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LOCAL SOLUTIONS FOR A BRINKMAN EQUATION COUPLED WITH HEAT-CONVECTIVE AND CONCENTRATION-DIFFUSIVE EQUATIONS AND A VOLUMETRIC MASS SOURCE

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ABSTRACT. In this article, we consider a model coupled with the Brinkman heat-convective and concentration-diffusive equations for a mixed gas flow in a porous media. The specificity of this model lies in the presence of a volumetric mass source depending on temperature and concentration in mass balance equation. We will prove the existence and uniqueness of the smooth local solutions for the 3-D Cauchy problem. As a byproduct, we show the convergence of the approximate solutions based on an iteration scheme.

1. Introduction

The fundamental model we study is based upon the Brinkman equation for the mixed gas flow and the concentration-diffusive equation for the gas concentrations in porous media. These make it possible to model the performance of the adsorption column in a portable oxygen concentrator (see [19]), and the heat-convective equation for the temperature. Thus, let $\rho(x,t), \mathbf{u}(x,t)$, $\theta(x,t)$ and $w_i(x,t)$ be the density, velocity, temperature of gas mixture and the concentration of i-gas component, respectively, where x is the position, t denotes time and $i = 1, 2, \dots, n$. Then, the balance equations of mass, momentum and concentrations are

$$\begin{gathered} \rho_t + \operatorname{div}(\rho\mathbf{u}) = Q_0(\mathbf{w},\theta), \\ (\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla P + \alpha^{-1}\mathbf{u} = 2\mu\operatorname{div}\mathbf{D}(\mathbf{u}) + \nu\nabla\operatorname{div}\mathbf{u}, \\ \mathbf{w}_t - d_f\Delta\mathbf{w} + (\mathbf{u},\nabla)\mathbf{w} + \mathbf{w}\operatorname{div}\mathbf{u} = \mathbf{S}(\mathbf{w},\theta), \end{gathered} \tag{1.1}$$

where $\mathbf{w} = (w_1, \dots, w_n)$. The balance equation of energy is

$$(\rho E)_t + \operatorname{div}(\rho\mathbf{u}E + P\mathbf{u}) = \kappa\Delta\theta + \operatorname{div}\Big(2\mu\mathbf{D}(\mathbf{u})\mathbf{u} + \nu\operatorname{div}\mathbf{u}\mathbf{u}\Big) + Q_1(\mathbf{w},\theta), \tag{1.2}$$

where $Q_0(\mathbf{w},\theta)$ is the mass source, $\alpha > 0$ denotes the permeability of the porous medium, $\mu > 0$ and $\nu \geq 0$ are the shear and bulk viscosities, respectively, $d_f > 0$ is the dispersion coefficient, $\mathbf{S}(\mathbf{w},\theta) = (S_1, \dots, S_n)$, S_i is the concentration deposit

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rate of i-gas component, $E = e + \frac{1}{2}\mathbf{u}^2$ is the specific total energy, e is the specific internal energy, $\kappa > 0$ is the heat-conducting coefficient, $Q_1(\mathbf{w}, \theta)$ is the heat source, and $\mathbf{D}(\mathbf{u})$ is the deformation tensor given by

$$\mathbf{D}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right),$$

where $(\nabla\mathbf{u})^T$ denotes the transpose of matrix $\nabla\mathbf{u}$. The mass source $Q_0(\mathbf{w}, \theta)$, the heat source $Q_1(\mathbf{w}, \theta)$ and the concentration deposit rate, $\mathbf{S}(\mathbf{w}, \theta)$ are the given smooth functions for the gas concentration $\mathbf{w}$ and the temperature θ. Also, for the pressure P and the internal energy e, we assume that

$$P = P(\rho, \theta; \mathbf{w}), \quad P_\rho(\rho, \theta; \mathbf{w}) > 0, \quad P_{w_i}(\rho, \theta; \mathbf{w}) > 0, \tag{1.3}$$

$$e = e(\rho, \theta), \quad e_\theta(\rho, \theta) > 0. \tag{1.4}$$

The specificity of the system (1.1) lies in the presence of the mass source Q_0. If we set $Q_0 = 0$ and $\rho = 1$, then the system $(1.1)_1$ and $(1.1)_2$ reduces to the Brinkman equation

$$\begin{gathered} \operatorname{div}\mathbf{u} = 0, \\ \mathbf{u}_t - \mu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla P + \alpha^{-1}\mathbf{u} = 0, \end{gathered} \tag{1.5}$$

which was first proposed by Brinkman [2] in 1947. The system (1.5) describing the viscous flow of incompressible fluids in porous media was extensively investigated in the last several decades (see, e.g., [20, 3, 22, 24, 18, 27, 13, 11, 14, 15, 17, 25, 26] and references therein). However, the studies in above works depend essentially on the fact that the velocity is solenoidal, that is, $\operatorname{div}\mathbf{u} = 0$.

Recently, the diffusive interface model for tumor growth have been developed and analyzed, which reduces to the system

$$\begin{gathered} \operatorname{div}\mathbf{u} = S_1(\varphi, \chi), \\ -\mu\Delta\mathbf{u} - \nu\nabla\operatorname{div}\mathbf{u} + \nabla P + \alpha^{-1}\mathbf{u} = (J + \varphi)\nabla\chi, \\ \chi_t + \operatorname{div}(\chi\mathbf{u}) = m\Delta J + S_2(\varphi, \chi), \quad J = -\Delta\chi + \Psi'(\chi) - \chi\varphi, \\ \varphi_t - d_f\Delta\varphi + \operatorname{div}(\varphi\mathbf{u}) = S_3(\varphi, \chi), \end{gathered} \tag{1.6}$$

which consists of the Brinkman equation with mass source (for fluid flow), Cahn-Hilliard (for the tumor density) and reaction-diffusion (for the nutrient or other chemical factors) equations. The model (1.6) is a description of the evolution of a two-phase cell mixture, containing tumour cells and healthy host cells, surrounded by a chemical species acting as nutrients only for the tumour cells, and is transported by a fluid velocity field. The variable χ denotes the difference in the volume fractions of the cells, with the region $\{\chi = 1\}$ representing the tumour cells and $\{\chi = -1\}$ representing the host cells, while φ denotes the concentration of the nutrient. The fluid velocity u is taken as the volume averaged velocity, with pressure p, and J denotes the chemical potential associated to χ. Also, the function Ψ is potential energy density and $S_i (i = 1, 2, 3)$ are generic source terms that can be specified depending on the application (see [8, 12] for more details in the model derivation).

An interesting feature of the system (1.6) is that the velocity is not solenoidal, that is, $\operatorname{div} u \neq 0$. Recently, considerable progress has been obtained for the theoretical and numerical studies of the system (1.6). The existence of weak solutions was established in [5] and the analysis of weak and stationary solutions of this system with singular potentials was considered in [7]. Numerical investigations can be found in [9]. A simplified version of this model was investigated in [6], where the time derivative and the convection term in the nutrient equation are neglected. For this model, the authors proved strong well-posedness and showed that the solutions converge to the corresponding Cahn-Hilliard-Darcy model, that is, the system (1.6) with $\mu = \nu = 0$.

On the other hand, there are a few results concerning the Brinkman equation with non-constant density. If we set $Q_0 = 0$, $\mu = \alpha = 1$ and $\nu = 0$, and assume that the terms u_t and $(u \cdot \nabla)u$ vanish, then the system $(1.1)_1$ and $(1.1)_2$ reduces to the system

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \quad \Delta \mathbf{u} - \mathbf{u} = \nabla p,$$

where $p = p(\rho)$, which is equivalent to

$$\rho_t = \operatorname{div}\left(\rho(1-\Delta)^{-1}\nabla p(\rho)\right). \tag{1.7}$$

For Cauchy problem of the system (1.7), the existence of the strong and smooth solutions was proved in [1] for 1-D case and in [16] for multi-D case, respectively. In [23], the existence of weak solutions was established for Cauchy problem of the following 1-D regularized Brinkman equation

$$\rho_t = \partial_{xx}\rho + \partial_x\left(\rho(1-\partial_{xx})^{-1}\partial_x(\rho^2)\right).$$

Although considerable progress has been obtained for the mathematical studies concerning the Brinkman equation in different settings, most of these results are obtained for the case where the fluid density is constant. Moreover, there is little result for the full system (1.1) except for [19], where they employed the model to simulate the microporous adsorption of nitrogen in a portable oxygen concentrator by using COMSOL Multiphysics. Thus, we will prove the existence and uniqueness of the smooth solutions to the 3-D Cauchy problem for the general system (1.1), (1.2).

More precisely, we consider the system (1.1), (1.2) in $\mathbb{R}^3$ with the initial conditions

$$(\rho, \mathbf{u}, \mathbf{w}, \theta)(x, 0) = (\rho_0, \mathbf{u}_0, \mathbf{w}_0, \theta_0)(x) \to (\bar\rho, \mathbf{0}, \bar{\mathbf{w}}, \bar\theta) \quad \text{as } |x| \to \infty, \tag{1.8}$$

where $\bar\rho, \bar{\mathbf{w}} = (\bar w_1, \dots, \bar w_n), \bar\theta$ are given positive constants.

Notation. $L^p(\mathbb{R}^3)$ and $W_p^k(\mathbb{R}^3)$ denote the usual Lebesgue and Sobolev spaces on $\mathbb{R}^3$, with norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W_p^k}$, respectively. When $p = 2$, we denote $W_p^k(\mathbb{R}^3)$ by $H^k(\mathbb{R}^3)$ with the norm $\|\cdot\|_{H^k}$ and $\|\cdot\|_{H^0} = \|\cdot\|$ will be used to denote the usual L^2-norm. The notation $\|(A_1, A_2, \dots, A_l)\|_{H^k}$ means the summation of $\|A_i\|_{H^k}$ from $i = 1$ to $i = l$. For an integer m, the symbol ∇^m denotes the summation of all terms D^α with the multi-index α satisfying $|\alpha| = m$. We omit the spatial domain $\mathbb{R}^3$ in integrals for convenience.

The main result of this paper can be stated as follows.

