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INEQUALITIES FOR FRACTIONAL DERIVATIVES VIA THE MARCHAUD DERIVATIVE

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Abstract. We study the Marchaud fractional derivative. We pay special attention to when the Marchaud fractional derivative is equal to the well-known Caputo or Riemann-Liouville fractional derivative. Conditions when this equality held were given in the interesting paper of Vainikko (2016). Several recent papers have used results from that paper in discussing inequalities that are useful in the study of stability. We have found some gaps in the proofs in the Vainikko paper but we give a proof of the most useful parts; in fact we also prove that equality holds under a more general condition. We use this equality to prove various inequalities for fractional derivatives, including a maximum principle, often under weaker conditions than previously given. In particular we prove strong versions of inequalities for differentiable convex functions that are useful in studying stability by Lyapunov's method.

1. Introduction

The study of fractional integrals and derivatives is an active research area. For fractional differential equations (FDEs) the two most commonly used and most applicable derivatives are the closely related Riemann-Liouville and Caputo ones. The Marchaud derivative is defined quite differently. It has the advantage that its definition does not involve any differentiation, but has the disadvantage that it requires a Hölder continuity condition in order to be well defined.

An important case arises when studying Caputo FDEs of the form $D_*^\alpha u(t) = f(t, u(t))$ for $0 < \alpha < 1$ in the space $C[0,T]$ when f is continuous. (Precise definitions are given later.) Then u and $D_*^\alpha u$ are continuous which implies that u has the form $u(t) = ct^\alpha + v(t)$ where v belongs to a Hölder space of order α. This case was treated in the interesting paper of Vainikko [34] and he claimed some equivalences including the assertion that the Caputo and Marchaud derivatives are equal for $t > 0$. Unfortunately there are some gaps in the proofs, discussed fully later in our paper, see Remark 4.4 (b). One of our main results, Theorem 4.2, proves that the Caputo and Marchaud derivatives are equal for $t > 0$, under conditions more general than mentioned above, which validates the part of the claims of [34] that we consider to be the most important, see Theorem 4.3. Since a good number of papers have used the result [34, Theorem 5.2] in their studies, for example Tuan

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and Trinh [33], Wu [41] and Wu, Fan, Tang, Shen [42], it is important to have a fully proved version of the results used.

We prove various inequalities for the Marchaud derivative whose special cases for the Caputo derivative prove such inequalities under weaker conditions than in the previous literature. In Section 5 we prove some inequalities similar to ones that were claimed in papers [12, 14, 28] but without proper proofs. We prove under weaker conditions a maximum principle as studied in Al-Refai [3] and Al-Refai and Luchko [5]. We then prove an inequality for a convex function that is useful in discussing Lyapunov stability of equations, as for example Tuan-Trinh [33]. Motivated by a paper of Fewster-Young [18] we prove a stronger version than is usually given, which proves the result of [18, Lemma 6] with a weaker condition. Special cases prove results that were previously given with stronger hypotheses.

2. Preliminaries

2.1. **Some function spaces.** In this paper all functions are assumed to be measurable, and all integrals are Lebesgue integrals, but possibly improper. We consider functions u that are defined at least almost everywhere (a.e.) on a given finite interval. For simplicity of expressions we consider an interval $[0,T]$, which, by a simple change of variable, is equivalent to any finite interval.

$L^p = L^p[0,T]$ $(1 \le p < \infty)$ denotes the usual space of functions whose p-th power is Lebesgue integrable; endowed with the norm $\|u\|_p = \left(\int_0^T |u(s)|^p\,ds\right)^{1/p}$ it is a Banach space. L^∞ denotes the essentially bounded functions with norm $\|u\|_\infty = \operatorname{esssup}_{t\in[0,T]} |u(t)|$.

The space of functions that are continuous on $[0,T]$ is denoted by $C[0,T]$ or often simply C and is a Banach space when endowed with the supremum norm $\|u\|_\infty := \max_{t\in[0,T]} |u(t)|$. $C^1 = C^1[0,T]$ denotes the continuously differentiable functions u, the derivative $u' \in C[0,T]$.

We will also use the space of absolutely continuous functions which is denoted $AC = AC[0,T]$.

If u is a continuous function and u' exists a.e. it does not follow that $u(t)-u(0) = \int_0^t u'(s)\,ds$, as shown, for example, by the well-known Lebesgue's singular function φ (also known as the Cantor-Vitali function, or Devil's staircase) where $\varphi(0) = 0$, $\varphi(1) = 1$ and $\varphi' = 0$ a.e.. In fact, we have the following equivalence.

$$\begin{aligned} &u \in AC[0,T] \text{ if and only if} \\ &u'(t) \text{ exists a.e.}, u' \in L^1[0,T] \text{ and} \\ &u(t) - u(0) = \int_0^t u'(s)\,ds \text{ for all } t \in [0,T]. \end{aligned} \tag{2.1}$$

For $\eta \ge -1$, we define a space of functions that are continuous except at 0, and which allows a pointwise integrable singularity at 0, For $\eta > -1$, the space C_η is defined by

$$C_\eta[0,T] := \{u : [0,T] \to \mathbb{R} : u \in L^1[0,T] \cap C(0,T] \text{ and } \lim_{t\to 0+} t^{-\eta}u(t) \text{ exists}\}. \tag{2.2}$$

When endowed with the norm $\|u\|_\eta = \sup_{t\in(0,T]} |t^{-\eta}u(t)|$, C_η is a Banach space. The spaces with a singularity at zero are $C_{-\eta}$ with $\eta > 0$. Functions u in $C_{-\eta}$ with $0 < \eta < 1$ are continuous on $(0,T]$ and $u(t) = t^\eta w(t),\ t > 0,$ for some $w \in C[0,T]$.

$C_{-\eta}$ is an appropriate class for the study of Riemann-Liouville fractional integrals and derivatives, see for example [7].

A note of caution: the space we denote $C_{-\eta}$ is often denoted by others as C_η. We use $C_{-\eta}$ since then we have the natural mapping property for fractional integrals, $I^\alpha : C_{-\eta} \to C_{\alpha-\eta}$.

In this paper Hölder spaces play a prominent role. Fractional integral have good properties in these spaces. The monograph by Samko, Kilbas and Marichev [32, §3.1,3.2,13.4] contains important facts and proofs, many of which were first proved by Hardy and Littlewood [21].

Definition 2.1. For an interval $[0,T]$, the Hölder space denoted $H^\lambda = H^\lambda[0,T]$, $0 < \lambda \le 1$, (also the notation $C^{0,\lambda}$ is often used) consists of all functions u defined on $[0,T]$ for which there is a positive real constant C_u, such that

$$|u(t) - u(s)| \le C_u |t-s|^\lambda, \text{ for all } t,s \in [0,T]. \tag{2.3}$$

The subspace $H^\lambda_\star[0,T]$ (there are other notations in the literature) is defined by

$$H^\lambda_\star := \big\{u \in H^\lambda : \sup_{0\le s<t\le T, t-s<h} \frac{|u(t)-u(s)|}{|t-s|^\lambda} \to 0 \text{ as } h \to 0\big\}. \tag{2.4}$$

That is, $u \in H^\lambda_\star[0,T]$ if for $0 \le t-h < t \le T$, $\frac{|u(t)-u(t-h)|}{h^\lambda} \to 0$ when $h \to 0$, uniformly in t. It is known, for example [32, Remark 1.1], that $H^\lambda_\star$ is a closed subspace of H^λ and the closure of C^1 in H^λ is the space $H^\lambda_\star$. Proofs can be found in [35, Lemma 2.1] and [35, Lemma 2.2]. It is clear that $u \in H^\lambda$ implies that $u \in H^\mu_\star$ for every $0 < \mu < \lambda$.

H^1 is the space of Lipschitz continuous functions, H^0 is the class of bounded functions and $H^0_\star$ is the space $C[0,T]$. H^λ_0 denotes the space $\{u \in H^\lambda : u(0) = 0\}$.

Example 2.2. For $c > 0$ and $t \in [0,T]$ let $u(t) = (t+c)^\lambda$ for $0 < \lambda < 1$. Then $u \in H^\lambda_\star[0,T]$. For $c = 0$, $u \in H^\lambda_0[0,T]$ but $u \notin H^\lambda_\star[0,T]$.

Proof. From the inequality $x^\lambda - y^\lambda \le (x-y)^\lambda$ for $0 \le y \le x$, we have $u(t) \in H^\lambda[0,T]$. Moreover, when $c > 0$, by L'Hôpital's rule

$$\lim_{h\to 0} \frac{(t+c)^\lambda - (t+c-h)^\lambda}{h^\lambda} = \lim_{h\to 0} \frac{h^{1-\lambda}}{(t+c-h)^{1-\lambda}} = 0.$$

For the case $c = 0$ let $s = 0$ and $t = h$. Then we have $|u(t)-u(s)|/(t-s)^\lambda = \frac{u(h)}{h^\lambda} = 1$ so the limit as $h \to 0+$ is not 0. □

A much deeper result is the following.

Example 2.3. If $0 < a < 1$, $b > 1$ then the function

$$C(x) = \sum_{k=1}^\infty a^k \cos(b^k \pi x),$$

does not have a finite derivative at any point when $ab \ge 1$. Further if $ab > 1$, and so $\lambda = \frac{\log(1/a)}{\log(b)} < 1$, then the above function belongs to H^λ but not to $H^\lambda_\star$.

The above examples was shown by Hardy [20, Theorems 1.31 and 1.33]. Note that in this paper log is the natural logarithm (base e) often denoted ln.

Remark 2.4. For $b > 1$ and $\lambda \le 1$ let $W_\lambda(x) = \sum_{n=0}^{\infty} b^{-n\lambda}\cos(b^n \pi x)$. Then W_1 is Hölder continuous of all orders $\alpha < 1$ but is not Lipschitz continuous. In the paper Ross, Samko and Love [31], the Weierstrass function $W(x) = W_1(x) - W_1(0)$, $x > 0$, is shown to have continuous Riemann-Liouville fractional derivatives of every order $\alpha < 1$, but nowhere has the first order derivative.

Remark 2.5. There is no inclusion relationship between Hölder spaces and AC. For example the Weierstrass function W is Hölder continuous but is not AC. Also there are AC functions that are not Hölder continuous, for example,

$$f(x) = \begin{cases} 1/\log x, & \text{if } x \in (0, 1/2], \\ 0 & \text{if } x = 0, \end{cases}$$

is AC but is not Hölder continuous for any $\lambda \in (0, 1)$.

We will use extensively the following weighted Hölder space; see [32, Definition 1.4] and [32, §3.2,13.4] for many properties of fractional integrals in such spaces. For a weight $\rho(t) = t^\beta$ we write

$$H^{\lambda,\beta}[0, T] := \{u \in C(0, T] : u(t)t^\beta \in H^\lambda[0, T]\}. \tag{2.5}$$

We will usually consider the case $0 \le \beta < 1$ when $H^{\lambda,\beta}[0, T] \subset L^1$, then functions have an integrable singularity at $t = 0$.

2.2. **Riemann-Liouville fractional integral.** In the study of fractional integrals and fractional derivatives the Gamma and Beta functions occur naturally and frequently. The Gamma function is, for $p > 0$, given by

$$\Gamma(p) := \int_0^\infty s^{p-1} \exp(-s)\, ds \tag{2.6}$$

which is an improper Riemann integral but is well defined as a Lebesgue integral, and is an extension of the factorial function: $\Gamma(n + 1) = n!$ for $n \in \mathbb{N}$. The Beta function is defined by

$$B(p, q) := \int_0^1 (1 - s)^{p-1} s^{q-1}\, ds \tag{2.7}$$

which is a well defined Lebesgue integral for $p > 0, q > 0$. It is well known, and proved in calculus texts, that $B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

The following simple lemma is classical, the proof follows simply by changing the variable of integration from s to σ where $s = \sigma t$. The result will be used several times.

