_\alpha) = m$, so $\Gamma_m \neq \emptyset$. Since $\Gamma_{m+1} \subset \Gamma_m$, we can see that $-\infty < c_{(m,\alpha)} \le c_{(m+1,\alpha)} < \infty$. $\square$

Lemma 3.17. *Assume that a functional $\chi : Y \to \mathbb{R}$ is sequentially weakly continuous and satisfies $\chi(0) = 0$. Then for any fixed $r > 0$,*

$$\lim_{n\to\infty} \sup_{u\in W_n, \|u\|_Y \le r} |\chi(u)| = 0. \tag{3.22}$$

Proof. Put $d_n = \sup_{u\in W_n,\|u\|_Y\le r} |\chi(u)|$. Then there exists $u_j \in W_n$ with $\|u_j\|_Y \le r$ such that $\lim_{j\to\infty} |\chi(u_j)| = d_n$. Since Y is a reflexive Banach space, there exist a subsequence $\{u_{j'}\}$ of $\{u_j\}$ and $u^{(n)} \in Y$ such that $u_{j'} \to u^{(n)}$ weakly in Y. Hence $\|u^{(n)}\|_Y \le \liminf_{j'\to\infty} \|u_{j'}\|_Y \le r$. Since W_n is a closed subspace of Y, we see that W_n is weakly closed, so $u^{(n)} \in W_n$. Since χ is sequentially weakly continuous, $|\chi(u_{j'})| \to |\chi(u^{(n)})|$ as $j' \to \infty$. Thereby $|\chi(u^{(n)})| = d_n$. Since $d_{n+1} \le d_n$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} d_n = d_0 \ge 0$ exists. Since $\{u^{(n)}\}$ satisfies $\|u^{(n)}\|_Y \le r$, there exists a subsequence $\{u^{(n')}\}$ of $\{u^{(n)}\}$ and $u_0 \in Y$ such that $u^{(n')} \to u_0$ weakly in Y, so $\|u_0\|_Y \le r$. Since again χ is sequentially weakly continuous, $|\chi(u^{(n')})| = d_{n'} \to |\chi(u_0)| = d_0$. Since Y is reflexive, we can look upon $u_0 \in Y^{**} = Y$. Therefore, for any $f_j \in Y^*$, since $u^{(n')} \in W_{n'}$, we have

$$\langle u_0, f_j\rangle_{Y^{**},Y^*} = \langle f_j, u_0\rangle_{Y^*,Y} = \lim_{n'\to\infty}\langle f_j, u^{(n')}\rangle_{Y^*,Y} = 0.$$

Thus we have $u_0 = 0$. Since $\chi(0) = 0$, we have $d_0 = 0$, that is, (3.22) holds. □

Proposition 3.18. *We have* $\lim_{n\to\infty} \inf_{u\in W_n\cap M_\alpha} \|u\|_Y = \infty$.

Proof. Suppose that the conclusion is false. Then there exist $c_1 > 0$ and $u_n \subset W_n \cap M_\alpha$ such that $\|u_n\|_Y \le c_1$ for large $n \in \mathbb{N}$. Then

$$\sup_{u\in W_n,\|u\|_Y\le c_1} |K(u)| \ge |K(u_n)| = \alpha.$$

Therefore,

$$\lim_{n\to\infty} \sup_{u\in W_n,\|u\|_Y\le c_1} |K(u)| \ge \lim_{n\to\infty} |K(u_n)| = \alpha > 0.$$

If we apply Lemma 3.17 with $\chi = K$, this is a contradiction. □

Proposition 3.19. *We have*

$$\lim_{n\to\infty} c_{(n,\alpha)} = \infty. \tag{3.23}$$

Proof. By Proposition 3.18, for any $c > 1$, there exists $n_0 \in \mathbb{N}$ such that for any $n \ge n_0$ and $u \in W_n \cap M_\alpha$, we have $\|u\|_Y > c$. For any $H \in \Sigma_\alpha$, we have $\gamma(H \cap Z_{n-1}) \le n - 1$. On the other hand, we have $\operatorname{codim} W_n = n - 1$. Hence for any $H \in \Sigma_\alpha$ with $\gamma(H) \ge n$, $H \cap W_n$ is non-empty. Indeed, since $H = (H \cap Z_{n-1}) \cup (H \cap W_n)$, it follows from Proposition 3.14 (iii) that

$$n \le \gamma(H) \le \gamma(H \cap Z_{n-1}) + \gamma(H \cap W_n) \le n - 1 + \gamma(H \cap W_n),$$

so $\gamma(H \cap W_n) \ge 1$. Hence $H \cap W_n \ne \emptyset$. For $n \ge n_0$, using (3.19), we have

$$\begin{aligned}
c_{(n,\alpha)} &= \inf_{H\in\Sigma_\alpha,\gamma(H)\ge n} \sup_{u\in H} \widetilde{\Psi}(u)\\
&= \inf_{H\in\Sigma_\alpha,\gamma(H)\ge n} \max\Big\{ \sup_{u\in H\cap(Y\setminus Z_{n-1})} \widetilde{\Psi}(u), \sup_{u\in H\cap Z_{n-1}} \widetilde{\Psi}(u)\Big\}\\
&\ge \inf_{H\in\Sigma_\alpha,\gamma(H)\ge n} \sup_{u\in H\cap(Y\setminus Z_{n-1})} \widetilde{\Psi}(u)\\
&= \inf_{H\in\Sigma_\alpha,\gamma(H)\ge n} \max\Big\{ \sup_{u\in H\cap((Y\setminus Z_{n-1})\setminus W_n)} \widetilde{\Psi}(u), \sup_{u\in H\cap W_n} \widetilde{\Psi}(u)\Big\}\\
&\ge \inf_{H\in\Sigma_\alpha,\gamma(H)\ge n} \sup_{u\in H\cap W_n} \widetilde{\Psi}(u)\\
&\ge \inf_{H\in\Sigma_\alpha,\gamma(H)\ge n} \sup_{u\in H\cap W_n} \frac{m_0}{l}\Big(\frac{k_0}{p^+}\|u\|_Y^{p^-}\Big)^l \ge \frac{m_0}{l}\Big(\frac{k_0}{p^+}c^{p^-}\Big)^l.
\end{aligned}$$