Theorem 1.1. *Assume* (1.3)*,* (1.4) *and*

$$Q_0(\bar{\mathbf{w}}, \bar\theta) = 0, \quad \mathbf{S}(\bar{\mathbf{w}}, \bar\theta) = \mathbf{0}, \quad Q_1(\bar{\mathbf{w}}, \bar\theta) = 0. \tag{1.9}$$

Suppose that the initial data $\rho_0, \mathbf{u}_0, \varphi_{i0}$ *satisfy*

$$\begin{gathered}(\rho_0-\bar{\rho}, \mathbf{u}_0, \mathbf{w}_0-\bar{\mathbf{w}}, \theta_0-\bar{\theta}) \in H^N(\mathbb{R}^3), \\ \inf_{x\in\mathbb{R}^3} \rho_0(x)>0, \quad \inf_{x\in\mathbb{R}^3} \mathbf{w}_0(x)>0, \quad \inf_{x\in\mathbb{R}^3} \theta_0(x)>0\end{gathered} \tag{1.10}$$

for an integer $N \geq 3$. *Moreover, there is a constant* M_0, *such that* $\rho_0, \mathbf{u}_0, \mathbf{w}_0, \theta_0$ *satisfy*

$$\|(\rho_0-\bar{\rho}, \mathbf{u}_0, \mathbf{w}_0-\bar{\mathbf{w}}, \theta_0-\bar{\theta})\|_{H^N} \leq M_0. \tag{1.11}$$

Then, there exist constants T_0 *and* C *depending on* M_0 *such that the unique smooth solution* $(\rho, \mathbf{u}, \mathbf{w}, \theta)$ *of the Cauchy problem* (1.1), (1.2) , (1.8) *exists on the time interval* $[0, T_0]$ *with the properties:*

$$\begin{gathered}(\rho-\bar{\rho}, \mathbf{u}, \mathbf{w}-\bar{\mathbf{w}}, \theta-\bar{\theta}) \in C([0,T_0]; H^N(\mathbb{R}^3)), \quad \nabla\rho \in L^2(0,T_0; H^N(\mathbb{R}^3)), \\ \mathbf{u} \in L^2(0,T_0; H^{N+1}(\mathbb{R}^3)), \quad \big(\nabla\mathbf{w}, \nabla\theta\big) \in L^2(0,T_0; H^N(\mathbb{R}^3))\end{gathered} \tag{1.12}$$

and

$$\begin{aligned}&\|(\rho-\bar{\rho}, \mathbf{u}, \mathbf{w}-\bar{\mathbf{w}}, \theta-\bar{\theta})(t)\|_{H^N}^2 + \int_0^t \|\nabla\rho\|_{H^{N-1}}^2 d\tau \\ &+ \int_0^t \|\mathbf{u}\|_{H^{N+1}}^2 d\tau + \int_0^t \|\big(\nabla\mathbf{w}, \nabla\theta\big)\|_{H^N}^2 d\tau \leq C\end{aligned} \tag{1.13}$$

for all $t \in [0, T_0]$.

Remark 1.2. Assumption (1.9) come from system (1.1) satisfying

$$\begin{gathered}Q_0(\mathbf{w}) = -\sum_{i=1}^n \alpha_i M_i (w_i - \bar{w}_i), \\ S_i = \alpha_i (w_i - \bar{w}_i), \ i=1,2,\dots,n, \quad \mathbf{S}(\mathbf{w}) = (S_1, S_2, \dots, S_n),\end{gathered} \tag{1.14}$$

where $\alpha_i > 0$ is the macroscopic mass transfer rate of i-gas component into zeolite particles, and $M_i > 0$ and $\bar{w}_i > 0$ are the molecular weight and the mean concentration of i-gas component, respectively (see [19, (4) and (12)]).

Next, we show the convergence of the approximate solutions based on the iteration scheme. To this end, we first reformulate the Cauchy problem (1.1), (1.2) , (1.8). By using $(1.1)_1$, $E = e(\rho,\theta) + \frac{\mathbf{u}^2}{2}$ and

$$\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) = \operatorname{div}(\rho\mathbf{u})\mathbf{u} + \rho(\mathbf{u},\nabla)\mathbf{u},$$

we rewrite $(1.1)_2$ and (1.2) as

$$\begin{aligned}&\rho\mathbf{u}_t + \rho(\mathbf{u},\nabla)\mathbf{u} + Q_0(\mathbf{w},\theta)\mathbf{u} + \nabla P + \alpha^{-1}\mathbf{u} = \mu\Delta\mathbf{u} + (\mu+\nu)\nabla\operatorname{div}\mathbf{u}, \\ &\rho e_t + \rho\mathbf{u}\cdot\nabla e + Q_0(\mathbf{w},\theta)\Big(e + \frac{\mathbf{u}^2}{2}\Big) + P\operatorname{div}\mathbf{u} \\ &= Q_0(\mathbf{w},\theta)\mathbf{u}^2 + \alpha^{-1}\mathbf{u}^2 + \kappa\Delta\theta + 2\mu\mathbf{D}(\mathbf{u}):\mathbf{D}(\mathbf{u}) + \lambda(\operatorname{div}\mathbf{u})^2 + Q_1(\mathbf{w},\theta).\end{aligned} \tag{1.15}$$

By using the thermodynamical relation

$$-\rho^2 e_\rho(\rho,\theta) = \theta P_\theta(\rho,\theta;\mathbf{w}) - P(\rho,\theta;\mathbf{w})$$

(see [10, (1.6)]), we rewrite $(1.15)_2$ as

$$
\begin{aligned}
&\theta_t + \mathbf{u}\cdot\nabla\theta + \frac{\theta P_\theta(\rho,\theta;\mathbf{w})}{\rho e_\theta(\rho,\theta)}\operatorname{div}\mathbf{u} - \frac{\kappa}{\rho e_\theta(\rho,\theta)}\Delta\theta \\
&= \frac{2\mu\mathbf{D}(\mathbf{u}):\mathbf{D}(\mathbf{u})}{\rho e_\theta(\rho,\theta)} + \frac{\lambda(\operatorname{div}\mathbf{u})^2}{\rho e_\theta(\rho,\theta)} + \frac{Q_1(\mathbf{w},\theta)}{\rho e_\theta(\rho,\theta)} \\
&\quad + \frac{1}{\rho e_\theta(\rho,\theta)}\Big(\frac{1}{2}Q_0(\mathbf{w},\theta)\mathbf{u}^2 + \alpha^{-1}\mathbf{u}^2 - Q_0(\mathbf{w},\theta)e(\rho,\theta)\Big).
\end{aligned} \tag{1.16}
$$

Setting

$$
\varphi = \rho - \bar\rho, \quad \mathbf{m} = \mathbf{w} - \bar{\mathbf{w}}, \quad \zeta = \theta - \bar\theta,
$$

and using assumption (1.9), we rewrite system $(1.1)_1$, $(1.15)_1$, $(1.1)_3$, (1.16) as follows:

$$
\begin{aligned}
&\varphi_t + \bar\rho\operatorname{div}\mathbf{u} + \mathbf{u}\cdot\nabla\varphi - \nabla_{\mathbf{w}}Q_0(\bar{\mathbf{w}},\bar\theta)\cdot\mathbf{m} - Q'_{0\theta}(\bar{\mathbf{w}},\bar\theta)\zeta = G_1(\varphi,\mathbf{u},\mathbf{m},\zeta), \\
&\qquad \mathbf{u}_t - \frac{\mu}{\bar\rho}\Delta\mathbf{u} - \frac{\mu+\lambda}{\bar\rho}\nabla\operatorname{div}\mathbf{u} + \frac{1}{\alpha\bar\rho}\mathbf{u} + \frac{P_\rho(\bar\rho,\bar\theta;\bar{\mathbf{w}})}{\bar\rho}\nabla\varphi \\
&\qquad + \frac{P_\theta(\bar\rho,\bar\theta;\bar{\mathbf{w}})}{\bar\rho}\nabla\zeta + \sum_{i=1}^n \frac{P_{w_i}(\bar\rho,\bar\theta;\bar{\mathbf{w}})}{\bar\rho}\nabla m_i \\
&\qquad = \mathbf{G}_2(\varphi,\mathbf{u},\mathbf{m},\zeta), \\
&\mathbf{m}_t - d_f\Delta\mathbf{m} + \bar{\mathbf{w}}\operatorname{div}\mathbf{u} - D_{\mathbf{w}}\mathbf{S}(\bar{\mathbf{w}},\bar\theta)\mathbf{m} - \mathbf{S}'_\theta(\bar{\mathbf{w}},\bar\theta)\zeta = \mathbf{G}_3(\varphi,\mathbf{u},\mathbf{m},\zeta), \\
&\zeta_t + \frac{\bar\theta P_\theta(\bar\rho,\bar\theta;\bar{\mathbf{w}})}{\bar\rho e_\theta(\bar\rho,\bar\theta)}\operatorname{div}\mathbf{u} \\
&= \frac{\kappa}{\bar\rho e_\theta(\bar\rho,\bar\theta)}\Delta\zeta + \frac{\nabla_{\mathbf{w}}Q_1(\bar{\mathbf{w}},\bar\theta)\cdot\mathbf{m} + Q'_{1\theta}(\bar{\mathbf{w}},\bar\theta)\zeta}{\bar\rho e_\theta(\bar\rho,\bar\theta)} \\
&\quad + \frac{e(\bar\rho,\bar\theta)}{\bar\rho e_\theta(\bar\rho,\bar\theta)}\Big(\nabla_{\mathbf{w}}Q_0(\bar{\mathbf{w}},\bar\theta)\cdot\mathbf{m} + Q'_{0\theta}(\bar{\mathbf{w}},\bar\theta)\zeta\Big) + G_4(\varphi,\mathbf{u},\mathbf{m},\zeta),
\end{aligned} \tag{1.17}
$$

where $D_{\mathbf{w}}\mathbf{S}(\bar{\mathbf{w}},\bar\theta) = \Big(\frac{\partial S_j(\bar{\mathbf{w}},\bar\theta)}{\partial w_i}\Big)_{i,j=1}^n$,

$$
\begin{aligned}
&G_1(\varphi,\mathbf{u},\mathbf{m},\zeta) \\
&= -\varphi\operatorname{div}\mathbf{u} + \Big(Q_0(\mathbf{w},\theta) - Q_0(\bar{\mathbf{w}},\bar\theta) - \nabla_{\mathbf{w}}Q_0(\bar{\mathbf{w}},\bar\theta)\cdot\mathbf{m} - Q'_{0\theta}(\bar{\mathbf{w}},\bar\theta)\zeta\Big),
\end{aligned} \tag{1.18}
$$

$$
\begin{aligned}
&\mathbf{G}_2(\varphi,\mathbf{u},\mathbf{m},\zeta) \\
&= -(\mathbf{u},\nabla)\mathbf{u} - (Q_0(\mathbf{w},\theta) - Q_0(\bar{\mathbf{w}},\bar\theta))\mathbf{u} + h_1(\varphi,\zeta,\mathbf{m})\nabla\varphi + h_2(\varphi,\zeta,\mathbf{m})\nabla\zeta \\
&\quad + \sum_{i=1}^n h_{3i}(\varphi,\zeta,\mathbf{m})\nabla m_i - h_4(\varphi)\Big(\mu\Delta\mathbf{u} + (\mu+\nu)\nabla\operatorname{div}\mathbf{u}\Big) + \alpha^{-1}h_4(\varphi)\mathbf{u}, \\
&\qquad h_1(\varphi,\zeta,\mathbf{m}) = \frac{P_\rho(\bar\rho,\bar\theta;\bar{\mathbf{w}})}{\bar\rho} - \frac{P_\rho(\rho,\theta;\mathbf{w})}{\rho}, \\
&\qquad h_2(\varphi,\zeta,\mathbf{m}) = \frac{P_\theta(\bar\rho,\bar\theta;\bar{\mathbf{w}})}{\bar\rho} - \frac{P_\theta(\rho,\theta;\mathbf{w})}{\rho}, \\
&\qquad h_{3i}(\varphi,\zeta,\mathbf{m}) = \frac{P_{w_i}(\bar\rho,\bar\theta;\bar{\mathbf{w}})}{\bar\rho} - \frac{P_{w_i}(\rho,\theta;\mathbf{w})}{\rho}, \quad h_4(\varphi) = \frac{1}{\bar\rho} - \frac{1}{\rho}
\end{aligned} \tag{1.19}
$$