Lemma 2.6. *Let $p > 0$, $q > 0$. Then we have*

$$\int_0^t (t - s)^{p-1} s^{q-1}\, ds = t^{p+q-1} B(p, q). \tag{2.8}$$

We always consider $\alpha \in (0, 1)$ in this paper.

Definition 2.7. The Riemann-Liouville (R-L) fractional integral of order $\alpha \in (0, 1)$ of a function $u \in L^1[0, T]$ is defined as an L^1 function by

$$I^\alpha u(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} u(s)\, ds.$$

The integral $I^\alpha u$ is the convolution of the L^1 functions h, u where $h(t) = t^{\alpha-1}/\Gamma(\alpha)$, so, by the well known results on convolutions, $I^\alpha u$ is defined as an L^1 function, in particular $I^\alpha u(t)$ is defined and finite for a.e. t. $I^\alpha u(0)$ is defined to be $\lim_{t\to 0+} I^\alpha u(t)$ if this limit exists. $I^\alpha u(0)$ is not necessarily defined for $u \in L^1$, for example if $0 < \alpha < \alpha + \varepsilon < 1$ and $u(t) = t^{-\alpha-\varepsilon}$ then $I^\alpha u(t) = \frac{\Gamma(1-\alpha-\varepsilon)}{\Gamma(1-\varepsilon)} t^{-\varepsilon}$ for $t > 0$ but the limit as $t \to 0+$ does not exist.

Interchanging the order of integration, using Fubini's theorem, shows that these fractional integral operators satisfy a semigroup property as follows:

Lemma 2.8 (Semigroup property). *Let $\alpha, \beta > 0$ and $u \in L^1[0,T]$. Then $I^\alpha I^\beta(u) = I^{\alpha+\beta}(u)$ as L^1 functions, in fact $I^\alpha I^\beta(u)(t) = I^{\alpha+\beta}(u)(t)$ for each t for which $I^{\alpha+\beta}|u|(t)$ exists (finite), that is for a.e. $t \in [0,T]$. If u is continuous, or if $u \in L^1$ and $\alpha + \beta \geq 1$, equality holds for all $t \in [0,T]$.*

The result is given in [32, (2.21)] and in [16, Theorem 2.2]. A detailed proof is given in [38, Lemma 3.4] where it is also shown that if $u \in C_{-\gamma}$ and $\alpha + \beta \geq \gamma$, this again holds for all $t \in [0,T]$.

We recall the following properties of fractional integral that we shall need. The mapping properties of fractional integro-differentiation within the framework of Hölder spaces are mainly due to Hardy and Littlewood [21].

Proposition 2.9. *Let $0 < \alpha < 1$.*

1. *For $1 \leq p \leq \infty$, I^α is a bounded operator from L^p into L^p.*
2. *For $1/p < \alpha < 1$, the fractional integral operator I^α is bounded from L^p into the Hölder space $H_\star^{\alpha-1/p}$. Moreover, $I^\alpha u(t) \to 0$ as $t \to 0+$, that is $I^\alpha u(0) = 0$.*
3. *$I^\alpha : L^\infty \to H^\alpha$. If $u \in C[0,T]$ and $u(0) = 0$ then $I^\alpha u \in H_\star^\alpha$. Hence, for $u \in C[0,T]$, $I^\alpha u(t) = u(0)t^\alpha/\Gamma(1+\alpha) + v(t)$ where $v \in H_\star^\alpha$.*
4. *I^α maps $AC[0,T]$ into $AC[0,T]$.*
5. *For $u \in C^1[0,T]$, $I^\alpha u \in C^1[0,T]$ if and only if $u(0) = 0$.*
6. *I^α does not map $C[0,T]$ into $AC[0,T]$.*

Proof. (1) The proof follows from Young's convolution theorem. A more precise result is given in [21, Theorem 4].

(2) This was proved by Hardy and Littlewood [21, Theorem 12]. The result is also proved in [32, Theorem 3.6 and Corollary].

(3) The first part is straightforward, for example this is proved in [39, Theorem 4.5] and was previously given in [32, Corollary 2, page 56] in less detail and with some misprints.

For the second part, it is necessary that $u(0) = 0$ since for a constant c, we have $I^\alpha c = ct^\alpha/\Gamma(1+\alpha)$, Example 2.2 then shows that the result fails when $u(0) \neq 0$. The result was proved by Hardy-Littlewood [21, Theorem 15]. A nice proof is given in the first part of Vainikko [34, Proposition 6.4] which uses the known fact that $H_\star^\alpha$ is a closed subset of H^α. For the last assertion, given $u \in C$, write $u(t) = u(0) + (u(t) - u(0))$ and use the fact that I^α is a linear operator.

(4) This is known, we give the easy proof. Since $u \in AC$, we have $u(t) = u(0) + Iu'$, where $u' \in L^1$. Therefore $I^\alpha u(t) = u(0)t^\alpha/\Gamma(1+\alpha) + I(I^\alpha u')(t)$ for all t by the semigroup property Lemma 2.8, and both terms on the right side are in AC.

(5) As above $u(t) = u(0) + Iu'$ gives $I^\alpha u(t) = u(0)t^\alpha/\Gamma(1+\alpha) + I(I^\alpha u')(t)$ where now $I^\alpha u'$ is continuous. Thus $I^\alpha u \in C^1$ if and only if $u(0) = 0$.

(6) This important fact has often been overlooked in the literature and has led to many mistaken claims. This was shown by Cichon-Salem [10, Counter-Example 1], and independently in [38, Addendum]. It follows simply from Remark 2.4. In fact, let $f_W(x) = D^\alpha W(x)$, then f_W is continuous and $I^\alpha f_W(x) = W(x)$ thus $I^\alpha f_W \notin AC$ since W is not differentiable at any point. □

Remark 2.10. The last result of part (3) is crucial in this paper.

2.3. **Riemann-Liouville and Caputo fractional derivatives.** Fractional derivatives are defined as 'inverses' of the fractional integral. We write D for the usual derivative operator, that is, $Du = u'$, with the usual one sided derivative at endpoints of an interval.

The Riemann-Liouville (R-L) fractional derivative of order $\alpha \in (0,1)$ is defined for functions that are more than integrable as follows.

Definition 2.11. For $\alpha \in (0,1)$ the R-L fractional derivative $D^\alpha u$ of an integrable function u is defined at a point $t \in [0,T]$ when $I^{1-\alpha}u$ is differentiable at t (one sided derivatives at 0 and T) by

$$D^\alpha u(t) := D\, I^{1-\alpha} u(t).$$

Note that $I^{1-\alpha}u(t)$ must be defined on an interval $(t-\delta, t+\delta)$ (when $0 < t < T$), this requires $u(s)$ to be defined for a.e. $s \in [0, t+\delta)$ and we require $u \in L^1[0, t+\delta)$.

The condition $I^{1-\alpha}u \in AC[0,T]$ must be imposed if we want to relate solutions of R-L fractional differential equations defined for a.e. $t \in [0,T]$ with solutions of a Volterra integral equation. It is not enough to assume that $I^{1-\alpha}u$ is differentiable for a.e. t. This has been noted long ago in the monograph [32], see [32, Definition 2.4] and the related comments in [32, Notes to §2.6].

As in the books Diethelm [16, Definition 3.2] and Kilbas, Srivastava and Trujillo [23, (2.4.1)], the Caputo fractional differential operator (or Caputo derivative) is defined via the R-L derivative as follows.

Definition 2.12. The Caputo fractional derivative $D_*^\alpha u$ of order $\alpha \in (0,1)$ is defined when $u(0)$ exists and $D^\alpha(u - u(0))$ exists at a point $t \in [0,T]$ by $D_*^\alpha u(t) := D^\alpha(u - u(0))(t)$.

The Caputo derivative is usually considered for continuous functions, then $u(0)$ does exist. If we want to relate Caputo fractional differential equations and integral equations then it is necessary to suppose that $I^{1-\alpha}(u - u(0)) \in AC[0,T]$ then $D_*^\alpha(t)$ is defined for a.e. t. If this AC condition is imposed, then for $f \in L^p$ with $p > 1/\alpha$ (for example f continuous) there is an equivalence between the fractional differential equation and an integral equation.

$$\begin{gathered} u \in C[0,T], I^{1-\alpha}u \in AC[0,T],\ D_*^\alpha u(t) = f(t),\ a.e.\ t \in [0,T],\ u(0) = u_0, \\ \text{is equivalent to } u \in C[0,T] \text{ and } u(t) = u_0 + I^\alpha f(t). \end{gathered} \tag{2.9}$$

This general case is proved in [25, Lemma 4], the case when f is continuous is well-known and can be found in Diethelm's book [16, Lemma 6.2]. We will make any necessary AC conditions explicit in this paper.

It is useful to note the following properties.

Lemma 2.13. *(1) If u and $D^\alpha u$ are continuous on $[0,T]$ then $I^{1-\alpha}u \in C^1$, $(I^{1-\alpha}u)(0) = 0$, and moreover $u(0) = 0$.*

(2) If $D_*^\alpha u$ is continuous on $[0,T]$ then u is continuous on $[0,T]$.

(3) If $u \in C^1[0,T]$ then $D_*^\alpha u$ is continuous. The converse is false.

Proof. (1) By definition, $D^\alpha u(t) = D(I^{1-\alpha}u)(t)$ so $D^\alpha u$ is continuous requires $I^{1-\alpha}u \in C^1$. Let $f(t) := D^\alpha u(t)$ with f continuous. Then integrating gives $(I^{1-\alpha}u)(t) - (I^{1-\alpha}u)(0) = If(t)$, that is $(I^{1-\alpha}u)(t) = If(t)$, using Proposition 2.9 (2). Applying the operator I^α and using the semigroup property gives $Iu(t) = I(I^\alpha u)(t)$ for all t. Since u and $I^\alpha u$ are continuous this gives $u(t) = I^\alpha f(t)$ which implies $u(0) = 0$.
(2) $D_*^\alpha u = f$ with f continuous gives $I^{1-\alpha}(u - u(0) \in C^1$ and $u(t) = u(0) + I^\alpha f(t)$ where $I^\alpha f$ is continuous by Proposition 2.9 (3), so u is continuous.
(3) Let $u \in C^1$. We have $D_*^\alpha u(t) = DI^{1-\alpha}(u(t) - u(0))$ where $I^{1-\alpha}(u(t) - u(0)) \in C^1$ by Proposition 2.9 (5). A simple counter-example for the converse is $u(t) = t^\alpha$, for which we have $I^{1-\alpha}u(t) = t\Gamma(1+\alpha) \in C^1$ but $u \notin C^1$. □

Remark 2.14. Part (1) means that studying the R-L fractional equation $D^\alpha u(t) = f(t, u(t))$ in the space $C[0,T]$ when f is continuous is equivalent to studying the Caputo fractional equation $D_*^\alpha u(t) = f(t, u(t))$ with $u(0) = 0$, so the Caputo case is more general in this case. R-L fractional equations should be studied in a larger space which allows singularities, such as $C_{-\eta}$ or L^1.

There is another frequently used definition of Caputo derivative, which uses the ordering $I^{1-\alpha}(Du)$. To distinguish between the definitions we will refer to this as the Caputo-C derivative.

Definition 2.15. *For $\alpha \in (0,1)$, if u is differentiable a.e. on $[0,T]$ and $Du = u' \in L^1[0,T]$ the Caputo-C fractional derivative, denoted $D_C^\alpha u(t)$, is defined for a.e. t by*

$$D_C^\alpha u(t) := I^{1-\alpha} Du(t).$$

This defines $D_C^\alpha u = I^{1-\alpha}(Du)$ as an L^1 function.

