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EXISTENCE AND MULTIPLICITY OF SOLUTIONS TO QUASILINEAR DIRAC-POISSON SYSTEMS

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Abstract. In this article, we study the existence and multiplicity of solutions of the quasilinear Dirac-Poisson system

$$i\sum_{k=1}^{3}\alpha_k\partial_k u - a\beta u - \omega u - \phi u = h(x,|u|)u, \quad x\in\mathbb{R}^3,$$
$$-\Delta\phi - \varepsilon^4\Delta_4\phi = u^2, \quad x\in\mathbb{R}^3,$$

where $\partial_k = \partial/\partial x_k$, $k=1,2,3$; $a>0$ is a constant; $\alpha_1,\alpha_2,\alpha_3$ and β are 4×4 Pauli-Dirac matrices; the operator Δ_4 is the 4-Laplacian operator, defined as $\Delta_4\phi := \operatorname{div}(|\nabla\phi|^2\nabla\phi)$; and $h(x,|u|)u$ describes the self-interaction. We prove the existence of the least energy solutions for the critical case and obtained that there exist finitely many critical points under certain conditions by variational methods. Additionally, we demonstrate the convergence behavior of solutions as ε tends to zero.

1. Introduction and results

This study considers the Dirac system

$$\begin{gathered} i\frac{\hbar}{c}\partial_t\psi + i\hbar\sum_{k=1}^{3}\alpha_k\partial_k\psi - mc\beta\psi - \varphi\beta\psi = f(x,|\psi|)\psi, \quad x\in\mathbb{R}^3, \\ -\operatorname{div}(|\nabla\varphi| - b|\nabla\varphi|^2)|\nabla\varphi| = (\beta\psi)\psi, \quad x\in\mathbb{R}^3, \end{gathered} \tag{1.1}$$

where ψ denotes the wave function of the state of an electron, φ is the gauge potential of the electromagnetic field, $\hbar$ symbolizes Planck's constant, $m>0$ means the mass of the electron, c is the speed of light, $\partial_k = \partial/\partial x_k$, $k=1,2,3$, b is a modify parameter, $\alpha_1,\alpha_2,\alpha_3$ and β are 4×4 Pauli-Dirac matrices:

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k=1,2,3;$$
$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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It is simple to verify that β, α_1, α_2 and α_3 satisfy the following anticommutative relation

$$\beta = \beta^*, \quad \alpha_k = \alpha_k^*,$$
$$\alpha_k \beta + \beta \alpha_k = 0, \quad \alpha_k \alpha_l + \alpha_l \alpha_k = 2\delta_{kl},$$

for $k, l = 1, 2, 3$.

The Dirac-Poisson system is fundamental to relativistic quantum electrodynamics. It describes the complex interaction of a spin-1/2 particle with its electromagnetic field. It plays a crucial role in quantum electrodynamics and is applied in various scientific fields, including quantum cosmology, nuclear physics, atomic physics, and gravitational physics ([29, 31]). This system plays a vital role in comprehending the quantum interactions between particles and their associated electromagnetic fields. It provides a theoretical foundation that has significantly advanced our comprehension of these phenomena. It has also been adapted to tackle specific issues in classical electrodynamics, especially those involving the infinities associated with point particles.

In this regard, the Born-Infeld electromagnetic theory [10, 22] provides a nonlinear alternative to Maxwell's theory to address these infinities. A quasi-linear Dirac-Poisson system is derived by replacing the standard Maxwell's Lagrangian density with that of Born-Infeld [23]. Born-Infeld's theory, parameterized by b, presents a Lagrangian density in square root form. It extends classical Maxwell's theory nonlinearly and ensures the finiteness of electric fields, thus avoiding the infinite field issues around point particles in classical electrodynamics. Additionally, the Dirac-Born-Infeld action is used to describe D-brane dynamics in superstring theory [1], demonstrating the system's adaptability and significance in modern theoretical physics, bridging quantum electrodynamics with advanced string theory concepts.

Many researchers have explored the solutions of Dirac equations and systems since Gross' groundbreaking study [25] on the local existence and uniqueness of solutions for autonomous systems. Over the following decades, the nonlinear Dirac equation has garnered significant attention due to its importance in theory and application. Researchers have used variational methods to explore solutions' existence, multiplicity, and other properties based on different assumptions about potential and nonlinearity, [7, 19, 16].

When coupled with some other theories, nonlinear Dirac systems always become a nonlocal challenge. Early on, Balabane et al. [6] paved the way by transforming the Dirac equation into a planar differential system and proving the existence of a sequence of solutions. Ding and Xu [20] further analyzed stationary semi-classical solutions with general subcritical self-coupling nonlinearity. Ding and Ruf [18] also studied the multiplicity of semi-classical solutions of a nonlinear Maxwell-Dirac system, with the number of solutions described by the ratio of maximum and behaviour at infinity of the potentials. Chen et al. [11] imposed local conditions on the potential V, assuming it is locally Hölder continuous, $\|V\|_{L^\infty} < a$, and there exists a bounded domain $\Lambda \in \mathbb{R}^3$ such that $\underline{\omega} := \min_\Lambda V < \min_{\partial\Lambda} V$. Moreover, they showed that a massive Dirac equation with critical growth has at least $\mathrm{cat}_{M_\delta}(M)$ solutions. [16] researched a nonlinear Dirac equation in space-dimension n, obtaining the existence of m pairs of solutions for any $\varepsilon < \varepsilon_m$. In [21], the authors studied the multiplicity of nonlinear Dirac-Klein-Gordon systems, also describing the number of solutions by the ratio of maximum and behaviour at infinity of the potentials. Benhassine [8] demonstrated the existence and multiplicity of stationary solutions in the asymptotically quadratic and super-quadratic cases using variational methods. Recently, Alves et al. [3] complemented the results found in [17], proving that the number of global minimum points of V is directly related to the number of solutions when ε is small.

The stationary wave solution of system (1.1) is a solution of the form

$$\psi(t,x) = u(x)e^{\frac{-i\xi t}{\hbar}},\\ \varphi(x) = \phi(x),$$

where ξ and t are real numbers, $w : \mathbb{R}^3 \to \mathbb{C}^4$. It is clear that (ψ, φ) solves (1.1) if and only if (u, ϕ) solves the system

$$\begin{gathered} i\sum_{k=1}^{3} \alpha_k \partial_k u - a\beta u - \omega u - \phi u = h(x, |u|)u, \quad x \in \mathbb{R}^3, \\ -\Delta\phi - \varepsilon^4 \Delta_4 \phi = u^2, \quad x \in \mathbb{R}^3, \end{gathered} \tag{1.2}$$

where, for simplicity, we take $b = \varepsilon^4$, $\hbar = 1$, $a = mc > 0$, $\omega = \frac{\xi}{c}$ to be constants, functional h satisfied $f(x, e^{i\theta}|u|) = e^{i\theta}h(x, |u|)$, and the operator Δ_4 is the 4-Laplacian operator, defined as $\Delta_4\phi := \operatorname{div}(|\nabla\phi|^2\nabla\phi)$.

As a modified version of the Dirac-Maxwell system, the existence and concentration of minimum energy solutions for subcritical nonlinearities were recently discussed in [32]. It is a logical next step to inquire whether similar results can be achieved for quasilinear Dirac-Poisson systems. We also note several research results concerning the multiplicity and concentration phenomena of solutions for quasilinear problems (see [5, 4]). These works prompt us to investigate whether analogous results can be established regarding the multiplicity of solutions for quasilinear Dirac-Poisson systems.

Consequently, we aim to explore the existence of minimum energy solutions and the multiplicity of solutions in quasilinear Dirac systems with critical nonlinearities. Here the critical exponent is 3, given by the relevant Sobolev embedding $H^{1/2}(\mathbb{R}^3, \mathbb{C}^4) \hookrightarrow L^3(\mathbb{R}^3, \mathbb{C}^4)$. To be specific, we consider the critical case:

$$h(x, |u|)u = K_1(x)g(|u|)u + K_2(x)|u|u. \tag{1.3}$$

Let $A_{\inf}$, $A_{\sup}$ denote the infimum and suprermum on the whole space, respectively, for any function A defined in $\mathbb{R}^3$. Writing $G(s) = \int_0^s g(t)tdt$, we assume the nonlinear potentials satisfy the following:

- (A1) $g(0) = 0$, $g \in C^1(0, \infty)$, $g'(s) > 0$ for $s > 0$, and there exist $p \in (2, 3)$, $c_1 > 0$ such that $g(s) \le c_1(1 + s^{p-2})$ for $s \ge 0$;
- (A2) There exist $q \ge 2$, $\theta > 2$, and $c_0 > 0$ such that $c_0 s^q \le G(s) \le \frac{1}{\theta} g(s)s^2$ for all $s > 0$;
- (A3) $K_j \in C^1(\mathbb{R}^3)$ with $K_j(x) \ge \lim_{|x|\to\infty} K_j(x) := k_{j,\infty} > 0$, for all $x \in \mathbb{R}^3$, $j = 1, 2$, and

$$1 < k_2 := \frac{K_{2,\sup}}{K_{2,\inf}} < \mathcal{R}(c_0, q, K_{1,\inf}, K_{2,\inf}),$$

where

$$\mathcal{R}(c_0, q, K_{1,\inf}, K_{2,\inf}) = \Big(\frac{S(a^2-(\omega^*)^2)}{a^2}\Big)^{1/2}\Big(\frac{(c_0 q K_{1,\inf})^{\frac{2}{q-2}}}{6\gamma_q K_{2,\inf}^2}\Big)^{1/3},$$

$\omega^* = \max\{\omega, 0\}$, S is the best Sobolev constant such that $S|u|_6^2 \le |\nabla u|_2^2$, and γ_q is the least energy (which is attained [17]) of the equation

$$i\sum_{k=1}^{3}\alpha_k\partial_k u - a\beta u - \omega u = |u|^{q-2}u.$$

Our first result concerns the existence and concentration behaviour of the least energy solutions for quasilinear Dirac-Poisson systems with critical growth. We also established the existence of the limit problem and the decay properties of the solutions.

Theorem 1.1. *Assume $\omega \in (-a, a)$, h is of the form (1.3), and conditions (A1)–(A3) are satisfied. Then the following Dirac-Poisson system admits at least one least energy solution (u_0, ϕ_0) in $\cap_{s\ge2,r\ge2} W^{1,r}_{\rm loc}(\mathbb{R}^3, \mathbb{C}^4) \times W^{1,s}_{\rm loc}(\mathbb{R}^3, \mathbb{R})$,*

$$\begin{gathered} i\alpha\cdot\nabla u - a\beta u - \omega u - \phi u = K_1(x)g(|u|)u + K_2(x)|u|u, \quad x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, \quad x \in \mathbb{R}^3. \end{gathered} \tag{1.4}$$

If additionally ∇K_j ,$j = 1, 2$ are bounded, there exist $C, c > 0$ such that $|u_0(x)| \le C\exp(-c|x|)$ for all $x \in \mathbb{R}^3$.