$$
\begin{aligned}
&\mathbf{G}_3(\varphi,\mathbf{u},\mathbf{m},\zeta)\\
&= -(\mathbf{u},\nabla)\mathbf{m} - \mathbf{m}\operatorname{div}\mathbf{u} + \Big(\mathbf{S}(\mathbf{w},\theta) - \mathbf{S}(\bar{\mathbf{w}},\bar\theta) - D_{\mathbf{w}}\mathbf{S}(\bar{\mathbf{w}},\bar\theta)\mathbf{m} - \mathbf{S}'_\theta(\bar{\mathbf{w}},\bar\theta)\zeta\Big),
\end{aligned}
\tag{1.20}
$$

and

$$
\begin{aligned}
&G_4(\varphi,\mathbf{u},\mathbf{m},\zeta)\\
&= -\mathbf{u}\cdot\nabla\zeta - \kappa h_5(\varphi,\zeta)\Delta\zeta - h_6(\varphi,\zeta,\mathbf{m})\operatorname{div}\mathbf{u}\\
&\quad + \frac{2\mu\mathbf{D}(\mathbf{u}):\mathbf{D}(\mathbf{u})}{\rho e_\theta(\rho,\theta)} + \frac{\lambda(\operatorname{div}\mathbf{u})^2}{\rho e_\theta(\rho,\theta)} + \frac{1}{\rho e_\theta(\rho,\theta)}\left(\frac12 Q_0(\mathbf{w},\theta) + \alpha^{-1}\right)\mathbf{u}^2\\
&\quad + h_5(\varphi,\zeta)\Big(Q_1(\mathbf{w},\theta) - Q_1(\bar{\mathbf{w}},\bar\theta)\Big) - h_7(\varphi,\zeta)\Big(Q_0(\mathbf{w},\theta) - Q_0(\bar{\mathbf{w}},\bar\theta)\Big)\\
&\quad + \frac{1}{\bar\rho e_\theta(\bar\rho,\bar\theta)}\Big(Q_1(\mathbf{w},\theta) - Q_1(\bar{\mathbf{w}},\bar\theta) - \nabla_{\mathbf{w}}Q_1(\bar{\mathbf{w}},\bar\theta)\cdot\mathbf{m} - Q'_{1\theta}(\bar{\mathbf{w}},\bar\theta)\zeta\Big)\\
&\quad + \frac{e(\bar\rho,\bar\theta)}{\bar\rho e_\theta(\bar\rho,\bar\theta)}\Big(Q_0(\mathbf{w},\theta) - Q_0(\bar{\mathbf{w}},\bar\theta) - \nabla_{\mathbf{w}}Q_0(\bar{\mathbf{w}},\bar\theta)\cdot\mathbf{m} - Q'_{0\theta}(\bar{\mathbf{w}},\bar\theta)\zeta\Big),\\
&\qquad h_5(\varphi,\zeta) = \frac{1}{\bar\rho e_\theta(\bar\rho,\bar\theta)} - \frac{1}{\rho e_\theta(\rho,\theta)},\\
&\qquad h_6(\varphi,\zeta,\mathbf{m}) = \frac{\bar\theta P_\theta(\bar\rho,\bar\theta;\bar{\mathbf{w}})}{\bar\rho e_\theta(\bar\rho,\bar\theta)} - \frac{\theta P_\theta(\rho,\theta;\mathbf{w})}{\rho e_\theta(\rho,\theta)},\\
&\qquad h_7(\varphi,\zeta) = \frac{e(\bar\rho,\bar\theta)}{\bar\rho e_\theta(\bar\rho,\bar\theta)} - \frac{e(\rho,\theta)}{\rho e_\theta(\rho,\theta)}.
\end{aligned}
\tag{1.21}
$$

Also, the initial condition (1.8) is reformulated as

$$
\begin{aligned}
(\varphi,\mathbf{u},\mathbf{m},\zeta)(x,0) &= (\varphi_0,\mathbf{u}_0,\mathbf{m}_0,\zeta_0)(x)\\
&\equiv \big(\rho_0(x) - \bar\rho, \mathbf{u}_0, \mathbf{w}_0 - \bar{\mathbf{w}}, \theta_0(x) - \bar\theta\big).
\end{aligned}
\tag{1.22}
$$

Based on the above reformulation, we construct the following iteration scheme:

$$
\begin{aligned}
&\varphi_t + \bar\rho\operatorname{div}\mathbf{u} + \mathbf{u}^{j-1}\cdot\nabla\varphi - \nabla_{\mathbf{w}}Q_0(\bar{\mathbf{w}},\bar\theta)\cdot\mathbf{m} - Q'_{0\theta}(\bar{\mathbf{w}},\bar\theta)\zeta\\
&= G_1(\varphi^{j-1},\mathbf{u}^{j-1},\mathbf{m}^{j-1},\zeta^{j-1}),\\
&\mathbf{u}_t - \frac{\mu}{\bar\rho}\Delta\mathbf{u} - \frac{\mu+\lambda}{\bar\rho}\nabla\operatorname{div}\mathbf{u} + \frac{1}{\alpha\bar\rho}\mathbf{u} + \frac{P_\rho(\bar\rho,\bar\theta;\bar{\mathbf{w}})}{\bar\rho}\nabla\varphi\\
&+ \frac{P_\theta(\bar\rho,\bar\theta;\bar{\mathbf{w}})}{\bar\rho}\nabla\zeta + \sum_{i=1}^{n}\frac{P_{w_i}(\bar\rho,\bar\theta;\bar{\mathbf{w}})}{\bar\rho}\nabla m_i\\
&= \mathbf{G}_2(\varphi^{j-1},\mathbf{u}^{j-1},\mathbf{m}^{j-1},\zeta^{j-1}),\\
&\mathbf{m}_t - d_f\Delta\mathbf{m} + \bar{\mathbf{w}}\operatorname{div}\mathbf{u} - D_{\mathbf{w}}\mathbf{S}(\bar{\mathbf{w}},\bar\theta)\mathbf{m} - \mathbf{S}'_\theta(\bar{\mathbf{w}},\bar\theta)\zeta\\
&= \mathbf{G}_3(\varphi^{j-1},\mathbf{u}^{j-1},\mathbf{m}^{j-1},\zeta^{j-1}),
\end{aligned}
$$

$$
\begin{aligned}
&\zeta_t + \frac{\bar\theta P_\theta(\bar\rho,\bar\theta;\bar{\mathbf{w}})}{\bar\rho e_\theta(\bar\rho,\bar\theta)}\operatorname{div}\mathbf{u}\\
&= \frac{\kappa}{\bar\rho e_\theta(\bar\rho,\bar\theta)}\Delta\zeta + \frac{\nabla_{\mathbf{w}}Q_1(\bar{\mathbf{w}},\bar\theta)\cdot\mathbf{m} + Q'_{1\theta}(\bar{\mathbf{w}},\bar\theta)\zeta}{\bar\rho e_\theta(\bar\rho,\bar\theta)}\\
&\quad + \frac{e(\bar\rho,\bar\theta)}{\bar\rho e_\theta(\bar\rho,\bar\theta)}\Big(\nabla_{\mathbf{w}}Q_0(\bar{\mathbf{w}},\bar\theta)\cdot\mathbf{m} + Q'_{0\theta}(\bar{\mathbf{w}},\bar\theta)\zeta\Big) + G_4(\varphi^{j-1},\mathbf{u}^{j-1},\mathbf{m}^{j-1},\zeta^{j-1}),
\end{aligned}
$$

$$(\varphi, \mathbf{u}, \mathbf{m}, \zeta)(x,0) = (\varphi_0, \mathbf{u}_0, \mathbf{m}_0, \zeta_0)(x), \quad x \in \Omega, \tag{1.23}$$

for each fixed $(\varphi^{j-1}, \mathbf{u}^{j-1}, \mathbf{m}^{j-1}, \zeta^{j-1}) \in X_N(0,t_0)$, $j = 1, 2, \dots$, where

$$(\varphi^0, \mathbf{u}^0, \mathbf{m}^0, \zeta^0) = (\varphi_0, \mathbf{u}_0, \mathbf{m}_0, \zeta_0)$$

and

$$\begin{aligned} X_N(0,t_0) = \big\{ &(\varphi, \mathbf{u}, \mathbf{m}, \zeta) \in C([0,t_0]; H^N(\mathbb{R}^3)) : \nabla\varphi \in L^2(0,t_0; H^{N-1}(\mathbb{R}^3)), \\ &\mathbf{u} \in L^2(0,t_0; H^{N+1}(\mathbb{R}^3)),\ (\nabla\mathbf{m}, \nabla\zeta) \in L^2(0,t_0; H^N(\mathbb{R}^3)) \big\}. \end{aligned}$$

Theorem 1.3. *Let $\{(\varphi^j, \mathbf{u}^j, \mathbf{m}^j, \zeta^j)\}_{i=1}^{\infty}$ be the sequence of the approximate solutions given by the iteration scheme* (1.23)*. Then, under the assumptions of Theorem 1.1, it holds that*

$$\begin{gathered} (\varphi^j, \mathbf{u}^j, \mathbf{m}^j, \zeta^j) \to (\varphi, \mathbf{u}, \mathbf{m}, \zeta) \quad \text{*-weak in } L^\infty(0,T_0; H^N(\mathbb{R}^3)), \\ \mathbf{u}^j \to \mathbf{u} \quad \text{weak in } L^2(0,T_0; H^{N+1}(\mathbb{R}^3)), \\ \big(\nabla\mathbf{m}^j, \nabla\zeta^j\big) \to \big(\nabla\mathbf{m}, \nabla\zeta\big) \quad \text{weak in } L^2(0,T_0; H^N(\mathbb{R}^3)), \\ (\varphi^j, \mathbf{u}^j, \mathbf{m}^j, \zeta^j) \to (\varphi, \mathbf{u}, \mathbf{m}, \zeta) \quad \text{strong in } L^2(0,T_0; H^{N-1}_{\rm loc}(\mathbb{R}^3)), \end{gathered}$$

where $H^{N-1}_{\rm loc}(\mathbb{R}^3)$ denotes the space $H^{N-1}(\Omega)$ for any bounded domain Ω of $\mathbb{R}^3$, and $(\varphi, \mathbf{u}, \mathbf{m}, \zeta)$ is the smooth unique solution to the Cauchy problem (1.17), (1.22) *on $[0,T_0]$ satisfying* (1.12).

Before finishing this section, we recall the following useful Lemmas which we will use extensively.

Lemma 1.4 ([4])**.** *Let $\Omega = \mathbb{R}^d$, $s_1 \geq s$ and $s_2 \geq s$ be such that either*

$$s_1 + s_2 - s \geq d\Big(\frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{q}\Big) \geq 0, \quad s_j - s > d\Big(\frac{1}{q_j} - \frac{1}{q}\Big), \quad j = 1, 2$$

or

$$s_1 + s_2 - s > d\Big(\frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{q}\Big) \geq 0, \quad s_j - s \geq d\Big(\frac{1}{q_j} - \frac{1}{q}\Big), \quad j = 1, 2,$$

then $(u,v) \mapsto u \cdot v$ is a continuous bilinear map from $W^{s_1}_{q_1}(\Omega) \times W^{s_1}_{q_1}(\Omega)$ into $W^s_q(\Omega)$.

