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CRITICAL FUJITA EXPONENTS FOR A CLASS OF QUASILINEAR COUPLED PARABOLIC EQUATIONS

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Abstract. This article concerns the critical Fujita exponents for a class of quasilinear coupled parabolic equations. Using energy estimates, suitable supersolutions, and the comparison principle, the blow-up theorem of Fujita type is established, and the critical Fujita exponent is obtained. Furthermore, we show that the critical case belongs to the blow-up case.

1. Introduction

In this article, we study the critical Fujita exponent for the Cauchy problem of quasilinear coupled parabolic equations

$$\frac{\partial u}{\partial t} = \Delta u^m + (|x|+1)^\lambda v^p, \quad x \in \mathbb{R}^n,\ t > 0, \tag{1.1}$$

$$\frac{\partial v}{\partial t} = \Delta v^m + (|x|+1)^\mu u^q, \quad x \in \mathbb{R}^n,\ t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \mathbb{R}^n, \tag{1.3}$$

where $p, q > m > 1$, $\lambda \geq 0$,

$$\mu = \frac{\lambda(q-m) + 2(q-p)}{p-m} \geq 0 \tag{1.4}$$

and $0 \leq u_0,\ v_0 \in C_0(\mathbb{R}^n)$ are nontrivial.

The earliest research on critical exponents of parabolic equations was published in 1966 by Fujita [5]. It was demonstrated that the Cauchy problem of the heat equation

$$\frac{\partial u}{\partial t} = \Delta u + u^p, \quad x \in \mathbb{R}^n,\ t > 0$$

admits no nontrivial nonnegative global solution when $1 < p < p_c = 1 + 2/n$, otherwise, it admits both nontrivial global (with small initial data) and nonglobal nonnegative (with large initial data) solutions when $p > p_c$. Subsequently, in [9, 13, 31], it was proved that any solution blows up in the critical case $p = p_c$. Herein, p_c is termed the critical Fujita exponent, and the corresponding results constitute the blow-up theorem of Fujita type.

Fujita's famous work reveals the relationship between the asymptotic behavior of the solutions to nonlinear partial differential equations and the exponents of the

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nonlinear internal sources. Since then, many important results have been obtained, including different types of equations and systems in various geometries with or without degeneracies or singularities, different boundary conditions, and different extension directions. For more detail, we refer the reader to works [1, 2, 3, 8, 9, 12, 15, 16, 17, 18, 11, 25, 35, 21, 23, 24, 26, 30, 31, 34, 36, 37, 38, 39, 40] and the references therein.

For the Cauchy problem, Galaktionov et al. [6, 7] studied the single slow diffusion equation

$$\frac{\partial u}{\partial t} = \Delta u^m + u^p, \quad x \in \mathbb{R}^n,\ t > 0,$$

where $p > m > 1$. It was proved that the critical Fujita exponent is $p_c = m + 2/n$. The Cauchy problem of the equation

$$|x|^{\lambda_1} \frac{\partial u}{\partial t} = \Delta u^m + |x|^{\lambda_2} u^p, \quad x \in \mathbb{R}^n,\ t > 0$$

with $p > m$ and $0 \le \lambda_1 \le \lambda_2 < p(\lambda_1 + 1) - 1$ was formulated as $p_c = m + (2 + \lambda_2)/(n + \lambda_1)$ in [29]. Furthermore, the authors proved that the critical case $p = p_c$ is also a blow-up case.

For the Cauchy problem of the following coupled semilinear parabolic system,

$$\frac{\partial u}{\partial t} = \Delta u + t^{\beta_1}|x|^{\alpha_1} v^p, \quad \frac{\partial v}{\partial t} = \Delta v + t^{\beta_2}|x|^{\alpha_2} u^q, \quad x \in \mathbb{R}^n,\ t > 0,$$

where α_1, α_2, β_1, $\beta_2 \ge 0$, and $p, q \ge 1$. Escobedo and Herrero in [4] considered this Cauchy problem with $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$, and they proved that the critical Fujita curve is

$$(pq)_c = 1 + \frac{2}{n} \max\{p + 1,\, q + 1\}.$$

More general, if $\beta_1 = \beta_2 = 0$, Mochizuki and Huang ([19]) proved that the critical Fujita curve is

$$(pq)_c = 1 + \frac{1}{n} \max\big\{(\alpha_1 + 2) + (\alpha_2 + 2)p,\, (\alpha_2 + 2) + (\alpha_1 + 2)q\big\}$$

with $0 < \alpha_1 < n(p - 1)$ and $0 < \alpha_2 < n(q - 1)$. If $\alpha_1 = \alpha_2 = 0$, it was proved in [27] that the critical Fujita curve is

$$(pq)_c = 1 + \frac{2}{n} \max\{(\beta_2 + 1)p + \beta_1 + 1,\, (\beta_1 + 1)q + \beta_2 + 1\}$$

with $pq > 1$. There are also some studies on the Cauchy problem of the coupled porous medium systems with fast diffusion

$$\frac{\partial u}{\partial t} = \Delta u^{m_1} + v^p, \quad \frac{\partial v}{\partial t} = \Delta v^{m_2} + u^q, \quad x \in \mathbb{R}^n,\ t > 0, \tag{1.5}$$

where $0 < m_1$, $m_2 < 1$, p, $q \ge 1$ and $pq > 1$. Qi and Levine ([22]) proved that the critical Fujita curve of the Cauchy problem of (1.5) is

$$(pq)_c = m_1 m_2 + \frac{2}{n} \max\{m_2 + p, m_1 + q\},$$

and proved that any nontrivial solution to the Cauchy problem of (1.5) must blow up in finite time if $pq < (pq)_c$, whereas both nonnegative nontrivial global and blowing-up solutions exist if $pq > (pq)_c$ with $m_1 = m_2$.

They pointed out that the method of constructing global supersolutions fails when $m_1 \ne m_2$ because of the different propagation rates of the two types of diffusion. Later in [10], it was shown that for the "very fast diffusions" case that $0 < m_1, m_2 < (n-2)_+/n$, the Cauchy problem of (1.5) admits nontrivial global solutions if the initial data is small enough although m_1 is not equal to m_2. Recently, the problem (1.1)–(1.3) with the special case $\lambda = 0$ was considered in [14], and it was shown that the critical Fujita exponent is $p_c = m + 2/n$. However, the result for the critical case $p = p_c$ remains unknown.

In this paper, we prove that the critical Fujita exponent of problem (1.1)–(1.3) is

$$p_c = m + \frac{\lambda + 2}{n}. \tag{1.6}$$

As in [20, 14, 21, 28, 29], we study the blowing-up properties of solutions by the integral estimates, and global existence of solutions by constructing a pair of suitable self-similar supersolutions. Note that the choice of μ is used to ensure that self-similar supersolutions have the same support set. This is still open to the other μ. Furthermore, we prove that the critical case $p = p_c$ can be classified as a blow-up case by analyzing the asymptotic behavior of the solutions.

This article consists of three sections. In §2, we introduce several basic definitions and theorems. Subsequently, in §3, we prove the blow-up theorems of Fujita type for the problem (1.1)–(1.3). Subsequently, the critical case $p = p_c$ is considered in §4.

2. Preliminaries

In this section, we introduce some basic definitions and relevant lemmas.

Definition 2.1. Assume that $0 < T \le +\infty$ and u, v are two nonnegative functions. If

$$u,\, v \in C([0,T),\, L^m_{\rm loc}(\mathbb{R}^n)) \cap L^\infty_{\rm loc}((0,T);\, L^\infty(\mathbb{R}^n))$$

and for any $0 \le \varphi, \psi \in C^{2,1}(\mathbb{R}^n \times [0,T))$ vanishing when t near T or $|x|$ being sufficiently large, the integral inequalities

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^n} u(x,t)\frac{\partial \varphi}{\partial t}(x,t)\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^n} u^m(x,t)\Delta\varphi(x,t)\mathrm{d}x\mathrm{d}t \\
&+ \int_0^T \int_{\mathbb{R}^n} (|x|+1)^\lambda v^p(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^n} u_0(x)\varphi(x,0)\mathrm{d}x \le (\ge)\, 0, \\
&\int_0^T \int_{\mathbb{R}^n} v(x,t)\frac{\partial \psi}{\partial t}(x,t)\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^n} v^m(x,t)\Delta\psi(x,t)\mathrm{d}x\mathrm{d}t \\
&+ \int_0^T \int_{\mathbb{R}^n} (|x|+1)^\mu u^q(x,t)\psi(x,t)\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^n} v_0(x)\psi(x,0)\mathrm{d}x \le (\ge)\, 0
\end{aligned}$$

hold, then (u,v) is said to be a super (sub) solution to (1.1)–(1.3) in $(0,T)$. Furthermore, (u,v) is said to be a solution to (1.1)–(1.3) in $(0,T)$ if it is both a supersolution and a subsolution.