Since $c > 1$ is arbitrary, we thus get (3.23). □

Theorem 3.20. *Assume that* (A2)–(A8) *hold and fix* $\alpha > 0$. *Then for every* $n \in \mathbb{N}$, $c_{(n,\alpha)}$ *defined by* (3.19) *is a critical value of* $\widetilde{\Psi}$ *with respect to the submanifold* M_α *such that*

$$0 < c_{(n,\alpha)} \le c_{(n+1,\alpha)} < \infty \quad \text{and} \quad c_{(n,\alpha)} \to \infty \text{ as } n \to \infty.$$

Moreover, (1.1) *has infinitely many eigenpair sequence* $\{(u_{(n,\alpha)}, \lambda_{(n,\alpha)})\}$ *such that*

$$K(\pm u_{(n,\alpha)}) = \alpha, \Psi(\pm u_{(n,\alpha)}) = c_{(n,\alpha)} \text{ and } 0 < \lambda_{(n,\alpha)} \to \infty \quad \text{as } n \to \infty.$$

Proof. Taking Proposition 3.15, (3.12), (3.13), Lemma 3.16, and Proposition 3.19 into consideration, it suffices to show that $\lambda_{(n,\alpha)} \to \infty$ as $n \to \infty$. It follows from (A8) that

$$\langle K'(u_{(n,\alpha)}), u_{(n,\alpha)}\rangle_{Y^*,Y} \le r^+ K(u_{(n,\alpha)}) = r^+\alpha.$$

Hence

$$\lambda_{(n,\alpha)} = \frac{\langle \Psi'(u_{(n,\alpha)}), u_{(n,\alpha)}\rangle_{Y^*,Y}}{\langle K'(u_{(n,\alpha)}), u_{(n,\alpha)}\rangle_{Y^*,Y}} \ge \frac{\langle \Psi'(u_{(n,\alpha)}), u_{(n,\alpha)}\rangle_{Y^*,Y}}{r^+\alpha}. \tag{3.24}$$

Assume that $\lambda_{(n,\alpha)} \le M$ for all $n \in \mathbb{N}$. Then by (3.24), $\langle \Psi'(u_{(n,\alpha)}), u_{(n,\alpha)}\rangle_{Y^*,Y} \le Mr^+\alpha =: c_2$. On the other hand, from (A5), we have

$$\begin{aligned}
\langle \Psi'(u_{(n,\alpha)}), u_{(n,\alpha)}\rangle_{Y^*,Y} &= M(\Phi(u_{(n,\alpha)}))\langle \Phi'(u_{(n,\alpha)}), u_{(n,\alpha)}\rangle_{Y^*,Y} \\
&\ge m_0 \left(\Phi(u_{(n,\alpha)})\right)^{l-1} \int_\Omega \mathbf{a}(x, \nabla u_{(n,\alpha)}(x)) \cdot \nabla u_{(n,\alpha)}(x)\, dx \\
&\ge m_0 \Big(\frac{1}{p^+} \int_\Omega \mathbf{a}(x, \nabla u_{(n,\alpha)}(x)) \cdot \nabla u_{(n,\alpha)}(x)\, dx\Big)^{l-1} \\
&\quad \times \int_\Omega \mathbf{a}(x, \nabla u_{(n,\alpha)}(x)) \cdot \nabla u_{(n,\alpha)}(x)\, dx \\
&= \frac{m_0}{(p^+)^{l-1}} \Big(\int_\Omega \mathbf{a}(x, \nabla u_{(n,\alpha)}(x)) \cdot \nabla u_{(n,\alpha)}(x)\, dx\Big)^l \\
&\ge \frac{m_0}{(p^+)^{l-1}} k_0^l \Big(\int_\Omega h_1(x) |\nabla u_{(n,\alpha)}(x)|^{p(x)} dx\Big)^l \\
&= \frac{m_0}{(p^+)^{l-1}} k_0^l (\widetilde{\rho}_{(p(\cdot),h_1(\cdot))}(u_{(n,\alpha)}))^l.
\end{aligned}$$

Therefore, we have $\widetilde{\rho}_{(p(\cdot),h_1(\cdot))}(u_{(n,\alpha)}) \le c_3$ for some constant c_3. In particular, $\|u_{(n,\alpha)}\|_Y \le c_4$ for all $n \in \mathbb{N}$ with some constant c_4. Then from Lemma 3.7 (i),

$$\Phi(u_{(n,\alpha)}) \le c(2\|h_0\|_{L^{p'(\cdot)}(\Omega)} \|u_{(n,\alpha)}\|_Y + \widetilde{\rho}_{(p(\cdot),h_1(\cdot))}(u_{(n,\alpha)})) \le c_5$$

for some constant $c_5 > 0$. Since $\widehat{M}$ is bounded for every bounded subset from (A1), we see that $c_{(n,\alpha)} = \Psi(u_{(n,\alpha)}) = \widehat{M}(\Phi(u_{(n,\alpha)}))$ is bounded from above. This contradicts Proposition 3.19. □

Remark 3.21. We do not know whether problem (1.1) only has eigenvalue sequences of the form $\{\lambda_{(n,\alpha)}\}$.