Theorem 1.2. *Assume $\omega \in (-a, a)$, h is of the form (1.3), and conditions (A1)–(A3) are satisfied. Then (1.2) admits at least one least energy solution $(u_\varepsilon, \phi_\varepsilon)$ in $\cap_{s\ge2,r\ge2} W^{1,r}_{\rm loc}(\mathbb{R}^3, \mathbb{C}^4) \times W^{1,s}_{\rm loc}(\mathbb{R}^3, \mathbb{R})$ for any $\varepsilon > 0$. If additionally ∇K_j, $j = 1, 2$ are bounded, these solutions have the following properties:*

1. *There exist $C, c > 0$ such that $|u_\varepsilon(x)| \le C\exp(-c|x|)$ for any $x \in \mathbb{R}^3$;*
2. *The solutions $(u_\varepsilon, \phi_\varepsilon) \to (u_0, \phi_0)$ in $H^1 \times D^{1,2}$ as $\varepsilon \to 0^+$.*

Regarding the existence of multiple solutions, we have the following results.

Theorem 1.3. *Assume $\omega \in (-a, a)$, h is of the form (1.3), and (A1)–(A3) are satisfied. For any positive integer N, there exist k_∞ and $m(c_0, q, N, K_{1,\inf}, K_{2,\inf})$, if*

$$k_\infty \le k_2 < m(c_0, q, N, K_{1,\inf}, K_{2,\inf}),$$

system (1.2) has at least N pairs of solutions $(u_{\varepsilon,n}, \phi_{\varepsilon,n})$ in $\cap_{s\ge2,r\ge2} W^{1,r}_{\rm loc}(\mathbb{R}^3, \mathbb{C}^4) \times W^{1,s}_{\rm loc}(\mathbb{R}^3, \mathbb{R})$ for any $\varepsilon > 0$. If additionally ∇K_j, $j = 1, 2$ are bounded, there exist $C, c > 0$ such that $|u_{\varepsilon,n}(x)| \le C\exp(-c|x|)$ for any $x \in \mathbb{R}^3$.

The mathematical challenges in quasi-linear Dirac-Poisson Systems are multifaceted. Firstly, the quasi-linearity of the second equation regarding ϕ adds another layer of difficulty, as its solution lacks an explicit formula and homogeneity. Secondly, the system's strong indefiniteness means the Dirac operator's spectrum is unbounded and contains essential spectrums, leading to a lack of a positive quadratic term in the energy functional of equation (1.2). Additionally, the Morse index and co-index are infinite at any critical point of this functional. Furthermore, critical growth and a lack of compactness further intensify the complexity of the problem. To overcome these challenges, we will employ the reduction method introduced by Ackermann within an appropriate variational framework. By using Ding's critical point theorems in [7], we consider the solutions of the equation as critical points of the energy functional Φ_ε associated with system (1.2), finally proving the existence and multiplicity of solutions.

The remainder part of this paper is organized as follows. In Section 2, we establish the variational framework, define the energy functionals, and recall the critical point theorems that are pivotal to our analysis. Subsequently, in Section 3, we demonstrate some preliminary results. Ultimately, in Section 4, we finish the proofs of our main results.

2. Variational framework

This section aims to establish an appropriate variational setting, introduce the energy functionals, and remind the reader of the critical point theorems. We will study the ground state solutions obtained as critical points of an energy functional Φ_ε associated with problem (1.2).

Let $D^{1,p} := D^{1,p}(\mathbb{R}^3,\mathbb{R})$ denote the Banach space defined as the completion of the test functions $C_0^\infty(\mathbb{R}^3,\mathbb{R})$ with respect to the L^p-norm of the gradient provided by

$$\|v\|_{D^{1,p}}^p = \int_{\mathbb{R}^3} |\nabla v|^p dx,$$

for $p \geq 2$.

Remembering the Sobolev inequality $S|v|_6^2 \leq |\nabla v|_2^2$, and considering the embeddings of $D^{1,2}(\mathbb{R}^3)$ and $D^{1,4}(\mathbb{R}^3)$ into $L^6(\mathbb{R}^3)$ and $C_0^\infty(\mathbb{R}^3)$ respectively, we can provide equivalent characterizations as follows:

$$D^{1,2}(\mathbb{R}^3) = \{v \in L^6(\mathbb{R}^3) : |\nabla v| \in L^2(\mathbb{R}^3)\},$$
$$D^{1,4}(\mathbb{R}^3) = \{v \in C_0^\infty(\mathbb{R}^3) : |\nabla v| \in L^4(\mathbb{R}^3)\}.$$

We define

$$\mathbb{D}(\mathbb{R}^3) := D^{1,2}(\mathbb{R}^3) \cap D^{1,4}(\mathbb{R}^3),$$

which is a Banach space equipped with the norm

$$\|\varphi\|_{\mathbb{D}} := |\nabla\varphi|_2 + |\nabla\varphi|_4.$$

For symbolic simplicity, let $\alpha := (\alpha_1, \alpha_2, \alpha_3)$ and $\alpha\cdot\nabla := \sum_{k=1}^3 \alpha_k\partial_k$. Then system (1.2) can be written as

$$\begin{gathered} i\alpha\cdot\nabla u - a\beta u - \omega u - \phi u = h(x,|u|)u, \quad x \in \mathbb{R}^3, \\ -\Delta\phi - \varepsilon^4\Delta_4\phi = u^2, \quad x \in \mathbb{R}^3. \end{gathered} \tag{2.1}$$

We will write $A_0 := i\alpha\cdot\nabla - a\beta$, $A_\omega := A_0 - \omega$ denote the self-adjoint operator on $L^2 := L^2(\mathbb{R}^3,\mathbb{C}^4)$ with domain $\mathcal{D}(A_\omega) \subset H^1 := H^1(\mathbb{R}^3,\mathbb{C}^4)$. Let $\sigma(A_\omega)$ and $\sigma_c(A_\omega)$ signify the spectrum and continuous spectrum of A_ω, respectively. Fourier analysis implies that $\sigma(A_\omega) = \sigma_c(A_\omega) = \mathbb{R}\backslash(-(a+\omega), a-\omega)$.

Notice that the space $L^2(\mathbb{R}^3,\mathbb{C}^4)$ has an orthogonal decomposition:

$$L^2(\mathbb{R}^3,\mathbb{C}^4) = L^+ \oplus L^-, \quad u = u^+ + u^-,$$

where A_0 is positive definite on L^+ and negative definite on L^-. Let $E := \mathcal{D}(|A_\omega|^{1/2}) = H^{1/2}$. Then E constitutes a Hilbert space equipped with a norm and inner product. For $u, v \in E$, the inner product is defined as

$$(u,v) := \Re(|A_\omega|^{1/2}u, |A_\omega|^{1/2}v)_2,$$

and the induced norm is $\|u\| = (u,u)^{1/2}$, where $|A_\omega|$ and $|A_\omega|^{1/2}$ represent the absolute value of A_ω and the square root of $|A_\omega|$ respectively.

Since $\sigma(A_\omega) = \mathbb{R}\backslash(-(a+\omega), a-\omega)$, we infer the inequality

$$(a \pm \omega)|u^{\mp}|_2^2 \le \|u^{\mp}\|^2, \quad \text{for all } u^{\pm} \in E^{\pm}.$$

The space E can be decomposed as

$$E = E^- \oplus E^+ \quad \text{with } E^{\pm} = E \cap L^{\pm}.$$

These subspaces are orthogonal with respect to both the $(\cdot,\cdot)$ and $(\cdot,\cdot)_2$ inner products. This decomposition further induces a natural decomposition of L^p for every $p \in (1,\infty)$ [19, Proposition 2.1], thus there exists $d_p > 0$ satisfying

$$d_p|u^{\pm}|_p^p \le |u|_p^p \quad \text{for all } u \in E \cap L^p. \tag{2.2}$$

For a proof of the next lemma we refer to [7, Lemma 3.4].

Lemma 2.1. *For any $q \in [2,3]$, the space E is continuously embedded in $L^q(\mathbb{R}^3,\mathbb{C}^4)$. For any $s \in [1,3)$, E is compactly embedded in $L^s_{\rm loc}(\mathbb{R}^3,\mathbb{C}^4)$. Namely, there exists constant $s_q > 0$ such that*

$$|u|_q \le s_q\|u\|, \quad \text{for all } u \in E.$$

It is easy to see that system (2.1) is variational, and its solutions are critical points of the C^2 functional $J_\varepsilon(u,\phi)$ on $E \times \mathbb{D}$, defined by

$$\begin{aligned} J_\varepsilon(u,\phi) &= \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \frac{1}{2}\int_{\mathbb{R}^3} \phi u^2 dx - F(u) \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^3} |\nabla\phi|^2 dx + \frac{1}{8}\varepsilon^4 \int_{\mathbb{R}^3} |\nabla\phi|^4 dx, \end{aligned} \tag{2.3}$$

where $u = u^+ + u^-$,

$$F(u) := \int_{\mathbb{R}^3} H(|u|)\,dx, \quad H(|u|) = \int_0^{|u|} h(x,t)t\,dt.$$

Observe that, for every $u \in E$, there exists a unique $\phi_\varepsilon^u \in \mathbb{D}$, which satisfies

$$-\Delta\phi_\varepsilon^u - \varepsilon^4\Delta_4\phi_\varepsilon^u = u^2. \tag{2.4}$$

For the rest of this article, ϕ_ε^u will denote the unique solution of equation (2.4), which satisfies the equation

$$\int_{\mathbb{R}^3} |\nabla\phi_\varepsilon^u|^2 dx + \varepsilon^4 \int_{\mathbb{R}^3} |\nabla\phi_\varepsilon^u|^4 dx = \int_{\mathbb{R}^3} \phi_\varepsilon^u u^2 dx. \tag{2.5}$$

For convenience, we define the operator $\phi_\varepsilon : E \to \mathbb{D}(\mathbb{R}^3)$ by $\phi_\varepsilon(u) = \phi_\varepsilon^u$ for any $\varepsilon > 0$ fixed. By Hölder inequality and Sobolev inequality, for every $u \in E$, we have

$$|\nabla\phi_\varepsilon(u)|_2^2 + \varepsilon^4|\nabla\phi_\varepsilon(u)|_4^4 = \int_{\mathbb{R}^3} \phi u^2 dx \le S^{-1/2}|\nabla\phi_\varepsilon|_2|u|^2_{\frac{12}{5}},$$

which implies that

$$|\nabla\phi_\varepsilon|_2 \le S^{-1/2}|u|^2_{\frac{12}{5}}.$$

Thus,

$$|\nabla\phi_\varepsilon(u)|_2^2 + \varepsilon^4|\nabla\phi_\varepsilon(u)|_4^4 \le S^{-1}|u|^4_{\frac{12}{5}} \le S^{-1}S^4_{\frac{12}{5}}\|u\|^4. \tag{2.6}$$

For ease of representation, we define the functional

$$\Gamma_\varepsilon : u \in E \mapsto \frac{1}{4}\int_{\mathbb{R}^3} |\nabla\phi_\varepsilon(u)|^2 dx + \frac{3}{8}\varepsilon^4 \int_{\mathbb{R}^3} |\nabla\phi_\varepsilon(u)|^4 dx \in \mathbb{R},$$

we have

$$\Gamma_\varepsilon(u) \le \frac{3}{8} S^{-1}|u|_{12/5}^4 \le \frac{3}{8} S^{-1} S_{12/5}^4 \|u\|^4.$$

By inserting equation (2.5) into the functional (2.3), we can express

$$\Phi_\varepsilon(u) := J_\varepsilon(u, \phi_\varepsilon(u)) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \Gamma_\varepsilon(u) - F(u).$$

In particular, we have

$$(\Phi_\varepsilon)'(u)v = \partial_u J_\varepsilon(u, \phi_\varepsilon(u))v + \partial_{\phi_\varepsilon} \phi_\varepsilon(u)\phi_\varepsilon'(u)v = \partial_u J_\varepsilon(u, \phi_\varepsilon(u))v.$$

We also deduced that

$$(\Phi_\varepsilon)'(u)v = (u^+ - u^-, v) - \Re \int_{\mathbb{R}^3} \phi_\varepsilon(u) u\bar{v}\, dx - \Re \int_{\mathbb{R}^3} h(x, |u|) u\bar{v}\, dx.$$

We collect some valuable properties for the nonlocal term ϕ_ε and Γ_ε blew. Their proofs can be found in [32].