Definition 2.2. Assume that (u,v) is a nontrivial solution to (1.1)–(1.3). If for some $0 < T < +\infty$,

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} + \|v(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \to +\infty \quad \text{as } t \to T^-,$$

then (u,v) is said to blow up in the finite time T. Otherwise, (u,v) is said to be a global solution.

Based on the classical theory of quasilinear parabolic equations (see [32, 33]), we have the following result.

Lemma 2.3. *For any* $0 \le u_0$, $v_0 \in L^1_{\rm loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, *there is at least one solution to* (1.1)–(1.3) *locally in time.*

Theorem 2.4. *Assume that* (u_1, v_1) *and* (u_2, v_2) *are two solutions to* (1.1)–(1.3) *in* $(0, T)$ *with nonnegative initial data* $(u_{0,1}(x), v_{0,1}(x))$ *and* $(u_{0,2}(x), v_{0,2}(x))$, *respectively. If* $(u_{0,1}, v_{0,1}) \le (u_{0,2}, v_{0,2})$ *a.e. in* $\mathbb{R}^n$, *then* $(u_1, v_1) \le (u_2, v_2)$ *a.e. in* $\mathbb{R}^n \times (0, T)$.

3. Blow-up theorems of Fujita type

In this section, we prove blow-up theorems of Fujita type for problem (1.1)–(1.3).

Theorem 3.1. *If* $m < p < p_c$, *any nontrivial solution to* (1.1)–(1.3) *blows up in finite time.*

Proof. We denote B_r as a ball in $\mathbb{R}^n$ with radius r centered at the origin. Assume that (u, v) is a nontrivial solution to (1.1)–(1.3). Set

$$w_l(t) = \int_{\mathbb{R}^n} \big(u(x,t) + l^\theta v(x,t)\big)\psi_l(x)\mathrm{d}x, \quad t > 0, \tag{3.1}$$

where θ is a constant to be determined, $l > 1$,

$$\psi_l(x) = \begin{cases} 1, & 0 \le |x| \le l, \\ \frac{1}{2}\big[1 + \cos\frac{\pi(|x|-l)}{l}\big], & l < |x| < 2l, \\ 0, & |x| \ge 2l, \end{cases} \qquad x \in \mathbb{R}^n.$$

Then for $x \in B_{2l} \setminus B_l$, we have

$$|\nabla\psi_l(x)| \le C_1 l^{-1}, \quad |\Delta\psi_l(x)| \le C_1 l^{-2}, \quad \Delta\psi_l(x) \ge -C_1 l^{-2}\psi_l(x).$$

where C_1 is a constant that depends only on n but is independent of l. By (1.1) and (1.2), we obtain

$$\begin{aligned} \frac{\mathrm{d}w_l(t)}{\mathrm{d}t} &= \int_{B_{2l}} u^m(x,t)\Delta\psi_l(x)\mathrm{d}x + \int_{\mathbb{R}^n} (|x|+1)^\lambda v^p(x,t)\psi_l(x)\mathrm{d}x \\ &\quad + l^\theta \int_{B_{2l}} v^m(x,t)\Delta\psi_l(x)\mathrm{d}x + l^\theta \int_{\mathbb{R}^n} (|x|+1)^\mu u^q(x,t)\psi_l(x)\mathrm{d}x. \end{aligned} \tag{3.2}$$

From the Hölder inequality we obtain

$$\begin{aligned} \int_{B_{2l}} u^m(x,t)\Delta\psi_l(x)\mathrm{d}x &= \int_{B_{2l}\setminus B_l} u^m(x,t)\Delta\psi_l(x)\mathrm{d}x \\ &\ge -C_1 l^{-2} \int_{B_{2l}\setminus B_l} u^m(x,t)\psi_l(x)\mathrm{d}x \\ &\ge -C_2 l^{n-2-(n+\mu)m/q}\Big(\int_{\mathbb{R}^n} (|x|+1)^\mu u^q(x,t)\psi_l(x)\mathrm{d}x\Big)^{m/q}, \end{aligned}$$

$$\begin{aligned}\int_{B_{2l}} v^m(x,t)\Delta\psi_l(x)\mathrm{d}x &= \int_{B_{2l}\setminus B_l} v^m(x,t)\Delta\psi_l(x)\mathrm{d}x\\ &\geq -C_1 l^{-2}\int_{B_{2l}\setminus B_l} v^m(x,t)\psi_l(x)\mathrm{d}x\\ &\leq -C_2 l^{n-2-(n+\lambda)m/p}\Big(\int_{\mathbb{R}^n}(|x|+1)^{\lambda}v^p(x,t)\psi_l(x)\mathrm{d}x\Big)^{m/p},\end{aligned}$$

where $C_2>0$ is a constant independent of l. From these two estimates and (3.2), one can obtain

$$\begin{aligned}&\frac{\mathrm{d}w_l(t)}{\mathrm{d}t}\\ &\geq \Big(\int_{\mathbb{R}^n}(|x|+1)^{\mu}u^q(x,t)\psi_l(x)\mathrm{d}x\Big)^{m/q}\\ &\quad\times\Big(l^{\theta}\Big(\int_{\mathbb{R}^n}(|x|+1)^{\mu}u^q(x,t)\psi_l(x)\mathrm{d}x\Big)^{(q-m)/q} - C_2 l^{n-2-(n+\mu)m/q}\Big)\\ &\quad+\Big(\int_{\mathbb{R}^n}(|x|+1)^{\lambda}v^p(x,t)\psi_l(x)\mathrm{d}x\Big)^{m/p}\\ &\quad\times\Big(\Big(\int_{\mathbb{R}^n}(|x|+1)^{\lambda}v^p(x,t)\psi_l(x)\mathrm{d}x\Big)^{(p-m)/p} - C_2 l^{n-2-(\lambda+n)m/p+\theta}\Big).\end{aligned}\tag{3.3}$$

It follows from the Hölder inequality that

$$\begin{aligned}&\int_{\mathbb{R}^n} u(x,t)\psi_l(x)\mathrm{d}x\\ &\leq \Big(\int_{\mathbb{R}^n}(|x|+1)^{-\mu/(q-1)}\psi_l(x)\mathrm{d}x\Big)^{(q-1)/q}\Big(\int_{\mathbb{R}^n}(|x|+1)^{\mu}u^q(x,t)\psi_l(x)\mathrm{d}x\Big)^{1/q},\\ &\int_{\mathbb{R}^n} v(x,t)\psi_l(x)\mathrm{d}x\\ &\leq \Big(\int_{\mathbb{R}^n}(|x|+1)^{-\lambda/(p-1)}\psi_l(x)\mathrm{d}x\Big)^{(p-1)/p}\Big(\int_{\mathbb{R}^n}(|x|+1)^{\lambda}v^p(x,t)\psi_l(x)\mathrm{d}x\Big)^{1/p},\end{aligned}$$