Lemma 1.5 ([21, Lemma 2.5])**.** *Let $f(\varphi)$ and $f(\varphi, w)$ be smooth functions of φ and (φ, w), respectively, with bounded derivatives of any order, and $\|\varphi\|_{L^\infty(\mathbb{R}^3)} + \|w\|_{L^\infty(\mathbb{R}^3)} \leq C$. Then for any integer $m \geq 1$, we have*

$$\begin{aligned} \|\nabla^m f(\varphi)\|_{L^p} &\leq C\|\nabla^m \varphi\|_{L^p}, \\ \|\nabla^m f(\varphi, w)\|_{L^p} &\leq C\|\nabla^m (\varphi, w)\|_{L^p}, \end{aligned} \tag{1.24}$$

for any $1 \leq p \leq \infty$, where C may depend on f and m.

Lemma 1.6 ([21, Lemma 2.6])**.** *Let α be any multi-index with $|\alpha| = k$ and $1 < p < \infty$. Then there exists a constant $C > 0$ such that*

$$\begin{aligned} \|D^\alpha(fg)\|_{L^p} &\leq C(\|f\|_{L^{p_1}}\|\nabla^k g\|_{L^{p_2}} + \|\nabla^k f\|_{L^{p_3}}\|g\|_{L^{p_4}}), \\ \|[D^\alpha, f]g\|_{L^p} &\leq C(\|\nabla f\|_{L^{p_1}}\|\nabla^{k-1} g\|_{L^{p_2}} + \|\nabla^k f\|_{L^{p_3}}\|g\|_{L^{p_4}}), \end{aligned} \tag{1.25}$$

where $f, g \in S$ is the Schwartz class, $1 \leq p_2, p_3 \leq \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$, and $[D^\alpha, f]g = D^\alpha(fg) - fD^\alpha g$.

2. Linearized Problem

In this section, we study the linearized system

$$
\begin{gathered}
\varphi_t + \bar\rho \operatorname{div}\mathbf{u} + \mathbf{A}\cdot\nabla\varphi - \bar{\mathbf{a}}_0\cdot\mathbf{m} - \bar b_0\zeta = G_1,\\
\mathbf{u}_t - \frac{\mu}{\bar\rho}\Delta\mathbf{u} - \frac{\mu+\lambda}{\bar\rho}\nabla\operatorname{div}\mathbf{u} + \frac{1}{\alpha\bar\rho}\mathbf{u} + \frac{\bar P_\rho}{\bar\rho}\nabla\varphi + \frac{\bar P_\theta}{\bar\rho}\nabla\zeta + \sum_{i=1}^n \frac{\bar P_{w_i}}{\bar\rho}\nabla m_i = \mathbf{G}_2,\\
\mathbf{m}_t - d_f\Delta\mathbf{m} + \bar{\mathbf{w}}\operatorname{div}\mathbf{u} - \bar{\mathbf{M}}\mathbf{m} - \bar{\mathbf{B}}\zeta = \mathbf{G}_3,\\
\zeta_t + \frac{\bar\theta\bar P_\theta}{\bar\rho\bar e_\theta}\operatorname{div}\mathbf{u} = \frac{\kappa}{\bar\rho\bar e_\theta}\Delta\zeta + \frac{\bar{\mathbf{a}}_1\cdot\mathbf{m} + \bar b_1\zeta}{\bar\rho\bar e_\theta} + \frac{\bar e}{\bar\rho\bar e_\theta}\Big(\bar{\mathbf{a}}_0\cdot\mathbf{m} + \bar b_0\zeta\Big) + G_4,
\end{gathered}
\tag{2.1}
$$

where $\mathbf{A}, G_1, \mathbf{G}_2, \mathbf{G}_3$ and G_4 are the given functions, and we used the following notations: $\bar{\mathbf{a}}_0 = \nabla_{\mathbf{w}} Q_0(\bar{\mathbf{w}},\bar\theta)$, $\bar b_0 = Q'_{0\theta}(\bar{\mathbf{w}},\bar\theta)$, $\bar P_\rho = P_\rho(\bar\rho,\bar\theta;\bar{\mathbf{w}})$, $\bar P_\theta = P_\theta(\bar\rho,\bar\theta;\bar{\mathbf{w}})$, $\bar P_{w_i} = P_{w_i}(\bar\rho,\bar\theta;\bar{\mathbf{w}})$, $\bar{\mathbf{M}} = D_{\mathbf{w}}\mathbf{S}(\bar{\mathbf{w}},\bar\theta)$, $\bar{\mathbf{B}} = \mathbf{S}'_\theta(\bar{\mathbf{w}},\bar\theta)$, $\bar e_\theta = e_\theta(\bar\rho,\bar\theta)$, $\bar e = e(\bar\rho,\bar\theta)$, $\bar{\mathbf{a}}_1 = \nabla_{\mathbf{w}} Q_1(\bar{\mathbf{w}},\bar\theta)$ and $\bar b_1 = Q'_{1\theta}(\bar{\mathbf{w}},\bar\theta)$.

We first consider an estimate for the linearized system (2.1).

Theorem 2.1. *Let $0 < T < \infty$. Assume that $(\varphi_0, \mathbf{u}_0, \mathbf{m}_0, \zeta_0) \in H^N(\mathbb{R}^3)$ for an integer $N \ge 3$. Also, suppose that*

$$
\begin{gathered}
\mathbf{A} \in L^\infty(0,T;H^N(\mathbb{R}^3)), \quad G_1 \in L^2(0,T;H^N(\mathbb{R}^3))\\
\mathbf{G}_2, \mathbf{G}_3, G_4 \in L^2(0,T;H^{N-1}(\mathbb{R}^3)).
\end{gathered}
\tag{2.2}
$$

Let

$$
\begin{gathered}
(\varphi,\mathbf{u},\mathbf{m},\zeta) \in C([0,T];H^N(\mathbb{R}^3)), \quad \nabla\varphi \in L^2(0,T;H^{N-1}(\mathbb{R}^3)),\\
\mathbf{u} \in L^2(0,T;H^{N+1}(\mathbb{R}^3)), \quad (\nabla\mathbf{m},\nabla\zeta) \in L^2(0,T;H^N(\mathbb{R}^3))
\end{gathered}
\tag{2.3}
$$

and $(\varphi,\mathbf{u},\mathbf{m},\zeta)$ be a solution of (2.1). Then there exist positive constants C_0 and c_0, independent of t, such that

$$
\begin{aligned}
&\|(\varphi,\mathbf{u},\mathbf{m},\zeta)(t)\|_{H^N}^2 + \int_0^t \|\nabla\varphi\|_{H^{N-1}}^2 d\tau + \int_0^t \|\mathbf{u}\|_{H^{N+1}}^2 d\tau + \int_0^t \|(\nabla\mathbf{m},\nabla\zeta)\|_{H^N}^2 d\tau\\
&\le C_0\Big[\|(\varphi,\mathbf{u},\mathbf{m},\zeta)(0)\|_{H^N}^2 + \int_0^T \big(\|G_1\|_{H^N}^2 + \|\big(\mathbf{G}_2,\mathbf{G}_3,G_4\big)\|_{H^{N-1}}^2\big)\, d\tau\Big]\\
&\quad\times\Big[1 + e^{c_0\int_0^T(1+\|\mathbf{A}\|_{H^N}^2)d\tau}\int_0^T \Big(1+\|\mathbf{A}\|_{H^N}^2\Big)d\tau\Big]
\end{aligned}
\tag{2.4}
$$

for each $t \in [0,T]$.

Proof. Applying ∇^k to (2.1) yields

$$
\begin{aligned}
&\nabla^k\varphi_t+\bar\rho\operatorname{div}\nabla^k\mathbf{u}+\nabla^k\Big(\mathbf{A}\cdot\nabla\varphi\Big)-\bar{\mathbf{a}}_0\cdot\nabla^k\mathbf{m}-\bar b_0\nabla^k\zeta=\nabla^kG_1,\\
&\nabla^k\mathbf{u}_t-\frac{\mu}{\bar\rho}\Delta\nabla^k\mathbf{u}-\frac{\mu+\lambda}{\bar\rho}\nabla\operatorname{div}\nabla^k\mathbf{u}+\frac{1}{\alpha\bar\rho}\nabla^k\mathbf{u}\\
&+\frac{\bar P_\rho}{\bar\rho}\nabla^{k+1}\varphi+\frac{\bar P_\theta}{\bar\rho}\nabla^{k+1}\zeta+\sum_{i=1}^n\frac{\bar P_{w_i}}{\bar\rho}\nabla^{k+1}m_i=\nabla^k\mathbf{G}_2,\\
&\nabla^k\mathbf{m}_t-d_f\Delta\nabla^k\mathbf{m}+\bar{\mathbf{w}}\operatorname{div}\nabla^k\mathbf{u}-\bar{\mathbf{M}}\nabla^k\mathbf{m}-\bar{\mathbf{B}}\nabla^k\zeta=\nabla^k\mathbf{G}_3,\\
&\nabla^k\zeta_t+\frac{\bar\theta\bar P_\theta}{\bar\rho\bar e_\theta}\operatorname{div}\nabla^k\mathbf{u}-\frac{\kappa}{\bar\rho\bar e_\theta}\Delta\nabla^k\zeta-\frac{\bar{\mathbf{a}}_1\cdot\nabla^k\mathbf{m}+\bar b_1\nabla^k\zeta}{\bar\rho\bar e_\theta}\\
&-\frac{\bar e}{\bar\rho\bar e_\theta}\Big(\bar{\mathbf{a}}_0\cdot\nabla^k\mathbf{m}+\bar b_0\nabla^k\zeta\Big)=\nabla^kG_4,
\end{aligned}
\tag{2.5}
$$

where $k=0,1,\dots,N$.