Remark 3.22. We assume the following more restrictive conditions instead of (A1) and (A5):

(A1') $M : [0,\infty) \to [0,\infty)$ is continuous and monotone non-decreasing, and there exist $0 < m_0 \le m_1 < \infty$ and $l \ge 1$ such that

$$m_0 s^{l-1} \le M(s) \le m_1 s^{l-1} \text{ for } s \ge 0.$$

(A5') $k_0 h_1(x)|\boldsymbol{\xi}|^{p(x)} \le \mathbf{a}(x,\boldsymbol{\xi})\cdot\boldsymbol{\xi} = p(x)A(x,\boldsymbol{\xi})$ for a.e. $x\in\Omega$ and all $\boldsymbol{\xi}\in\mathbb{R}^N$.

We note that (i) in Example 3.2 satisfies (A5'), but (ii) does not satisfy this condition.

Under assumptions (A1)–(A4), (A6)–(A8), (A1'), and (A5'), we have

$$\lambda_{(n+1,\alpha)} \ge \frac{p^- r^- m_0^2}{p^+ r^+ m_1^2}\lambda_{(n,\alpha)}. \tag{3.25}$$

In particular, if $p(x)=p$ (a constant), $r(x)=r$ (a constant) and $m_0=m_1$, then we have $\lambda_{(n+1,\alpha)}\ge\lambda_{(n,\alpha)}$.

Proof. Let u_n be the eigenfuntion associated with the eigenvalue $\lambda_{(n,\alpha)}$ for $n=1,2,\dots$. From assumption(A5'), (A8) and Theorem 3.20, we have

$$\begin{aligned}
\lambda_{(n+1,\alpha)} &= \frac{\langle\Psi'(u_{n+1}),u_{n+1}\rangle_{Y^*,Y}}{\langle K'(u_{n+1}),u_{n+1}\rangle_{Y^*,Y}} = \frac{M(\Phi(u_{n+1}))\langle\Phi'(u_{n+1}),u_{n+1}\rangle_{Y^*,Y}}{\int_{\Gamma_2}K'(u_{n+1}),u_{n+1}\rangle_{Y^*,Y}}\\
&= \frac{M(\Phi(u_{n+1}))\int_\Omega \mathbf{a}(x,\boldsymbol{\nabla}u_{n+1}(x))\cdot\boldsymbol{\nabla}u_{n+1}(x)\,dx}{\int_{\Gamma_2}g(x,u_{n+1}(x))u_{n+1}(x)d\sigma_x}\\
&\ge \frac{m_0\Phi(u_{n+1})^{l-1}\int_\Omega p(x)A(x,\boldsymbol{\nabla}u_{n+1}(x))\,dx}{\int_{\Gamma_2}r(x)G(x,u_{n+1}(x))d\sigma_x}\\
&\ge \frac{m_0p^-}{r^+\alpha}\Phi(u_{n+1})^l\\
&\ge \frac{m_0p^-}{r^+\alpha}\frac{l}{m_1}\widehat{M}(\Phi(u_{n+1}))\\
&= \frac{m_0p^-l}{r^+\alpha m_1}c_{(n+1,\alpha)} \ge \frac{m_0lp^-}{r^+\alpha m_1}c_{(n,\alpha)}.
\end{aligned}$$

The last inequality follows from Theorem 3.20.

On the other hand, from (A1'), (A5') and (A8), we have

$$\begin{aligned}
c_{(n,\alpha)} &= \alpha\frac{\Psi(u_n)}{K(u_n)}\\
&= \alpha\frac{\widehat{M}(\Phi(u_n))}{\int_{\Gamma_2}G(x,u_n(x))d\sigma_x}\\
&\ge \alpha\frac{\frac{m_0}{l}\Phi(u_n)^l}{\int_{\Gamma_2}\frac{1}{r(x)}g(x,u_n(x))u_n(x)d\sigma_x}\\
&\ge \frac{\alpha\frac{m_0}{l}\Phi(u_n)^{l-1}\Phi(u_n)}{\frac{1}{r^-}\langle K'(u_n),u_n\rangle_{Y^*,Y}}\\
&\ge \frac{\alpha\frac{m_0}{l}\frac{1}{m_1}M(\Phi(u_n))\int_\Omega\frac{1}{p(x)}\mathbf{a}(x,\boldsymbol{\nabla}u_n(x))\cdot\boldsymbol{\nabla}u_n(x)\,dx}{\frac{1}{r^-}\langle K'(u_n),u_n\rangle_{Y^*,Y}}\\
&\ge \frac{\alpha\frac{m_0}{l}\frac{1}{m_1}\frac{1}{p^+}M(\Phi(u_n))\langle\Phi'(u_n),u_n\rangle_{Y^*,Y}}{\frac{1}{r^-}\langle K'(u_n),u_n\rangle_{Y^*,Y}}\\
&= \frac{\alpha m_0r^-}{lm_1p^+}\frac{\langle\Psi'(u_n),u_n\rangle_{Y^*,Y}}{\langle K'(u_n),u_n\rangle_{Y^*,Y}}\\
&= \frac{\alpha m_0r^-}{lm_1p^+}\lambda_{(n,\alpha)}.
\end{aligned}$$

Thus we obtain the estimate (3.25). □

4. The infimum of all the eigenvalues

In this section, we consider the infimum of all the eigenvalues of the problem (1.1). We show that there exist two cases where the infimum is equal to zero, and positive according to the hypoetheses on the variable exponent.