Remark 2.16. This definition is not adequate. In order to prove any result about a Caputo fractional equation with the definition D_C^α, it is not sufficient to suppose that $u' \in L^1$ but it is necessary to have $u \in AC$. In fact there exist functions such as, for example, Lebesgue's singular function φ which is (Hölder) continuous but not AC, $\varphi(0) = 0, \varphi(1) = 1$, and $\varphi'(t) = 0$ for a.e. t. Thus we would have $D_C^\alpha \varphi(t) = 0$, which shows that nothing useful about u can be deduced from $D_C^\alpha u(t) = f(t)$ without the AC condition, since also $D_C^\alpha(u + k\varphi) = f$ for all constants k.

When $u \in AC$, if u satisfies the equation $D_C^\alpha u(t) = f(t)$ a.e., then $u(t) = u(0) + I^\alpha f(t)$ for all t. However, the reverse is often claimed, namely that for $0 < \alpha < 1$ and f continuous

$$D_C^\alpha u(t) = f(t), \text{ for a.e. } t, \; u(0) = u_0, \text{ is equivalent to } u(t) = u_0 + I^\alpha f(t), \text{ for all } t.$$

However, 'solution' means different things on each side of the equation. Usually solution of the integral equation is a function in $C[0,T]$, and it has never been shown that u continuous and $u(t) = u_0 + I^\alpha f(t)$ for f continuous gives $u \in AC[0,T]$ for the very good reason that it is false in general, as shown above in Proposition 2.9 (6). A comprehensive discussion is given in [25]. A correct equivalence is given in (2.9) above.

As shown in Diethelm [16, Theorem 3.1], also in [38, Proposition 4.4], for $0 < \alpha < 1$ the two definitions $D_*^\alpha u$ and $D_C^\alpha u$ coincide when $u \in AC$, so there is usually no reason to consider $D_C^\alpha u$.

To avoid errors, the definition $D^\alpha_C u$ should not be used, except as an alternative for AC functions.

2.4. **The Marchaud fractional derivative.** We recall here some definitions from the monograph by Samko, Kilbas and Marichev [32, Section 13.1].

For $0 < \alpha < 1$, the Marchaud fractional derivative (MFD) is defined for a 'sufficiently good' function f defined on a finite interval $[0, T]$ by the formula

$$\mathrm{D}^\alpha_M f(t) := \frac{f(t)}{\Gamma(1-\alpha)t^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{f(t)-f(s)}{(t-s)^{1+\alpha}} ds. \tag{2.10}$$

The MFD is well defined for every $t \in (0, T]$ for functions f that are continuously differentiable or more generally that satisfy a Hölder condition of order $\lambda > \alpha$.

For more general functions, the definition is extended as follows in [32, page 226]. For $\varepsilon > 0$, the truncated fractional derivative is defined for $t \in (0, T]$ by

$$\mathrm{D}^\alpha_{M,\varepsilon} f(t) = \frac{f(t)}{\Gamma(1-\alpha)t^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \psi_\varepsilon(t), \tag{2.11}$$

where

$$\psi_\varepsilon(t) := \begin{cases} \int_0^{t-\varepsilon} \frac{f(t)-f(s)}{(t-s)^{1+\alpha}} ds, & t \geq \varepsilon, \\ f(t) \int_0^{t-\varepsilon} \frac{1}{(t-s)^{1+\alpha}} ds, & 0 \leq t < \varepsilon. \end{cases} \tag{2.12}$$

The last integral can be evaluated but it is not important to us. The fractional Marchaud derivative is then understood as

$$\mathrm{D}^\alpha_M f(t) = \lim_{\varepsilon \to 0, L^p} \mathrm{D}^\alpha_{M,\varepsilon} f(t) = \frac{f(t)}{\Gamma(1-\alpha)t^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \lim_{\varepsilon \to 0, L^p} \psi_\varepsilon(t), \tag{2.13}$$

defined when this limit exists, where the limit is with respect to the norm of the space L^p when considering fractional integrals of functions in L^p. Many results using this definition can be found in [32, Section 13.1].

We will consider another method of taking the limit, namely a pointwise limit as an improper integral.

Definition 2.17. For f defined on $[0, T]$ we define $D^\alpha_M f(t)$ and $D^\alpha_{*,M} f(t)$ by

$$\begin{gathered} D^\alpha_M f(t) := \frac{f(t)}{\Gamma(1-\alpha)t^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \lim_{\varepsilon \to 0+} \int_0^{t-\varepsilon} \frac{f(t)-f(s)}{(t-s)^{1+\alpha}} ds, \\ D^\alpha_{*,M} f(t) = D^\alpha_M (f(t) - f(0)) = \frac{f(t)-f(0)}{\Gamma(1-\alpha)t^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \lim_{\varepsilon \to 0+} \int_0^{t-\varepsilon} \frac{f(t)-f(s)}{(t-s)^{1+\alpha}} ds, \end{gathered} \tag{2.14}$$

for those $t > 0$ for which these limits exist. It is to be understood that one only considers $\varepsilon < t$. For a function g we will use the notation $\int_0^{t-} g(s)\, ds = \lim_{\varepsilon \to 0+} \int_0^{t-\varepsilon} g(s)\, ds$.

The definition 2.17 is motivated by the paper of Vainniko [34]. He claimed [34, Theorem 2.1] several deep equivalences when both u and $D^\alpha_* u$ are continuous, which corresponds to considering $u \in I^\alpha(C[0, T])$. In particular he claimed [34, Theorem 5.2] that $D^\alpha_* u(t) = D^\alpha_{*,M} u(t)$ for $0 < t \leq T$ in that case. Unfortunately there are some gaps in the proofs, we give details in Remark 4.4. We will prove that the important assertion, given above, of [34, Theorem 2.1] is correct in our Theorem 4.3.

We give a result that can be used to motivate the definition of Marchaud derivative. It's disadvantage is that $u \in AC$ is required which usually does not occur for solutions of Caputo fractional differential equations.

Lemma 2.18. *Let $0 < \alpha < 1$, $t > 0$, and suppose that $f \in AC[0,t] \cap H^{\alpha}_{\star}[0,t]$. Then $D^{\alpha}_{C} f$ exists at the point t if and only if $D^{\alpha}_{*,M} f$ exists at t and*

$$D^{\alpha}_{*,M} f(t) = D^{\alpha}_{C} f(t) = D^{\alpha}_{*} f(t).$$

Proof. For $t > 0$, since $f \in AC[0,t]$, we can integrate by parts to get the improper integral in the definition of $D^{\alpha}_{*,M} f(t)$ is given by

$$\begin{aligned}\int_0^{t-} \frac{f(t)-f(s)}{(t-s)^{1+\alpha}} ds = \lim_{s\to t-} \frac{1}{\alpha(t-s)^{\alpha}}(f(t)-f(s)) - \frac{1}{\alpha t^{\alpha}}(f(t)-f(0)) \\ + \int_0^{t-} \frac{1}{\alpha}(t-s)^{-\alpha} f'(s) ds.\end{aligned}$$

The first term on the right side is zero since $f \in H^{\alpha}_{\star}$, the second terms exists for $t > 0$ and the third term is $\frac{\Gamma(1-\alpha)}{\alpha} D^{\alpha}_{C} f(t)$ when it exists. Moving the term $\frac{1}{\alpha t^{\alpha}}(f(t)-f(0))$ to the left side and multiplying by $\frac{\alpha}{\Gamma(1-\alpha)}$, we see that if one of $D^{\alpha}_{*,M} f(t)$ and $D^{\alpha}_{C} f(t)$ exists then so does the other and they are equal. □

3. Existence of Marchaud derivative in weighted Hölder space

We will prove a result on existence of the Marchaud derivative. We will motivate the result by a new example that will prove to be very useful.

Lemma 3.1. *For $0 < \alpha < 1$ and $0 < \beta < 1$, let $u(t) = t^{-\beta}$ for $0 < t \le T$. Then $D^{\alpha}_{M} u(t)$ exists for $t > 0$ and is given by*

$$\begin{aligned} D^{\alpha}_{M} u(t) &= t^{-\beta-\alpha}(1-\alpha-\beta)\frac{\Gamma(1-\beta)}{\Gamma(2-\alpha-\beta)}, \text{ when } \alpha+\beta \ge 1, \\ D^{\alpha}_{M} u(t) &= t^{-\beta-\alpha}\frac{\Gamma(1-\beta)}{\Gamma(1-\alpha-\beta)}, \text{ when } \alpha+\beta < 1. \end{aligned} \tag{3.1}$$

Proof. We first show that for $t > 0$ each term exists in the expression for the MFD, which is

$$D^{\alpha}_{M} u(t) = \frac{t^{-\beta}}{\Gamma(1-\alpha)t^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{t-} \frac{t^{-\beta}-s^{-\beta}}{(t-s)^{1+\alpha}} ds. \tag{3.2}$$

The first term on the right side obviously exists for $t > 0$. For the improper integral it is important not to split the integrand. We let $s = t\sigma$ and write the integral as

$$t^{-\beta-\alpha} \int_0^{1-} \frac{1-\sigma^{-\beta}}{(1-\sigma)^{1+\alpha}} d\sigma.$$

The integrand is negative so we consider its negative. Temporarily ignoring the fixed term $t^{-\beta-\alpha}$ we therefore consider the integral

$$\int_0^{1-} \sigma^{-\beta} \frac{1-\sigma^{\beta}}{(1-\sigma)^{1+\alpha}} d\sigma.$$

Let $\gamma \in (0,1)$ be such that $\gamma \ge \beta$ and $\gamma > \alpha$, then $\sigma^{\beta} \ge \sigma^{\gamma}$ for $\sigma \in (0,1)$. Then, using the well known inequality $1-\sigma^{\gamma} \le (1-\sigma)^{\gamma}$ for $0 \le \sigma \le 1$, $0 \le \gamma \le 1$, we have

$$\sigma^{-\beta} \frac{1-\sigma^{\beta}}{(1-\sigma)^{1+\alpha}} \le \sigma^{-\beta} \frac{1-\sigma^{\gamma}}{(1-\sigma)^{1+\alpha}} \le \sigma^{-\beta}(1-\sigma)^{\gamma-1-\alpha}.$$

The integral $\int_0^1 \sigma^{-\beta}(1-\sigma)^{\gamma-1-\alpha}\,d\sigma = B(1-\beta,\gamma-\alpha)$ exists as a Lebesgue integral, therefore $\int_0^1 \sigma^{-\beta}\frac{1-\sigma^\beta}{(1-\sigma)^{1+\alpha}}\,d\sigma$ exists. This shows that the MFD exists for $t>0$.