which indicate that

$$\begin{aligned}&\int_{\mathbb{R}^n}(|x|+1)^{\mu}u^q(x,t)\psi_l(x)\mathrm{d}x\\ &\geq \begin{cases} C_3\Big(\int_{\mathbb{R}^n} u(x,t)\psi_l(x)\mathrm{d}x\Big)^q l^{-qn+n+\mu}, & \text{if } -qn+n+\mu<0,\\ C_3\Big(\int_{\mathbb{R}^n} u(x,t)\psi_l(x)\mathrm{d}x\Big)^q (\ln l)^{1-q}, & \text{if } -qn+n+\mu=0,\\ C_3\Big(\int_{\mathbb{R}^n} u(x,t)\psi_l(x)\mathrm{d}x\Big)^q, & \text{if } -qn+n+\mu>0,\end{cases}\end{aligned}\tag{3.4}$$

$$\begin{aligned}&\int_{\mathbb{R}^n}(|x|+1)^{\lambda}v^p(x,t)\psi_l(x)\mathrm{d}x\\ &\geq \begin{cases} C_3\Big(\int_{\mathbb{R}^n} v(x,t)\psi_l(x)\mathrm{d}x\Big)^p l^{-pn+n+\lambda}, & \text{if } -pn+n+\lambda<0,\\ C_3\Big(\int_{\mathbb{R}^n} v(x,t)\psi_l(x)\mathrm{d}x\Big)^p (\ln l)^{1-p}, & \text{if } -pn+n+\lambda=0,\\ C_3\Big(\int_{\mathbb{R}^n} v(x,t)\psi_l(x)\mathrm{d}x\Big)^p, & \text{if } -pn+n+\lambda>0,\end{cases}\end{aligned}\tag{3.5}$$

where C_3 is a constant independent of l.

First, consider the case in which $-qn+n+\mu<0$ and $-pn+n+\lambda<0$. It follows from (3.3)–(3.5) that

$$\begin{aligned}
\frac{\mathrm{d}w_l(t)}{\mathrm{d}t} &\geq C_3^{m/q} l^{(-qn+n+\mu)m/q}\Big(\int_{\mathbb{R}^n} u(x,t)\psi_l(x)\mathrm{d}x\Big)^m \\
&\quad \times \Big[C_3^{1-m/q} l^{(-qn+n+\mu)(q-m)/q+\theta}\Big(\int_{\mathbb{R}^n} u(x,t)\psi_l(x)\mathrm{d}x\Big)^{q-m} \\
&\quad - C_2 l^{n-2-(n+\mu)m/q}\Big] \\
&\quad + C_3^{m/p} l^{(-pn+n+\lambda)m/p}\Big(\int_{\mathbb{R}^n} v(x,t)\psi_l(x)\mathrm{d}x\Big)^m \\
&\quad \times \Big[C_3^{1-m/p} l^{(-pn+n+\lambda)(p-m)/p}\Big(\int_{\mathbb{R}^n} v(x,t)\psi_l(x)\mathrm{d}x\Big)^{p-m} \\
&\quad - C_2 l^{n-2-(n+\lambda)m/p+\theta}\Big] \\
&\geq -C_4 l^{\kappa(\theta)} w_l^m(t) + C_3 l^{-qn+n+\mu+\theta}\Big(\int_{\mathbb{R}^n} u(x,t)\psi_l(x)\mathrm{d}x\Big)^q \\
&\quad + C_3 l^{-pn+n+\lambda-p\theta}\Big(\int_{\mathbb{R}^n} l^\theta v(x,t)\psi_l(x)\mathrm{d}x\Big)^p,
\end{aligned} \tag{3.6}$$

where

$$C_4=\max\{C_2C_3^{m/q}, C_2C_3^{m/p}\}, \quad \kappa(\theta)=\max\big\{-mn+n-2,\ -mn+n-2-(m-1)\theta\big\}.$$

We choose

$$\theta=\frac{q-p}{p+1}\Big(n-\frac{\lambda+2}{p-m}\Big).$$

It is clear that $-qn+n+\mu+\theta=-np+n+\lambda-p\theta=\Theta$ with

$$\Theta=\frac{-p^2qn-p^2n+pqmn+pmn+(\lambda+2)(pq-p^2)}{(p+1)(p-m)}+\lambda+n.$$

Then we obtain

$$\begin{aligned}
\frac{\mathrm{d}w_l(t)}{\mathrm{d}t} &\geq -C_4 l^{\kappa(\theta)} w_l^m(t) \\
&\quad + C_3 l^{\Theta}\Big[\Big(\int_{\mathbb{R}^n} u(x,t)\psi_l(x)\mathrm{d}x\Big)^q+\Big(\int_{\mathbb{R}^n} l^\theta v(x,t)\psi_l(x)\mathrm{d}x\Big)^p\Big] \\
&\geq w_l^m(t)\Big[-C_4 l^{\kappa(\theta)}+2^{-(p+q)}C_3 l^{\Theta}\cdot\min\big\{w_l^{p-m}(t),\ w_l^{q-m}(t)\big\}\Big].
\end{aligned} \tag{3.7}$$

It follows from $p<p_c$ that $\kappa(\theta)<\Theta$. Because $w_l(0)$ is nondecreasing with respect to $l\in(0,+\infty)$ with $\sup\big\{w_l(0): l\in(0,+\infty)\big\}>0$, there exists $l_1>1$ such that

$$C_4 l_1^{\kappa(\theta)} \leq 2^{-(p+q+1)}C_3 l_1^{\Theta}\min\big\{w_{l_1}^{p-m}(0), w_{l_1}^{q-m}(0)\big\}. \tag{3.8}$$

From (3.7) and (3.8), we obtain

$$\frac{\mathrm{d}w_{l_1}(t)}{\mathrm{d}t}\geq 2^{-(p+q+1)}C_3 l_1^{\Theta}\min\big\{w_{l_1}^p(t),\ w_{l_1}^q(t)\big\}, \quad t>0.$$

Due to p, $q>m>1$, there exists a constant $T_*\in(0,+\infty)$ such that

$$w_{l_1}(t)=\int_{\mathbb{R}^n}\big(u(x,t)+l_1^\theta v(x,t)\big)\psi_{l_1}(x)\mathrm{d}x\to+\infty \quad \text{as } t\to T_*^-.$$

Since $\operatorname{supp}\psi_{l_1}(x) = B_{2l_1}$, we have

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} + \|v(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \to +\infty \quad \text{as } t \to T_*^-.$$

That is, (u,v) blows up in finite time.

Let us consider the case in which $-qn+n+\mu = 0$ and $-pn+n+\lambda < 0$. It is assumed that $\theta = 0$. It follows from (3.3)–(3.5) that

$$\begin{aligned}
\frac{\mathrm{d}w_l(t)}{\mathrm{d}t} \geq\, & C_3^{m/q}(\ln l)^{(1-q)m/q}\Big(\int_{\mathbb{R}^n} u(x,t)\psi_l(x)\mathrm{d}x\Big)^m \\
& \times \Big[C_3^{1-m/q}(\ln l)^{(1-q)(q-m)/q}\Big(\int_{\mathbb{R}^n} u(x,t)\psi_l(x)\mathrm{d}x\Big)^{q-m} \\
& - C_2 l^{n-2-m(n+\mu)/q}\Big] \\
& + C_3^{m/p} l^{(-pn+n+\lambda)m/p}\Big(\int_{\mathbb{R}^n} v(x,t)\psi_l(x)\mathrm{d}x\Big)^m \\
& \times \Big(C_3^{(p-m)/p} l^{(-pn+n+\lambda)(p-m)/p}\Big(\int_{\mathbb{R}^n} v(x,t)\psi_l(x)\mathrm{d}x\Big)^{p-m} \\
& - C_2 l^{n-2-m(n+\lambda)/p}\Big).
\end{aligned} \tag{3.9}$$