Put $\Lambda = \{\lambda$ is an eigenvalue of problem (1.1)$\}$ and $\lambda_* = \inf \Lambda$. For a subset $A \subset \overline{\Omega}$ and $\delta > 0$, put

$$B(A,\delta) = \{x \in \mathbb{R}^N; \operatorname{dist}(x,A) < \delta\}, \quad B_\Omega(A,\delta) = B(A,\delta) \cap \Omega,$$
$$B_{\Gamma_2}(A,\delta) = B(A,\delta) \cap \Gamma_2.$$

Here, for $x_0 \in \overline{\Omega}$, if $A = \{x_0\}$, then we simply write $B(\{x_0\},\delta)$, $B_\Omega(\{x_0\},\delta)$ and $B_{\Gamma_2}(\{x_0\},\delta)$ by $B(x_0,\delta)$, $B_\Omega(x_0,\delta)$ and $B_{\Gamma_2}(x_0,\delta)$, respectively. Assume that (A1)–(A8), hold.

Lemma 4.1. *For $\delta, \alpha > 0$, if we define*

$$\beta_\delta(u) = \int_{B_\Omega(\overline{\Gamma_2},\delta)} h_1(x)|\boldsymbol{\nabla} u(x)|^{p(x)}\,dx \text{ for } u \in Y,$$

then we have

$$\beta_{(\delta,\alpha)} := \inf_{u \in M_\alpha} \beta_\delta(u) > 0.$$

Proof. First we consider $Y(B_\Omega(\overline{\Gamma_2},\delta))$. We extend the function $b(x)$ on Γ_2 in (A8) to a function $\widetilde{b}(x)$ on $\widetilde{\Gamma_2}$, where $\widetilde{\Gamma_2} := \Gamma_2 \cup (\partial B_\Omega(\overline{\Gamma_2},\delta) \setminus \Gamma)$ by a positive constant outside $\partial B_\Omega(\overline{\Gamma_2},\delta) \setminus \Gamma$, and define $\widetilde{G}$ and $\widetilde{K}$ as in (3.10) and (3.12), respectively. Since $\delta > 0$, we have $\widetilde{\Gamma_1} := \partial B_\Omega(\overline{\Gamma_2},\delta) \cap \Gamma_1 \neq \emptyset$, so $Y(B_\Omega(\overline{\Gamma_2},\delta))$ is the same properties as Y, if we replace Γ_1 in Y with $\widetilde{\Gamma_1}$. Thus $Y \hookrightarrow Y(B_\Omega(\overline{\Gamma_2},\delta))$ and β_δ is a modular on $Y(B_\Omega(\overline{\Gamma_2},\delta))$.

Assume that $\beta_{(\delta,\alpha)} = 0$. Then there exist $\{u_n\} \subset M_\alpha$ such that $\beta_\delta(u_n) \to 0$ as $n \to \infty$. Hence $\|u_n\|_{Y(B_\Omega(\overline{\Gamma_2},\delta))} \to 0$ as $n \to \infty$, where $\|u\|_{Y(B_\Omega(\overline{\Gamma_2},\delta))} = \inf\left\{\tau > 0; \beta_\delta\left(\frac{u}{\tau}\right) \le 1\right\}$.

On the other hand, we have

$$\widetilde{K}(u_n) = \int_{\partial B_\Omega(\overline{\Gamma_2},\delta))} \widetilde{G}(x,u_n(x))d\sigma_x \ge \int_{\Gamma_2} G(x,u_n(x))d\sigma_x = K(u_n) = \alpha > 0.$$

Since $\widetilde{K}$ is continuous on $Y(B_\Omega(\overline{\Gamma_2},\delta))$, we can see that $\widetilde{K}(u_n) \to \widetilde{K}(0) = 0$. This is a contradiction. □

Lemma 4.2. *For $\alpha > 0$, let u_0 be an eigenfunction associated with $\lambda_{(1,\alpha)}$. Then*

$$\Psi(u_0) = c_{(1,\alpha)} = \inf\{\Psi(u); u \in M_\alpha\}.$$

Proof. Put $b_\alpha = \inf\{\Psi(u); u \in M_\alpha\}$. Since $c_{(1,\alpha)} = \inf_{H \in \Sigma_\alpha, \gamma(H) \ge 1} \sup_{u \in H} \widetilde{\Psi}(u)$, if $u \in H$ and $H \in \Sigma_\alpha \subset M_\alpha$ with $\gamma(H) \ge 1$, then $\widetilde{\Psi}(u) = \Psi(u) \ge b_\alpha$. Thus $c_{(1,\alpha)} \ge b_\alpha$.