The mathematical software Maple gives the integral to be given by

$$\int_0^t \frac{t^{-\beta}-s^{-\beta}}{(t-s)^{1+\alpha}}ds = -\frac{t^{-\beta-\alpha}}{\alpha} + (1-\alpha-\beta)\frac{t^{-\beta-\alpha}}{\alpha}\frac{\Gamma(1-\beta)\Gamma(1-\alpha)}{\Gamma(2-\alpha-\beta)}. \tag{3.3}$$

The second term in (3.2) is then

$$\frac{\alpha}{\Gamma(1-\alpha)}\int_0^t \frac{t^{-\beta}-s^{-\beta}}{(t-s)^{1+\alpha}}ds = -\frac{t^{-\beta-\alpha}}{\Gamma(1-\alpha)} + (1-\alpha-\beta)\frac{\Gamma(1-\beta)}{\Gamma(2-\alpha-\beta)}t^{-\beta-\alpha}. \tag{3.4}$$

A term in (3.2) cancels and this gives (3.1). The last term is equal to $t^{-\beta-\alpha}\frac{\Gamma(1-\beta)}{\Gamma(1-\alpha-\beta)}$ when $\alpha+\beta<1$ by the property $\Gamma(a+1)=a\Gamma(a)$ for $a>0$. $\square$

Remark 3.2. For $u(t)=t^{-\beta}$ $(0<\beta<1)$, the formula in (3.1) agrees with the calculation of the R-L fractional derivative $D^\alpha(u)(t)$ for $t>0$. In fact we have, using Lemma 2.6,

$$\begin{aligned}
I^{1-\alpha}u(t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}s^{-\beta}\,ds \\
&= t^{1-\alpha-\beta}\frac{1}{\Gamma(1-\alpha)}B(1-\alpha,1-\beta) \\
&= t^{1-\alpha-\beta}\frac{\Gamma(1-\beta)}{\Gamma(2-\alpha-\beta)}.
\end{aligned}$$

Therefore

$$D^\alpha u(t) = D\big(I^{1-\alpha}u(t)\big) = (1-\alpha-\beta)t^{-\alpha-\beta}\frac{\Gamma(1-\beta)}{\Gamma(2-\alpha-\beta)}.$$

Note that $u\notin AC[0,T]$. $I^{1-\alpha}u(t)=t^{1-\alpha-\beta}\frac{\Gamma(1-\beta)}{\Gamma(2-\alpha-\beta)}$ is $AC[0,T]$ when $\alpha+\beta\le 1$, u is not in $C^1[0,T]$, but is in $C^1(0,T]$.

Example 3.3. For the special case when $\alpha=1/2$ and $\beta=1/2$ the integral in (3.2) can be calculated by standard methods. It can be readily checked that

$$\int \frac{1-x^{-1/2}}{(1-x)^{3/2}}\,dx = 2\frac{1-x^{1/2}}{(1-x)^{1/2}}.$$

Hence, using L'Hôpital's rule to calculate $\lim_{x\to 1-}\frac{1-x^{1/2}}{(1-x)^{1/2}}=0$, we obtain $\int_0^{1-}\frac{1-x^{-1/2}}{(1-x)^{3/2}}\,dx=-2$ which gives, for $u(t)=t^{-1/2}$,

$$D_M^{1/2}u(t) = \frac{1}{\Gamma(1/2)} - 2\frac{1/2}{\Gamma(1/2)} = 0.$$

This agrees with (3.1).

Example 3.4. For $u(t)=t^\beta$ where $\beta\ge 0$ we see that $D_M^\alpha u(t)$ exists for $t>0$, since similarly to the above, we have

$$\int_0^t \frac{t^\beta-s^\beta}{(t-s)^{1+\alpha}}\,ds = t^{\beta-\alpha}\int_0^1 \frac{1-x^\beta}{(1-x)^{1+\alpha}}\,dx.$$

As before, take $\gamma \in (0,1)$ with $\gamma > \alpha$ and $\gamma \geq \beta$ to get

$$\int_0^1 \frac{1-\sigma^\beta}{(1-\sigma)^{1+\alpha}}\,d\sigma \leq \int_0^1 \frac{1-\sigma^\gamma}{(1-\sigma)^{1+\alpha}}\,d\sigma \leq \frac{1}{\gamma-\alpha},$$

so the integral exists for $t > 0$. Maple gives the formula

$$\int_0^{t-} \frac{t^\beta - s^\beta}{(t-s)^{1+\alpha}}\,ds = -\frac{t^{\beta-\alpha}}{\alpha} + t^{\beta-\alpha}\frac{\Gamma(1-\alpha)\Gamma(1+\beta)}{\alpha\Gamma(1-\alpha+\beta)}.$$

This then gives

$$D_M^\alpha u(t) = \frac{t^\beta}{\Gamma(1-\alpha)t^\alpha} - \frac{t^{\beta-\alpha}}{\Gamma(1-\alpha)} + t^{\beta-\alpha}\frac{\Gamma(1+\beta)}{\Gamma(1-\alpha+\beta)} = t^{\beta-\alpha}\frac{\Gamma(1+\beta)}{\Gamma(1-\alpha+\beta)}$$

which agrees with the calculation of $D^\alpha u(t)$. The special case $\beta = \alpha$ gives $D_M^\alpha t^\alpha = \Gamma(1+\alpha)$, which gives another proof of the existence result in §3.2 of Vainikko's paper [34].

Example 3.5. The special case $\alpha = \beta = 1/2$ can be found by standard methods.

$$\text{The indefinite integral is } \int \frac{1-t^{1/2}}{(1-t)^{3/2}}\,dt = 2\sin^{-1}(t^{1/2}) + 2\frac{1-t^{1/2}}{(1-t)^{1/2}},$$

$$\text{hence } \int_0^1 \frac{1-t^{1/2}}{(1-t)^{3/2}}\,dt = \pi - 2.$$

We can now prove the following result.

Theorem 3.6. *Let $0 < \alpha < 1$, let $0 \leq \beta < 1$, and suppose that $u \in H^{\lambda,\beta}$ for some $\lambda \in (\alpha, 1]$. Then $D_M^\alpha u(t)$ exists for $t \in (0,T]$.*

Proof. Let $v(t) = u(t)t^\beta$ so that $v \in H^\lambda$ and $u(t) = v(t)t^{-\beta}$ for $t > 0$. We will show that all terms exist for $t > 0$ in the expression for $D_M^\alpha u(t)$, namely

$$D_M^\alpha u(t) = \frac{t^{-\beta}v(t)}{\Gamma(1-\alpha)t^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{t-} \frac{t^{-\beta}v(t) - s^{-\beta}v(s)}{(t-s)^{1+\alpha}}ds.$$

Clearly the first term exists for $t > 0$. We write the integral as follows.

$$\int_0^{t-} \frac{t^{-\beta}v(t) - s^{-\beta}v(s)}{(t-s)^{1+\alpha}}ds = v(t)\int_0^{t-} \frac{t^{-\beta} - s^{-\beta}}{(t-s)^{1+\alpha}}ds + \int_0^{t-} \frac{s^{-\beta}(v(t)-v(s))}{(t-s)^{1+\alpha}}ds$$

The first term on the right exists for $t > 0$ by Lemma 3.1 above. For the second term we have

$$\begin{aligned}\int_0^t \frac{s^{-\beta}|v(t)-v(s)|}{(t-s)^{1+\alpha}}ds &\leq \int_0^t \frac{s^{-\beta}C_v|t-s|^\lambda}{(t-s)^{1+\alpha}}ds \\ &= C_v\int_0^t s^{-\beta}(t-s)^{\lambda-\alpha-1}\,ds = C_v t^{\lambda-\alpha-\beta}B(1-\beta, \lambda-\alpha).\end{aligned}$$

Thus, for $t > 0$, the integrand is dominated by an integrable function and is therefore integrable so the second integral exists. □

Remark 3.7. There are continuous functions that are not Hölder continuous for any $\lambda \in (0,1)$ so the set of functions $H^{\lambda,\beta}$ does not include all functions in the space $C_{-\beta}$. Of course the set includes functions that are not continuous at 0. The result just proved complements the paper of Vainikko [34] which requires u and $D^\alpha u$ to be continuous. For functions that are simple powers of t, $u(t) = t^\gamma$, in order that

u and $D^\alpha u$ are continuous, it is necessary that $\gamma \geq \alpha$. Our result allows a greater range of power functions as we now show.

Lemma 3.8. *For $0 < \alpha < 1$ and $\alpha - 1 < \gamma \leq 1$, let $u(t) = t^\gamma$; note that γ may be negative. Then $u \in H^{\lambda,\beta}$ for some $\alpha < \lambda \leq 1$ and some $\beta \in [0, 1)$.*

Proof. We can write $u(t) = t^{-\beta} t^{\gamma+\beta}$ and this is in $H^{\lambda,\beta}$ for $\lambda = \gamma + \beta$, where we want to have $\alpha < \gamma + \beta \leq 1$. If $\gamma \geq 0$ we may choose $\beta \in (\alpha - \gamma, 1 - \gamma]$. If $\gamma < 0$, then we must choose $\beta > 0$ such that $\alpha - \gamma < \beta < 1$, this choice is possible if and only if $\alpha - \gamma < 1$, that is $\gamma > \alpha - 1$. $\square$

Corollary 3.9. *If $f(t) = t^\rho$ for $-1 < \rho < 1 - \alpha$ then $I^\alpha f \in H^{\lambda,\beta}$ for some $\lambda \in (\alpha, 1]$ and some $\beta \in [0, 1)$. If $\rho \geq 1 - \alpha$ and $f(t) = t^\rho$ then $I^\alpha f \in C^1 \subset H^1$*

Proof. For the first part, we have $I^\alpha f(t) = \frac{\Gamma(1+\rho)}{\Gamma(1+\alpha+\rho)} t^{\alpha+\rho}$ where $\alpha - 1 < \alpha + \rho < 1$. Lemma 3.8 now applies. The second part is immediate since $t^{\alpha+\rho} \in C^1$ when $\alpha + \rho \geq 1$. $\square$

4. Equality of fractional derivatives in weighted Hölder space

We will deal with a function $v \in H^\alpha_\star$. Such a function has the property that, for arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that $|v(t) - v(s)| < \varepsilon |t - s|^\alpha$ for $|t - s| < \delta$. We will be considering a fixed $t > 0$. Without loss of generality we can suppose that $\delta < t/2$. The following estimate is very useful.

Lemma 4.1. *Let $0 < \alpha < 1$. For $t > 0$ and $0 < h < \eta \leq t$, we have*

$$
\begin{aligned}
0 &\leq \int_{t-\eta}^{t-h} \big((t-s)^{-1-\alpha} - (t+h-s)^{-1-\alpha}\big)(t-s)^\alpha \, ds \\
&\leq \frac{1}{\alpha}\Big(\frac{\eta}{h+\eta}\Big)^\alpha - \frac{1}{\alpha 2^\alpha} + \log\big(\frac{2\eta}{h+\eta}\big) \\
&\leq \frac{1}{\alpha}(1 - \frac{1}{2^\alpha}) + \log(2) := c_\alpha.
\end{aligned}
\tag{4.1}
$$

In particular for $0 \leq \beta < 1$, $v \in H^\alpha_\star$ and $0 < h < \eta < \delta < t/2$,

$$
0 \leq \int_{t-\eta}^{t-h} \big((t-s)^{-1-\alpha} - (t+h-s)^{-1-\alpha}\big) s^{-\beta} |v(t) - v(s)| \, ds < \varepsilon c_\alpha (2/t)^\beta.
$$

Proof. The first part of the proof is the same as the proof of [34, Lemma 6.1] with a change of the lower limit of integration. Integrating by parts and estimating from below we get

$$
\begin{aligned}
\int_{t-\eta}^{t-h} (t+h-s)^{-1-\alpha}(t-s)^\alpha \, ds &= \frac{1}{\alpha}\big(2^{-\alpha} - (\frac{\eta}{h+\eta})^\alpha\big) \\
&\quad + \int_{t-\eta}^{t-h} (t+h-s)^{-\alpha}(t-s)^{\alpha-1} \, ds \\
&\geq \frac{1}{\alpha}\big(2^{-\alpha} - (\frac{\eta}{h+\eta})^\alpha\big) + \int_{t-\eta}^{t-h} (t+h-s)^{-1} \, ds.
\end{aligned}
$$

Then we have

$$\begin{aligned}0 &\le \int_{t-\eta}^{t-h} \big((t-s)^{-1-\alpha} - (t+h-s)^{-1-\alpha}\big)(t-s)^{\alpha}\, ds\\ &\le \frac{1}{\alpha}\Big(\frac{\eta}{h+\eta}\Big)^{\alpha} - \frac{1}{\alpha 2^{\alpha}} + \int_{t-\eta}^{t-h} (t-s)^{-1} - (t+h-s)^{-1}\, ds\\ &= \frac{1}{\alpha}\Big(\frac{\eta}{h+\eta}\Big)^{\alpha} - \frac{1}{\alpha 2^{\alpha}} + \log\big(\frac{\eta}{h}\big) - \log\big(\frac{h+\eta}{2h}\big) \le c_{\alpha}.\end{aligned}$$

For the second part, when $0 < h < \eta < \delta < t/2$, we have $h < t-s < \eta < \delta$, therefore for $v \in H^{\alpha}_{\star}$ we have $|v(t)-v(s)| \le \varepsilon(t-s)^{\alpha}$ and $s^{-\beta} < (t-\delta)^{-\beta} < (2/t)^{\beta}$, and the result follows directly from the first part. □

We now have the important result which proves an equality of the Marchaud and R-L/Caputo fractional derivatives, which will be used to justify the most useful claim of [34].