Thanks to

$$n-2-m(n+\mu)/q < 0, \quad \text{and} \quad n-2-m(n+\lambda)/p < (-np+n+\lambda)(p-m)/p,$$

there exists sufficiently large $l_2 > 1$, such that

$$\begin{aligned}
\frac{\mathrm{d}w_{l_2}(t)}{\mathrm{d}t} \geq\, & C_3^{m/q}(\ln l_2)^{m(1-q)/q}\Big(\int_{\mathbb{R}^n} u(x,t)\psi_{l_2}(x)\mathrm{d}x\Big)^m \\
& \times \frac{1}{2} C_3^{(q-m)/q}(\ln l_2)^{(1-q)(q-m)/q}\Big(\int_{\mathbb{R}^n} u(x,t)\psi_{l_2}(x)\mathrm{d}x\Big)^{q-m} \\
& + C_3^{m/p} l_2^{-mn+m(n+\lambda)/p}\Big(\int_{\mathbb{R}^n} v(x,t)\psi_{l_2}(x)\mathrm{d}x\Big)^m \\
& \times \frac{1}{2} C_3^{(p-m)/p} l_2^{(-np+n+\lambda)(p-m)/p}\Big(\int_{\mathbb{R}^n} v(x,t)\psi_{l_2}(x)\mathrm{d}x\Big)^{p-m} \\
\geq\, & C_5\Big(\Big(\int_{\mathbb{R}^n} u(x,t)\psi_{l_2}(x)\mathrm{d}x\Big)^q + \Big(\int_{\mathbb{R}^n} v(x,t)\psi_{l_2}(x)\mathrm{d}x\Big)^p\Big) \\
\geq\, & 2^{-(p+q)} C_5 \min\{w_{l_2}^p(t), w_{l_2}^q(t)\},
\end{aligned}$$

where C_5 denotes a constant that depends only on l_2. The same discussion as above shows that (u,v) blows up in finite time.

For the other cases, we still assume that $\theta = 0$. Similar to the second case, we can prove that (u,v) blows up in finite time. □

Now we construct self-similar supersolutions to (1.1) and (1.2). Set

$$u(x,t) = \frac{U((t+1)^{-\beta}(|x|+1))}{(t+1)^\alpha}, \quad v(x,t) = \frac{V((t+1)^{-\beta}(|x|+1))}{(t+1)^\alpha}, \tag{3.10}$$

for $(x,t) \in \mathbb{R}^n \times [0,+\infty)$, where U, $V \in C^1([0,+\infty))$ with U^m, $V^m \in C^1([0,+\infty))$ and

$$\alpha = \frac{\lambda+2}{2(p-1)+\lambda(m-1)}, \quad \beta = \frac{(p-m)\alpha}{\lambda+2}.$$

If the following inequalities

$$(U^m)''(r)+\frac{n-1}{r}\cdot\frac{|x|+1}{|x|}(U^m)'(r)+\beta rU'(r)+\alpha U(r)+r^\lambda V^p(r)\le 0, \quad (3.11)$$

$$(V^m)''(r)+\frac{n-1}{r}\cdot\frac{|x|+1}{|x|}(V^m)'(r)+\beta rV'(r)+\alpha V(r)+r^\mu U^q(r)\le 0 \quad (3.12)$$

hold for $r>0$ and $x\in\mathbb{R}^n$, then (u,v) given in (3.10) is a supersolution to (1.1) and (1.2).

Lemma 3.2. *Assume that*

$$U(r)=V(r)=\big(\eta-Ar^2\big)_+^{1/(m-1)},\quad r>0, \quad (3.13)$$

where $s_+=\max\{0,\,s\}$*,* $\eta>0$ *and*

$$A=\frac{(m-1)(p-m)\alpha}{mn(p+p_c-2m)}.$$

Then, when $p>p_c$*, there exists sufficiently small* $\eta>0$*, such that* (u,v) *defined by* (3.10) *and* (3.13) *is a supersolution to* (1.1) *and* (1.2)*.*

Proof. It is not difficult to cheek that $U,V\in C^1([0,+\infty))$ satisfy $U^m,V^m\in C^1([0,+\infty))$ and (3.11), (3.12) hold for $r\ge(\eta/A)^{1/2}$ and $x\in\mathbb{R}^n$.

For $0<r<(\frac{\eta}{A})^{1/2}$, following from direction calculations, it yields that

$$\begin{aligned}&(U^m)''(r)+\frac{n-1}{r}(U^m)'(r)+\beta rU'(r)+\alpha U(r)\\&=\Big(\frac{2A}{m-1}\Big(\frac{2Am}{m-1}-\beta\Big)r^2U^{1-m}(r)+\Big(\alpha-\frac{2Amn}{m-1}\Big)\Big)U(r)\end{aligned}$$

and

$$\begin{aligned}&(V^m)''(r)+\frac{n-1}{r}(V^m)'(r)+\beta rV'(r)+\alpha V(r)\\&=\Big(\frac{2A}{m-1}\Big(\frac{2Am}{m-1}-\beta\Big)r^2V^{1-m}(r)+\Big(\alpha-\frac{2Amn}{m-1}\Big)\Big)V(r).\end{aligned}$$

It follows from $\frac{2Am}{m-1}<\beta$ that there exists some $\eta_1>0$ such that for any $0<\eta<\eta_1$,

$$(U^m)''(r)+\frac{n-1}{r}(U^m)'(r)+\beta rU'(r)+\alpha U(r)<-\frac{(p-p_c)\alpha}{2(p+p_c-2m)}U(r)$$

and

$$(V^m)''(r)+\frac{n-1}{r}(V^m)'(r)+\beta rV'(r)+\alpha V(r)<-\frac{(p-p_c)\alpha}{2(p+p_c-2m)}V(r).$$

Owing to $(U^m)'(r)=-\frac{2Amr}{m-1}U(r)\le 0$ and $(V^m)'(r)=-\frac{2Amr}{m-1}V(r)\le 0$, we obtain that for any $0<\eta<\eta_1$, it holds

$$\begin{aligned}&(U^m)''(r)+\frac{n-1}{r}\cdot\frac{|x|+1}{|x|}(U^m)'(r)+\beta rU'(r)+\alpha U(r)\\&<-\frac{(p-p_c)\alpha}{2(p+p_c-2m)}U(r),\end{aligned} \quad (3.14)$$

$$\begin{aligned}&(V^m)''(r)+\frac{n-1}{r}\cdot\frac{|x|+1}{|x|}(V^m)'(r)+\beta rV'(r)+\alpha V(r)\\&<-\frac{(p-p_c)\alpha}{2(p+p_c-2m)}V(r).\end{aligned} \quad (3.15)$$

From the definition of U, V and λ, $\mu \geq 0$, there exists $\eta_2 \in (0, \eta_1)$ such that for any $0 < \eta < \eta_2$,

$$r^{\mu} U^{q-1}(r) = r^{\lambda} V^{p-1}(r) \leq A^{-\lambda/2} \eta^{(p-1)/(m-1)+\lambda/2} < \frac{(p-p_c)\alpha}{2(p+p_c-2m)},$$

which together with (3.14) and (3.15), shows that for and $0 < \eta < \eta_2$, (3.11) and (3.12) hold for $0 < r < \left(\frac{\eta}{A}\right)^{1/2}$ and $x \in \mathbb{R}^n$. □

Theorem 3.3. *For $p > p_c$, there exist both nontrivial global and blow-up solutions to* (1.1)–(1.3).

Proof. It follows from Lemma 2.4 and Lemma 3.2 that (1.1)–(1.3) has a nontrivial global solution with small initial data. Below, we prove the blowing-up properties of solutions to (1.1)–(1.3) with large initial data. Set

$$\tilde{w}_l(t) = \int_{\mathbb{R}^n} \big(u(x,t) + v(x,t)\big) \psi_l(x) \mathrm{d}x, \quad t \geq 0,$$

where $l > 1$ and (u, v) is a solution to (1.1)–(1.3). According to (3.3) and the Hölder inequality, we have