Multiplying $(2.5)_2$ by $\nabla^k\mathbf{u}$ and using $(2.5)_1$, we have

$$
\begin{aligned}
&\frac12\frac{d}{dt}\Big(\|\nabla^k\mathbf{u}\|^2+\frac{\bar P_\rho}{\bar\rho^2}\|\nabla^k\varphi\|^2\Big)+\frac{1}{\bar\rho}\int\big(\mu|\nabla^{k+1}\mathbf{u}|^2+(\lambda+\mu)|\operatorname{div}\nabla^k\mathbf{u}|^2\big)\,dx\\
&+\frac{1}{\alpha\bar\rho^2}\|\nabla^k\mathbf{u}\|^2-\frac{\bar P_\theta}{\bar\rho}\int\nabla^k\zeta\operatorname{div}\nabla^k\mathbf{u}dx-\sum_{i=1}^n\frac{\bar P_{w_i}}{\bar\rho}\int\nabla^km_i\operatorname{div}\nabla^k\mathbf{u}dx\\
&-\frac{\bar P_\rho}{\bar\rho^2}\int\nabla^k\varphi\Big(\bar{\mathbf{a}}_0\cdot\nabla^k\mathbf{m}+\bar b_0\nabla^k\zeta\Big)dx\\
&=\int\nabla^k\mathbf{G}_2\cdot\nabla^k\mathbf{u}\,dx+\frac{\bar P_\rho}{\bar\rho^2}\int\nabla^k\varphi\nabla^kG_1dx-\frac{\bar P_\rho}{\bar\rho^2}\int\operatorname{div}\mathbf{A}|\nabla^k\varphi|^2\,dx\\
&\quad-\frac{\bar P_\rho}{\bar\rho^2}\int\big[\nabla^k,\mathbf{A}\big]\cdot\nabla\varphi\nabla^k\varphi dx,
\end{aligned}
\tag{2.6}
$$

where we used

$$\nabla^k\left(\mathbf{A}\cdot\nabla\varphi\right)=\mathbf{A}\cdot\nabla\nabla^k\varphi-\big[\nabla^k,\mathbf{A}\big]\cdot\nabla\varphi.$$

Multiplying $(2.5)_3$ by $\big(\frac{\bar P_{w_1}}{\bar\rho\bar w_1}\nabla^km_1,\dots,\frac{\bar P_{w_n}}{\bar\rho\bar w_n}\nabla^km_n\big)$, we have

$$
\begin{aligned}
&\sum_{i=1}^n\frac{\bar P_{w_i}}{\bar\rho\bar w_i}\Big(\frac{d}{2dt}\|\nabla^km_i\|^2+d_f\|\nabla^{k+1}m_i\|^2\Big)+\sum_{i=1}^n\frac{\bar P_{w_i}}{\bar\rho}\int\operatorname{div}\nabla^k\mathbf{u}\,\nabla^km_i\,dx\\
&-\int\Big(\bar{\mathbf{M}}\nabla^k\mathbf{m}+\bar{\mathbf{B}}\nabla^k\zeta\Big)\cdot\Big(\frac{\bar P_{w_1}}{\bar\rho\bar w_1}\nabla^km_1,\dots,\frac{\bar P_{w_n}}{\bar\rho\bar w_n}\nabla^km_n\Big)\,dx\\
&=\int\nabla^k\mathbf{G}_3\cdot\Big(\frac{\bar P_{w_1}}{\bar\rho\bar w_1}\nabla^km_1,\dots,\frac{\bar P_{w_n}}{\bar\rho\bar w_n}\nabla^km_n\Big)dx.
\end{aligned}
\tag{2.7}
$$

Multiplying $(2.5)_4$ by $\frac{\bar e_\theta}{\bar\theta}\nabla^k\zeta$, we have

$$
\begin{aligned}
&\frac{\bar e_\theta}{2\bar\theta}\frac{d}{dt}\|\nabla^k\zeta\|^2+\frac{\kappa}{\bar\rho\bar\theta}\|\nabla^{k+1}\zeta\|^2+\frac{\bar P_\theta}{\bar\rho}\int\operatorname{div}\nabla^k\mathbf{u}\nabla^k\zeta\,dx\\
&-\frac{1}{\bar\rho\bar\theta}\int\Big(\bar{\mathbf{a}}_1\cdot\nabla^k\mathbf{m}+\bar b_1\nabla^k\zeta\Big)\nabla^k\zeta\,dx+\frac{\bar e}{\bar\rho\bar\theta}\int\Big(\bar{\mathbf{a}}_0\cdot\nabla^k\mathbf{m}+\bar b_0\nabla^k\zeta\Big)\nabla^k\zeta\,dx\\
&=\frac{\bar e_\theta}{\bar\theta}\int\nabla^kG_2\nabla^k\zeta dx.
\end{aligned}
\tag{2.8}
$$

Adding (2.6)-(2.8) yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\nabla^k\mathbf{u}\|^2+\frac{\bar{P}_\rho}{\bar{\rho}^2}\|\nabla^k\varphi\|^2+\sum_{i=1}^n\frac{\bar{P}_{w_i}}{\bar{\rho}\bar{w}_i}\|\nabla^k m_i\|^2+\frac{\bar{e}_\theta}{\bar{\theta}}\|\nabla^k\zeta\|^2\Big)\\
&+\frac{1}{\bar{\rho}}\int\big(\mu|\nabla^{k+1}\mathbf{u}|^2+(\lambda+\mu)|\operatorname{div}\nabla^k\mathbf{u}|^2\big)\,dx+\frac{1}{\alpha\bar{\rho}^2}\|\nabla^k\mathbf{u}\|^2\\
&+\sum_{i=1}^n\frac{d_f\bar{P}_{w_i}}{\bar{\rho}\bar{w}_i}\|\nabla^{k+1}m_i\|^2+\frac{\kappa}{\bar{\rho}\bar{\theta}}\|\nabla^{k+1}\zeta\|^2\\
&=J_k^1+J_k^2+J_k^3,
\end{aligned}\tag{2.9}$$

where

$$\begin{aligned}
J_k^1=&-\int\Big(\bar{\mathbf{M}}\nabla^k\mathbf{m}+\bar{\mathbf{B}}\nabla^k\zeta\Big)\cdot\Big(\frac{\bar{P}_{w_1}}{\bar{\rho}\bar{w}_1}\nabla^k m_1,\dots,\frac{\bar{P}_{w_n}}{\bar{\rho}\bar{w}_n}\nabla^k m_n\Big)\,dx\\
&-\frac{\bar{P}_\rho}{\bar{\rho}^2}\int\nabla^k\varphi\Big(\bar{\mathbf{a}}_0\cdot\nabla^k\mathbf{m}+\bar{b}_0\nabla^k\zeta\Big)dx-\frac{1}{\bar{\rho}\bar{\theta}}\int\Big(\bar{\mathbf{a}}_1\cdot\nabla^k\mathbf{m}+\bar{b}_1\nabla^k\zeta\Big)\nabla^k\zeta\,dx\\
&+\frac{\bar{e}}{\bar{\rho}\bar{\theta}}\int\Big(\bar{\mathbf{a}}_0\cdot\nabla^k\mathbf{m}+\bar{b}_0\nabla^k\zeta\Big)\nabla^k\zeta\,dx,\\
J_k^2=&-\frac{\bar{P}_\rho}{\bar{\rho}^2}\int\operatorname{div}\mathbf{A}|\nabla^k\varphi|^2dx-\frac{\bar{P}_\rho}{\bar{\rho}^2}\int\Big[\nabla^k,\mathbf{A}\Big]\cdot\nabla\varphi\nabla^k\varphi\,dx,\\
J_k^3=&\int\nabla^k\mathbf{G}_2\cdot\nabla^k\mathbf{u}\,dx+\frac{\bar{P}_\rho}{\bar{\rho}^2}\int\nabla^k\varphi\nabla^k G_1dx+\frac{\bar{e}_\theta}{\bar{\theta}}\int\nabla^k G_4\nabla^k\zeta\,dx\\
&+\int\nabla^k\mathbf{G}_3\cdot\Big(\frac{\bar{P}_{w_1}}{\bar{\rho}\bar{w}_1}\nabla^k m_1,\dots,\frac{\bar{P}_{w_n}}{\bar{\rho}\bar{w}_n}\nabla^k m_n\Big)dx.
\end{aligned}\tag{2.10}$$

Summing (2.9) for $k=0,1,\dots,N$ we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\mathbf{u}\|_{H^N}^2+\frac{\bar{P}_\rho}{\bar{\rho}^2}\|\varphi\|_{H^N}^2+\sum_{i=1}^n\frac{\bar{P}_{w_i}}{\bar{\rho}\bar{w}_i}\|m_i\|_{H^N}^2+\frac{\bar{e}_\theta}{\bar{\theta}}\|\zeta\|_{H^N}^2\Big)\\
&+\frac{1}{\bar{\rho}}\|\nabla\mathbf{u}\|_{H^N}^2+\frac{1}{\alpha\bar{\rho}^2}\|\mathbf{u}\|_{H^N}^2+\frac{d_f\bar{P}_{min}}{\bar{\rho}\bar{w}_{min}}\|\nabla\mathbf{m}\|_{H^N}^2+\frac{\kappa}{\bar{\rho}\bar{\theta}}\|\nabla\zeta\|_{H^N}^2\\
&\le\sum_{k=0}^N J_k^1+\sum_{k=0}^N J_k^2+\sum_{k=0}^N J_k^3,
\end{aligned}\tag{2.11}$$

where $\bar{w}_{min}=\min\{\bar{w}_1,\dots,\bar{w}_n\}>0$ and $\bar{P}_{min}=\min\{\bar{P}_1,\dots,\bar{P}_n\}>0$ because of (1.3).

Using Hölder inequality and (1.25), we obtain from (2.10) that

$$\begin{aligned}
\sum_{k=0}^N J_k^1&\le C\sum_{k=0}^N\Big(\|\nabla^k\mathbf{m}\|+\|\nabla^k\zeta\|\Big)\Big(\|\nabla^k\mathbf{m}\|+\|\nabla^k\varphi\|+\|\nabla^k\zeta\|\Big)\\
&\le C\Big(\|\mathbf{m}\|_{H^N}^2+\|\varphi\|_{H^N}^2+\|\zeta\|_{H^N}^2\Big),
\end{aligned}\tag{2.12}$$

$$\begin{aligned}
\sum_{k=0}^N J_k^2&\le C\sum_{k=0}^N\Big(\|\nabla\mathbf{A}\|_{L^\infty}\|\nabla^k\varphi\|+\|\nabla\varphi\|_{L^\infty}\|\nabla^k\mathbf{A}\|\Big)\|\nabla^k\varphi\|\\
&\le C\|\mathbf{A}\|_{H^N}\|\varphi\|_{H^N}^2
\end{aligned}\tag{2.13}$$

and

$$\begin{aligned}
\sum_{k=0}^{N} J_k^3 \le{}& C\sum_{k=0}^{N-1}\Big(\|\nabla^k\mathbf{G}_2\|\|\nabla^k\mathbf{u}\|+\|\nabla^k\mathbf{G}_3\|\|\nabla^k\mathbf{m}\|+\|\nabla^k G_4\|\|\nabla^k\zeta\|\Big)\\
&+C\sum_{k=0}^{N}\|\nabla^k G_1\|\|\nabla^k\varphi\|+C\Big|\int\nabla^{N-1}\mathbf{G}_2\cdot\nabla^{N+1}\mathbf{u}d\mathbf{x}\Big|\\
&+C\Big|\int\nabla^{N-1}G_4\nabla^{N+1}\zeta dx\Big|\\
&+C\Big|\int\nabla^{N-1}\mathbf{G}_3\cdot\Big(\frac{\bar P_{w_1}}{\bar\rho\bar w_1}\nabla^{N+1}m_1,\dots,\frac{\bar P_{w_n}}{\bar\rho\bar w_n}\nabla^{N+1}m_n\Big)dx\Big|\\
\le{}& C\Big(\|G_1\|_{H^N}\|\varphi\|_{H^N}+\|(\mathbf{G}_2,\mathbf{G}_3,G_4)\|_{H^{N-1}}\|\nabla(\mathbf{u},\mathbf{m},\zeta)\|_{H^N}\Big).
\end{aligned}\tag{2.14}$$