Theorem 4.2. *Assume that $0 < \alpha < 1$, $0 \le \beta < 1$, and $u(t) = t^{-\beta}v(t)$ for $v \in H^{\alpha}_{\star}[0,T]$. Then the R-L derivative $D^{\alpha}u(t) = (I^{1-\alpha}u)'(t)$ exists for some $t \in (0,T)$ if and only if the Marchaud derivative exists and we then have*

$$D^{\alpha}u(t) = D^{\alpha}_{M}u(t) = \frac{1}{\Gamma(1-\alpha)}\Big(u(t)t^{-\alpha} + \alpha\int_0^{t-} (t-s)^{-\alpha-1}(u(t)-u(s))\, ds\Big). \tag{4.2}$$

Proof. The case $\beta = 0$ is somewhat simpler but with the exact same method so we do not give the simpler version separately. The case $\beta > 0$ also has extra complications but they come only at the end of the proof. Let $t \in (0,T)$ be fixed throughout the proof. Let $h > 0$ be such that $t+h < T$. We consider the expression

$$\Gamma(1-\alpha)\frac{I^{1-\alpha}u(t+h) - I^{1-\alpha}u(t)}{h} = \frac{1}{h}\Big(\int_0^{t+h}(t+h-s)^{-\alpha}u(s)\, ds - \int_0^{t}(t-s)^{-\alpha}u(s)\, ds\Big).$$

We have $u(s) = s^{-\beta}v(s)$ where $v \in H^{\alpha}_{\star}$. We write

$$\begin{aligned}&\int_0^{t+h}(t+h-s)^{-\alpha}u(s)\, ds - \int_0^{t}(t-s)^{-\alpha}u(s)\, ds\\ &= \int_0^{t+h}(t+h-s)^{-\alpha}s^{-\beta}(v(s)-v(t))\, ds - \int_0^{t}(t-s)^{-\alpha}s^{-\beta}(v(s)-v(t)\, ds \\ &\quad + v(t)\Big(\int_0^{t+h}(t+h-s)^{-\alpha}s^{-\beta}\, ds - \int_0^{t}(t-s)^{-\alpha}s^{-\beta}\, ds\Big).\end{aligned} \tag{4.3}$$

By Lemma 2.6, the last term is

$$v(t)\big((t+h)^{1-\alpha-\beta} - t^{1-\alpha-\beta}\big)B(1-\alpha, 1-\beta).$$

Then, since $t > 0$, by the definition of derivative we obtain

$$v(t)\lim_{h\to 0}\big(\frac{(t+h)^{1-\alpha-\beta} - t^{1-\alpha-\beta}}{h}\big)B(1-\alpha,1-\beta) = v(t)t^{-\alpha-\beta}(1-\alpha-\beta)B(1-\alpha,1-\beta).$$

We now study the other terms in (4.3). We write

$$\begin{aligned}
&\int_0^{t+h}(t+h-s)^{-\alpha}s^{-\beta}(v(s)-v(t))\,ds-\int_0^t(t-s)^{-\alpha}s^{-\beta}(v(s)-v(t)\,ds\\
&=\int_0^t\big((t-s)^{-\alpha}-(t+h-s)^{-\alpha}\big)s^{-\beta}(v(t)-v(s))\,ds\\
&\quad+\int_t^{t+h}(t+h-s)^{-\alpha}s^{-\beta}(v(s)-v(t))\,ds.
\end{aligned}\tag{4.4}$$

For the second integral, since $v\in H^\alpha_\star$, for $h<\delta$ we have

$$\begin{aligned}
\Big|\int_t^{t+h}(t+h-s)^{-\alpha}s^{-\beta}(v(s)-v(t))\,ds\Big| &\le \varepsilon\int_t^{t+h}(t+h-s)^{-\alpha}s^{-\beta}(t-s)^\alpha\,ds\\
&\le \varepsilon\int_t^{t+h}\Big(\frac{t-s}{t+h-s}\Big)^\alpha t^{-\beta}\,ds\le \varepsilon t^{-\beta}h.
\end{aligned}$$

Therefore, for $t>0$,

$$\frac{1}{h}\Big|\int_t^{t+h}(t+h-s)^{-\alpha}s^{-\beta}(v(s)-v(t))\,ds\Big|\le\varepsilon t^{-\beta}. \tag{4.5}$$

The final term to deal with is $\frac{1}{h}\int_0^t\big((t-s)^{-\alpha}-(t+h-s)^{-\alpha}\big)s^{-\beta}(v(t)-v(s))\,ds$. For $t>0$, we split the integral into the sum $\int_0^{t-h}+\int_{t-h}^t$. For the second term, as above, we use $|v(t)-v(s)|\le\varepsilon(t-s)^\alpha$, and for $h<\delta<t/2$ we get

$$\begin{aligned}
&\Big|\frac{1}{h}\int_{t-h}^t\big((t-s)^{-\alpha}-(t+h-s)^{-\alpha}\big)s^{-\beta}(v(s)-v(t))\,ds\Big|\\
&\le\varepsilon\frac{1}{h}\int_{t-h}^t\Big(1-\big(\frac{(t-s)}{(t+h-s)}\big)^\alpha\Big)(t-h)^{-\beta}\,ds\le\varepsilon(2/t)^\beta.
\end{aligned}$$

For the first term, we claim that

$$\int_0^{t-h}\Big(\alpha(t-s)^{-1-\alpha}-\frac{\big((t-s)^{-\alpha}-(t+h-s)^{-\alpha}\big)}{h}\Big)s^{-\beta}(v(t)-v(s))\,ds\to0, \tag{4.6}$$

as $h\to0$. Since $t-s>0$ for $0\le s\le t-h$, by the mean value theorem we have

$$\frac{(t-s)^{-\alpha}-(t+h-s)^{-\alpha}}{h}=\alpha(t+\hat h-s)^{-1-\alpha},\ \text{where } 0<\hat h=\hat h(s,h)<h.$$

Therefore we consider

$$\int_0^{t-h}\big((t-s)^{-1-\alpha}-(t+\hat h-s)^{-1-\alpha}\big)s^{-\beta}(v(t)-v(s))\,ds.$$

Let $\eta>0$ be such that $0<h<\eta<\delta<t/2$ and then write the integral as $\int_0^{t-h}=\int_0^{t-\eta}+\int_{t-\eta}^{t-h}$. For $\int_{t-\eta}^{t-h}$ we have, using Lemma 4.1,

$$\begin{aligned}
&\Big|\int_{t-\eta}^{t-h}\big((t-s)^{-1-\alpha}-(t+\hat h-s)^{-1-\alpha}\big)s^{-\beta}(v(t)-v(s))\,ds\Big|\\
&\le\int_{t-\eta}^{t-h}\big((t-s)^{-1-\alpha}-(t+h-s)^{-1-\alpha}\big)s^{-\beta}|v(t)-v(s)|\,ds<\varepsilon c_\alpha(2/t)^\beta.
\end{aligned}$$

In the integral $\int_0^{t-\eta}$ we have $t-s\geq\eta$, and $|v(t)-v(s)|\leq C_v(t-s)^\alpha$ since $v\in H^\alpha$. Therefore we have for $s\in[0,t-\eta]$,

$$|(t+\hat{h}-s)^{-1-\alpha}s^{-\beta}||v(t)-v(s)|\leq \eta^{-1-\alpha}C_v t^\alpha s^{-\beta}\in L^1[0,t-\eta].$$

By the dominated convergence theorem, we get

$$\lim_{h\to 0}\int_0^{t-\eta}(t+\hat{h}-s)^{-1-\alpha}s^{-\beta}(v(t)-v(s))\,ds=\int_0^{t-\eta}(t-s)^{-1-\alpha}s^{-\beta}(v(t)-v(s))\,ds,$$

which, by the result for $\int_{t-\eta}^{t-h}$ proved above, shows that (4.6) holds. The part above applies when $\beta=0$ and for this case $u=v$. We have then proved that, for a fixed $t\in(0,T)$, the right hand derivative $D^\alpha u(t)$ exists if and only if the improper integral $\int_0^{t-}\alpha(t-s)^{-1-\alpha}(u(t)-u(s))\,ds$ exists. In that case we have

$$\begin{aligned} D^\alpha u(t)&=\frac{1}{\Gamma(1-\alpha)}u(t)t^{-\alpha}\\ &\quad+\lim_{h\to 0}\frac{\alpha}{\Gamma(1-\alpha)}\int_0^{t-h}(t-s)^{-1-\alpha}(u(t)-u(s))\,ds, \end{aligned}\tag{4.7}$$

that is $D^\alpha u(t)=D_M^\alpha u(t)$.

For the case $\beta>0$ we need an extra step. We write

$$\begin{aligned} &\int_0^{t-}(t-s)^{-1-\alpha}(u(t)-u(s))\,ds=\int_0^{t-}(t-s)^{-1-\alpha}(t^{-\beta}v(t)-s^{-\beta}v(s))\,ds\\ &=\int_0^{t-}(t-s)^{-1-\alpha}s^{-\beta}(v(t)-v(s)\,ds+v(t)\int_0^{t-}(t-s)^{-1-\alpha}(t^{-\beta}-s^{-\beta})\,ds\\ &=\int_0^{t-}(t-s)^{-1-\alpha}s^{-\beta}(v(t)-v(s)\,ds\\ &\quad+v(t)\big(-\frac{t^{-\beta-\alpha}}{\alpha}+(1-\alpha-\beta)\frac{t^{-\beta-\alpha}}{\alpha}B(1-\beta,1-\alpha)\big). \end{aligned}\tag{4.8}$$

where we have used Lemma 3.1.