$$\begin{aligned}
\frac{\mathrm{d}\tilde{w}(t)}{\mathrm{d}t} \geq &\Big(\int_{\mathbb{R}^n} u(x,t)\psi_l(x)\mathrm{d}x\Big)^m \Big(\int_{\mathbb{R}^n} (|x|+1)^{\mu/(1-q)}\psi_l(x)\mathrm{d}x\Big)^{m(1-q)/q} \\
&\times \Big[\Big(\int_{\mathbb{R}^n} u(x,t)\psi_l(x)\mathrm{d}x\Big)^{q-m} \Big(\int_{\mathbb{R}^n} (|x|+1)^{\mu/(1-q)}\psi_l(x)\mathrm{d}x\Big)^{(1-q)(q-m)/q} \\
&- C_2 l^{n-2-m(n+\mu)/q}\Big] \\
&+ \Big(\int_{\mathbb{R}^n} v(x,t)\psi_l(x)\mathrm{d}x\Big)^m \Big(\int_{\mathbb{R}^n} (|x|+1)^{\lambda/(1-p)}\psi_l(x)\mathrm{d}x\Big)^{m(1-p)/p} \\
&\times \Big[\Big(\int_{\mathbb{R}^n} v(x,t)\psi_l(x)\mathrm{d}x\Big)^{p-m} \Big(\int_{\mathbb{R}^n} (|x|+1)^{\lambda/(1-p)}\psi_l(x)\mathrm{d}x\Big)^{(1-p)(p-m)/p} \\
&- C_2 l^{n-2-m(n+\lambda)/p}\Big] \\
\geq\ & \tilde{w}_l^m(t)\big(-C_2C_6 + 2^{-(p+q)} C_7 \min\big\{\tilde{w}_l^{p-m}(t),\, \tilde{w}_l^{q-m}(t)\big\}\big), \quad t \geq 0,
\end{aligned} \tag{3.16}$$

where

$$\begin{gathered}
C_6 = \max\Big\{ l^{n-2-m(n+\mu)/q}\Big(\int_{\mathbb{R}^n} (|x|+1)^{\mu/(1-q)}\psi_l(x)\mathrm{d}x\Big)^{m(1-q)/q}, \\
l^{n-2-m(n+\lambda)/p}\Big(\int_{\mathbb{R}^n} (|x|+1)^{\lambda/(1-p)}\psi_l(x)\mathrm{d}x\Big)^{m(1-p)/p}\Big\}, \\
C_7 = \min\Big\{\Big(\int_{\mathbb{R}^n} (|x|+1)^{\mu/(1-q)}\psi_l(x)\mathrm{d}x\Big)^{1-q},\, \Big(\int_{\mathbb{R}^n} (|x|+1)^{\lambda/(1-p)}\psi_l(x)\mathrm{d}x\Big)^{1-p}\Big\}
\end{gathered}$$

are positive constants depending on λ, m, n, p, q and l. If (u_0, v_0) is so large that

$$C_2C_6 \leq 2^{-(p+q+1)} C_7 \min\big\{\tilde{w}_l^{p-m}(0),\, \tilde{w}_l^{q-m}(0)\big\},$$

then (3.16) indicates that

$$\frac{\mathrm{d}\tilde{w}_l(t)}{\mathrm{d}t} \geq 2^{-(p+q+1)} C_7 \min\big\{\tilde{w}_l^p(t),\, \tilde{w}_l^q(t)\big\}, \quad t > 0.$$

It can be proven by the same process as in Theorem 3.1 that (u, v) blows up in finite time. □

4. Critical case

In this section, we consider the critical case

$$p = p_c = m + \frac{2+\lambda}{n}. \tag{4.1}$$

For this case, one obtains

$$-qn + n + \mu = -p_c n + n + \lambda = -mn + n - 2 < 0, \tag{4.2}$$

and (3.6) and (3.7) still hold. The proof is based on the following three lemmas.

Lemma 4.1. *Assume that (u, v) is a nontrivial global solution to (1.1)–(1.3) with $p = p_c$, then there exists a constant $M_0 > 0$ independent of t, such that*

$$\int_{\mathbb{R}^n} \big(u(x,t) + v(x,t)\big)\mathrm{d}x \le M_0, \quad t > 0. \tag{4.3}$$

Furthermore, it holds that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} \big(u(x,t) + v(x,t)\big)\mathrm{d}x \\ &\ge \frac{1}{2}\int_{\mathbb{R}^n} \big((|x|+1)^{\mu} u^q(x,t) + (|x|+1)^{\lambda} v^{p_c}(x,t)\big)\mathrm{d}x.\end{aligned} \tag{4.4}$$

Proof. For any sufficiently large $l > 1$, (3.7) implies

$$\frac{\mathrm{d}w_l(t)}{\mathrm{d}t} \ge w_l^m(t) l^{-mn+n-2}\Big(-C_4 + 2^{-(p_c+q)} C_3 \min\big\{w_l^{p_c-m}(t),\, w_l^{q-m}(t)\big\}\Big),$$

where w_l is defined by (3.1) with $\theta = 0$. Similar to the end of the proof of Theorem 3.1, there exists some $l_3 > 1$, such that for any $l > l_3$,

$$2^{-(p_c+q+1)} C_3 \min\big\{w_l^{p_c-m}(t),\, w_l^{q-m}(t)\big\} \le C_4,$$

i.e.

$$w_l(t) \le \max\Big\{\big(2^{p_c+q+1} C_3^{-1} C_4\big)^{1/(p_c-m)},\, \big(2^{p_c+q+1} C_3^{-1} C_4\big)^{1/(q-m)}\Big\}. \tag{4.5}$$

Let $l \to +\infty$ in (4.5); then, we can obtain (4.3).

From the Hölder inequality and (4.3), we obtain that for any $l \ge 1$,

$$\begin{aligned}&\int_{B_{2l}} u^m(x,t)\Delta\psi_l(x)\mathrm{d}x = \int_{B_{2l}\setminus B_l} u^m(x,t)\Delta\psi_l(x)\mathrm{d}x \\ &\ge -\frac{C_1}{l^2}\int_{B_{2l}\setminus B_l} u^m(x,t)\psi_l(x)\mathrm{d}x \\ &\ge -\frac{C_1}{l^2}\Big(\int_{B_{2l}\setminus B_l} u^q(x,t)\psi_l(x)\mathrm{d}x\Big)^{(m-1)/(q-1)}\Big(\int_{B_{2l}\setminus B_l} u(x,t)\psi_l(x)\mathrm{d}x\Big)^{(q-m)/(q-1)} \\ &\ge -\frac{C_1}{l^2}\Big(\int_{\mathbb{R}^n} (|x|+1)^{\mu} u^q(x,t)\psi_l(x)\mathrm{d}x\Big)^{(m-1)/(q-1)} \\ &\quad\times\Big(\int_{\mathbb{R}^n} \big(u(x,t)+v(x,t)\big)\mathrm{d}x\Big)^{(q-m)/(q-1)} \\ &\ge -\frac{C_8}{l^2}\Big(\int_{\mathbb{R}^n} (|x|+1)^{\mu} u^q(x,t)\psi_l(x)\mathrm{d}x\Big)^{(m-1)/(q-1)}\end{aligned}$$

and

$$\int_{B_{2l}} v^m(x,t)|\Delta\psi_l(x)|\mathrm{d}x \ge -\frac{C_8}{l^2}\Big(\int_{\mathbb{R}^n} (|x|+1)^{\lambda} v^{p_c}(x,t)\psi_l(x)\mathrm{d}x\Big)^{(m-1)/(p_c-1)},$$

where $C_8 > 0$ is a constant independent of l. Substituting the above two inequalities into (3.2) yields that for $t > 0$,

$$\begin{aligned}\frac{\mathrm{d}w_l(t)}{\mathrm{d}t} \geq &\Big(\int_{\mathbb{R}^n}(|x|+1)^\mu u^q(x,t)\psi_l(x)\mathrm{d}x\Big)^{(m-1)/(q-1)}\\ &\times\Big(-\frac{C_8}{l^2}+\Big(\int_{\mathbb{R}^n}(|x|+1)^\mu u^q(x,t)\psi_l(x)\mathrm{d}x\Big)^{(q-m)/(q-1)}\Big)\\ &+\Big(\int_{\mathbb{R}^n}(|x|+1)^\lambda v^{p_c}(x,t)\psi_l(x)\mathrm{d}x\Big)^{(m-1)/(p_c-1)}\\ &\times\Big(-\frac{C_8}{l^2}+\Big(\int_{\mathbb{R}^n}(|x|+1)^\lambda v^{p_c}(x,t)\psi_l(x)\mathrm{d}x\Big)^{(p_c-m)/(p_c-1)}\Big).\end{aligned}$$