Lemma 2.2. *For each $\varepsilon > 0$, ϕ_ε and Γ_ε we have the following properties:*

(1) *ϕ_ε maps bounded sets into bounded sets;*

(2) *The map $u \mapsto \Gamma_\varepsilon(u)$ is of class C^2 in E, and its derivative satisfies*

$$\Gamma_\varepsilon'(u)v = \int_{\mathbb{R}^3} \phi_\varepsilon(u) uv dx, \quad \text{for all } u, v \in E;$$

(3) *Γ_ε is non-negative, weakly sequentially lower semi-continuous, Γ_ε' is weakly sequentially continuous;*

(4) *If $u_n \to u$ in E, then, for every fixed $\varepsilon > 0$, $\Gamma_\varepsilon(u_n) \to \Gamma_\varepsilon(u)$ and $\Gamma_\varepsilon'(u_n)u_n \to \Gamma_\varepsilon'(u)u$.*

It is easy to see that $\Phi_\varepsilon \in C^2(E, \mathbb{R})$ and critical points of Φ_ε are weak solutions of system (2.1). To study the asymptotic behaviour of the solutions, similarly, for the limit system

$$\begin{gathered} \mathrm{i}\alpha \cdot \nabla u - \alpha\beta u - \omega u - \phi u = h(x, |u|)u, \\ -\Delta\phi = u^2, \end{gathered} \tag{2.7}$$

we define functional Φ_0 as

$$\Phi_0(u) := \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \Gamma_0(u) - F(u),$$

where $\Gamma_0(u) := \frac{1}{4}\int_{\mathbb{R}^3} \phi_0(u) u^2 dx$, and $\phi_0(u) = \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy = \frac{1}{|x|} * u^2$ is the unique solution in $D^{1,2}$ such that

$$\int_{\mathbb{R}^3} |\nabla \phi_0|^2 dx = \int_{\mathbb{R}^3} \phi_0 u^2 dx.$$

Regarding the non-local term in the limit system (2.7), there are also the following properties, and their proofs can be referred to in [20].

Lemma 2.3. *ϕ_0 and Γ_0 have the following properties:*

(1) *ϕ_0 maps bounded sets into bounded sets;*
(2) *The map $u \mapsto \Gamma_0(u)$ is of class C^2 in E, and its derivative satisfies*

$$\Gamma_0'(u)v = \int_{\mathbb{R}^3} \phi_0(u) uv dx, \quad \text{for all } u, v \in E; \tag{2.8}$$

(3) Γ_0 *is non-negative, weakly sequentially lower semi-continuous,* Γ_0' *is weakly sequentially continuous;*
(4) *If* $u_n \to u$ *in* E*, then* $\Gamma_0(u_n) \to \Gamma_0(u)$ *and* $\Gamma_0'(u_n)u_n \to \Gamma_0'(u)u$.

To establish our results, we recall some abstract critical point theorems, see [7, 15]. Assume X, Y are Banach Spaces with X being separable and reflexive and set $E = X \oplus Y$. Let $\mathcal{S} \subset X^*$ be a countable dense subset. Let $\mathcal{P}$ be the family of semi-norms on E consisting of all semi-norms

$$p_s : E = X \oplus Y \to \mathbb{R}, \quad p_s(x+y) = |s(x)| + \|y\|, \quad s \in \mathcal{S}.$$

Denote by $\mathcal{T}_{\mathcal{P}}$ the topology on E induced by $\mathcal{P}$. Let $\mathcal{T}_{w^*}$ be the weak*-topology of E^*.

For a functional $\Phi : E \to \mathbb{R}$ and numbers $a, b \in \mathbb{R}$ we write $\Phi^a := \{u \in E : \Phi(u) \le a\}$, $\Phi_a := \{u \in E : \Phi(u) \ge a\}$, and $\Phi_a^a := \Phi^a \cup \Phi_a$. Assume

(A4) $\Phi \in C^1(E, \mathbb{R})$, $\Phi : (E, \mathcal{T}_{\mathcal{P}}) \to \mathbb{R}$ is upper semi-continuous, and $\Phi' : (\Phi_a, \mathcal{T}_{\mathcal{P}}) \to (E^*, \mathcal{T}_{w^*})$ is continuous for every $a \in \mathbb{R}$;
(A5) For any $c > 0$, there exists $\gamma > 0$, such that $\|u\| \le \gamma \|u^+\|$ for all $u \in \Phi_c$;
(A6) There exists $r > 0$ with $\rho := \inf \Phi(S_r Y) > \Phi(0) = 0$ where $S_r Y := \{y \in Y : \|y\| = r\}$;
(A7) For any $e \in Y \backslash \{0\}$, there exists R with $R > r > 0$, such that $\sup \Phi(\partial Q) \le \rho$, where $Q := \{y = x + te : x \in X, t > 0, \|y\| < R\}$;
(A8) There exist a finite dimensional subspace $Y_0 \subset Y$ and $R > r$ such that, for $E_0 := X \times Y_0$ and $B_0 := \{u \in E_0 : \|u\| \le R\}$, we have $\sup \Phi(E_0) < \infty$ and $\sup \Phi(E_0 \backslash B_0) < \inf \Phi(B_r Y)$.

We say sequence $\{u_n\} \subset E$ is a $(C)_c$ sequence for $\Phi \in C^1(E, \mathbb{R})$, if $\Phi(u_n) \to c$ and $(1 + \|u_n\|)\Phi'(u_n) \in 0$. We say Φ satisfies the $(C)_c$ condition if any $(C)_c$ sequence for Φ has a convergent subsequence. A sequence $\{u_n\}$ is considered a $(PS)_c$-sequence of functional Φ if $\Phi(u_n)$ tends to c and $\Phi'(u_n)$ tends to 0. We say Φ satisfies the $(PS)_c$ condition if any $(PS)_c$-sequence has a convergent subsequence.

To prove the existence of the ground state solution, we will use the following critical point theorem.

Lemma 2.4 ([15, Theorem 4.5])**.** *Assume that conditions* (A4)–(A7) *are satisfied, then the functional* Φ *possesses a* $(C)c$*-sequence with* $\rho \le c \le \sup \Phi(Q)$.

Now we consider the set $\mathcal{M}(\Phi^c)$ of maps $g : \Phi^c \to E$ with the properties:

(1) g is $\mathcal{P}$-continuous and odd;
(2) $g(\Phi^a) \subset \Phi^a$ for all $a \in [\rho, b]$;
(3) each $u \in \Phi^c$ has a $\mathcal{P}$-open neighborhood $O \subset E$ such that the set $(id - g)(O \cap \Phi^c)$ is contained in a finite dimensional linear subspace.

We define the pseudo-index of Φ^c by

$$\psi(c) := \min\{\operatorname{gen} n(g(\Phi^c) \cap S_r Y) : g \in \mathcal{M}(\Phi^c)\} \in \mathbb{N}_0 \cup \{\infty\},$$

where $\operatorname{gen}(\cdot)$ denotes the usual symmetric index. Additionally, set for $d > 0$ fixed

$$\mathcal{M}_0(\Phi^d) := \{g \in \mathcal{M}(\Phi^d) : g \text{ is a homeomorphism from } \Phi^d \text{ to } g(\Phi^d)\}.$$

We define for $c \in [0, d]$, $\psi_d(c) := \min\{\operatorname{gen}(\Phi^c \cap S_r Y) : g \in \mathcal{M}_0(\Phi^d)\}$. Then, by definition, we have $\psi(c) \le \psi_d(c)$ for all $c \in [0, d]$.

The following theorem plays a crucial role in proving the existence of multiple solutions.

Theorem 2.5 ([15, Theorem 4.6]). *Let the assumptions* (A4)–(A6), (A8) *be satisfied, and assume that* Φ *is even and satisfy the* $(C)_c$*-condition for* $c \in [\rho, \sup \Phi(E_0)]$. *Then* Φ *has at least* $n := \dim Y_0$ *pairs of critical points with critical values given by*

$$c_i = \inf\{c \ge 0 : \psi(c) \ge i\} \in [\rho, b], \quad i = 1, \dots, n.$$

If Φ *has only finitely many critical points in* $\Phi_\rho^{\sup \Phi(E_0)}$, *then* $\rho < c_1 < c_2 < \dots < c_n \le \sup \Phi(E_0)$.

We are going to use these theorems. For our purposes, we set $\mathcal{P} = X^*$, thereby making $\mathcal{T}_\mathcal{P}$ the product topology on $E = X \oplus Y$, which is defined by the weak topology on X and the strong topology on Y.

3. Preliminaries

Throughout this section, we always let the hypotheses of Theorem 1.2 be satisfied. Next, we only prove these lemmas for the problem (1.2) for critical case (1.3), the proof of the limit problem (1.4) is similar, and most of these can be checked easily in [32].

Note that (A1) and (A2) imply that for each $\delta > 0$, there is $C_\delta > 0$ such that

$$g(s) \le \delta + C_\delta t^{p-2}, \quad G(s) \le \delta s^2 + C_\delta s^p, \quad \text{for all } s \ge 0.$$

Moreover, we deduce that

$$\hat{G}(s) := \frac{1}{2} g(s) s^2 - G(s) \ge 0, \quad \text{for all } s \ge 0.$$

Then, we will check the assumptions in the critical theorems before.

Lemma 3.1. *Under assumption* (A5), *there exists* $\gamma > 0$, *satisfying* $\|u\| \le \gamma \|u^+\|$ *for all* $u \in (\Phi_\varepsilon)_c$ *with* $c > 0$.