Collecting all the information together, two terms cancel and we get

$$\begin{aligned} D^\alpha u(t)&=\lim_{h\to 0}\frac{I^{1-\alpha}u(t+h)-I^{1-\alpha}u(t)}{h}\\ &=\frac{\alpha}{\Gamma(1-\alpha)}\Big(\int_0^{t-}(t-s)^{-1-\alpha}s^{-\beta}(v(t)-v(s))\,ds\\ &\quad+v(t)t^{-\alpha-\beta}(1-\alpha-\beta)\frac{B(1-\alpha,1-\beta)}{\Gamma(1-\alpha)}\Big)\\ &=\frac{\alpha}{\Gamma(1-\alpha)}\int_0^{t-}(t-s)^{-1-\alpha}(u(t)-u(s))\,ds+\frac{1}{\Gamma(1-\alpha)}v(t)t^{-\alpha-\beta}\\ &=\frac{u(t)t^{-\alpha}}{\Gamma(1-\alpha)}+\frac{\alpha}{\Gamma(1-\alpha)}\int_0^{t-}(t-s)^{-1-\alpha}(u(t)-u(s))\,ds=D_M^\alpha u(t). \end{aligned}$$

Thus we have $D^\alpha u(t)=D_M^\alpha u(t)$ for $0<t<T$ whenever one of these exists. The left hand derivative for $0<t\leq T$ can be treated similarly, by studying

$$\frac{1}{\Gamma(1-\alpha)}\lim_{h\to 0}\int_0^{t-h}\frac{\big((t-h-s)^{-\alpha}-(t-s)^{-\alpha}\big)}{h}s^{-\beta}(v(t)-v(s))\,ds.$$

with the same methods, as indicated in Vainikko [34]. □

There are some important special cases.

Theorem 4.3. *(1) Suppose that the Caputo derivative $D_*^\alpha u$ is continuous on $[0,T]$. Then the Marchaud derivative also exists for all $t \in (0,T]$ and $D_*^\alpha u(t) = D_{*,M}^\alpha u(t)$. If u and the R-L derivative $D^\alpha u(t)$ are continuous then $D^\alpha u(t) = D_M^\alpha u(t)$ for all $t \in (0,T]$.*

(2) Assume that $0 < \alpha < 1$, $0 < \beta < 1$, and $u \in H^{\lambda,\beta}$ for some $\lambda > \alpha$. Then $D^\alpha u(t)$ exists for all $t \in (0,T]$ and $D^\alpha u(t) = D_M^\alpha u(t)$.

(3) Let $0 < \alpha < 1$, and suppose that, for some $\lambda > \alpha$, either $u - u(0) \in H^{\lambda,\beta}$ for some $0 < \beta < 1$ or that $u \in H^{\lambda,\beta}$ and some $\beta \geq \lambda$. Then $D_*^\alpha u(t)$ exists for all $t \in (0,T]$ and $D_*^\alpha u(t) = D_{*,M}^\alpha u(t)$.

Proof. (1) When $D_*^\alpha u(t) = f(t)$ with f continuous on [0,T], we have

$$u(t) - u(0) = I^\alpha f(t) = (I^\alpha f(0))(t) + I^\alpha (f - f(0))(t) = f(0)\frac{t^\alpha}{\Gamma(1+\alpha)} + v(t),$$

where $v \in H_\star^\alpha$, by Proposition 2.9 (3) (which again shows that u is continuous). Since we have $D^\alpha(f(0)\frac{t^\alpha}{\Gamma(1+\alpha)}) = f(0) = D_M^\alpha(f(0)\frac{t^\alpha}{\Gamma(1+\alpha)})$, by Example 3.4, the result follows from the case $\beta = 0$ of Theorem 4.2. For the R-L case we note that, by Lemma 2.13, $u(0) = 0$ and the R-L derivative equals the Caputo derivative.
(2) Since $D_M^\alpha u(t)$ exists for $t > 0$ when $u \in H^{\lambda,\beta}$ by Theorem 3.6 and since $H^\lambda \subset H_\star^\alpha$, the result follows from Theorem 4.2.
(3) $D_*^\alpha u(t) = D^\alpha(u(t) - u(0))$ and the first assertion follows from part (2). For the second assertion, $u(0) \in H^{\lambda,\beta}$ if $u(0)t^\beta \in H^\lambda$, that is for $\beta \geq \lambda$ and then $u(t) - u(0) \in H^{\lambda,\beta}$; part (2) again applies. ☐

Remark 4.4. (a) Case (1) is probably the most useful since it corresponds to studying the fractional differential equation $D_*^\alpha u(t) = f(t, u(t))$ in the space of continuous functions. For continuous functions this Caputo case is more general than the R-L case where $u(0) = 0$ is implied; see Lemma 2.13. For case (2) it is known [32, Lemma 13.2] that $I^\alpha : H_0^{\gamma,\beta} \to H_0^{\alpha+\gamma,\beta}$ if $\alpha + \gamma < 1$. The proof uses connections with L^p spaces, and leaves many items to be proved by the reader. This result would require equations to be studied in a weighted Hölder space.
(b) The case $\beta = 0$ is the case studied by Vainikko [34, Theorem 5.2]. There is a gap in the proof of Lemma 6.2 of Vainikko [34], the last part (end of page 477) uses $|v(t) - v(s)| < \varepsilon |t - s|^\alpha$ which is only valid for $|t - s| < \delta$ not for all $s \in [0, t - h]$. This gap is filled by our argument. Some extra properties are asserted in the main theorem of Vainikko [34, Theorem 2.1]. These properties should be treated with caution since their proofs use a wrong assertion. We do not need these extra properties and we do not have a correction. In the proof of of [34, Proposition 6.4, page 482], the Banach-Steinhaus theorem is cited as saying that, because C^1 is dense in C, convergence as $\theta \to 1$ of operators $A_\theta u \to Au$ in C for each function $u \in C^1$ implies the convergence for each continuous function u. This is not correct, the Banach-Steinhaus theorem applies for convergence on a non-meagre subset, but C^1 is a meagre (first Baire category) subset of C, so this argument fails. In fact the set of functions which are differentiable at even a single point is meagre in $C[0,1]$ so also AC functions form a meagre subset of C. This is proved using the Baire Category Theorem, see for example [22, Theorem 17.8] or [8, Page 184]
(c) Our part (2) of Corollary 4.3 and Remark 3.2 shows that $D^\alpha u(t) = D_M^\alpha u(t)$, for all $t > 0$, is possible for functions that are not continuous at 0.

5. A fractional differential inequality and a maximum principle

In proving a comparison theorem for fractional differential equations, the following result was given by Lakshmikantham and Vatsala [24, Lemma 2.1].

Lemma 5.1 ([24, Lemma 2.1])**.** *Let $m : \mathbb{R}_+ \to \mathbb{R}$ be continuous. If for some $t_1 > 0$, we have*

$$m(t) \le 0 \text{ for } 0 < t \le t_1, \text{ and } m(t_1) = 0,$$

and m is locally λ-Hölder continuous (that is, near t_1) for some $\lambda > \alpha$, then it follows that $D^\alpha m(t_1) \ge 0$.

The proof assumes that $D^\alpha m(t_1)$ exists which was not stated and does not seem to follow from the *local* Hölder continuity, it does follow from λ-Hölder continuity on all of $[0, t_1]$. There are some typos in the given proof.

It was claimed in Denton-Vatsala [12, Lemma 2.7] and by Devi-McRae-Drici [14, Lemma 2.3], and also by Ma and Yan [28, Corollary 2.1], that an improved version held.

Lemma 5.2. [12, 14, 28]. *Let $m \in C_{\alpha-1}$ be such that for some $t_1 \in (0, T]$, we have*

$$m(t) \le 0 \text{ for } 0 < t \le t_1, \text{ and } m(t_1) = 0,$$

then $D^\alpha m(t_1) \ge 0$.

The proofs are not correct. It is assumed that $D^\alpha m(t_1)$ exists, which is not valid under these assumptions since, even for a function $g \in C$, the fractional integral $I^{1-\alpha} g$ need not be differentiable at t_1. For example let $g(t) = D^{1-\alpha} W(t)$, as in the proof of Proposition 2.9 (6).

Moreover the proof in [12] uses the following inequality

$$|t_1^{1-\alpha} m(t) - s^{1-\alpha} m(s)| \le h K_h, \text{ for } h \text{ small and } t_1 - h \le s \le t_1,$$

where K_h are uniformly bounded constants, which is only valid for locally Lipschitz functions. The proof in [14] is similar with a variant of the local Lipschitz condition. The proof in [28] is close to the one of Denton-Vatsala. The comparison result [28, Lemma 2.2] is therefore not correctly proved.

Under a weak Hölder condition we will give a correct versions of Lemma 5.2 in Corollary 5.5 (3).

Remark 5.3. Cong-Tuan-Trinh [11, Lemma 25] claim the result holds for the Caputo derivative assuming only that $D_*^\alpha u$ exists on the interval $(0, T]$, they write "proof of this lemma is obtained by using arguments as in the proof of [29, Lemma 2.1]". However that Lemma refers for the proof to Denton-Vatsala [12, Lemma 2.7] which proof does not give this as noted above. Our result will prove the result holds when $D_*^\alpha u$ is continuous on $[0, T]$.

We first give a result for the Marchaud derivative.

Theorem 5.4. *Suppose that $t_1 \in (0, T)$ and $u(t) \le 0$ for $0 < t < t_1$ and $u(t_1) = 0$. If $D_M^\alpha u(t_1)$ exists then $D_M^\alpha u(t_1) \ge 0$. Moreover, if also u is continuous and $u(0) < 0$ then $D_M^\alpha u(t_1) > 0$.*

Proof. If $D_M^\alpha(t_1)$ exists then it is given by

$$D_M^\alpha u(t_1) = \frac{u(t_1)}{\Gamma(1-\alpha) t_1^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{t_1-} \frac{u(t_1) - u(s)}{(t_1 - s)^{1+\alpha}} ds.$$

Since $u(t_1)=0$ and the integrand is non-negative, we have $D^\alpha_M u(t_1)\ge 0$. When u is continuous and $u(0)<0$ then $u(s)<0$ on an interval so the integral is strictly positive and $D^\alpha_M u(t_1)>0$. □

Corollary 5.5. *(1) Suppose that $D^\alpha_* u$ is continuous and that $u(t)\le 0$ for $0<t<t_1$ and $u(t_1)=0$. Then $D^\alpha_* u(t_1)\ge \frac{-u(0)}{t_1^\alpha\Gamma(1-\alpha)}\ge 0$ and if $u(0)<0$ then $D^\alpha_* u(t_1)>0$.*
(2) Suppose that u and $D^\alpha u$ are continuous and that $u(t)\le 0$ for $0<t<t_1$ and $u(t_1)=0$. Then $D^\alpha u(t_1)\ge 0$.
(3) Let $u\in H^{\lambda,\beta}$ for some $\beta\in[0,1)$ and some $\lambda>\alpha$. Suppose that $t_1\in(0,T)$ and $u(t)\le 0$ for $0<t<t_1$ and $u(t_1)=0$. Then $D^\alpha u(t_1)\ge 0$. If u is continuous and $u(0)<0$ then $D^\alpha u(t_1)>0$.

Proof. For the Caputo case (1)

$$\begin{aligned} D^\alpha_* u(t_1) &= D^\alpha_{*,M} u(t_1)\\ &= \frac{u(t_1)-u(0)}{t_1^\alpha\Gamma(1-\alpha)}+\frac{\alpha}{\Gamma(1-\alpha)}\int_0^{t_1-}\frac{u(t_1)-u(s)}{(t_1-s)^{1+\alpha}}\,ds\\ &\ge \frac{-u(0)}{t_1^\alpha\Gamma(1-\alpha)}, \end{aligned}$$

since $u(t_1)=0$ and the integrand is non-negative. Case (2) is case (1) with $u(0)=0$. For case (3), $D^\alpha u(t_1)$ exists and equals $D^\alpha_M u(t_1)$ by Theorem 4.3, the result follows by Theorem 5.4. □

Remark 5.6. The observation that we get a strict inequality in Theorem 5.4 is new. Part (3) is an improved version of Lemma 5.1 since we have a weight. Parts (1), (2) are more readily applicable versions of that Lemma since no explicit Hölder continuity condition needs to be assumed. Al-Refai and Luchko [5, Theorem 2.2] proved similar results of R-L and Caputo-C type for so called general fractional derivatives where the general fractional integral has $(t-s)^{\alpha-1}$ replaced by a Sonine type kernel. They use the class of functions $u\in C[0,T]\cap C^1(0,T]$ and $u'\in L^1$ which implies $u\in AC[0,T]$ by [38, Proposition 2.2].