By letting $l \to +\infty$, (4.4) is obtained. □

Lemma 4.2. *Under the assumption of Lemma 4.1, there exist three positive constants M_1, M_2, $M_3 > 0$ independent of l and t, such that for any sufficiently large $l > 1$,*

$$\begin{aligned}\frac{\mathrm{d}w_l(t)}{\mathrm{d}t} \geq &M_1^{m-\tau} l^{-mn+n-2} w_l^{m-\tau}(t)\Big(-M_2\Big(\int_{B_{2l}\setminus B_l}(u(x,t)+v(x,t))\psi_l\mathrm{d}x\Big)^\tau\\ &+M_1^{-(m-\tau)}M_3\cdot\min\big\{w_l^{p_c-m+\tau}(t), w_l^{q-m+\tau}(t)\big\}\Big),\end{aligned} \tag{4.6}$$

where

$$0 < \tau < \min\Big\{\frac{p_c-m}{p_c-1}, \frac{q-m}{q-1}\Big\}.$$

Proof. It is easy to verify that

$$n-2-\frac{m(n+\mu)}{q}+\frac{\tau(\mu-(q-1)n)}{q} = \frac{(-qn+n+\mu)(q-m+\tau)}{q}, \tag{4.7}$$

$$n-2-\frac{m(n+\lambda)}{p_c}+\frac{\tau(\lambda-(p_c-1)n)}{p_c} = \frac{(-p_cn+n+\lambda)(p_c-m+\tau)}{p_c}. \tag{4.8}$$

Using the Hölder inequality, we obtain that for any sufficiently large $l > 1$,

$$\begin{aligned}&\int_{B_{2l}} u^m(x,t)\Delta\psi_l(x)\mathrm{d}x = \int_{B_{2l}\setminus B_l} u^m(x,t)\Delta\psi_l(x)\mathrm{d}x\\ &\geq -C_1 l^{-2}\int_{B_{2l}\setminus B_l} u^m(x,t)\psi_l(x)\mathrm{d}x\\ &\geq -C_1 l^{-2}\Big(\int_{B_{2l}\setminus B_l}(|x|+1)^{-(m-\tau)\mu/(q-m-(q-1)\tau)}\mathrm{d}x\Big)^{(q-m-(q-1)\tau)/q}\\ &\quad\times\Big(\int_{B_{2l}\setminus B_l}(|x|+1)^\mu u^q(x,t)\psi_l(x)\mathrm{d}x\Big)^{(m-\tau)/q}\Big(\int_{B_{2l}\setminus B_l}u(x,t)\psi_l(x)\mathrm{d}x\Big)^\tau\\ &\geq -C_9 l^{n-2-(n+\mu)m/q+\tau(\mu-(q-1)n)/q}\Big(\int_{\mathbb{R}^n}(|x|+1)^\mu u^q(x,t)\psi_l(x)\mathrm{d}x\Big)^{(m-\tau)/q}\\ &\quad\times\Big(\int_{B_{2l}\setminus B_l}u(x,t)\psi_l(x)\mathrm{d}x\Big)^\tau,\end{aligned}$$

$$\int_{B_{2l}} v^m(x,t)|\Delta\psi_l(x)|\mathrm{d}x$$

$$
\begin{aligned}
&\geq -C_9 l^{n-2-(n+\lambda)m/p_c+\tau(\lambda-(p_c-1)n)/p_c}\Big(\int_{\mathbb{R}^n}(|x|+1)^\lambda v^{p_c}(x,t)\psi_l(x)\mathrm{d}x\Big)^{(m-\tau)/p_c}\\
&\quad\times\Big(\int_{B_{2l}\setminus B_l} v(x,t)\psi_l(x)\mathrm{d}x\Big)^{\tau},
\end{aligned}
$$

where $C_9>0$ is a constant independent of l. Substituting the above two inequalities into (3.2) with $\theta=0$, then using (3.4), (3.5), (4.2), (4.7) and (4.8), one gets that

$$
\begin{aligned}
&\frac{\mathrm{d}w_l(t)}{\mathrm{d}t}\\
&\geq -C_3^{(m-\tau)/q}C_9 l^{-qn+n+\mu}\Big(\int_{\mathbb{R}^n} u(x,t)\psi_l(x)\mathrm{d}x\Big)^{m-\tau}\Big(\int_{B_{2l}\setminus B_l} u(x,t)\psi_l(x)\mathrm{d}x\Big)^{\tau}\\
&\quad - C_3^{(m-\tau)/p_c}C_9 l^{-p_c n+n+\lambda}\Big(\int_{\mathbb{R}^n} v(x,t)\psi_l(x)\mathrm{d}x\Big)^{m-\tau}\Big(\int_{B_{2l}\setminus B_l} v(x,t)\psi_l(x)\mathrm{d}x\Big)^{\tau}\\
&\quad + C_3 l^{-qn+n+\mu}\Big(\int_{\mathbb{R}^n} u(x,t)\psi_l(x)\mathrm{d}x\Big)^{q} + C_3 l^{-p_c n+n+\lambda}\Big(\int_{\mathbb{R}^n} v(x,t)\psi_l(x)\mathrm{d}x\Big)^{p_c}\\
&\geq -M_1^{m-\tau}C_9 l^{-mn+n-2}w_l^{m-\tau}(t)\Big(\int_{B_{2l}\setminus B_l}\big(u(x,t)+v(x,t)\big)\psi_l(x)\mathrm{d}x\Big)^{\tau}\\
&\quad + 2^{-(p_c+q)}C_3 l^{-mn+n-2}\min\{w_l^{p_c}(t),w_l^{q}(t)\},\quad t>0.
\end{aligned}
$$

Thus, (4.6) holds for $M_1=\max\{C_3^{1/p_c},C_3^{1/q}\}$, $M_2=C_9$ and $M_3=2^{-(p_c+q)}C_3$. $\square$

Lemma 4.3. *Under the assumption of Lemma* 4.1*, there exists a constant $M_4>0$ independent of l and t, such that for any sufficiently large $l>1$,*

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}\big(u(x,t)+v(x,t)\big)\phi_l(x)\mathrm{d}x\\
&\leq 2\int_{\mathbb{R}^n}\big((|x|+1)^\mu u^q(x,t)+(|x|+1)^\lambda v^{p_c}(x,t)\big)\mathrm{d}x\\
&\quad + M_4 l^{(p_c(n-2)-m(n+\lambda_1))/(p_c-m)},
\end{aligned}
\tag{4.9}
$$

where

$$
\phi_l(x)=1-\psi_l(x),\quad x\in\mathbb{R}^n.
$$

Proof. Let $\xi\in C_0^\infty(\mathbb{R}^n)$ satisfy $0\leq\xi(x)\leq 1$ for $x\in\mathbb{R}^n$, $\xi(x)=1$ if $|x|<2$ and $\xi(x)=0$ if $|x|>3$. For $k\geq l>0$, denote

$$
\xi_k(x)=\xi\left(\frac{x}{k}\right),\quad x\in\mathbb{R}^n.
$$

Obviously,

$$
|\nabla\phi_l(x)|\leq\frac{C_{10}}{l},\quad |\Delta\phi_l(x)|\leq\frac{C_{10}}{l^2},\quad |\Delta\xi_k(x)|\leq\frac{C_{10}}{k^2},\quad x\in\mathbb{R}^n,
$$

where $C_{10} > 0$ is a constant independent of l and k. From Definition 2.1, we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}\big(u(x,t)+v(x,t)\big)\phi_l(x)\xi_k(x)\mathrm{d}x\\
&\le \int_{\mathbb{R}^n\setminus B_l} u^m(x,t)\Delta(\phi_l(x)\xi_k(x))\mathrm{d}x + \int_{\mathbb{R}^n}(|x|+1)^{\mu}u^q(x,t)\phi_l(x)\xi_k(x)\mathrm{d}x\\
&\quad + \int_{\mathbb{R}^n\setminus B_l} v^m(x,t)\Delta(\phi_l(x)\xi_k(x))\mathrm{d}x + \int_{\mathbb{R}^n}(|x|+1)^{\lambda}v^{p_c}(x,t)\phi_l(x)\xi_k(x)\mathrm{d}x\\
&\le \int_{B_{2l}\setminus B_l} u^m(x,t)|\Delta\phi_l(x)|\mathrm{d}x + \int_{B_{3k}\setminus B_{2k}} u^m(x,t)|\Delta\xi_k(x)|\mathrm{d}x\\
&\quad + \int_{\mathbb{R}^n}(|x|+1)^{\mu}u^q(x,t)\mathrm{d}x + \int_{B_{2l}\setminus B_l} v^m(x,t)|\Delta\phi_l(x)|\mathrm{d}x\\
&\quad + \int_{B_{3k}\setminus B_{2k}} v^m(x,t)|\Delta\xi_k(x)|\mathrm{d}x + \int_{\mathbb{R}^n}(|x|+1)^{\lambda}v^{p_c}(x,t)\mathrm{d}x.
\end{aligned}
\tag{4.10}
$$