Substituting (2.12)-(2.14) into (2.11), we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(\|\mathbf{u}\|_{H^N}^2+\frac{\bar P_\rho}{\bar\rho^2}\|\varphi\|_{H^N}^2+\sum_{i=1}^n\frac{\bar P_{w_i}}{\bar\rho\bar w_i}\|m_i\|_{H^N}^2+\frac{\bar e_\theta}{\bar\theta}\|\zeta\|_{H^N}^2\Big)\\
&+\frac{1}{\bar\rho}\|\nabla\mathbf{u}\|_{H^N}^2+\frac{2}{\alpha\bar\rho^2}\|\mathbf{u}\|_{H^N}^2+\frac{d_f\bar P_{min}}{\bar\rho\bar w_{min}}\|\nabla\mathbf{m}\|_{H^N}^2+\frac{\kappa}{\bar\rho\bar\theta}\|\nabla\zeta\|_{H^N}^2\\
&\le C\Big(\|\mathbf{m}\|_{H^N}^2+\|\varphi\|_{H^N}^2+\|\zeta\|_{H^N}^2\Big)+C\|\mathbf{A}\|_{H^N}\|\varphi\|_{H^N}^2\\
&\quad+C\Big(\|G_1\|_{H^N}^2+\|(\mathbf{G}_2,\mathbf{G}_3,G_4)\|_{H^{N-1}}^2\Big).
\end{aligned}\tag{2.15}$$

On the other hand, noticing that

$$\begin{aligned}
&\int\nabla^k\mathbf{u}_t\cdot\nabla^{k+1}\varphi\,dx\\
&=\frac{d}{dt}\int\nabla^k\mathbf{u}\cdot\nabla^{k+1}\varphi\,dx-\bar\rho\|\nabla^k\operatorname{div}\mathbf{u}\|^2-\int\nabla^k\operatorname{div}\mathbf{u}\nabla^k\Big(\mathbf{A}\cdot\nabla\varphi\Big)\,dx\\
&\quad+\int\nabla^k\operatorname{div}\mathbf{u}\Big(\bar{\mathbf{a}}_0\cdot\nabla^k\mathbf{m}+\bar b_0\nabla^k\zeta\Big)\,dx+\int\nabla^k\operatorname{div}\mathbf{u}\nabla^k G_1dx
\end{aligned}$$

because of $(2.5)_1$, and multiplying $(2.5)_2$ by $\nabla^{k+1}\varphi$ we have

$$\frac{d}{dt}\int\nabla^k\mathbf{u}\cdot\nabla^{k+1}\varphi\,dx+\frac{\bar P_\rho}{\bar\rho}\|\nabla^{k+1}\varphi\|^2=J_k^4+J_k^5+\bar\rho\|\nabla^k\operatorname{div}\mathbf{m}\|^2,\tag{2.16}$$

where

$$\begin{aligned}
J_k^4={}&\int\nabla^k\mathbf{G}_2\cdot\nabla^{k+1}\varphi\,dx-\frac{\bar P_\theta}{\bar\rho}\int\nabla^{k+1}\zeta\cdot\nabla^{k+1}\varphi\,dx\\
&-\frac{1}{\alpha\bar\rho}\int\nabla^k\mathbf{u}\cdot\nabla^{k+1}\varphi\,dx\\
&+\frac{\mu}{\bar\rho}\int\Delta\nabla^k\mathbf{u}\cdot\nabla^{k+1}\varphi\,dx+\frac{\mu+\lambda}{\bar\rho}\int\nabla\operatorname{div}\nabla^k\mathbf{u}\cdot\nabla^{k+1}\varphi\,dx\\
&-\int\nabla^k\operatorname{div}\mathbf{u}\nabla^k G_1dx-\sum_{i=1}^n\frac{\bar P_{w_i}}{\bar\rho}\int\nabla^{k+1}m_i\cdot\nabla^{k+1}\varphi\,dx\\
&-\int\nabla^k\operatorname{div}\mathbf{u}\Big(\bar{\mathbf{a}}_0\cdot\nabla^k\mathbf{m}+\bar b_0\nabla^k\zeta\Big)\,dx
\end{aligned}\tag{2.17}$$

and

$$J_k^5 = \int \nabla^k \operatorname{div} \mathbf{u} \nabla^k \Big(\mathbf{A} \cdot \nabla \varphi \Big)\, dx. \tag{2.18}$$

Using Hölder inequality and (1.25), we obtain from (2.17) and (2.18) that

$$\begin{aligned}\sum_{k=0}^{N-1} J_k^4 &\le C\Big(\|\mathbf{G}_2\|_{H^{N-1}} + \|(\zeta, \mathbf{u})\|_{H^N} + \|\nabla \mathbf{u}\|_{H^N} \Big) \|\varphi\|_{H^N} \\ &\quad + C\|\mathbf{u}\|_{H^N} \|G_1\|_{H^{N-1}} + C\|\mathbf{m}\|_{H^N} \|\varphi\|_{H^N} \\ &\quad + C\|\mathbf{u}\|_{H^N} (\|\mathbf{m}\|_{H^{N-1}} + \|\zeta\|_{H^{N-1}}) \\ &\le C\Big(\|G_1\|_{H^N}^2 + \|\mathbf{G}_1\|_{H^{N-1}}^2 + \|(\varphi, \mathbf{u}, \mathbf{m}, \zeta)\|_{H^N}^2 \Big) \\ &\quad + C\|\nabla \mathbf{u}\|_{H^N} \|\varphi\|_{H^N}\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}\sum_{k=0}^{N-1} J_k^3 &\le C \sum_{k=0}^{N-1} \|\nabla^{k+1} \mathbf{u}\| \Big(\|\mathbf{A}\|_{L^\infty} \|\nabla^{k+1} \varphi\| + \|\nabla \varphi\|_{L^\infty} \|\nabla^k \mathbf{A}\| \Big) \\ &\le C\|\mathbf{A}\|_{H^N} \|\nabla \mathbf{u}\|_{H^{N-1}} \|\varphi\|_{H^N},\end{aligned} \tag{2.20}$$

respectively.

Summing (2.16) for $k = 0, 1, \dots, N-1$, and using (2.19) and (2.20), we obtain

$$\begin{aligned}&\frac{d}{dt} \int \nabla^k \mathbf{u} \cdot \nabla^{k+1} \varphi \, dx + \frac{\bar{P}_\rho}{\bar{\rho}} \|\nabla^{k+1} \varphi\|^2 \\ &\le \|\nabla \mathbf{u}\|_{H^N}^2 + C\Big(1 + \|\mathbf{A}\|_{H^N}^2 \Big) \|\varphi\|_{H^N}^2 \\ &+ C\Big(\|G_1\|_{H^N}^2 + \|\mathbf{G}_1\|_{H^{N-1}}^2 + \|(\varphi, \mathbf{u}, \mathbf{m}, \zeta)\|_{H^N}^2 \Big).\end{aligned} \tag{2.21}$$

We can assume $0 < \epsilon \le 1$ without loss of generality. And we choose $\beta \in (0, 1]$ to be suitably small. Then, adding (2.15) and $\beta \times$ (2.21), we obtain

$$\begin{aligned}&\frac{d}{dt} E(t) + \frac{\beta \bar{P}_\rho}{\bar{\rho}} \|\nabla \varphi\|_{H^{N-1}}^2 + (\bar{\rho}^{-1} - \beta) \|\nabla \mathbf{m}\|_{H^N}^2 \\ &+ \frac{2}{\alpha \bar{\rho}^2} \|\mathbf{u}\|_{H^N}^2 + \frac{d_f \bar{P}_{min}}{\bar{\rho} \bar{w}_{min}} \|\nabla \mathbf{m}\|_{H^N}^2 + \frac{\kappa}{\bar{\rho} \bar{\theta}} \|\nabla \zeta\|_{H^N}^2 \\ &\le C\Big(\|G_1\|_{H^N}^2 + \|(\mathbf{G}_2, \mathbf{G}_3, G_4)\|_{H^{N-1}}^2 \Big) \\ &\quad + C\Big(1 + \|\mathbf{A}\|_{H^N}^2 \Big) \|(\varphi, \mathbf{u}, \mathbf{m}, \zeta)\|_{H^N}^2,\end{aligned} \tag{2.22}$$

where

$$\begin{aligned}E(t) &:= \|\mathbf{u}\|_{H^N}^2 + \beta \sum_{k=0}^{N-1} \int \nabla^k \mathbf{u} \cdot \nabla^{k+1} \varphi dx + \frac{\bar{P}_\rho}{\bar{\rho}^2} \|\varrho\|_{H^N}^2 \\ &\quad + \sum_{i=1}^{n} \frac{\bar{P}_{w_i}}{\bar{\rho} \bar{w}_i} \|m_i\|_{H^N}^2 + \frac{\bar{e}_\theta}{\bar{\theta}} \|\zeta\|_{H^N}^2.\end{aligned} \tag{2.23}$$

By (2.23) and (1.3), we can choose a small $\beta \in (0, \frac{1}{2\bar{\rho}}]$ independent $\epsilon \in (0, 1]$ such that

$$E(t) \backsimeq \|(\varphi, \mathbf{u}, \mathbf{m}, \zeta)(t)\|_{H^N}^2 \tag{2.24}$$

uniformly for all $t \in [0, T]$.

Integrating (2.22) over $t \in [0,T]$, and using (2.24), (1.3) and the smallest of β, we have

$$\begin{aligned} &\|(\varphi,\mathbf{u},\mathbf{m},\zeta)(t)\|_{H^N}^2 + \int_0^t \|\nabla\varphi\|_{H^{N-1}}^2 d\tau + \int_0^t \|\mathbf{u}\|_{H^{N+1}}^2 d\tau + \int_0^t \|\big(\nabla\mathbf{m},\nabla\zeta\big)\|_{H^N}^2 d\tau \\ &\le \|(\varphi,\mathbf{u},\mathbf{m},\zeta)(0)\|_{H^N}^2 + C\int_0^t \Big(1+\|\mathbf{A}\|_{H^N}^2\Big)\|(\varphi,\mathbf{u},\mathbf{m},\zeta)\|_{H^N}^2 d\tau \\ &\quad + C\int_0^t \Big(\|G_1\|_{H^N}^2 + \|(\mathbf{G}_2,\mathbf{G}_3,G_4)\|_{H^{N-1}}^2\Big) d\tau. \end{aligned} \tag{2.25}$$

Applying Gronwall inequality to (2.25) yields

$$\|(\varphi,\mathbf{u},\mathbf{m},\zeta)(t)\|_{H^N}^2 \le C\Lambda_1(T)e^{C\Lambda_2(T)}, \tag{2.26}$$

where

$$\begin{gathered} \Lambda_1(T) = \|(\varphi,\mathbf{u},\mathbf{m},\zeta)(0)\|_{H^N}^2 + \int_0^T \Big(\|G_1\|_{H^N}^2 + \|(\mathbf{G}_2,\mathbf{G}_3,G_4)\|_{H^{N-1}}^2\Big) d\tau, \\ \Lambda_2(T) = \int_0^T \Big(1+\|\mathbf{A}\|_{H^N}^2\Big) d\tau. \end{gathered}$$

By (2.25) and (2.26), we obtain (2.4). The proof of Theorem 2.1 is complete. □

Next, applying Theorem 2.1, it is easy to prove the existence and uniqueness of the smooth solution for the linearized system (2.1) by the standard methods. We will omit the proof for brevity.