Example 5.7. We illustrate item (3). Take $\alpha=1/2$, $0<\gamma<1/2$, let $u(t)=1-t^{-\gamma}$ on $[0,2]$ with $t_1=1$. Then $u(t)\in H^{\lambda,\beta}$ for some $\lambda>\alpha$ and $\beta\in(\alpha+\gamma,1]$ by Lemma 3.8. We have

$$I^{1/2}u(t)=\frac{1}{\Gamma(1/2)}\int_0^t (t-s)^{-1/2}(1-s^{-\gamma})\,ds=\frac{1}{\Gamma(1/2)}\big(2t^{1/2}-t^{1/2-\gamma}B(1/2,1-\gamma)\big).$$

Thus $D^{1/2}u(t)=D(I^{1/2}u)(t)=t^{-1/2}/\Gamma(1/2)-(1/2-\gamma)\Gamma(1-\gamma)t^{-1/2-\gamma}/\Gamma(3/2-\gamma)$. At $t_1=1$ we have $D^{1/2}u(t_1)=\frac{1}{\Gamma(1/2)}-\frac{\Gamma(1-\gamma)}{\Gamma(1/2-\gamma)}$ which can be checked to be (strictly) positive for every $\gamma\in(0,1/2)$. In this case $u(t)$ is continuous on $(0,2)$, negative on $(0,1)$ and zero at $t=1$.

We now use the preceding results to prove a maximum principle.

Theorem 5.8. *Let $D^\alpha_* u$ be continuous on $[0,T]$ and suppose that u attains its maximum at $t_1\in(0,T)$. Then $D^\alpha_* u(t_1)\ge\frac{u(t_1)-u(0)}{t_1^\alpha\Gamma(1-\alpha)}\ge 0$. If u and $D^\alpha u$ are continuous on $[0,T]$ then $D^\alpha u(t_1)\ge\frac{u(t_1)}{t_1^\alpha\Gamma(1-\alpha)}$.*

Proof. Let $v(t) := u(t) - u(t_1)$. Then v is continuous and $D_*^\alpha v(t) = D_*^\alpha u(t)$ is therefore also continuous. Moreover on the interval $[0, t_1]$, $v(t) \le v(t_1) = 0$. By Corollary 5.5 we have $D_*^\alpha v(t_1) \ge \frac{-v(0)}{t_1^\alpha \Gamma(1-\alpha)}$, which is the result. For u and $D^\alpha u$ continuous, the R-L case is the special case $u(0) = 0$. □

Al-Refai [3, Theorems 2.1,2.4] proved the equivalent results for the case of a minimum assuming that $u \in C^1$ for Caputo-C and R-L derivatives; at least $u \in AC$ is required for those proofs because integration by parts is used. Al-Refai and Luchko [4, Theorem 2.1] proved the result for a maximum assuming $u \in C^1$ by the same method as [3]. Our result is more general since by Lemma 2.13 (3) we use a weaker condition. Luchko-Yamamoto [27, Theorem 3.1] has a similar result for the general fractional derivative of R-L and Caputo-C type for $u \in C[0,T] \cap C^1(0,T]$ and $u' \in L^1$.

We can now prove a comparison theorem as given under Hölder continuity assumptions in Lakshmikantham-Vatsala [24, Theorem 2.3]. Al-Refai and Luchko [5, Theorems 2.5, 2.6] proved a result for the general fractional derivatives using a Caputo-C type of definition and used the space mentioned above where functions are AC.

Theorem 5.9. *Let $f \in C([0,T] \times \mathbb{R})$.*
(1) Let $D_*^\alpha u$ and $D_*^\alpha v$ be continuous. Suppose that

$$(i)\ D_*^\alpha u(t) \le f(t, u(t)),\quad (ii)\ D_*^\alpha v(t) \ge f(t, v(t)),\ \text{for all}\ 0 \le t \le T. \tag{5.1}$$

Then $u(0) < v(0)$ implies that $u(t) < v(t)$, for $0 \le t \le T$.
(2) Let $u, v \in H^{\lambda,\beta}$ for some $\beta \in [0,1)$ and some $\lambda > \alpha$. If also u is continuous then the same result holds.

Proof. (1) Let $w(t) := u(t) - v(t)$, then $D_*^\alpha w$ is continuous and $w(0) < 0$. If w becomes zero then there exists $0 < t_1 \le T$ such that $w(t) < 0$ for $0 < t < t_1$ and $w(t_1) = 0$. By Corollary 5.5 (1) this implies $D_*^\alpha w(t_1) > 0$. Thus we have

$$f(t_1, u(t_1)) \ge D_*^\alpha u(t_1) > D_*^\alpha v(t_1) \ge f(t_1, v(t_1)) = f(t_1, u(t_1)).$$

This contradiction proves the result.
(2) Now $w = u - v \in H^{\lambda,\beta}$ and Corollary 5.5 (3) applies. □

Remark 5.10. Part (1) is likely to be most applicable since a solution of Caputo FDE satisfies $D_*^\alpha u$ is continuous. In Lakshmikantham-Vatsala [24, Theorem 2.3] it was assumed that $u, v \in H^\lambda$ for some $\lambda > \alpha$ and that one of the inequalities in (5.1) is strict. We showed in part (2) that the strict hypothesis can be removed and weaker hypotheses are possible. Cong-Tuan-Trinh [11, Proposition 26] claimed a similar comparison result assuming only that $D_*^\alpha v(t), D_*^\alpha w(t)$ exist on $(0,T]$ and applying their version of Lemma 5.2, which, as noted above, is not valid. Wu [41, Theorem 3.2] proved a comparison result in terms of a maximal solution. Wu used the D_C^α definition in the paper, so needs to have an extra AC hypothesis, but his proof actually uses the D_*^α definition since it used Vainikko's claimed result, our Theorem 4.3 (1), showing that the Caputo derivative equals the Marchaud derivative.

6. Lyapunov inequality for a convex function

We will use the following known inequality. For completeness we include the proof. By an interval J we mean either a finite interval, with or without endpoints, or $[0,\infty)$ or $\mathbb{R}$.

Lemma 6.1. *Let J be an interval, $V : J \to \mathbb{R}$ be convex and differentiable at a point x (one sided derivative at an endpoint). Then for $x, y \in J$,*

$$V(x) - V(y) \le V'(x)(x-y). \tag{6.1}$$

Proof. For $x, y \in J$, if $x = y$ the result is trivial. If $x \neq y$, by convexity we get for $\lambda \in (0,1)$,

$$\begin{gathered} V(\lambda y + (1-\lambda)x) \le \lambda V(y) + (1-\lambda)V(x), \text{ that is} \\ V(x + \lambda(y-x)) \le V(x) + \lambda(V(y) - V(x)). \end{gathered} \tag{6.2}$$

This gives

$$V(y) - V(x) \ge \frac{V(x+\lambda(y-x)) - V(x)}{\lambda} = \frac{V(x+\lambda(y-x)) - V(x)}{\lambda(y-x)}(y-x). \tag{6.3}$$

Letting $\lambda \to 0+$, differentiability of V at x shows that the right side has a limit and therefore

$$V(y) - V(x) \ge V'(x)(y-x), \text{ for } y \neq x,$$

which gives (6.1). □

Remark 6.2. For V convex and differentiable on an open interval J, the derivative is continuous, see Rockafellar [30] page 246, Theorem 25.5 and Corollary 25.5.1, hence V satisfies a Lipschitz condition on any closed subinterval of J.

We first give a result appropriate for the Caputo derivative case.

Theorem 6.3. *Let V be convex and differentiable on an interval J. For a function $u : [0,T] \to J$, define $V_u(t) := V(u(t))$. If $D^\alpha_{*,M} u$ exists at a point $t > 0$ then $D^\alpha_{*,M} V_u(t)$ also exists and we have*

$$\begin{aligned} D^\alpha_{*,M} V_u(t) \le V'(u(t)) D^\alpha_{*,M} u(t) + \frac{1}{\Gamma(1-\alpha)t^\alpha} V(u(t)) - V(u(0)) \\ - V'(u(t))(u(t) - u(0)). \end{aligned} \tag{6.4}$$

In particular we have the weaker version

$$D^\alpha_{*,M} V_u(t) \le V'(u(t)) D^\alpha_{*,M} u(t). \tag{6.5}$$

Proof. We have

$$D^\alpha_{*,M} V_u(t) := \frac{V(u(t)) - V(u(0))}{\Gamma(1-\alpha)t^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \lim_{\varepsilon\to 0} \int_0^{t-\varepsilon} \frac{V(u(t)) - V(u(s))}{(t-s)^{1+\alpha}} ds.$$

Our assumption on u means that

$$\lim_{\varepsilon\to 0} \int_0^{t-\varepsilon} \frac{u(t) - u(s)}{(t-s)^{1+\alpha}} ds \text{ exists.}$$

Thus, for $0 < \eta < t$, there exists $\delta > 0$ such that for every $0 < \varepsilon_1 < \varepsilon_2 < \delta$,

$$\Big| \int_{t-\varepsilon_2}^{t-\varepsilon_1} \frac{u(t) - u(s)}{(t-s)^{1+\alpha}} ds \Big| < \eta.$$

Then, using Lemma 6.1, we get

$$\begin{aligned}\Big|\int_{t-\varepsilon_2}^{t-\varepsilon_1} \frac{Vu(t)-Vu(s)}{(t-s)^{1+\alpha}}ds\Big| &\le \Big|\int_{t-\varepsilon_2}^{t-\varepsilon_1} V'(u(t))\frac{u(t)-u(s)}{(t-s)^{1+\alpha}}ds\Big|\\ &\le |V'(u(t))|\Big|\int_{t-\varepsilon_2}^{t-\varepsilon_1} \frac{u(t)-u(s)}{(t-s)^{1+\alpha}}ds\Big| < |V'(u(t)|\,\eta.\end{aligned}$$

This proves that $D^\alpha_{*,M}V_u(t)$ exists for this t. Moreover we have the bounds

$$\begin{aligned}D^\alpha_{*,M}V_u(t) &= \frac{V(u(t))-V(u(0))}{\Gamma(1-\alpha)t^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{t-}\frac{V(u(t))-V(u(s))}{(t-s)^{1+\alpha}}ds\\ &\le \frac{V(u(t))-V(u(0))}{\Gamma(1-\alpha)t^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)}V'(u(t))\int_0^{t-}\frac{u(t))-u(s))}{(t-s)^{1+\alpha}}ds\\ &= \frac{V(u(t))-V(u(0))}{\Gamma(1-\alpha)t^\alpha} + V'(u(t))\Big(D^\alpha_{*,M}u(t) - \frac{u(t)-u(0)}{\Gamma(1-\alpha)t^\alpha}\Big)\\ &= V'(u(t))D^\alpha_{*,M}u(t)\\ &\quad + \frac{1}{\Gamma(1-\alpha)t^\alpha}\big(V(u(t))-V(u(0))-V'(u(t))(u(t)-u(0))\big).\end{aligned}$$

Since $V(u(t))-V(u(0))-V'(u(t))(u(t)-u(0))\le 0$ by Lemma 6.1, this completes the proof. ☐

For the R-L type case we have the following result.

Corollary 6.4. *Let V be convex and differentiable on an interval J containing* 0, *and $V(0)=0$. For a function $u:[0,T]\to J$, defined at every point of $[0,T]$, define $V_u(t):=V(u(t))$. If $D^\alpha_M u$ exists at a point $t>0$, then also $D^\alpha_M V_u(t)$ exists and we have*

$$D^\alpha_M V_u(t) \le V'(u(t))D^\alpha_M u(t). \tag{6.6}$$

The proof of the above corolary is almost identical to that of Theorem 6.3 replacing $u(0)$ by 0.