Owing to the Hölder inequality, one obtains

$$
\begin{aligned}
&\int_{B_{2l}\setminus B_l} u^m(x,t)|\Delta\phi_l(x)|\mathrm{d}x\\
&\le \Big(\int_{B_{2l}\setminus B_l}(|x|+1)^{-m\mu/(q-m)}|\Delta\phi_l(x)|^{q/(q-m)}\mathrm{d}x\Big)^{(q-m)/q}\\
&\quad\times\Big(\int_{B_{2l}\setminus B_l}(|x|+1)^{\mu}u^q(x,t)\mathrm{d}x\Big)^{m/q}\\
&\le C_{11}l^{n-2-m(n+\mu)/q}\Big(\int_{B_{2l}\setminus B_l}(|x|+1)^{\mu}u^q(x,t)\mathrm{d}x\Big)^{m/q},
\end{aligned}
\tag{4.11}
$$

$$
\begin{aligned}
&\int_{B_{2l}\setminus B_l} v^m(x,t)|\Delta\phi_l(x)|\mathrm{d}x\\
&\le C_{11}l^{n-2-m(n+\lambda)/p_c}\Big(\int_{B_{2l}\setminus B_l}(|x|+1)^{\lambda}v^{p_c}(x,t)\mathrm{d}x\Big)^{m/p_c},
\end{aligned}
\tag{4.12}
$$

where $C_{11} > 0$ is a constant independent of l and k. Similarly, one gets

$$
\begin{aligned}
&\int_{B_{3k}\setminus B_{2k}} u^m(x,t)|\Delta\xi_k(x)|\mathrm{d}x\\
&\le \frac{C_{10}}{k^2}\int_{B_{3k}\setminus B_{2k}} u^m(x,t)\mathrm{d}x\\
&\le \frac{C_{10}}{k^2}\Big(\int_{B_{3k}\setminus B_{2k}} u^q(x,t)\mathrm{d}x\Big)^{(m-1)/(q-1)}\Big(\int_{B_{3k}\setminus B_{2k}} u(x,t)\mathrm{d}x\Big)^{(q-m)/(q-1)}\\
&\le \frac{C_{10}}{k^2}\Big(\int_{\mathbb{R}^n}(|x|+1)^{\mu}u^q(x,t)\mathrm{d}x\Big)^{(m-1)/(q-1)}\Big(\int_{\mathbb{R}^n} u(x,t)\phi_l(x)\mathrm{d}x\Big)^{(q-m)/(q-1)},
\end{aligned}
\tag{4.13}
$$

$$
\begin{aligned}
&\int_{B_{3k}\setminus B_{2k}} v^m(x,t)|\Delta\xi_k(x)|\mathrm{d}x\\
&\le \frac{C_{10}}{k^2}\Big(\int_{\mathbb{R}^n}(|x|+1)^{\lambda}v^{p_c}(x,t)\mathrm{d}x\Big)^{(m-1)/(p_c-1)}\Big(\int_{\mathbb{R}^n} v(x,t)\phi_l(x)\mathrm{d}x\Big)^{(p_c-m)/(p_c-1)}.
\end{aligned}
\tag{4.14}
$$

Substituting (4.11)–(4.14) into (4.10) implies that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}(u(x,t)+v(x,t))\phi_l(x)\xi_k(x)\mathrm{d}x\\
&\le C_{11}l^{n-2-m(n+\mu)/q}\Big(\int_{B_{2l}\setminus B_l}(|x|+1)^{\mu}u^q(x,t)\mathrm{d}x\Big)^{m/q}\\
&\quad+\frac{C_{10}}{k^2}\Big(\int_{\mathbb{R}^n}(|x|+1)^{\mu}u^q(x,t)\mathrm{d}x\Big)^{(m-1)/(q-1)}\Big(\int_{\mathbb{R}^n}u(x,t)\phi_l(x)\mathrm{d}x\Big)^{(q-m)/(q-1)}\\
&\quad+\int_{\mathbb{R}^n}(|x|+1)^{\mu}u^q(x,t)\mathrm{d}x\\
&\quad+C_{11}l^{n-2-m(n+\lambda)/p_c}\Big(\int_{B_{2l}\setminus B_l}(|x|+1)^{\lambda}v^{p_c}(x,t)\mathrm{d}x\Big)^{m/p_c}\\
&\quad+\frac{C_{10}}{k^2}\Big(\int_{\mathbb{R}^n}(|x|+1)^{\lambda}v^{p_c}(x,t)\mathrm{d}x\Big)^{(m-1)/(p_c-1)}\Big(\int_{\mathbb{R}^n}v(x,t)\phi_l(x)\mathrm{d}x\Big)^{(p_c-m)/(p_c-1)}\\
&\quad+\int_{\mathbb{R}^n}(|x|+1)^{\lambda}v^{p_c}(x,t)\mathrm{d}x.
\end{aligned}
$$

Letting $k\to+\infty$ in the above inequality yields

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}\big(u(x,t)+v(x,t)\big)\phi_l(x)\mathrm{d}x\\
&\le C_{11}l^{n-2-m(n+\mu)/q}\Big(\int_{B_{2l}\setminus B_l}(|x|+1)^{\mu}u^q(x,t)\mathrm{d}x\Big)^{m/q}+\int_{\mathbb{R}^n}(|x|+1)^{\mu}u^q(x,t)\mathrm{d}x\\
&\quad+C_{11}l^{n-2-m(n+\lambda)/p_c}\Big(\int_{B_{2l}\setminus B_l}(|x|+1)^{\lambda}v^{p_c}(x,t)\mathrm{d}x\Big)^{m/p_c}\\
&\quad+\int_{\mathbb{R}^n}(|x|+1)^{\lambda}v^{p_c}(x,t)\mathrm{d}x.
\end{aligned}
$$

From the Young inequality and

$$
(q(n-2)-m(n+\mu))/(q-m)=(p_c(n-2)-m(n+\lambda))/(p_c-m),
$$

we obtain that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}\big(u(x,t)+v(x,t)\big)\phi_l(x)\mathrm{d}x\\
&\le\frac{m}{q}\int_{B_{2l}\setminus B_l}(|x|+1)^{\mu}u^q(x,t)\mathrm{d}x+\frac{q-m}{q}C_{11}^{q/(q-m)}l^{(q(n-2)-m(n+\mu))/(q-m)}\\
&\quad+\int_{\mathbb{R}^n}(|x|+1)^{\mu}u^q(x,t)\mathrm{d}x+\frac{m}{p_c}\int_{B_{2l}\setminus B_l}(|x|+1)^{\lambda}v^{p_c}(x,t)\mathrm{d}x\\
&\quad+\frac{p_c-m}{p_c}C_{11}^{p_c/(p_c-m)}l^{(p_c(n-2)-m(n+\lambda))/(p_c-m)}+\int_{\mathbb{R}^n}(|x|+1)^{\lambda}v^{p_c}(x,t)\mathrm{d}x\\
&\le 2\int_{\mathbb{R}^n}\big((|x|+1)^{\mu}u^q(x,t)+(|x|+1)^{\lambda}v^{p_c}(x,t)\big)\mathrm{d}x+M_4l^{(p_c(n-2)-m(n+\lambda))/(p_c-m)},
\end{aligned}
$$

where

$$
M_4=\max\Big\{\frac{q-m}{q}C_{11}^{q/(q-m)},\ \frac{p_c-m}{p_c}C_{11}^{p_c/(p_c-m)}\Big\}.
$$