Theorem 2.2. *Let $0 < T < \infty$. Assume that $(\varphi_0,\mathbf{u}_0,\mathbf{m}_0,\zeta_0) \in H^N(\mathbb{R}^3)$ for an integer $N \ge 3$. Also, suppose that* (2.2) *holds. Then, there exists a unique solution $(\varphi,\mathbf{m},\mathbf{u},\zeta)$ of the linearized system* (2.1) *satisfying* (2.3) *and* (2.4).

3. Proof of main results

We first prove the existence of the smooth local solutions to the reformulated system (1.17), (1.22) using Theorem 2.1. To this end, we define a set

$$\begin{aligned} \mathbf{Z}_M(0,t_0) = \big\{ &(\varphi,\mathbf{u},\mathbf{m},\zeta) \in X_N(0,t_0) : \|(\varphi,\mathbf{u},\mathbf{m},\zeta)\|_{X_N(0,t_0)}^2 \le M, \\ &0 < m_0^{-1} \le \bar\rho + \varphi(x,t),\ \bar{\mathbf{w}} + \mathbf{m}(x,t),\ \bar\theta + \zeta(x,t) \le m_0 \big\} \end{aligned}$$

for a positive constant $m_0 > 1$ and an integer $N \ge 3$, where

$$M = 4C_0\|(\varphi_0,\mathbf{u}_0,\mathbf{m}_0,\zeta_0)\|_{H^N}^2,\ C_0 \text{ is the constant determined in (2.4)}, \tag{3.1}$$

and

$$\begin{aligned} \|(\varphi,\mathbf{u},\mathbf{m},\zeta)\|_{X_N(0,t_0)}^2 = &\sup_{0\le t\le t_0} \|(\varphi,\mathbf{u},\mathbf{m},\zeta)(t)\|_{H^N}^2 + \int_0^{t_0} \|\nabla\varphi\|_{H^{N-1}}^2 d\tau \\ &+ \int_0^{t_0} \|\mathbf{u}\|_{H^{N+1}}^2 d\tau + \int_0^t \|\big(\nabla\mathbf{m},\nabla\zeta\big)\|_{H^N}^2 d\tau. \end{aligned} \tag{3.2}$$

Noticing that

$$Q_0(\mathbf{w},\theta) - Q_0(\bar{\mathbf{w}},\bar\theta) - \nabla_{\mathbf{w}} Q_0(\bar{\mathbf{w}},\bar\theta)\cdot\mathbf{m} - Q_{0\theta}'(\bar{\mathbf{w}},\bar\theta)\zeta = O(\mathbf{m}^2 + \zeta^2),$$

and using Lemma 1.4 and (1.24), we have

$$\int_0^{t_0} \|\varphi^{j-1} \operatorname{div} \mathbf{u}^{j-1}\|_{H^N}^2 d\tau \le C \sup_{0\le t\le t_0} \|\varphi^{j-1}\|_{H^N}^2 \int_0^{t_0} \|\nabla \mathbf{u}^{j-1}\|_{H^N}^2 d\tau \le CM^2,$$

and

$$\begin{aligned}
&\int_0^{t_0} \|Q_0(\mathbf{w}^{j-1}, \theta^{j-1}) - Q_0(\bar{\mathbf{w}}, \bar{\theta}) - \nabla_{\mathbf{w}} Q_0(\bar{\mathbf{w}}, \bar{\theta}) \cdot \mathbf{m}^{j-1} - Q'_{0\theta}(\bar{\mathbf{w}}, \bar{\theta})\zeta^{j-1}\|_{H^N}^2 d\tau \\
&\le C \sup_{0\le t\le t_0} \|(\mathbf{m}^{j-1}, \zeta^{j-1})\|_{H^N}^2 \int_0^{t_0} \|(\mathbf{m}^{j-1}, \zeta^{j-1})\|_{H^N}^2 d\tau \le CM^2 t_0
\end{aligned}$$

for $(\varphi^{j-1}, \mathbf{u}^{j-1}, \mathbf{m}^{j-1}, \zeta^{j-1}) \in \mathbf{Z}_N(0, t_0)$, $j = 1, 2, \dots$. Therefore, from (1.18) we obtain that

$$\begin{gathered}
G_1(\varphi^{j-1}, \mathbf{u}^{j-1}, \mathbf{m}^{j-1}, \zeta^{j-1}) \in L^2(0, t_0; H^N(\mathbb{R}^3)), \\
\int_0^{t_0} \|G_1(\varphi^{j-1}, \mathbf{u}^{j-1}, \mathbf{m}^{j-1}, \zeta^{j-1})\|_{H^N}^2 d\tau \le CM^2(1 + t_0).
\end{gathered} \tag{3.3}$$

By using Lemma 1.4 and (1.24), we have

$$\begin{aligned}
&\int_0^{t_0} \|(\mathbf{u}^{j-1}, \nabla)\mathbf{u}^{j-1}\|_{H^{N-1}}^2 d\tau \le C \sup_{0\le t\le t_0} \|\mathbf{u}^{j-1}\|_{H^N}^2 \int_0^{t_0} \|\nabla \mathbf{u}^{j-1}\|_{H^N}^2 d\tau \le CM^2, \\
&\int_0^{t_0} \|(Q_0(\mathbf{w}, \theta) - Q_0(\bar{\mathbf{w}}, \bar{\theta}))\mathbf{u}\|_{H^{N-1}}^2 d\tau \\
&\le C \sup_{0\le t\le t_0} \|(\mathbf{m}^{j-1}, \zeta^{j-1})\|_{H^N}^2 \sup_{0\le t\le t_0} \|\mathbf{u}^{j-1}\|_{H^{N-1}}^2 t_0 \le CM^2 t_0, \\
&\int_0^{t_0} \|h_1(\varphi^{j-1}, \zeta^{j-1}, \mathbf{m}^{j-1})\nabla\varphi^{j-1}\|_{H^{N-1}}^2 d\tau \\
&\le C \sup_{0\le t\le t_0} \|(\varphi^{j-1}, \zeta^{j-1}, \mathbf{m}^{j-1})\|_{H^N}^2 \int_0^{t_0} \|\nabla\varphi^{j-1}\|_{H^N}^2 d\tau \le CM^2, \\
&\int_0^{t_0} \|h_2(\varphi^{j-1}, \zeta^{j-1}, \mathbf{m}^{j-1})\nabla\zeta^{j-1}\|_{H^{N-1}}^2 d\tau \\
&\le C \sup_{0\le t\le t_0} \|(\varphi^{j-1}, \zeta^{j-1}, \mathbf{m}^{j-1})\|_{H^N}^2 \int_0^{t_0} \|\nabla\zeta^{j-1}\|_{H^N}^2 d\tau \le CM^2, \\
&\sum_{i=1}^n \int_0^{t_0} \|h_{3i}(\varphi^{j-1}, \zeta^{j-1}, \mathbf{m}^{j-1})\nabla m_i^{j-1}\|_{H^{N-1}}^2 d\tau \\
&\le C \sup_{0\le t\le t_0} \|(\varphi^{j-1}, \zeta^{j-1}, \mathbf{m}^{j-1})\|_{H^N}^2 \int_0^{t_0} \|\nabla \mathbf{m}^{j-1}\|_{H^N}^2 d\tau \le CM^2, \\
&\int_0^{t_0} \|h_4(\varphi^{j-1})\Big(\mu\Delta\mathbf{u}^{j-1} + (\mu + \nu)\nabla \operatorname{div} \mathbf{u}^{j-1}\Big)\|_{H^{N-1}}^2 d\tau \\
&\le C \sup_{0\le t\le t_0} \|\varphi^{j-1}\|_{H^N}^2 \int_0^{t_0} \|\nabla \mathbf{u}^{j-1}\|_{H^N}^2 d\tau \le CM^2, \\
&\int_0^{t_0} \|h_4(\varphi^{j-1})\mathbf{u}^{j-1}\|_{H^{N-1}}^2 d\tau \le C \sup_{0\le t\le t_0} \|\varphi^{j-1}\|_{H^N}^2 \sup_{0\le t\le t_0} \|\mathbf{u}^{j-1}\|_{H^{N-1}}^2 t_0 \le CM^2 t_0
\end{aligned}$$

for $(\varphi^{j-1},\mathbf{u}^{j-1},\mathbf{m}^{j-1},\zeta^{j-1}) \in \mathbf{Z}_N(0,t_0)$, $j=1,2,\dots$. Therefore, from (1.19) we obtain

$$\begin{gathered}\mathbf{G}_2(\varphi^{j-1},\mathbf{u}^{j-1},\mathbf{m}^{j-1},\zeta^{j-1}) \in L^2(0,t_0;H^{N-1}(\mathbb{R}^3)),\\ \int_0^{t_0}\|\mathbf{G}_2(\varphi^{j-1},\mathbf{u}^{j-1},\mathbf{m}^{j-1},\zeta^{j-1})\|_{H^{N-1}}^2 d\tau \le CM^2(1+t_0).\end{gathered} \tag{3.4}$$

By similar arguments, we obtain from (1.20) and (1.21), respectively, that

$$\begin{gathered}\mathbf{G}_3(\varphi^{j-1},\mathbf{u}^{j-1},\mathbf{m}^{j-1},\zeta^{j-1}),\\ \int_0^{t_0}\|\mathbf{G}_3(\varphi^{j-1},\mathbf{u}^{j-1},\mathbf{m}^{j-1},\zeta^{j-1})\|_{H^{N-1}}^2 d\tau \le CM^2(1+t_0)\end{gathered} \tag{3.5}$$

and

$$\begin{gathered}G_4(\varphi^{j-1},\mathbf{u}^{j-1},\mathbf{m}^{j-1},\zeta^{j-1}),\\ \int_0^{t_0}\|G_4(\varphi^{j-1},\mathbf{u}^{j-1},\mathbf{m}^{j-1},\zeta^{j-1})\|_{H^{N-1}}^2 d\tau \le CM^3(1+t_0).\end{gathered} \tag{3.6}$$

Also, we have

$$\mathbf{u}^{j-1} \in L^\infty(0,t_0;H^N(\mathbb{R}^3)), \quad \sup_{0\le t\le t_0}\|\mathbf{u}^{j-1}\|_{H^N} \le M. \tag{3.7}$$