By the definition of b_α, there exists a sequence $\{u_n\} \subset M_\alpha$ such that $b_\alpha = \lim_{n\to\infty} \Psi(u_n)$. For large n, $b_\alpha + 1 \ge \Psi(u_n) = \widehat{M}(\Phi(u_n)) \ge \frac{m_0}{l}(\frac{k_0}{p^+}\|u_n\|_Y^{p^+} \wedge \|u_n\|_Y^{p^-})^l$. Thus $\{u_n\}$ is bounded in Y. So there exist a subsequence $\{u_{n'}\}$ of $\{u_n\}$ and $u_* \in Y$ such that $u_{n'} \to u_*$ weakly in Y. Then $\Psi(u_*) \le \liminf_{n'\to\infty} \Psi(u_{n'}) =$

b_α. Since M_α is a weakly closed subset of Y, $u_* \in M_\alpha$, so $\Psi(u_*) \ge b_\alpha$. Thus we have $\Psi(u_*) = b_\alpha$. By (A7), $\Psi(\pm u_*) = b_\alpha$. Let $H_0 = \{\pm u_*\}$, then clearly $\gamma(H_0) = 1$. Therefore, $c_{(1,\alpha)} \le \sup_{u\in H_0} \Psi(u) = b_\alpha$. Thus we have $c_{(1,\alpha)} = b_\alpha$. □

From now on, we suppose that the following more restrictive assumption than (A8) on the given function g hold.

(A8') (A8) holds with $r(x) = lp(x)$, where l is a constant in (A1), that is, $g(x,t) = b(x)|t|^{lp(x)-2}t$ with a function $b(x)$ satisfying the condition in (A8) with $r(x) = lp(x)$.

Theorem 4.3. *Assume that* (A1)–(A7), (A8') *hold, moreover, suppose that there exists* $\delta > 0$ *such that* $p(x) = p$ *(a constant) for all* $x \in B_\Omega(\overline{\Gamma_2}, \delta)$. *Then we have* $\lambda_* > 0$.

Proof. Let u be the eigenfunction of problem (1.1), associated with λ. Then $K(u) > 0$. In fact, let $K(u) = 0$. Since $\langle \Psi'(u), u\rangle_{Y^*,Y} = \lambda\langle K'(u), u\rangle_{Y^*,Y}$, it follows from (A8') that

$$\begin{aligned} &M(\Phi(u))\int_\Omega \mathbf{a}(x, \nabla u(x))\cdot \nabla u(x)\,dx\\ &= \lambda\int_{\Gamma_2} g(x,u(x))u(x)d\sigma_x\\ &= \lambda lp\int_{\Gamma_2} G(x,u(x))d\sigma_x\\ &= \lambda lpK(u) = 0.\end{aligned}$$

Hence from (A5) and $M(\Phi(u)) > 0$, we have

$$0 = \int_\Omega \mathbf{a}(x,\nabla u(x))\cdot\nabla u(x)\,dx \ge k_0\int_\Omega h_1(x)|\nabla u(x)|^{p(x)}\,dx.$$

Thus we have $\nabla u(x) = \mathbf{0}$ a.e. $x \in \Omega$. From Proposition 2.14, we have $u = 0$ a.e. in Ω. This is a contradiction.

We show that there exists $t_0 > 0$ such that $u_1 := \frac{1}{t_0}u \in M_1$. Indeed, since $g(x,t)t = lpG(x,t)$ for σ-a.e. $x \in \Gamma_2$ and all $t \in \mathbb{R}$, it follows from (3.4) that $K(\frac{u}{t}) = t^{-lp}K(u)$ for $t > 0$. Here we can see that $K(\frac{u}{t}) \to 0$ as $t \to \infty$ and $K(\frac{u}{t}) \to \infty$ as $t \to +0$. Since $K(\frac{u}{t})$ is continuous with respect to $t \in (0,\infty)$, it follows from the intermediate value theorem that there exists $t_0 > 0$ such that $K(\frac{u}{t_0}) = 1$, so $u_1 := \frac{u}{t_0} \in M_1$.

Now since $K(u_1) = 1$, it follows from (A1), (A5), (A8'), and Lemma 4.1 that

$$\begin{aligned}\lambda &= \frac{\langle\Psi'(u),u\rangle_{Y^*,Y}}{\langle K'(u),u\rangle_{Y^*,Y}}\\ &= \frac{M(\Phi(u))\int_\Omega \mathbf{a}(x,\nabla u(x))\cdot\nabla u(x)\,dx}{\int_{\Gamma_2} g(x,u(x))u(x)d\sigma_x}\\ &\ge \frac{m_0\big(\int_\Omega \frac{1}{p(x)}\mathbf{a}(x,\nabla u(x))\cdot\nabla u(x)\,dx\big)^{l-1}\int_\Omega \mathbf{a}(x,\nabla u(x))\cdot\nabla u(x)\,dx}{\int_{\Gamma_2} g(x,u(x))u(x)d\sigma_x}\\ &\ge \frac{m_0}{(p^+)^{l-1}}\frac{\big(\int_{B_\Omega(\overline{\Gamma_2},\delta)} \mathbf{a}(x,\nabla u(x))\cdot\nabla u(x)\,dx\big)^l}{lp\int_{\Gamma_2} G(x,u(x))d\sigma_x}\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{m_0 k_0^l}{lp(p^+)^{l-1}} \frac{\big(\int_{B_\Omega(\overline{\Gamma_2},\delta)} h_1(x)|\nabla u(x)|^p dx\big)^l}{\int_{\Gamma_2} G(x,u(x))d\sigma_x} \\
&= \frac{m_0 k_0^l}{lp(p^+)^{l-1}} \frac{\big(\int_{B_\Omega(\overline{\Gamma_2},\delta)} h_1(x)|t_0\nabla u_1(x)|^p dx\big)^l}{\int_{\Gamma_2} G(x,t_0u_1(x))d\sigma_x} \\
&= \frac{m_0 k_0^l}{lp(p^+)^{l-1}} \frac{t_0^{lp}\big(\int_{B_\Omega(\overline{\Gamma_2},\delta)} h_1(x)|\nabla u_1(x)|^p dx\big)^l}{t_0^{lp}\int_{\Gamma_2} G(x,u_1(x))d\sigma_x} \\
&\geq \frac{m_0 k_0^l}{lp(p^+)^{l-1}} \beta_{(\delta,1)}^l > 0.
\end{aligned}
$$