□

Theorem 4.4. *The nontrivial solution to* (1.1)–(1.3) *with* $p = p_c$ *blows up in finite time.*

Proof. We prove this theorem by contradiction. Assume that (u, v) is a nontrivial global solution to (1.1)–(1.3) with $p = p_c$. Set

$$\Lambda = \sup_{l>1,t>0} w_l(t) = \sup_{t>0} \int_{\mathbb{R}^n} \big(u(x,t) + v(x,t)\big)\mathrm{d}x. \tag{4.15}$$

From (4.3) and the nontriviality of (u, v), $0 < \Lambda < +\infty$. Owing to (4.15), for any $0 < \varepsilon < \Lambda$, there exist $l_0 > 1$ and $t_0 > 0$ such that

$$w_{l_0/2}(t_0) \geq \Lambda - \varepsilon.$$

From (4.4) and (4.15), we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{t_0}^{+\infty}\int_{\mathbb{R}^n} \big((|x|+1)^{\mu}u^q(x,t) + (|x|+1)^{\lambda}v^{p_c}(x,t)\big)\mathrm{d}x\mathrm{d}t \\
&\leq \int_{\mathbb{R}^n} \big(u(x,t) + v(x,t)\big)\mathrm{d}x\Big|_{t=t_0}^{t=+\infty} \leq \Lambda - w_{l_0/2}(t_0) \leq \varepsilon.
\end{aligned}$$

For any $s \geq t_0$, (4.9) gives

$$\begin{aligned}
&\int_{\mathbb{R}^n} \big(u(x,s) + v(x,s)\big)\phi_{l_0/2}(x)\mathrm{d}x \\
&\leq 2\int_{t_0}^{s}\int_{\mathbb{R}^n} \big((|x|+1)^{\mu}u^q(x,t) + (|x|+1)^{\lambda}v^{p_c}(x,t)\big)\mathrm{d}x\mathrm{d}t \\
&\quad + M_4\Big(\frac{1}{2}l_0\Big)^{(p_c(n-2)-m(n+\lambda))/(p_c-m)}(s-t_0) \\
&\quad + \int_{\mathbb{R}^n} (u(x,t_0) + v(x,t_0))\phi_{l_0/2}(x)\mathrm{d}x \\
&\leq 4\varepsilon + M_4\Big(\frac{1}{2}l_0\Big)^{(p_c(n-2)-m(n+\lambda))/(p_c-m)}(s-t_0) \\
&\quad + \int_{\mathbb{R}^n} \big(u(x,t_0) + v(x,t_0)\big)\mathrm{d}x - w_{l_0/2}(t_0). \\
&\leq 5\varepsilon + M_4\Big(\frac{1}{2}l_0\Big)^{(p_c(n-2)-m(n+\lambda))/(p_c-m)}(s-t_0).
\end{aligned}$$

Letting $l = l_0$ in (4.6) and combining it with the above inequality, we have

$$\begin{aligned}
\frac{\mathrm{d}w_{l_0}(t)}{\mathrm{d}t} &\geq M_1^{m-\tau} l_0^{-mn+n-2} w_{l_0}^{m-\tau}(t) \\
&\quad \times \Big(-M_2\Big(5\varepsilon + M_4\Big(\frac{1}{2}l_0\Big)^{(p_c(n-2)-m(n+\lambda))/(p_c-m)}(s-t_0)\Big)^{\tau} \\
&\quad + M_1^{-(m-\tau)}M_3 \cdot \min\big\{w_{l_0}^{p_c-m+\tau}(t), w_{l_0}^{q-m+\tau}(t)\big\}\Big).
\end{aligned}$$

Taking $\varepsilon_0 \in (0, \Lambda)$ and $M_5 > 0$ such that

$$M_2(5\varepsilon_0 + M_5)^{\tau} \leq \frac{1}{2}M_1^{-(m-\tau)}M_3 \min\big\{(\Lambda-\varepsilon_0)^{p_c-m+\tau},\, (\Lambda-\varepsilon_0)^{q-m+\tau}\big\}$$

with $0 < \tau < \min\Big\{\frac{p_c-m}{p_c-1},\, \frac{q-m}{q-1}\Big\}$, one obtains

$$\frac{\mathrm{d}w_{l_0}(t)}{\mathrm{d}t} \geq \frac{1}{2}M_3 l_0^{-mn+n-2} \cdot \min\big\{w_{l_0}^{p_c}(t),\, w_{l_0}^{q}(t)\big\}, \quad t_0 < t < t_1, \tag{4.16}$$

where

$$t_1 = t_0 + \frac{M_5}{M_4}\Big(\frac{1}{2}l_0\Big)^{-(p_c(n-2)-m(n+\lambda))/(p_c-m)}.$$

Integrating (4.16) over (t_0, t_1) with respect to t leads to

$$\begin{aligned} w_{l_0}(t_1) &\geq w_{l_0}(t_0) + \frac{1}{2}M_3 l_0^{-mn+n-2}\min\big\{(\Lambda-\varepsilon_0)^{p_c},\, (\Lambda-\varepsilon_0)^q\big\}(t_1-t_0) \\ &\geq w_{l_0}(t_0) + 2^{(p_c(n-2)-m(n+\lambda))/(p_c-m)}\min\big\{(\Lambda-\varepsilon_0)^{p_c},\, (\Lambda-\varepsilon_0)^q\big\} \\ &\quad\times \frac{M_3M_5}{2M_4} l_0^{-mn+n-2-(p_c(n-2)-m(n+\lambda))/(p_c-m)}. \end{aligned}$$

Noting that

$$-mn+n-2-(p_c(n-2)-m(n+\lambda))/(p_c-m) = 0,$$

one has

$$\int_{\mathbb{R}^n}\big(u(x,t_1)+v(x,t_1)\big)\mathrm{d}x \geq w_{l_0}(t_1) \geq w_{l_0}(t_0) + \delta_0 \geq \Lambda - \varepsilon_0 + \delta_0$$

with

$$\delta_0 = \frac{M_3M_5}{2M_4}\min\big\{(\Lambda-\varepsilon_0)^{p_c},\, (\Lambda-\varepsilon_0)^q\big\}2^{(p_c(n-2)-m(n+\lambda))/(p_c-m)}$$

being a positive constant independent of l_0. Obviously,

$$w_{(2l_0)/2}(t_1) = w_{l_0}(t_1) \geq \Lambda - \varepsilon_0 + \delta_0 \geq \Lambda - \varepsilon_0.$$

The same argument leads to

$$\int_{\mathbb{R}^n}\big(u(x,t_2)+v(x,t_2)\big)\mathrm{d}x \geq w_{2l_0}(t_2) \geq w_{2l_0}(t_1) + \delta_0 \geq \Lambda - \varepsilon_0 + 2\delta_0,$$

where

$$t_2 = t_1 + \frac{M_5}{M_4}(l_0)^{-(p_c(n-2)-m(n+\lambda))/(p_c-m)}.$$

Repeating the above process, one gets that for any positive integer i,

$$\int_{\mathbb{R}^n}\big(u(x,t_i)+v(x,t_i)\big)\mathrm{d}x \geq w_{2^{i-1}l_0}(t_i) \geq w_{2^{i-1}l_0}(t_{i-1}) + \delta_0 \geq \Lambda - \varepsilon_0 + i\delta_0$$

with

$$t_i = t_{i-1} + \frac{M_5}{M_4}(2^{i-2}l_0)^{-(p_c(n-2)-m(n+\lambda))/(p_c-m)}.$$

Letting $i \to +\infty$ implies

$$\sup_{t>0}\int_{\mathbb{R}^n}\big(u(x,t)+v(x,t)\big)\mathrm{d}x = +\infty,$$

and this contradicts (4.3). □

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