From (3.3)-(3.7) and Theorem 2.2, there exists a unique solution $(\varphi^j,\mathbf{u}^j,\mathbf{m}^j,\zeta^j) \in X_N(0,t_0)$ to the linearized system (1.23) satisfying

$$\begin{aligned}&\|(\varphi^j,\mathbf{u}^j,\mathbf{m}^j,\zeta^j)(t)\|_{H^N}^2 + \int_0^t \|\nabla\varphi^j\|_{H^{N-1}}^2 d\tau + \int_0^t \|\nabla\mathbf{u}^j\|_{H^{N+1}}^2 d\tau\\ &+ \int_0^t \|\big(\nabla\mathbf{m}^j,\nabla\zeta^j\big)\|_{H^N}^2 d\tau\\ &\le C_0\Big[\|(\varphi_0,\mathbf{u}_0,\mathbf{m}_0,\zeta_0)\|_{H^N}^2 + \int_0^{t_0}\Big(\|G_1^{j-1}\|_{H^N}^2 + \|(\mathbf{G}_2^{j-1},\mathbf{G}_3^{j-1},G_4^{j-1})\|_{H^{N-1}}^2\Big)d\tau\Big]\\ &\quad\times\Big[1+t_0\Big(1+\sup_{0\le t\le t_0}\|\mathbf{u}^{j-1}\|_{H^N}^2\Big)e^{c_0t_0\Big(1+\sup_{0\le t\le t_0}\|\mathbf{u}^{j-1}\|_{H^N}^2\Big)}\Big]\end{aligned} \tag{3.8}$$

for any $t\in[0,t_0]$, where

$$\begin{gathered}G_1^{j-1} = G_1(\varphi^{j-1},\mathbf{u}^{j-1},\mathbf{m}^{j-1},\zeta^{j-1}), \quad \mathbf{G}_2^{j-1} = \mathbf{G}_2(\varphi^{j-1},\mathbf{u}^{j-1},\mathbf{m}^{j-1},\\ \zeta^{j-1}), \quad \mathbf{G}_3^{j-1} = \mathbf{G}_3(\varphi^{j-1},\mathbf{u}^{j-1},\mathbf{m}^{j-1},\zeta^{j-1}),\\ G_4^{j-1} = G_4(\varphi^{j-1},\mathbf{u}^{j-1},\mathbf{m}^{j-1},\zeta^{j-1}),\end{gathered}$$

and we used that

$$\int_0^{t_0}\Big(1+\|\mathbf{u}^{j-1}\|_{H^N}^2\Big)d\tau \le t_0\Big(1+\sup_{0\le t\le t_0}\|\mathbf{u}^{j-1}\|_{H^N}^2\Big).$$

Therefore, choosing t_0 small such that

$$t_0\Big(1+\sup_{0\le t\le t_0}\|\mathbf{u}^{j-1}\|_{H^N}^2\Big)e^{c_0t_0\Big(1+\sup_{0\le t\le t_0}\|\mathbf{u}^{j-1}\|_{H^N}^2\Big)} \le 1$$

and

$$\int_0^{t_0}\Big(\|G_1^{j-1}\|_{H^N}^2 + \|(\mathbf{G}_2^{j-1},\mathbf{G}_3^{j-1},G_4^{j-1})\|_{H^{N-1}}^2\Big)d\tau \le \|(\varphi_0,\mathbf{u}_0,\mathbf{m}_0,\zeta_0)\|_{H^N}^2,$$

which is possible because of (3.3)-(3.7), from (3.8) we obtain

$$\begin{aligned} &\|(\varphi^j,\mathbf{u}^j,\mathbf{m}^j,\zeta^j)(t)\|_{H^N}^2+\int_0^t\|\nabla\varphi^j\|_{H^{N-1}}^2d\tau+\int_0^t\|\nabla\mathbf{u}^j\|_{H^{N+1}}^2d\tau\\ &+\int_0^t\|\big(\nabla\mathbf{m}^j,\nabla\zeta^j\big)\|_{H^N}^2d\tau\\ &\le 4C_0\|(\varphi_0,\mathbf{u}_0,\mathbf{m}_0,\zeta_0)\|_{H^N}^2 \end{aligned} \tag{3.9}$$

for any $t\in[0,t_0]$. By using (1.10), $\bar\rho+\varphi(x,t)=\rho_0(x)+(\varphi(x,t)-\varphi(x,0))$, $\bar{\mathbf{w}}+\mathbf{m}(x,t)=\mathbf{w}_0(x)+(\mathbf{m}(x,t)-\mathbf{m}(x,0))$, and $\bar\zeta+\varphi(x,t)=\theta_0(x)+(\zeta(x,t)-\zeta(x,0))$, we can choose the small $t_0>0$ such that

$$m_0^{-1}\le\bar\rho+\varphi(x,t),\quad \bar{\mathbf{w}}+\mathbf{m}(x,t),\ \bar\theta+\zeta(x,t)\le m_0. \tag{3.10}$$

By (3.9), (3.10), (3.1) and (3.2), we obtain

$$(\varphi^j,\mathbf{u}^j,\mathbf{m}^j,\zeta^j)\in\mathbf{Z}_N(0,t_0)$$

for $j=1,2,\dots$. Moreover, by using (3.9) and (1.11), we have

$$\begin{gathered} \{(\varphi^j,\mathbf{u}^j,\mathbf{m}^j,\zeta^j)\}_{j=1}^\infty\quad\text{is bounded in } L^\infty(0,t_0;H^N(\mathbb{R}^3)),\\ \{\nabla\varphi^j\}_{j=1}^\infty\quad\text{is bounded in } L^2(0,t_0;H^{N-1}(\mathbb{R}^3)),\\ \{\mathbf{u}^j\}_{j=1}^\infty\quad\text{is bounded in } L^2(0,t_0;H^{N+1}(\mathbb{R}^3)),\\ \{\big(\nabla\mathbf{m}^j,\nabla\zeta^j\big)\}_{j=1}^\infty\quad\text{is bounded in } L^2(0,t_0;H^N(\mathbb{R}^3)). \end{gathered} \tag{3.11}$$

Also, by using (3.11) and (3.3)-(3.7), from (1.23) we obtain

$$\{\partial_t(\varphi^j,\mathbf{u}^j,\mathbf{m}^j,\zeta^j)\}_{j=1}^\infty\quad\text{is bounded in } L^2(0,t_0;H^{N-1}(\mathbb{R}^3)). \tag{3.12}$$

By using (3.11), (3.12) and the compactness result, there exist the subsequences of $\{(\varphi^j,\mathbf{u}^j,\mathbf{m}^j,\zeta^j)\}$ (denoting them as $\{(\varphi^j,\mathbf{u}^j,\mathbf{m}^j,\zeta^j)\}$ still) such that when $j\to\infty$, it holds that

$$\begin{gathered} (\varphi^j,\mathbf{u}^j,\mathbf{m}^j,\zeta^j)\to(\varphi,\mathbf{u},\mathbf{m},\zeta)\quad *\text{-weak in } L^\infty(0,t_0;H^N(\mathbb{R}^3)),\\ \nabla\varphi^j\to\nabla\varphi\quad\text{weak in } L^2(0,t_0;H^{N-1}(\mathbb{R}^3)),\\ \mathbf{u}^j\to\mathbf{u}\quad\text{weak in } L^2(0,t_0;H^{N+1}(\mathbb{R}^3)),\\ \big(\nabla\mathbf{m}^j,\nabla\zeta^j\big)\to\big(\nabla\mathbf{m},\nabla\zeta\big)\quad\text{weak in } L^2(0,t_0;H^N(\mathbb{R}^3)),\\ (\varphi^j,\mathbf{u}^j,\mathbf{m}^j,\zeta^j)\to(\varphi,\mathbf{u},\mathbf{m},\zeta)\quad\text{strong in } L^2(0,t_0;H^{N-1}_{\rm loc}(\mathbb{R}^3)), \end{gathered} \tag{3.13}$$

and

$$\begin{gathered} (\varphi,\mathbf{u},\mathbf{m},\zeta)\in L^\infty(0,t_0;H^N(\mathbb{R}^3)),\quad \nabla\varphi\in L^2(0,t_0;H^{N-1}(\mathbb{R}^3)),\\ \mathbf{u}\in L^2(0,t_0;H^{N+1}(\mathbb{R}^3)),\quad \big(\nabla\mathbf{m},\nabla\zeta\big)\in L^2(0,t_0;H^N(\mathbb{R}^3)). \end{gathered} \tag{3.14}$$

By using (3.13) and (1.19)-(1.21), we have

$$\begin{gathered}
\int_0^{t_0}\int \mathbf{u}^{j-1}\cdot\nabla\varphi^j z\eta(t)\,dx\,dt \to \int_0^{t_0}\int \mathbf{u}\cdot\nabla\varphi z\eta(t)\,dx\,dt,\\
\int_0^{t_0}\int G_1(\varphi^{j-1},\mathbf{u}^{j-1},\mathbf{m}^{j-1},\zeta^{j-1})z\eta(t)\,dx\,dt \to \int_0^{t_0}\int G_1(\varphi,\mathbf{u},\mathbf{m},\zeta)z\eta(t)\,dx\,dt,\\
\int_0^{t_0}\int \mathbf{G}_2(\varphi^{j-1},\mathbf{u}^{j-1},\mathbf{m}^{j-1},\zeta^{j-1})\cdot\mathbf{z}\eta(t)\,dx\,dt \to \int_0^{t_0}\int \mathbf{G}_2(\varphi,\mathbf{u},\mathbf{m},\zeta)\cdot\mathbf{z}\eta(t)d\mathbf{x}dt,\\
\int_0^{t_0}\int \mathbf{G}_3(\varphi^{j-1},\mathbf{u}^{j-1},\mathbf{m}^{j-1},\zeta^{j-1})\cdot\mathbf{z}\eta(t)\,dx\,dt \to \int_0^{t_0}\int \mathbf{G}_3(\varphi,\mathbf{u},\mathbf{m},\zeta)\cdot\mathbf{z}\eta(t)d\mathbf{x}dt,\\
\int_0^{t_0}\int G_4(\varphi^{j-1},\mathbf{u}^{j-1},\mathbf{m}^{j-1},\zeta^{j-1})z\eta(t)\,dx\,dt \to \int_0^{t_0}\int G_4(\varphi,\mathbf{u},\mathbf{m},\zeta)z\eta(t)\,dx\,dt
\end{gathered} \tag{3.15}$$

for all $\mathbf{z}\in\{C_0^\infty(\mathbb{R}^3)\}^3$, $z\in C_0^\infty(\mathbb{R}^3)$ and $\eta\in C_0^\infty(0,t_0)$. Moreover, by using (3.13) and (3.15), it is easy to check that $(\varphi,\mathbf{u},\mathbf{m},\zeta)$ is a solution to the system (1.17), (1.22). Then, setting

$$\rho=\bar\rho+\varphi,\quad \mathbf{w}=\bar{\mathbf{w}}+\mathbf{m},\quad \theta=\bar\theta+\zeta,$$

and by using (1.17), (1.22), (1.18)-(1.21), and (3.14), we know that $(\rho,\mathbf{u},\mathbf{w},\theta)$ is a solution to the Cauchy problem (1.1), (1.2) , (1.8) satisfying (1.12). Also, the estimate (1.13) follows from (3.9) and (3.13). The proof for the uniqueness of the local solution is standard, so we will omit it for brevity. The proof of Theorem 1.1 is complete.

Theorem 1.3 follows form (3.13) and the uniqueness of the solutions to the Cauchy problem (1.1), (1.2) , (1.8).

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