Thus we have $\lambda_* = \inf \Lambda \geq \frac{m_0 k_0^l}{lp(p^+)^{l-1}}\beta_{(\delta,1)}^l > 0$. □

Next we will treat the case $\lambda_* = 0$. From the absolute continuity of integral, we obtain the following lemma which is needed later.

Lemma 4.4. *Let $u \in Y$ be given. Then for any $\varepsilon > 0$, there exists $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$,*

$$
\beta_u(\delta) := \int_{B_\Omega(\overline{\Gamma_2},\delta)} A(x, \nabla u(x))\, dx < \varepsilon.
$$

Theorem 4.5. *Assume that* (A2)–(A7), (A1'), (A8') *hold. Moreover, suppose that there exist $\delta > 0$ and $x_0 \in \Gamma_2$ such that the following hold:*

(i) *$p(x) = p$ (a constant) for all $x \in B_{\Gamma_2}(x_0,\delta)$.*
(ii) *$p(x) < p$ for all $x \in B_\Omega(x_0,\delta)$.*
(iii) *$h_1 \in L^1(B_\Omega(x_0,\delta))$, where h_1 is the function of* (A3)–(A5)*.*

Then we have $\lim_{\alpha\to\infty} \lambda_{(1,\alpha)} = 0$, so $\lambda_* = 0$.

Proof. Replacing $\delta > 0$ with smaller one, if necessary, we may assume that $B(x_0,\delta)\cap \Gamma \subset \Gamma_2$. Choose $0 \leq u \in C^\infty(\overline{\Omega})$ such that $u(x) = 1$ for $x \in B_\Omega(x_0,\delta/4)$ and $u(x) = 0$ for $x \in \Omega \setminus B_\Omega(x_0,\delta/2)$. We note that from (iii) it follows that $u \in Y$. By Lemma 4.4, for any $\varepsilon > 0$, there exists $\delta_0 \in (0,\delta/4)$ such that for each $\delta_1 \in (0,\delta_0)$,

$$
\frac{\big(\int_{B_\Omega(\overline{\Gamma_2},\delta_1)} A(x,\nabla u(x))\, dx\big)^l}{K(u)} < \varepsilon/(2c), \quad \text{where } c = \frac{m_1^2 p^+}{l m_0 p^-} 2^{l-1}.
$$

Since $p \in C(\overline{\Omega})$, it follows from (ii) that for any $x \in B_\Omega(x_0,\delta/2) \setminus B(\Gamma_2,\delta_0)$, we have

$$
p(x) - p \leq p^+\big(\overline{B_\Omega(x_0,\delta/2)} \setminus B(\Gamma_2,\delta_0)\big) - p := -\varepsilon_0 < 0.
$$

We note that $p(x) = p$ on $\operatorname{supp} u \cap \Gamma_2$. If we define $h(t) = K(tu) = t^{lp}K(u)$, then h is differentiable in $(0,\infty)$ and $h'(t) = lpt^{lp-1}K(u) > 0$, so h is strictly monotone increasing and clearly $h(t) \to 0$ as $t \to +0$ and $h(t) \to \infty$ as $t \to \infty$. Hence for any $\alpha > 0$, there exists unique $t(\alpha) > 0$ such that $t(\alpha)u \in M_\alpha$. Clearly $t(\alpha) \to 0$ as $\alpha \to +0$ and $t(\alpha) \to \infty$ as $\alpha \to \infty$. So there exists $\alpha_0 > 1$ such that for any $\alpha \in (\alpha_0,\infty)$, $\max\{1, (2\varepsilon^{-1}c\frac{\Phi(u)^l}{K(u)})^{1/(l\varepsilon_0)}\} < t(\alpha)$. Let u_0 be the eigenfunction associated with $\lambda_{(1,\alpha)}$. Then from (A1') we have

$$
\lambda_{(1,\alpha)} = \frac{M(\Phi(u_0)) \int_\Omega \mathbf{a}(x,\nabla u_0(x)) \cdot \nabla u_0(x)\, dx}{\int_{\Gamma_2} g(x,u_0(x))u_0(x)\, d\sigma_x}
$$

$$
\begin{aligned}
&\le \frac{m_1\Phi(u_0)^{l-1}\int_\Omega p(x)A(x,\nabla u_0(x))\,dx}{lp^-\int_{\Gamma_2}G(x,u_0(x))d\sigma_x}\\
&\le \frac{m_1p^+\Phi(u_0)^l}{lp^-\alpha}\\
&\le \frac{m_1p^+\Psi(u_0)}{m_0p^-\alpha}.
\end{aligned}
$$

By Lemma 4.2, since $\Psi(u_0)=c_{(1,\alpha)}=\inf\{\Psi(u);u\in M_\alpha\}$, we have $\Psi(u_0)\le\Psi(t(\alpha)u)$. Hence

$$
\lambda_{(1,\alpha)}\le\frac{m_1p^+\Psi(t(\alpha)u)}{m_0p^-\alpha}=\frac{m_1p^+}{m_0p^-}\frac{\Psi(t(\alpha)u)}{K(t(\alpha)u)}.
$$