Proof. We argue by contradiction. Assume that there is a positive constant c and a sequence $\{u_n\} \subset (\Phi_\varepsilon)_c$ such that $\|u_n\|^2 \ge n \|u_n^+\|^2$, for any $j \in \mathbb{N}$. Then $0 \ge (2-n)\|u_n^-\|^2 \ge (n-1)(\|u_n^+\|^2 - \|u_n^-\|^2)$, for $n \ge 2$. Hence

$$\begin{aligned}
\Phi_\varepsilon(u_n) &= \frac{1}{2}(\|u_n^+\|^2 - \|u_n^-\|^2) - \Gamma_\varepsilon(u_n) - \int_{\mathbb{R}^3} K_1(x) G(|u_n|)\, dx - \frac{1}{3}\int_{\mathbb{R}^3} K_2(x)|u_n|^3 dx \\
&\le \frac{1}{2}(\|u_n^+\|^2 - \|u_n^-\|^2) \le 0.
\end{aligned}$$

But we know that $\Phi_\varepsilon(u_n) \ge c > 0$, which is a contradiction. □

Lemma 3.2. *Let* Φ_ε *satisfy* (A6) *and* (A7), *that is,* Φ_ε *possess the linking structure. Then*

1. *there exist* $r > 0$, $\rho > 0$ *(independent of* ε*), such that* $\Phi_\varepsilon|_{B_r^+} \ge 0$ *and* $\Phi_\varepsilon|_{\partial B_r^+} \ge \rho$ *where* $B_r^+ = B_r \cap E^+ = \{u \in E^+ : \|u\| \le r\}$;
2. *for each* $e \in E^+ \backslash \{0\}$, *there exist* $R = R_e > 0$, $C = C_e > 0$ *(both independent of* ε*), such that* $\Phi_\varepsilon(u) < 0$ *for any* $u \in E_e \backslash B_R$, *and* $\sup \Phi_\varepsilon(E_e) \le C$.

Proof. (1) For all $u \in E^+$ and any $\varepsilon > 0$, we know that $\Gamma_\varepsilon(u) \le \frac{3}{8} S^{-1} S_{\frac{12}{5}}^4 \|u\|^4$, we have

$$\begin{aligned}
\Phi_\varepsilon(u) &= \frac{1}{2}\|u\|^2 - \Gamma_\varepsilon(u) - \int_{\mathbb{R}^3} K_1(x) G(|u|)\, dx - \frac{1}{3}\int_{\mathbb{R}^3} K_2(x)|u|^3\, dx \\
&\ge \frac{1}{2}\|u\|^2 - \frac{3}{8} S^{-1} S_{\frac{12}{5}}^4 \|u\|^4 - K_{2\sup}(\delta |u|_2^2 + C_\delta |u|_p^p) - \frac{K_{2\sup}}{3} |u|_3^3
\end{aligned}$$

$$\geq \frac{1}{4}\|u\|^2 - \frac{3}{8}S^{-1}S_{\frac{12}{5}}^4\|u\|^4 - C_\delta s_p^p K_{2\sup}\|u\|^p - \frac{s_3^3 K_{2\sup}}{3}\|u\|^3.$$

(2) For any $e \in E^+\backslash\{0\}$, by virtue of (2.2), for $u = se + v$, we obtain

$$\begin{aligned}\Phi_\varepsilon(u) &= \frac{1}{2}(\|se\|^2 - \|v\|^2) - \Gamma_\varepsilon(u) - \int_{\mathbb{R}^3} K_1(x)G(|u|)\,dx - \frac{1}{3}\int_{\mathbb{R}^3} K_2(x)|u|^3 dx \\ &\leq \frac{1}{2}(s^2\|e\|^2 - \|v\|^2) - K_{1,\inf}\int_{\mathbb{R}^3} G(|u|)\,dx - \frac{K_{2,\inf}}{3}\int_{\mathbb{R}^3}|u|^3 dx \\ &\leq \frac{s^2}{2}\|e\|^2 - \frac{d_3 s^3 K_{2,\inf}}{3}|e|_3^3.\end{aligned}$$

The proof is complete. □

Next, we turn to study the $(C)_c$ sequence of Φ_ε.

Lemma 3.3. *For all $c > 0$, the $(C)_c$ sequences Φ_ε is bounded in E uniformly in ε.*

Proof. Given $\{u_n\}$ satisfies $\Phi_\varepsilon(u_n) \to c$ and $(1 + \|u_n\|)\Phi'_\varepsilon(u_n) \to 0$ as $n \to \infty$. Without loss of generality, assume that $\|u_n\| \geq 1$. When n is sufficiently large, we have

$$\begin{aligned}c + 1 &\geq \Phi_\varepsilon(u_n) - \frac{1}{2}(\Phi_\varepsilon)'(u_n)u_n \\ &= \frac{1}{4}\int_{\mathbb{R}^3}|\nabla\phi_\varepsilon|^2 dx + \frac{\varepsilon^4}{8}\int_{\mathbb{R}^3}|\nabla\phi_\varepsilon|^4 dx + \int_{\mathbb{R}^3} K_1(x)\hat{G}(|u_n|)\,dx \\ &\quad + \frac{1}{6}\int_{\mathbb{R}^3} K_2(x)|u_n|^3 dx \\ &\geq K_{1,\inf}(\frac{1}{2} - \frac{1}{\theta})\theta c_0|u_n|_q^q + \frac{K_{2,\inf}}{6}|u|_3^3.\end{aligned}$$

The sequence $\{u_n\}$ is bounded in the spaces L^2, L^q and L^3 with an upper bound denoted by C_1, where C_1 depends only on c, $K_{1,\inf}$, $K_{2,\inf}$ and q. Further deductions lead to

$$\begin{aligned}1 &\geq \|u_n\|^2 - \Gamma'_\varepsilon(u_n)(u_n^+ - u_n^-) - \Re\int_{\mathbb{R}^3} K_1(x)g(u_n)u_n\overline{u_n^+ - u_n^-}\,dx \\ &\quad - \Re\int_{\mathbb{R}^3} K_2(x)|u_n|^2\overline{u_n^+ - u_n^-}\,dx.\end{aligned} \tag{3.1}$$

Set $v_n = \frac{u_n}{\|u_n\|}$, snd recall that $\phi_\varepsilon^{u_n}$ satisfies

$$-\Delta\phi_\varepsilon^{u_n} - \varepsilon^4\Delta_4\phi_\varepsilon^{u_n} = u_n^2.$$

Hence, for each $\psi \in \mathbb{D}(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3}\nabla\frac{\phi_\varepsilon^{u_n}}{\|u_n\|}\nabla\psi\,dx + \varepsilon^4\int_{\mathbb{R}^3}|\nabla\phi_\varepsilon^{u_n}|^2\frac{\nabla\phi_\varepsilon^{u_n}}{\|u_n\|}\nabla\psi\,dx = \int_{\mathbb{R}^3}\psi u_n v_n\,dx.$$

For $\|v_n\| = 1$, choose $s = \frac{6q}{5q-6}$ such that $\frac{1}{q} + \frac{1}{s} + \frac{1}{6} = 1$. Then, since $2 < s < 3$, it follows that

$$\begin{aligned}\Big|\int_{\mathbb{R}^3}(1 + \varepsilon^4|\nabla\phi_\varepsilon^{u_n}|^2)\nabla\frac{\phi_\varepsilon^{u_n}}{\|u_n\|}\nabla\psi\,dx\Big| &\leq \Big|\int_{\mathbb{R}^3} u_n v_n\psi\,dx\Big| \\ &\leq |u_n|_q|v_n|_s|\psi|_6 \\ &\leq S^{-1/2}|u_n|_q|v_n|_s|\nabla\psi|_2.\end{aligned}$$

Then

$$\Big|\nabla \frac{\phi_\varepsilon^{u_n}}{\|u_n\|}\Big|_2 \le \Big|(1+\varepsilon^4|\nabla\phi_\varepsilon^{u_n}|^2)\nabla\frac{\phi_\varepsilon^{u_n}}{\|u_n\|}\Big|_2 \le S^{-1/2}s_s|u_n|_q$$

Therefore $|\phi_\varepsilon^{u_n}|_6 \le \frac{C_1 s_s}{S}\|u_n\|$, where S is the best Sobolev constant mentioned earlier, and s_s is the constant in Lemma 2.1. Note that, by Hölder inequality, when $2 < s \le q < 3$, we have

$$|u_n|_s^s \le |u_n|_2^{\frac{3(q-s)}{q-2}} |u_n|_q^{\frac{q(s-2)}{q-2}} \le C_1^{\frac{3(q-s)}{q-2}} |u_n|_q^{\frac{q(s-2)}{q-2}},$$

and when $2 < q < s < 3$, we have

$$|u_n|_s^s \le |u_n|_3^{\frac{3(q-s)}{q-3}} |u_n|_q^{\frac{q(s-3)}{q-3}} \le C_1^{\frac{3(q-s)}{q-3}} |u_n|_q^{\frac{q(s-3)}{q-3}}.$$

By Lemma 2.2, we obtain that

$$\begin{aligned} \big|\Gamma_\varepsilon'(u_n)(u_n^+ - u_n^-)\big| &= \Big|\Re\int_{\mathbb{R}^3} \phi_\varepsilon u_n \overline{u_n^+ - u_n^-}\,dx\Big| \\ &\le |\phi_\varepsilon|_6 |u_n|_s |u_n^+ - u_n^-|_q \\ &\le C_2\|u_n\|^{1+t}, \end{aligned} \tag{3.2}$$

where $t = \frac{q(s-2)}{s(q-2)}$ when $s \le q$, and $t = \frac{q(s-3)}{s(q-3)}$ when $s > q$. It is easy to see that $0 < t < 1$.

Notice that by (A1), there exists $r_1, r_2 > 0$ such that $g(s) \le \frac{a-|\omega|}{2K_{1\sup}}$, for every $s < r_1$, and $g(s) \le r_2 s^{p-2}$, for $s \ge r_1$. By (A2), set $\delta_0 := \frac{p}{p-2}$ and $r_3 := \frac{2\theta r_2^{\delta_0-1}}{\theta-2}$, for all $s \ge r_1$ we have $g^{\delta_0}(s) \le r_2^{\delta_0-1}g(s)s^2 \le r_3\hat{G}(s)$. Then, for $l := \frac{pq}{2q-p}$ such that $\frac{1}{\delta_0} + \frac{1}{q} + \frac{1}{l} = 1$, we can estimate

$$\begin{aligned} &\Big|\Re\int_{\mathbb{R}^3} K_1(x)g(u_n)u_n\overline{(u_n^+ - u_n^-)}\,dx\Big| \\ &\le \frac{a-|\omega|}{2}|u_n|_2^2 + K_{1\sup}\Big(\int_{|u|\ge r_1} g^{\delta_0}(|u_n|)\,dx\Big)^{\frac{1}{\delta_0}}|u_n|_q|u_n^+ - u_n^-|_l \\ &\le \frac{a-|\omega|}{2a}\|u_n\|^2 + C_3\|u_n\|, \end{aligned} \tag{3.3}$$

where C_3 is independent of ε. Moreover, we have

$$\Big|\Re\int_{\mathbb{R}^3} K_2(x)|u_n|u_n\overline{u_n^+ - u_n^-}dx\Big| \le K_{2\sup}|u_n|_3^3 \le C_4. \tag{3.4}$$

Then, the combination of estimates (3.1)-(3.4) shows that

$$\frac{a-|\omega|}{2a}\|u_n\|^2 \le 1 + C_2\|u_n\|^{1+t} + C_3\|u_n\| + C_4.$$

Consequently, there exists a constant $\Lambda \ge 1$ such that $\|u_n\| \le \Lambda$ as desired. The value of Λ is independent of ε. □

Let $\mathcal{K}_\varepsilon := \{u \in E : \Phi_\varepsilon'(u) = 0\}$ be the critical set of Φ_ε. Due to the presence of critical terms in system (2.1), the standard bootstrap argument fails to establish the regularity of finite action weak solutions. We obtain the following regularity result using the similar argument in [26].