Corollary 6.5. *Let V be convex and differentiable on an interval J and suppose that $u:[0,T]\to J$ is continuous and $D^\alpha_* u$ is continuous. Then, for every $t>0$ we have*

$$\begin{aligned}D^\alpha_* V_u(t) &\le V'(u(t))D^\alpha_* u(t)\\ &\quad + \frac{1}{\Gamma(1-\alpha)t^\alpha}\big(V(u(t))-V(u(0))-V'(u(t))(u(t)-u(0))\big)\\ &\le V'(u(t))D^\alpha_* u(t).\end{aligned}\tag{6.7}$$

Proof. $D^\alpha_* u = D^\alpha_{*,M}u$ by Theorem 4.3 so the latter exists for all $t>0$. Theorem 6.3 now applies. ☐

Remark 6.6. Chen-Dai-Song-Zhang [9] proved the weaker form of Corollary 6.5 for u and V continuously differentiable using the Caputo-C definition. Gomoyunov [19, Lemma 4.1] proved the weaker version (6.5) holds for a.e. t for the R-L fractional derivative when $u\in I^\alpha(L^\infty[0,T])$, by first giving a proof for functions $u\in \mathrm{Lip}_0$ (functions u satisfying a Lipschitz condition and with $u(0)=0$) and then using a non-trivial approximation argument. Note that $u\in I^\alpha(L^\infty[0,T])$ implies $u\in H^\alpha$ (but not necessarily $H^\alpha_\star$) and $u(0)=0$. Tuan-Trinh [33, Theorem 2] also gave the weaker version of the result for the Caputo derivative D_*^α for functions in $I^\alpha(C[0,T])$ using the characterization of continuous Caputo derivatives claimed by Vainikko [34] which shows it is equal to the Marchaud derivative $D_{*,M}^\alpha$. Their proof used some of the claimed extra properties in Vainikko [34] which are unclear. Li and Liu [26, Proposition 3.11] proved similar weaker versions for a generalized Caputo derivative, defined in terms of distributions, assuming that V is a convex C^1 function and $u \in C[0,T] \cap C^1(0,T]$.

We now give two useful examples of Lyapunov functions that occur frequently in the literature.

Proposition 6.7. *Suppose that u is nonnegative on $[0,T]$ and $D_{*,M}^\alpha u(t)$ exists for some $t > 0$. Then for $r \geq 1$ we have (strong version)*

$$\begin{aligned} &D_{*,M}^\alpha u^r(t) \\ &\leq r u^{r-1}(t) D_{*,M}^\alpha u(t) - \frac{1}{\Gamma(1-\alpha)t^\alpha}\big((r-1)u^r(t) + u^r(0) - r u^{r-1}(t)u(0)\big). \end{aligned} \tag{6.8}$$

If $D_*^\alpha u$ is continuous, then

$$D_*^\alpha u^r(t) \leq r u^{r-1}(t) D_*^\alpha u(t) - \frac{1}{\Gamma(1-\alpha)t^\alpha}\big((r-1)u^r(t) + u^r(0) - r u^{r-1}(t)u(0)\big), \quad t > 0. \tag{6.9}$$

If u and $D^\alpha u$ are continuous, then

$$D^\alpha u^r(t) \leq r u^{r-1}(t) D^\alpha u(t) - \frac{1}{\Gamma(1-\alpha)t^\alpha}(r-1)u^r(t), \tag{6.10}$$

for $t > 0$. In particular under these conditions we have the corresponding weaker versions

$$D_*^\alpha u^r(t) \leq r u^{r-1}(t) D_*^\alpha u(t), \ t > 0. \tag{6.11}$$

$$D^\alpha u^r(t) \leq r u^{r-1}(t) D^\alpha u(t), \ t > 0. \tag{6.12}$$

Proof. $V(u) = u^r$ is well defined on $J = [0,\infty)$ and is convex for $r \geq 1$. Apply Theorem 6.3 and Corollary 6.5. □

Remark 6.8. This result (6.10) is an improved version of Fewster-Young [17, Lemma 6] who did the case $r = 2$ with the R-L derivative assuming that $u \in C^1$, which seems to be the first time the stronger inequality is given. The proof in [17] actually uses a definition via Hadamard's finite-part integral (see Diethelm's book [16, Lemma 2.21, p.38], but this was not stated.

Aguila-Camacho, Duarte-Mermoud and Gallegos [1, Lemma 1] proved the weaker version (6.11) for $r = 2$ and the Caputo derivative D_C^α assuming that u is continuously differentiable. Alikhanov [2, Lemma 1] gave the result for $r = 2$ for an AC function u, but the proof requires more, it used differentiability of $I^{1-\alpha}u'$. Díaz, Pierantozzi and Vázquez [15, Lemma 3.1] have the weaker inequality for a.e. t when $u \in AC$ is either increasing or decreasing. Alsaedi, Ahmad and Kirane [6, Lemma 1] proved a result for a product uv of two functions whose special case $u = v$ would prove the result (6.12) for $r = 2$ when $u \in H^\lambda$ for some $\lambda > \alpha$. Their proof assumes both $D^\alpha u$ and $D^\alpha v$ exist which does not follow from their hypotheses. For the Caputo case the D_C^α definition is used in [6] which requires an AC condition and restricts its applicability.

The second example is a type known as a Volterra Lyapunov function.

Proposition 6.9. *Let $V(u) := u - w_0 - w_0\log(\frac{u}{w_0})$ for $u > 0$, $w_0 > 0$. Then V is convex and differentiable for $u > 0$. If u is always positive and $D^\alpha_{*,M}u(t)$ exists at a point $t > 0$ then we have*

$$\begin{aligned}
&D^\alpha_{*,M}V_u(t)\\
&\le (1-\frac{w_0}{u(t)})D^\alpha_{*,M}u(t) + \frac{1}{\Gamma(1-\alpha)t^\alpha}\big(u(t)-w_0-w_0\log(\frac{u(t)}{w_0})\\
&\quad -(1-\frac{w_0}{u(t)})(u(t)-w_0)\big)\\
&= (1-\frac{w_0}{u(t)})D^\alpha_{*,M}u(t) + \frac{1}{\Gamma(1-\alpha)t^\alpha}\Big((1-\frac{w_0}{u(t)})w_0 - w_0\log(\frac{u(t)}{w_0})\Big),\\
&\qquad \textit{(strong form)}\\
&\le (1-\frac{w_0}{u(t)})D^\alpha_{*,M}u(t) \quad \textit{(weak form).}
\end{aligned} \tag{6.13}$$

The inequalities hold for all $t > 0$ with $D^\alpha_{*,M}$ replaced by D^α_* when D^α_*u is continuous.

Proof. This follows at once from Theorem 6.3 and Corollary 6.5. □

Remark 6.10. Vargas-De-León [36, Lemma 3.1], with a longer argument, gave the weak form of this result for the Caputo derivative $D^\alpha_C u$ assuming that u is differentiable. Our results shows that this holds for $D^\alpha_* u$ under weaker conditions.

7. Comments on higher order cases

Vainkko's paper also states results for higher order fractional derivatives. The result used is that for $m \in \mathbb{N}$, and $m < \beta < m+1$, if the ordinary derivative $D^m u$ and the fractional derivative $D^\beta_* u$ are continuous, then the equality $D^\beta_* u = D^{\beta-m}_* D^m u$ holds. Then the result for $0 < \alpha < 1$ can be applied to $D^m u$. We prove this equality for the case $D^{1+\alpha}_* u$ with $0 < \alpha < 1$, similar arguments prove the general case.

Lemma 7.1. *Let $0 < \alpha < 1$. If $u \in C^1[0,T]$ and $I^{1-\alpha}u' \in AC[0,T]$ then $D^{1+\alpha}_* u(t) = D^\alpha_* u'(t)$ for a.e. t. Moreover, if $D^\alpha_* u' = D\big(I^{1-\alpha}(u'-u'(0))\big) \in C$ then $D^{1+\alpha}_* u(t) = D^\alpha_* u'(t)$ for all t.*

Proof. Using the semigroup property we have

$$I^{1-\alpha}(u-u(0)-tu'(0)) = I^{1-\alpha}I(u'-u'(0)) = I\,I^{1-\alpha}(u'-u'(0)), \text{ for all } t.$$

Therefore, when $I^{1-\alpha}u' \in AC[0,T]$, we obtain

$$D^{1+\alpha}_* u(t) = D^2 I^{1-\alpha}(u-u(0)-tu'(0))(t) = D\,I^{1-\alpha}(u'-u'(0))(t) = D^\alpha_* u'(t), \text{ for a.e. } t.$$

When $I^{1-\alpha}(u'-u'(0)) \in C^1$ then $I^{1-\alpha}(u-u(0)-tu'(0)) \in C^2$ and the result holds for all t. □

Remark 7.2. Theorem 4.3 can be applied to get the result that $D^\alpha_* u' = D^\alpha_{*,M} u'$ when u' and $D^{1+\alpha}_* u$ are continuous, as in the paper of Vainikko [34]. Derhab and Imakhlav [13, Theorem 2.11] used this result to prove a maximum principle for a function $u \in C^1$ with $D^{1+\alpha}_* u \in C$. This maximum principle was proved by Al-Refai [3, Theorem 2.11] assuming the stronger condition that $u \in C^2[0,T]$ which implies that $D^{1+\alpha}_* u$ is continuous.

For the Caputo derivative, $D_*^{1+\alpha}u$ continuous implies that u' is continuous. For when $D_*^{1+\alpha}u = f$ with f continuous we have $u(t) = u(0) + tu'(0) + I^{1+\alpha}f$, and $u'(t) = u'(0) + I^{\alpha}f(t)$, thus u' is continuous. This is not true in the R-L case. We have the following observation.

Lemma 7.3. *Let $0 < \alpha < 1$ and suppose that $D^{1+\alpha}u = f$ where f is continuous. Then $u \in C[0,T]$ only if $u(0) = 0$. Moreover $u \in C^1$ only if $u(0) = 0$ and $u'(0) = 0$.*

Proof. From $D^{1+\alpha}u = f$, we obtain

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha} + I^{1+\alpha}f(t),$$

where $c_1 = I^{1-\alpha}u(0)/\Gamma(\alpha)$ and $c_2 = (D^{\alpha}u)(0)/\Gamma(1+\alpha)$, see for example [38, Theorem 6.8]. Thus, in general, u is not continuous. The term $I^{1+\alpha}f \in C^1$, hence $u \in C$ requires $c_1 = 0$ which then gives $u(0) = 0$. To have $u \in C^1$ we must have $c_1 = c_2 = 0$ so $u(t) = I^{1+\alpha}f(t)$ and then $u'(t) = I^{\alpha}f(t)$. Proposition 2.9 (2) gives $u(0) = 0$ and $u'(0) = 0$. □

Remark 7.4. This means that the R-L case with a continuous fractional derivative can only use the equality with the Marchaud derivative involving u' if $u' \in C^1$ and both $u(0) = 0$ and $u'(0) = 0$, when it is then a special case of the Caputo derivative.

8. Conclusion

We have proved that the Marchaud derivative is equal to the much better known R-L or Caputo derivatives under some more general conditions than previous works. As a special case our result proves that the important assertion in the paper of Vainikko [34], whose proof had gaps, is valid. We have shown how the Marchaud derivative is convenient to prove inequalities that are useful in the study of properties of solutions of FDEs including their Lyapunov stability.

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