Thus using (3.1), we have

$$
\begin{aligned}
\lambda_{(1,\alpha)} &\le \frac{m_1p^+}{m_0p^-}\frac{\Psi(t(\alpha)u)}{K(t(\alpha)u)}\\
&\le \frac{m_1p^+}{m_0p^-}\frac{\frac{m_1}{l}\big(\int_\Omega A(x,t(\alpha)\nabla u(x))\,dx\big)^l}{\int_{\Gamma_2}G(x,t(\alpha)u(x))d\sigma_x}\\
&\le c\frac{\big(\int_{B_\Omega(x_0,\delta/2)\setminus B_\Omega(\Gamma_2,\delta_1)}A(x,t(\alpha)\nabla u(x))\,dx\big)^l}{\int_{B_{\Gamma_2}(x_0,\delta/2)}G(x,t(\alpha)u(x))d\sigma_x}\\
&\quad + c\frac{\big(\int_{B_\Omega(x_0,\delta/2)\cap B_\Omega(\Gamma_2,\delta_1)}A(x,t(\alpha)\nabla u(x))\,dx\big)^l}{\int_{B_{\Gamma_2}(x_0,\delta/2)}G(x,t(\alpha)u(x))d\sigma_x}\\
&\le c\frac{\Big(\int_{B_\Omega(x_0,\delta/2)\setminus B_\Omega(\Gamma_2,\delta_1)}t(\alpha)^{p(x)}A(x,\nabla u(x))\,dx\Big)^l}{t(\alpha)^{lp}\int_{B_{\Gamma_2}(x_0,\delta/2)}G(x,u(x))d\sigma_x}\\
&\quad + c\frac{\big(\int_{B_\Omega(x_0,\delta/2)\cap B_\Omega(\Gamma_2,\delta_1)}t(\alpha)^{p(x)}A(x,\nabla u(x))\,dx\big)^l}{t(\alpha)^{lp}\int_{B_{\Gamma_2}(x_0,\delta/2)}G(x,u(x))d\sigma_x}\\
&= c\frac{\big(\int_{B_\Omega(x_0,\delta/2)\setminus B_\Omega(\Gamma_2,\delta_1)}t(\alpha)^{p(x)-p}A(x,\nabla u(x))\,dx\big)^l}{\int_{B_{\Gamma_2}(x_0,\delta/2)}G(x,u(x))d\sigma_x}\\
&\quad + c\frac{\big(\int_{B_\Omega(x_0,\delta/2)\cap B_\Omega(\Gamma_2,\delta_1)}t(\alpha)^{p(x)-p}A(x,\nabla u(x))\,dx\big)^l}{\int_{B_{\Gamma_2}(x_0,\delta/2)}G(x,u(x))d\sigma_x}\\
&\le ct(\alpha)^{-l\varepsilon_0}\frac{\Phi(u)^l}{K(u)}+c\frac{\Big(\int_{B_\Omega(\Gamma_2,\delta_1)}A(x,\nabla u(x))\,dx\Big)^l}{K(u)}\\
&< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}
$$

Therefore, $0<\lambda_{(1,\alpha)}<\varepsilon$ for all $\alpha>\alpha_0$. Since $\varepsilon>0$ is arbitrary, we have $\lim_{\alpha\to\infty}\lambda_{(1,\alpha)}=0$. □

Remark 4.6. (1) If $p(x)=p$ (a constant) in Ω, then it is well known that $\lambda_*=\lambda_{(1,\alpha)}=\lambda_1$ and so λ_* is a principal eigenvalue.

(2) For a variable exponent $p(x)$, under some assumptions, $\lambda_* = 0$. This means that under some assumptions, there does not exist a principal eigenvalue and the set of eigenvalues is not closed.

(3) For a variable exponent $p(x)$, under some assumptions, we have $\lambda_* > 0$.

References

[1] G. A. Afrouzi, M. Mirzapour; *Eigenvalue probelms for $p(x)$-Kirchhoff type equations*, Electr. J. Differ. Equa., Vol. 2013(253), (2013), 1-10.

[2] C. Alves, A. Moussoui, L. Tavares; *An elliptic system with logarithmic nonlinearity*, Advances in Nonl. Anal., Vol. 8, (2019), 928-945.

[3] C. Alves, L. S. Tavares; *A Hardy-Littlewood-Sobolev-type inequality for variable exponents and applications to quasilinear Choquard equations involving variable exponent*, Mediterranean J. Math., Vol. 16(2), (2019), Paper No. 55: 1-27.

[4] A. Anane; *Simplicité et isolation de la premiére valeur propre de p-Laplacian avec poids*, C. R. Acad. Sci. Paris, Sér. I. Math., Vol. 305, (1987), 725-728.

[5] J. Aramaki; *Existence of three weak solutions for a class of nonlinear operators involving $p(x)$-Laplacian with mixed boundary conditions.* Nonlinear Funct. Anal. Appl., Vol. 26(3), (2021), 531-551.

[6] J. Aramaki; *Mixed boundary value problem for a class of quasi-linear elliptic operators containing $p(\cdot)$-Laplacian in a variable exponent Sobolev space*, Adv. Math. Sci. Appl., Vol. 31(2), (2022), 207-239.