Lemma 3.4. *Suppose $u \in \mathcal{K}_\varepsilon$ is a critical point of Φ_ε. Then the pair (u, ϕ_ε) is in the space $\cap_{s\geq 2, r\geq 2} W^{1,s}_{\rm loc}(\mathbb{R}^3, \mathbb{C}^4) \times W^{1,r}_{\rm loc}(\mathbb{R}^3, \mathbb{R})$. Besides this, (u, ϕ_ε) also belongs to the space $L^\infty(\mathbb{R}^3, \mathbb{C}^4) \times L^\infty(\mathbb{R}^3, \mathbb{R})$.*

Proof. Set $x \in \mathbb{R}^3$ fixed, let $\bar\rho \in C_0^\infty(B_2(x))$ be arbitrary. Choose $\bar\eta \in C_0^\infty(B_2(x))$ such that $\bar\eta = 1$ on $\operatorname{supp}\bar\rho$. Define the operator D to be $D = i\alpha \cdot \nabla$, we deduce

$$D(\bar\rho u) = \bar\rho Du + D\bar\rho \cdot u = \bar\eta \cdot \bar\rho Du + D\bar\rho \cdot \bar\eta.$$

Noting that

$$Du = a\beta u + \omega u + \phi_\varepsilon u + K_1(x)g(|u|)u + K_2(x)|u|u,$$

we have

$$D\bar\rho \cdot u = A_\omega(\bar\rho u) - T_{\varepsilon,u}(\bar\rho u), \tag{3.5}$$

where $A_\omega := A_0 - \omega$. For $1 < t < 3$, $T_{\varepsilon,u} : W^{1,t}(B_2(x)) \to L^t(B_2(x))$ is defined by

$$w \mapsto \bar\eta \cdot [\phi_\varepsilon + K_1(x)g(|u|) + K_2(x)|u|]w.$$

By applying the Gagliardo-Nirenberg inequality, it follows that

$$|\phi_\varepsilon|_\infty \leq C|\phi_\varepsilon|_6^{1/3} |\nabla\phi_\varepsilon|_4^{2/3}.$$

Through Sobolev embedding and the inequality (2.6), we can conclude that $\phi_\varepsilon \in L^\infty(\mathbb{R}^3, \mathbb{C}^4)$. By $\phi_\varepsilon \in D^{1,4}$, in value of [14], it follows that $\phi_\varepsilon \in C^{1,\alpha}_{\rm loc}$ for $0 < \alpha < 1$. Hence, we derive that

$$\lim_{|x|\to\infty} |\nabla\phi_\varepsilon| = 0.$$

Note that $a(|\nabla u|) := 1 + |\nabla u|^2$ belongs to the class $C^1(0, \infty)$ and satisfies the inequalities

$$-1 < \inf_{t>0} \frac{ta'(t)}{a(t)} \leq \sup_{t>0} \frac{ta'(t)}{a(t)} < \infty,$$

and

$$t^3 \leq ta(t) \leq C(t^3 + 1)$$

for $t > 0$. In [12, Theorem 3.1] it was shown that

$$\|\nabla\phi_\varepsilon\|_{L^\infty_{\rm loc}(\mathbb{R}^3)} \leq C\|u^2\|^{\frac{1}{p-1}}_{L^{3,1}_{\rm loc}(\mathbb{R}^3)}.$$

The embedding theorems in Lorentz space show that $L^{p,q}_{\rm loc}$ is continuously embedded into $L^{n,s}_{\rm loc}$, for any $0 < n < p < \infty$ and $0 < q, s < \infty$. Combining this with $L^{q,q}_{\rm loc} = L^q_{\rm loc}$ and $u^2 \in L^\infty_{\rm loc}$, we deduce that

$$|D\phi_\varepsilon(x)| \leq C, \quad \text{for all } x \in \mathbb{R}^3,$$

where C is independent of ε.

Using the Sobolev embedding $W^{1,t}(B_2(x)) \hookrightarrow L^{\frac{3t}{3-t}}(B_2(x))$ and Hölder inequality, it follows that $T_{\varepsilon,u}(w) \in L^t(B_2(x))$ for $w \in W^{1,t}(B_2(x))$ and the above map is well defined. By Minkowski and Hölder inequality, the operator norm is estimated by

$$\|T_{\varepsilon,u}\|_{W^{1,t}\to L^t} \leq C_1(|u|_{L^3(B)} + |B|^{t/3})$$

for some constant C_1 (depending only on t), where $B := \operatorname{supp}\bar\eta$.

Since $0 \notin \sigma(A_\omega)$,

$$A_\omega - T_{\varepsilon,u} : W^{1,t}(B_2(x)) \to L^t(B_2(x))$$

is invertible when $|B|$ is small.

Thus, from (3.5), there exists a unique solution $w \in W^{1,t}(B_2(x))$ to the equation $A_\omega w - T_{\varepsilon,u}(w) = D\bar{\rho} \cdot u$. Let us show that there is a well-defined map

$$T_{\varepsilon,u} : L^3(B_2(x)) \to W^{-1,3}(B_2(x)).$$

In fact, by Hölder inequality, we can confirm that $T_{\varepsilon,u}(w)$ is belongs to $L^{3/2}(B_2(x))$. Moreover, considering $L^{3/2}(B_2(x)) \subset W^{-1,3}(B_2(x))$ by the Sobolev embedding theorem, the map above remains well defined. The operator norm can be estimated as follows:

$$\|T_{\varepsilon,u}\|_{L^3 \to W^{-1,3}} \leq C_2(|u|_{L^3(B)} + |B|^{1/3}).$$

Thus,

$$A_\omega - T_{\varepsilon,u} : L^3(B_2(x)) \to W^{-1,3}(B_2(x))$$

becomes invertible if $|B|$ is small and there exists a unique solution $\tilde{w} \in L^3(B_2(x))$ solves the equation

$$A_\omega \tilde{w} - T_{\varepsilon,u}(\tilde{w}) = D\bar{\rho} \cdot u. \tag{3.6}$$

Consequently, we have $\tilde{w} = \bar{\rho} \cdot u$ based on (3.5). Additionally, by $W^{1,s}(B_2(x)) \hookrightarrow L^3(B_2(x))$ for $\frac{3}{2} \leq s \leq 3$, we conclude that $w \in W^{1,s}$ is also a L^3-solution to (3.6), given $\frac{3}{2} \leq s < 3$. As a result, the uniqueness of the solution leads to $w = \bar{\rho} \cdot u$. This implies that $\bar{\rho} \cdot u \in W^{1,s}(B_2(x))$ for any $s \in [\frac{3}{2}, 3)$, provided that $B = \operatorname{supp} \bar{\eta}$ is sufficiently small. Since $\bar{\rho}$ and $\bar{\eta}$ are arbitrary, it follows that $u \in W^{1,s}(B_1(x))$ for any $s \in [3/2, 3)$.

Therefore, by Sobolev embedding, we obtain $u \in \cap_{s \geq 2} L^s_{\rm loc}(\mathbb{R}^3)$ and this implies $u \in \cap_{s \geq 2} W^{1,s}_{\rm loc}(\mathbb{R}^3)$. Moreover, regarding equation (2.4) and $u^2 \in W^{1,n}_{\rm loc}(\mathbb{R}^3)$, using elliptic regularity theory [24], we deduce that $\phi_\varepsilon \in W^{2,n}_{\rm loc}$ for any integer $n \geq 2$. Using the regularity of the 4-Laplacian, we have $\phi_\varepsilon \in W^{1,4}_{\rm loc}$. Consequently, $\phi_\varepsilon \in \cap_{r \geq 2} W^{1,r}_{\rm loc}$. Finally, by elliptic estimates, we have $u \in L^\infty(\mathbb{R}^3)$. □

Next, we state the minimax scheme and recall the mountain-pass type reduction.

For each $\varepsilon > 0$ and $e \in E^+ \setminus \{0\}$, let c_ε denote the minimax level of Φ_ε deduced by the linking structure [30]:

$$c_\varepsilon := \inf_{e \in E^+/\{0\}} \max_{u \in E_e} \Phi_\varepsilon(u) = \inf_{e \in E^+/\{0\}} \max_{u \in \hat{E}_e} \Phi_\varepsilon(u),$$

where $E_e = E^- \oplus \mathbb{R}e$ and $\hat{E}_e = E^- \oplus \mathbb{R}^+ e$.

For $u = u^+ + u^- \in E$ fixed, define the reduction map $h_\varepsilon : E^+ \to E^-$ by

$$\Phi_\varepsilon(u + h_\varepsilon(u)) = \max_{v \in E^-} \Phi_\varepsilon(u + v).$$

Then

$$v \neq h_\varepsilon(u) \Leftrightarrow \Phi_\varepsilon(u + v) < \Phi_\varepsilon\left(u + h_\varepsilon(u)\right).$$

By differentiating the functional and using the convexity of the nonlinear terms, it can be verified that h_ε is uniquely determined. Moreover, the following is known (see [2]):

(1) $h_\varepsilon \in C^1(E^+, E^-)$, $h_\varepsilon(0) = 0$;
(2) h_ε is a bounded map;
(3) If $u_n \rightharpoonup u$ in E^+, then $h_\varepsilon(u_n) - h_\varepsilon(u_n - u) \to h_\varepsilon(u)$ and $h_\varepsilon(u_n) \rightharpoonup h_\varepsilon(u)$.

We define the reduced functional $I_\varepsilon : E^+ \to \mathbb{R}$ by

$$I_\varepsilon(u) = \Phi_\varepsilon(u + h_\varepsilon(u)),$$

and set the Nehari-Pankov manifold

$$\mathcal{N}_\varepsilon := \left\{ u \in E^+ \backslash \{0\} : I'_\varepsilon(u)u = 0 \right\}.$$

Then $I_\varepsilon \in C^2(E^+, \mathbb{R})$ and $u \in E^+$ is a critical point of I_ε if and only if $u + h_\varepsilon(u)$ is a critical point of Φ_ε. We will call $(h_\varepsilon(\cdot), I_\varepsilon(\cdot), \mathcal{N}_\varepsilon)$ the Mountain-Pass reduction of system (2.1). Clearly,

$$c_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon} I_\varepsilon(u).$$

For the reduction fuctional, we can verify the following result.

Lemma 3.5. *For any $\varepsilon > 0$, we have:*

(1) *I_ε possesses the mountain pass structure: $I_\varepsilon(0) = 0$ and there exist $r, \rho > 0$ and $e_0 \in E^+$ satisfy $\|e_0\| > r$ such that $\inf I_\varepsilon(S_r^+) > 0$ and $\sup I_\varepsilon(e_0) < 0$;*

(2) *For any $u \in E^+ \setminus \{0\}$, there is a unique $t_\varepsilon = t_\varepsilon(u) > 0$ such that $t_\varepsilon u \in \mathcal{N}_\varepsilon$. Moreover, $\{t_\varepsilon(u)\}_{\varepsilon \leq 1}$ is bounded.*

Proof. (1) For all $u \in E^+$, by the definition of h_ε we have $\Phi_\varepsilon(u + h_\varepsilon(u)) \geq \Phi_\varepsilon(u)$. Hence

$$\begin{aligned}
I_\varepsilon(u) &= \Phi_\varepsilon(u + h_\varepsilon(u)) \\
&= \frac{1}{2}\|u\|^2 + (\Phi_\varepsilon(u + h_\varepsilon(u)) - \Phi_\varepsilon(u)) - \Gamma_\varepsilon(u) - F(u) \\
&\geq \frac{1}{2}\|u\|^2 - \Gamma_\varepsilon(u) - \int_{\mathbb{R}^3} K_1(x) G(|u|)\, dx - \frac{1}{3}\int_{\mathbb{R}^3} K_2(x)|u|^3 dx \\
&\geq \frac{1}{4}\|u\|^2 - \frac{3}{8} S^{-1} S^4_{\frac{12}{5}} \|u\|^4 - C_\delta s_p^p K_{2\sup} \|u\|^p - \frac{s_3^3 K_{2\sup}}{3}\|u\|^3,
\end{aligned}$$

where δ was chosen such that $\delta \leq (2 s_2^2 K_{2\sup})^{-1}$ and $2 < p < 3$.

For any $e \in E^+$ and $s > 0$, we have

$$\begin{aligned}
I_\varepsilon(se) &= \Phi_\varepsilon(se + h_\varepsilon(u)) \\
&= \frac{1}{2}(\|se\|^2 - \|h_\varepsilon(u)\|^2) - \Gamma_\varepsilon(u + h_\varepsilon(u)) - \int_{\mathbb{R}^3} K_1(x) G(|u + h_\varepsilon(u)|)\, dx \\
&\quad - \frac{1}{3}\int_{\mathbb{R}^3} K_2(x) |u + h_\varepsilon(u)|^3 dx \\
&\leq \frac{1}{2} s^2 \|e\|^2 - \frac{K_{2,\inf}}{3}\int_{\mathbb{R}^3} |u|^3 dx \leq \frac{s^2}{2}\|e\|^2 - \frac{d_3 s^3 K_{2,\inf}}{3}|e|_3^3.
\end{aligned}$$

(2) We just repeat the arguments in [32] to have the results. □

When $K_j(x) = k_{j,\infty}$, system (2.1) becomes a autonomous problem

$$\begin{gathered}
i\alpha \cdot \nabla u - a\beta u - \omega u - \phi u = k_{1,\infty} g(|u|)u + k_{2,\infty}|u|u, \quad x \in \mathbb{R}^3, \\
-\Delta\phi - \varepsilon^4 \Delta_4 \phi = u^2, \quad x \in \mathbb{R}^3.
\end{gathered}$$

In this case we use the notation $\Phi_\infty(u)$ and c_∞, respectively, for the functional and the least energy. For

$$\Phi_\varepsilon(u) := \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \Gamma_\varepsilon(u) - k_{1,\infty}\int_{\mathbb{R}^3} G(|u_n|)\, dx - \frac{k_{2,\infty}}{3}\int_{\mathbb{R}^3} |u_n|^3 dx,$$

by a similar argument as in [20], it is straightforward to verify that c_∞ is attained if

$$k_{2,\infty}^2 < \Big(\frac{S(a^2-(\omega^*)^2)}{a^2}\Big)^{3/2}\frac{(c_0 q k_{1,\infty})^{\frac{2}{q-2}}}{6\gamma_q},$$

where $\omega^* = \max\{\omega, 0\}$.

Lemma 3.6. *Φ_ε satisfied the $(C)_c$-condition for any $0 < c \le c_\infty$. Moreover, c is attained if*

$$c < \Big(\frac{a^2-(\omega^*)^2}{a^2}\Big)^{3/2}\frac{S^{3/2}}{6k_2 K_{2,\inf}^2},$$

where $\omega^* = \max\{\omega, 0\}$.

Proof. Suppose that $\{u_n\}$ is a $(C)_c$-sequence of Φ_ε. Lemma 3.3 shown that $\{u_n\}$ is bounded. It is easy to check that $\{u_n\}$ is relatively compact for all $0 < c \le c_\infty$. Without loss of generality, let us assume that there is a u_ε such that u_n converges weakly to u_ε in E. Now we are going to show that $u_\varepsilon \neq 0$ for all small $\varepsilon > 0$. Assume by contradiction that u_n is vanishing, then $u_n \to 0$ in L^q for $q \in (2, 3)$. Notice that

$$\begin{aligned}
c + o(1) &\ge \Phi_\varepsilon(u_n) - \frac{1}{3}(\Phi_\varepsilon)'(u_n)u_n \\
&= \frac{1}{6}(\|u_n^+\|^2 - \|u_n^-\|^2) + \frac{1}{12}\int_{\mathbb{R}^3}|\nabla\phi_\varepsilon|^2 dx - \frac{\varepsilon^4}{24}\int_{\mathbb{R}^3}|\nabla\phi_\varepsilon|^4 dx \\
&\quad - \int_{\mathbb{R}^3} K_1(x)(\frac{1}{3}g(|u_n|)u_n^2 - G(|u_n|))\,dx \\
&\ge \frac{1}{6}||A_\omega|^{1/2}u_n|_2^2 + o(1),
\end{aligned}$$

we have $||A_\omega|^{1/2}u_n|_2^2 \le 6c + o(1)$. Similarly,

$$|u_n|_3^3 \le \frac{6c}{K_{2,\inf}} + o(1).$$

Moreover,

$$\begin{aligned}
o(1) \ge& (\Phi_\varepsilon)'(u_n)(u_n^+ - u_n^-) \\
=& \int_{\mathbb{R}^3}\langle A_\omega u_n, u_n^+ - u_n^-\rangle\,dx - \Gamma_\varepsilon'(u_n)(u_n^+ - u_n^-) \\
&- \int_{\mathbb{R}^3} K_1(x)g(|u_n|)u_n(u_n^+ - u_n^-)\,dx - \int_{\mathbb{R}^3} K_2(x)|u_n|u_n(u_n^+ - u_n^-)\,dx.
\end{aligned}$$

Then

$$\int_{\mathbb{R}^3}\langle A_\omega u_n, u_n^+ - u_n^-\rangle\,dx \le K_{2\sup}|u_n|_3^2|u_n^+ - u_n^-|_3 + o(1).$$

By Calderón-Lions interpolation theorem [28], we have

$$S^{1/2}|u_n|_3^2 \le ||i\alpha\cdot\nabla|^{1/2}u_n|_2^2.$$

We denote the Fourier transform by $\mathscr{F} : L^2 \to L^2$, recall from [19] that

$$\int_{\mathbb{R}^3}\langle i\alpha\cdot\nabla u_n, u_n\rangle\,dx = \int_{\mathbb{R}^3}|\xi||\mathscr{F}u(\xi)|^2\,d\xi,$$

$$\int_{\mathbb{R}^3}\langle A_\omega u_n, u_n\rangle\,dx = \int_{\mathbb{R}^3}\Big((a^2+|\xi|^2)^{1/2} - \omega\Big)|\mathscr{F}u(\xi)|^2\,d\xi.$$

Taking into account that

$$\inf_{|\xi|>0}\frac{(a^2+|\xi|^2)^{1/2}-\omega}{|\xi|}=\begin{cases}\big(\frac{a^2-\omega^2}{a^2}\big)^{1/2}, & \text{if}\omega>0,\\ 1, & \text{if}\omega\le 0,\end{cases}$$

we have

$$\int_{\mathbb{R}^3}\langle A_\omega u_n,u_n\rangle\,dx\ge\Big(\frac{a^2-(\omega^*)^2}{a^2}\Big)^{1/2}\int_{\mathbb{R}^3}\langle i\alpha\cdot\nabla u_n,u_n\rangle\,dx,$$

where $\omega^*=\max\{\omega,0\}$. Finally we obtain

$$c\ge\Big(\frac{a^2-(\omega^*)^2}{a^2}\Big)^{3/2}\frac{S^{3/2}K_{2,\inf}}{6K_{2\sup}^3}=\Big(\frac{a^2-(\omega^*)^2}{a^2}\Big)^{3/2}\frac{S^{3/2}}{6k_2K_{2,\inf}^2},$$

which contradicts the hypothesis. Therefore, $u_\varepsilon\neq 0$ and $(u_\varepsilon,\phi_\varepsilon)$ is a solution of system (2.1). □

Similarly, for system (2.7), we have a Mountain-Pass reduction $(h_0(\cdot),I_0(\cdot),\mathcal{N}_0)$ and the similar results as before. Denote $\mathcal{L}_0$ be the set of all least energy solutions. Now, we need to analyze the energy levels using the reduced functional to obtain the asymptotic behaviour of the least energy solutions.

Lemma 3.7. $\lim_{\varepsilon\to 0}c_\varepsilon=c_0$.

Proof. First we prove that $\liminf_{\varepsilon\to 0^+}c_\varepsilon\ge c_0$. Arguing indirectly, assume that $\liminf_{\varepsilon\to 0^+}c_\varepsilon<c_0$. By definition and Lemma 3.5 we can choose $e_j\in\mathcal{N}_{\varepsilon_j}$ and $\delta>0$ such that

$$\max_{u\in E_{e_j}}\Phi_{\varepsilon_j}(u)\le c_0-\delta$$

as $j\to\infty$. By [9, Lemma 3.2], for all $u\in E$, we have

$$\Phi_{\varepsilon_j}(u)-\Phi_0(u)=\Gamma_0(u)-\Gamma_{\varepsilon_j}(u)\to 0.$$

Note that

$$c_0\le I_0\left(e_j\right)\le\max_{u\in E_{e_j}}\Phi_0(u).$$

Therefore for all j sufficiently large such that $|\Gamma_0(u)-\Gamma_{\varepsilon_j}(u)|<\frac{\delta}{2}$, we have

$$c_0-\delta\ge\max_{u\in E_{e_j}}\Phi_{\varepsilon_j}(u)\ge\max_{u\in E_{e_j}}\Phi_0(u)+\Gamma_0(u)-\Gamma_{\varepsilon_j}(u)\ge c_0+\Gamma_0(u)-\Gamma_{\varepsilon_j}(u)>c_0-\frac{\delta}{2},$$

which is a contradiction.