[7] J. Aramaki; *Existence of nontrivial weak solutions for nonuniformly elliptic equation with mixed boundary condition in a variable exponent Sobolev space*, Electronic J. Qualitative Theory Differ. Eq., Vol. 2023(12), (2023), 1-22.

[8] J. Aramaki; *Existence of three weak solutions for the Kirchhoff-type problem with mixed boundary condition in a variable exponent Sobolev space*, East-West J. Math., Vol. 24(2), (2023), 89-117.

[9] J. Aramaki; *Existence of three weak solutions for a nonlinear problem with mixed boundary condition in a variable exponent Soboev space*, J. Analysis, Vol. 32(2), (2024), 733-755.

[10] P. G. Ciarlet, G. Dinca; *A Poincaré inequality in a Sobolev space with a variable exponent*, Chin. Ann. Math., Vol. 32B(3), (2011), 333-342.

[11] S. G. Deng; *Eigenvalues of the $p(x)$-Laplacian Steklov problem*, J. Math. Anal. Appl., Vol. 339, (2008), 925-937.

[12] L. Diening; *Theoretical and numerical results for electrorheological fluids*, ph. D. thesis, University of Frieburg, Germany 2002.

[13] L. Diening, P. Harjulehto, P. Hästö, M. Růžička; *Lebesgue and Sobolev Spaces with Variable Exponent*, Lecture Notes in Math. Springer, 2017.

[14] X. L. Fan; *Solutions for $p(x)$-Laplacian Dirichlet problems with singular coefficients*, J. Math. Anal. Appl., Vol. 312, (2005), 464-477.

[15] X. L. Fan; *Eigenvalues of the $p(x)$-Laplacian Neumann problem*, Nonlinear Anal., Vol. 67, (2007), 2982-2992.

[16] X. L. Fan; *Boundary trace embedding theorems for variable exponent Sobolev spaces*, J. Math. Anal. Appl., Vol. 339, (2008), 1395-1412.

[17] X. L. Fan, Q. H Zhang; *Existence of solutions for $p(x)$-Laplacian Dirichlet problem*, Nonlinear Anal., Vol. 52, (2003), 1843-1852.

[18] X. L. Fan, D. Zhao; *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl., Vol. 263, (2001), 424–446.

[19] X. L. Fan, Q. Zhang, D. Zhao; *Eigenvalues of $p(x)$-Laplacian Dirichlet problem*, J. Math. Anal. Appl., Vol. 302, (2015), 306–317.

[20] L. Friedlander; *Asymptotic behavior of the eigenvalues of the p-Laplacian*, Comm. Partial Differential Equations, Vol. 14, (1989), 1059-1069.

[21] T. C. Halsey; *Electrorheological fluids*, Science, Vol. 258, (1992), 761–766.

[22] G. Kirchhoff; *Mechanik*, Teubner, Leipzig, 1883.

[23] O. Kováčik, J. Rákosník; *On spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$*, Czechoslovak Math. J., Vol. 41(116), (1991), 592–618.

[24] A. Lê; *Eigenvalue problems for the p-Laplacian*, Nonlinear Anal., Vol. 64 (2006), 1057-1099.

[25] L. Ljusternik, L. Schnirelmann; *Méthodes topologiques dans les problémes variationels*, Hermann, Paris, 1934.
[26] R. A. Mashiyev, B. Cekic, M. Avci, Z. Yucedag; *Existence and multiplicity of weak solutions for nonuniformly elliptic equations with nonstandard growth condition*, Complex Variables Elliptic Equa., Vol. 57(5), (2012), 579-595.
[27] O. Méndez; *On the eigenvalue problem for a class of Kirchhoff-type equations*, J. Math. Anal. Appl., Vol. 494, (2021), 124671.
[28] M. Mihăilescu, V. Rădulescu; *A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids*, Proceeding of the Royal Society A., Vol. 462, (2006), 2625–2641.
[29] M. Mihăilescu, V. Rădulescu; *On a nonhomogenuous quasilinear eigenvalue problem in Sobolev cpaces with variable exponent*, Proc. Amer. Math. Soc., Vol. 135, (2007), 2929-2937.
[30] P. H. Rabinowitz; *Minimax Methods in Critical Point Theory with Application to Differential Equations*, CBMS Reg. Conf. Ser. in Math., Vol. 65, Am. Math. Soc., Providence, 1986.
[31] M. Růžička; *Electrotheological fluids: Modeling and Mathematical Theory*, Lecture Notes in Mathematics, Vol. 1784, Berlin, Springer, 2000.
[32] A. Szulkin; *Ljusternik-Schnirelmann theory on C^1-manifolds*, Ann. Inst. Henri Poincaré, Vol. 5(2), (1988), 119-139.
[33] J. Yao; *Solutions for Neumann boundary value problem involving $p(x)$-Laplace operators*, Nonlinear Anal., Vol. 68, (2008), 1271-1283.
[34] E. Zeidler; *Nonlinear Functional Analysis and its Applications I: Fixed-Point Theorems, III: Variational Methods and Optimization*, Springer-Verlag, Now York, Berlin, Heidelberg, London, Paris, Tokyo, 1990.
[35] D. Zhao, WJ. Qing, X. L. Fan; *On generalized Orlicz space $L^{p(x)}(\Omega)$*, J. Gansu Sci., Vol. 9(2), (1996), 1–7. (in Chinese).
[36] VV. Zhikov; *Averaging of functionals of the calculus of variation and elasticity theory*, Math. USSR, Izv., Vol. 29, (1987), 33–66.

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