Next, we turn to show that $\limsup_{\varepsilon\to 0^+}c_\varepsilon\le c_0$. Let $u=u^++u^-\in\mathcal{L}_0$, and set $e=u^+$. It is evident that $e\in\mathcal{N}_0$, $h_0(e)=u^-$, and $I_0(e)=c_0$. There exist a unique $t_\varepsilon>0$ such that $t_\varepsilon e\in\mathcal{N}_\varepsilon$, and we have

$$c_\varepsilon\le I_\varepsilon(t_\varepsilon e).\tag{3.7}$$

By Lemma 3.5, t_ε is bounded. Hence, without loss of generality we can assume $t_\varepsilon\to t_0$ as $\varepsilon\to 0^+$. Setting $u_\varepsilon=t_\varepsilon e+h_0\left(t_\varepsilon e\right)$, $w_\varepsilon=t_\varepsilon e+h_\varepsilon\left(t_\varepsilon e\right)$ and $v_\varepsilon=u_\varepsilon-w_\varepsilon$, we deduce that

$$\begin{aligned}\frac{1}{2}\left\|v_\varepsilon\right\|^2+(I)&=\Phi_\varepsilon\left(w_\varepsilon\right)-\Phi_\varepsilon\left(u_\varepsilon\right)\\&=\Phi_0\left(w_\varepsilon\right)-\Phi_0\left(u_\varepsilon\right)-\Gamma_\varepsilon\left(w_\varepsilon\right)+\Gamma_\varepsilon\left(u_\varepsilon\right)+\Gamma_0\left(w_\varepsilon\right)-\Gamma_0\left(u_\varepsilon\right),\end{aligned}$$

where

$$(I) := \int_0^1 (1-s)\left(\Gamma''_\varepsilon(w_\varepsilon + sv_\varepsilon)[v_\varepsilon, v_\varepsilon] + \Gamma''_0(w_\varepsilon + sv_\varepsilon)[v_\varepsilon, v_\varepsilon]\right) ds.$$

Considering

$$\Gamma_\varepsilon(u_\varepsilon) - \Gamma_\varepsilon(w_\varepsilon) = \Gamma'_\varepsilon(w_\varepsilon) v_\varepsilon + \int_0^1 (1-s)\Gamma''_\varepsilon(w_\varepsilon + sv_\varepsilon)[v_\varepsilon, v_\varepsilon]\, ds,$$
$$\Gamma_0(w_\varepsilon) - \Gamma_0(u_\varepsilon) = -\Gamma'_0(u_\varepsilon) v_\varepsilon + \int_0^1 (1-s)\Gamma''_0(u_\varepsilon - sv_\varepsilon)[v_\varepsilon, v_\varepsilon]\, ds,$$

we have

$$\frac{1}{2}\|v_\varepsilon\|^2 + (I) + (II) \le \Gamma'_\varepsilon(w_\varepsilon) v_\varepsilon + \int_0^1 (1-s)F''(w_\varepsilon + sv_\varepsilon)[v_\varepsilon, v_\varepsilon]\, ds - \Gamma'_0(u_\varepsilon) v_\varepsilon,$$

where

$$(II) := \int_0^1 (1-s)\Gamma''_0(u_\varepsilon - sv_\varepsilon)[v_\varepsilon, v_\varepsilon]\, ds.$$

So we deduce that

$$\frac{1}{2}\|v_\varepsilon\|^2 + \int_0^1 (1-s)\Gamma''_0(w_\varepsilon + sv_\varepsilon)[v_\varepsilon, v_\varepsilon]\, ds \le |\Gamma'_\varepsilon(w_\varepsilon) v_\varepsilon| + |\Gamma'_0(u_s) v_\varepsilon|. \tag{3.8}$$

Since $t_\varepsilon \to t_0$, it is clear that $\{u_\varepsilon\}$, $\{w_\varepsilon\}$ and $\{v_\varepsilon\}$ are bounded in E. Moreover, we have

$$\Gamma_\varepsilon(z_\varepsilon) = o(1), \quad \|\Gamma_\varepsilon(z_\varepsilon)\| = o(1)$$

as $\varepsilon \to 0^+$ for $z_\varepsilon = u_\varepsilon, w_\varepsilon, v_\varepsilon$. By (2.8), noting that

$$|\Gamma'_0(u_\varepsilon) v_\varepsilon| \to 0.$$

Thus from (3.8), it follows that $\|v_\varepsilon\|^2 \to 0$, that is, $h_\varepsilon(t_\varepsilon e) \to h_0(t_0 e)$. This, jointly with the definitions,implies

$$\Phi_\varepsilon(w_\varepsilon) = \Phi_0(w_\varepsilon) + o(1) = \Phi_0(u_\varepsilon) + o(1),$$

that is

$$I_\varepsilon(t_\varepsilon e) = I_0(t_0 e) + o(1)$$

as $\varepsilon \to 0^+$. Then, since

$$I_0(t_0 e) \le \max_{v \in E_e} \Phi_0(v) = I_0(e) = c_0,$$

we obtain by (3.7) that

$$\limsup_{\varepsilon \to 0^+} c_\varepsilon \le \lim_{\varepsilon \to 0^+} I_\varepsilon(t_\varepsilon e) \le c_0.$$

Above all, we have

$$c_0 \le \liminf_{\varepsilon \to 0^+} c_\varepsilon \le \limsup_{\varepsilon \to 0^+} c_\varepsilon \le c_0.$$

□

4. Proof of main results

Given that the proof for the limit system (2.7) parallels that of the more complex system (2.1), we choose to address the existence of the least energy solutions for system (2.1) first. This approach establishes a foundation for the subsequent proof of the limit system (2.7), as outlined in Theorem 1.1. We will thus begin by proving Theorem 1.2, which pertains to system (2.1).

Proof of Theorem 1.2. Part 1. Existence of least energy solutions for system (2.1). For any $\varepsilon > 0$, by Lemma 2.2, assumption (A4) is satisfied. And by Lemma 3.1 and Lemma 3.2, Φ_ε satisfies all the assumptions of Lemma 2.4. Hence Φ_ε has a $(C)_{c_\varepsilon}$-sequence $\{u_n\}$ with $\rho \le c_\varepsilon \le \sup \Phi_\varepsilon(E_e \cap B_R)$. By Lemma 3.3, $\{u_n\}$ is bounded in E. Therefore, up to a subsequence, there is a point u_ε such that $u_n \rightharpoonup u_\varepsilon$ in E. Since we have assumed

$$k_2 := \frac{K_{2\sup}}{K_{2,\inf}} < \Big(\frac{S(a^2-(\omega^*)^2)}{a^2}\Big)^{1/2}\Big(\frac{(c_0qK_{1,\inf})^{\frac{2}{q-2}}}{6\gamma_q K_{2,\inf}^2}\Big)^{1/3},$$

it follows that

$$\gamma_q < \Big(\frac{a^2-(\omega^*)^2}{a^2}\Big)^{3/2}\frac{S^{3/2}(c_0qK_{1,\inf})^{\frac{2}{q-2}}}{6k_2^3K_{2,\inf}^2}.$$

Consider the least energy $\gamma_{c_0qK_{1,\inf},q}$ of the following equation

$$i\alpha\cdot\nabla u - a\beta u - \omega u = c_0qK_{1,\inf}|u|^{q-2}u.$$

It is easy to see that

$$\gamma_{c_0qK_{1,\inf},q} = (c_0qK_{1,\inf})^{\frac{-2}{q-2}}\gamma_q,$$

and $\gamma_{c_0qK_{1,\inf},q}$ satisfies

$$\gamma_{c_0qK_{1,\inf},q} < \Big(\frac{a^2-(\omega^*)^2}{a^2}\Big)^{3/2}\frac{S^{3/2}}{6k_2^3K_{2,\inf}^2}.$$

Observe that

$$c_\varepsilon \le \gamma_{c_0qK_{1,\inf},q},$$

Lemma 3.6 shows that c_ε is attained by some point, denoted as u_ε. By Lemma 3.4 we see that solution $(u_\varepsilon,\phi_\varepsilon)$ is in $\cap_{s\ge2,r\ge2}W^{1,s}_{\rm loc}\times W^{1,r}_{\rm loc}$.

Part 2. Decay estimate of solutions for system (2.1). By Lemma 3.4 it can be observed that $u\in L^\infty(\mathbb{R}^3,\mathbb{C}^4)$. Expressing (2.1) as

$$Du = a\beta u + \omega u + \phi_\varepsilon u + K_1(x)g(|u|)u + K_2(x)|u|u.$$

Operating the operator D on both sides and using the property that $D^2=-\Delta$, we derive a relation

$$\begin{aligned}\Delta u &= a^2u - (\omega+\phi_\varepsilon+K_1(x)g(|u|)+K_2(x)|u|)^2u\\ &\quad - D(\phi_\varepsilon+K_1(x)g(|u|)+K_2(x)|u|)u.\end{aligned}$$

Let

$$\operatorname{sgn} u = \begin{cases}\bar u/|u|, & \text{if } u\ne0,\\ 0, & \text{if } u=0,\end{cases}$$

and referring to Kato's inequality [13], it can be found that

$$\begin{aligned}\Delta|u| &\geqslant \Re(\triangle u\cdot\operatorname{sgn} u)\\ &= \Re((a^2u-(\omega+\phi_\varepsilon+K_1(x)g(|u|)+K_2(x)|u|)^2u\end{aligned}$$

$$- D(\phi_\varepsilon + K_1(x)g(|u|) + K_2(x)|u|)u) \cdot \operatorname{sgn} u).$$

Further, observing that

$$\Re[D(K_1(x)g(|u|) + K_2(x)|u|)u(\operatorname{sgn} u)] = 0,$$

we obtain

$$\Delta|u| \geq a^2|u| - (\omega + \phi_\varepsilon(x) + K_1(x)g(|u|) + K_2(x)|u|)^2|u| - |D\phi_\varepsilon| \cdot |u|. \tag{4.1}$$

Recall that in Lemma 3.4, we have $|\phi_\varepsilon|_\infty \leq C$ and $|\nabla\phi_\varepsilon| \leq C$, where C is independent of ε. Subsequently, it follows from (4.1) that there exists a constant $M > 0$ (independent of ε) such that

$$\Delta|u| \geq -M|u|.$$

Then applying the maximum principle (see [27]), we can conclude that

$$|u_\varepsilon(x)| \leq C\exp(-c|x|)$$

for all $x \in \mathbb{R}^3$, and C, c is independent of ε.

Part 3. Asymptotic behaviour of solutions for system (2.1). Suppose $(u_\varepsilon, \phi_\varepsilon)$ is a pair of least energy solution for system (2.1), then u_ε satisfies $\Phi_\varepsilon(u_\varepsilon) = c_\varepsilon$, $(1 + \|u_\varepsilon\|)(\Phi_\varepsilon)'(u_\varepsilon) \to 0$ and $\rho \leq c_\varepsilon \leq \sup\Phi_\varepsilon(E_e \cap B_R)$. With the independence of ε in Lemma 3.3, $\{\|u_\varepsilon\|\}$ is bounded uniformly in ε, and there exists a point $u_0 \in E$ such that $u_\varepsilon \rightharpoonup u_0$ in E as $\varepsilon \to 0^+$. As the proof above, we see that $u_\varepsilon \to u_0$ in E, then we also have $u_\varepsilon \to u_0$ in L^q for all $q \in [2, 3]$.

Recall that

$$\begin{aligned} |A_\omega(u_\varepsilon - u_0)|_2 &\leq |\phi_\varepsilon u_\varepsilon - \phi_0 u_0|_2 + |K_1(x)(g(|u_\varepsilon|)u_\varepsilon \\ &\quad - g(|u_0|)u_0)|_2 + |K_2(x)(|u_\varepsilon|u_\varepsilon - |u_0|u_0)|_2. \end{aligned}$$

Since

$$|\phi_\varepsilon u_\varepsilon - \phi_0 u_0|_2 \leq |u_\varepsilon|_3|\phi_\varepsilon - \phi_0|_6 + |u_\varepsilon - u_0|_3|\phi_\varepsilon|_6,$$

and

$$\begin{aligned} &|K_1(x)(g(|u_\varepsilon|)u_\varepsilon - g(|u_0|)u_0)|_2 \\ &\leq K_{1\sup}|(g(u_\varepsilon) - g(u_0))u_\varepsilon|_2 + \delta K_{1\sup}|u_\varepsilon - u_o|_2 + C_\delta K_{1\sup}|u_0|_\infty^{2(p-2)}|u_\varepsilon - u_0|_2, \end{aligned}$$

we deduce $|A_\omega(u_\varepsilon - u_0)|_2 \to 0$ as $\varepsilon \to 0^+$, that is, $u_\varepsilon \to u$ in H^1. From [9, Lemma 3.2] we have that $\phi_\varepsilon(u_\varepsilon) \to \phi_0(u_0)$ in $D^{1,2}(\mathbb{R}^3)$, $\varepsilon\phi_\varepsilon(u_\varepsilon) \to 0$ in $D^{1,4}(\mathbb{R}^3)$. Supposing $v \in C_0^\infty(\mathbb{R}^3)$, $\operatorname{supp}(v) \subset K$, and K is compact. Recall that

$$\begin{aligned} &(u_\varepsilon^+ - u_\varepsilon^-, v) \\ &= \Re\int_{\mathbb{R}^3} \phi_\varepsilon(u_\varepsilon)u_\varepsilon\bar{v}\,dx + \Re\int_{\mathbb{R}^3} K_1(x)g(|u_\varepsilon|)u_\varepsilon\bar{v}\,dx + \Re\int_{\mathbb{R}^3} K_2(x)|u_\varepsilon|u_\varepsilon\bar{v}\,dx. \end{aligned}$$

We will pass to the limit as $\varepsilon \to 0^+$ in the above identity. Let us examine each term individually.

Of course

$$(u_\varepsilon^+ - u_\varepsilon^-, v) \to (u_0^+ - u_0^-, v).$$

Since $\phi_\varepsilon(u_\varepsilon) \to \phi_0(u_0)$ in $L^6(\mathbb{R}^3)$, $u_\varepsilon \to u_0$ in $L^{12/5}(K)$ and $v \in L^{12/5}(K)$. It is easy to see that

$$\Re\int_{\mathbb{R}^3} \phi_\varepsilon(u_\varepsilon)u_\varepsilon\bar{v}\,dx \to \Re\int_{\mathbb{R}^3} \phi_0(u_0)u_0\bar{v}\,dx.$$

We also have

$$\Re\int_{\mathbb{R}^3} K_1(x)g(|u_\varepsilon|)u_\varepsilon\bar v\,dx \to \Re\int_{\mathbb{R}^3} K_1(x)g(|u_0|)u_0\bar v\,dx,$$
$$\Re\int_{\mathbb{R}^3} K_2(x)|u_\varepsilon|u_\varepsilon\bar v\,dx \to \Re\int_{\mathbb{R}^3} K_2(x)|u_0|u_0\bar v\,dx.$$

As a result, we deduce that

$$\begin{aligned}&(u_0^+ - u_0^-, v) - \Re\int_{\mathbb{R}^3}\phi_0(u_0)u_0\bar v\,dx - \Re\int_{\mathbb{R}^3} K_1(x)g(|u_0|)u_0\bar v\,dx\\ &- \Re\int_{\mathbb{R}^3} K_2(x)|u_0|u_0\bar v\,dx = 0.\end{aligned}$$

This means that (u_0, ϕ_0) solves system (2.7) with energy

$$\begin{aligned}\Phi_0(u_0) &= \Phi_0(u_0) - \frac{1}{2}(\Phi_0)'(u_0)u_0\\ &= \frac{1}{4}\int_{\mathbb{R}^3}|\nabla\phi_0|^2dx + \int_{\mathbb{R}^3} K_1(x)\hat G(|u_0|)\,dx + \frac{1}{6}\int_{\mathbb{R}^3} K_2(x)|u_0|^3dx.\end{aligned}$$

By applying Fatou's lemma, we deduce that

$$\begin{aligned}c_0 &\le \frac{1}{4}\int_{\mathbb{R}^3}|\nabla\phi_0|^2dx + \int_{\mathbb{R}^3} K_1(x)\hat G(|u_0|)\,dx + \frac{1}{6}\int_{\mathbb{R}^3} K_2(x)|u_0|^3dx\\ &\le \liminf_{\varepsilon\to 0^+}\Big(\frac{1}{4}\int_{\mathbb{R}^3}|\nabla\phi_\varepsilon|^2dx + \frac{\varepsilon^4}{8}\int_{\mathbb{R}^3}|\nabla\phi_\varepsilon|^4dx + \int_{\mathbb{R}^3} K_1(x)\hat G\Big(|u_\varepsilon|\Big)\,dx\\ &\quad + \frac{1}{6}\int_{\mathbb{R}^3} K_2(x)\,|u_\varepsilon|^3\,dx\Big)\\ &= \liminf_{\varepsilon\to 0^+}\Phi_\varepsilon(u_\varepsilon)\\ &\le \limsup_{\varepsilon\to 0^+}\Phi_\varepsilon(u_\varepsilon) \le c_0.\end{aligned}$$

Consequently, Lemma 3.7 shows that

$$\lim_{\varepsilon\to 0^+}\Phi_\varepsilon(u_\varepsilon) = \lim_{\varepsilon\to 0^+} c_\varepsilon = \Phi_0(u_0) = c_0.$$

Now we can conclude that $(u_\varepsilon, \phi_\varepsilon)$ converges in $H^1\times D^{1,2}(\mathbb{R}^3)$ to (u_0, ϕ_0) which is a least energy solution of the system (2.7). □

Proof of Theorem 1.1. Part 1. Existence of least energy solutions for system (2.7). From the explicit expression of Γ_0, in comparison with the non-local term Γ_ε of the original problem (2.1), the non-local term Γ_0 in the limit problem (2.7) possesses the same or better properties, as demonstrated in Lemma 2.4. Therefore, the series of lemmas proven for the functional Φ_ε also hold the same conclusions for Φ_0. In line with the proof of Theorem 1.2, we similarly have the functional Φ_0 satisfies all the assumptions of Lemma 2.4. By assumptions (K_1), (K_2) and Lemma 3.6, it follows that functional Φ_0 has at least one nontrivial solution that achieves the least energy.

Part 2. Decay of solutions for system (2.7). Expressing (2.7) as

$$Du = a\beta u + \omega u + \phi_0 u + K_1(x)g(|u|)u + K_2(x)|u|u.$$

Operating the operator D on both sides, we obtain the relation

$$\Delta u = a^2u - (\omega + \phi_0 + K_1(x)g(|u|) + K_2(x)|u|)^2u$$

$$- D\phi_0 u - D(K_1(x)g(|u|) + K_2(x)|u|)u.$$

Let

$$\operatorname{sgn} u = \begin{cases} \bar{u}/|u|, & \text{if } u \neq 0, \\ 0, & \text{if } u = 0. \end{cases}$$

By Kato's inequality [13], it can be found that

$$\begin{aligned} \Delta|u| &\geq \Re(\triangle u \cdot \operatorname{sgn} u) \\ &= \Re((a^2 u - (\omega + \phi_0 + K_1(x)g(|u|) + K_2(x)|u|)^2 u \\ &\quad - D(\phi_0 + K_1(x)g(|u|) + K_2(x)|u|)u) \cdot \operatorname{sgn} u). \end{aligned}$$

Further, observing

$$\Re[D(K_1(x)g(|u|) + K_2(x)|u|)u(\operatorname{sgn} u)] = 0,$$

we obtain

$$\Delta|u| \geq a^2|u| - (\omega + \phi_0(x) + K_1(x)g(|u|) + K_2(x)|u|)^2|u| - |D\phi_0| \cdot |u|. \tag{4.2}$$

Then, following the analogous arguments in [20], we can readily deduce from (4.2) that there exists a constant $M > 0$ such that

$$\Delta|u| \geq -M|u|.$$

Let Γ be a fundamental solution to $-\Delta + \tau$. We may choose $\Gamma(x)$ such that $|u_\varepsilon(x)| \leq \tau\Gamma(x)$ on $B_R(0)$. Denote $z = |u_\varepsilon| - \tau\Gamma$, then

$$\Delta z = \Delta|u_\varepsilon| - \tau\Delta\Gamma \geq \tau(|u_\varepsilon| - \tau\Gamma) = \tau z$$

for $|x| \geq R$. The application of the maximum principle [27] brings us to the conclusion that

$$|u_\varepsilon(x)| \leq C\exp(-c|x|),$$

where C, c is independent of ε. □

Finally, we proceed with the proof of the existence of multiple solutions.

Proof of Theorem 1.3. Obviously, Φ_ε is even in u, and in virtue of Lemma 2.2 , 3.1 and 3.2, the assumptions (A4), (A5) and (A6) are satisfied. It remains to verify (A8) and the $(C)_c$-conditon.

For any $N \in \mathbb{N}^+$, let $\{e_n\} \subset E^+$ be a standard orthogonal basis, $E_N := \operatorname{span}\{e_1, e_2, \dots, e_N\}$. Since E_N is a finite dimensional subspace, there exits $c_N > 0$ such that for every $u \in E_N$, $|u|_q \geq c_N\|u\|$. Then for any $u = u^+ + u^- \in E_N \oplus E^-$,

$$\begin{aligned} \Phi_\varepsilon(u) &= \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \Gamma_\varepsilon(u) - \int_{\mathbb{R}^3} F(x,|u|)\,dx \\ &\leq \|u^+\|^2 - \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^3} K_1(x)G(|u|)\,dx \\ &\leq \frac{1}{c_N^2}|u^+|_q^2 - c_0 K_{1,\inf}|u^+|_q^q - \frac{1}{2}\|u\|^2 \\ &\leq \frac{q-2}{q} c_N^{-\frac{2q}{q-2}} \left(\frac{2}{qc_0K_{1,\inf}}\right)^{\frac{2}{q-2}} - \frac{1}{2}\|u\|^2. \end{aligned}$$

Clearly, there exists

$$R_N = (\frac{2q-4}{q})^{1/2} c_N^{-\frac{q}{q-2}} \Big(\frac{2}{qc_0K_{1,\inf}}\Big)^{\frac{1}{q-2}},$$

such that

$$\sup_{u\in E_N\oplus E^-,\|u\|>R_N}\Phi_\varepsilon(u)<0 \quad\text{and}\quad \sup_{u\in E_N\oplus E^-}\Phi_\varepsilon(u)\le 2R_N^2.$$

We denote

$$m(c_0,q,N,K_{1,\inf},K_{2,\inf}):=\frac{q}{6(q-2)K_{2,\inf}}\Big(\frac{K_{1,\inf}c_N^q c_0 q}{2}\Big)^{\frac{2}{q-2}}\Big(\frac{S(a^2-(\omega^*)^2)}{a^2}\Big)^{3/2}$$

and $T(s):=\Big(\frac{S(a^2-(\omega^*)^2)}{a^2}\Big)^{3/2}\frac{1}{6K_{2,\inf}^2 s}$ for $s>1$. Note that, $T(1)>c_\infty$ and $T(s)\to 0$ as $s\to\infty$. There exists $k_\infty>1$ such that $T(k_\infty)=c_\infty$. Hence for $k_\infty\le k_2<m(c_0,q,N,K_{1,\inf},K_{2,\inf})$, by Lemma 3.6 and Lemma 2.5, Φ_ε has at least N distinct critical values. Finally, repeat the proof of Theorem 1.1, we obtain the decay estimate. □

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