Electronic Journal of Differential Equations, Vol. 2025 (2025), No. 24, pp. 1–21.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2025.24

EXISTENCE OF POSITIVE SOLUTIONS FOR SYSTEMS OF QUASILINEAR SCHRÖDINGER EQUATIONS

AYESHA BAIG, ZHOUXIN LI

Abstract. In this article, we study the existence of standing wave solutions for the quasilinear Schrödinger system

$$\begin{gathered} -\varepsilon^2\Delta u + W(x)u - \kappa\varepsilon^2\Delta(u^2)u = Q_u(u,v) \quad \text{in } \mathbb{R}^N, \\ -\varepsilon^2\Delta v + V(x)v - \kappa\varepsilon^2\Delta(v^2)v = Q_v(u,v) \quad \text{in } \mathbb{R}^N, \\ u,v>0 \quad \text{in } \mathbb{R}^N, \quad u,v \in H^1(\mathbb{R}^N). \end{gathered}$$

where $N \geq 3$, $\kappa > 0$, $\varepsilon > 0$, $W, V : \mathbb{R}^N \to \mathbb{R}$ are continuous functions that fall into two classes of potentials. To overcome the lack of differentiability, we use the dual approach developed by Colin–Jeanjean. The existence of solutions is obtained using Del Pino–Felmer's penalization technique with an adaptation of Alves' arguments [1].

1. Introduction

In this article, we examine the existence of solutions to the system of quasilinear Schrödinger equations (QLSE):

$$\begin{gathered} -\varepsilon^2\Delta u + W(x)u - \kappa\varepsilon^2\Delta(u^2)u = Q_u(u,v) \quad \text{in } \mathbb{R}^N, \\ -\varepsilon^2\Delta v + V(x)v - \kappa\varepsilon^2\Delta(v^2)v = Q_v(u,v) \quad \text{in } \mathbb{R}^N, \\ u,v>0 \quad \text{in } \mathbb{R}^N, \quad u,v \in H^1(\mathbb{R}^N). \end{gathered} \tag{1.1}$$

where $\mathbb{N} \geq 3$, $\kappa > 0$, $\varepsilon > 0$, $W, V : \mathbb{R}^N \to \mathbb{R}$ are continuous functions that fall into two classes of potentials introduced in [1]. The functions $Q_u, Q_v : {\mathbb{R}_+}^2 \to \mathbb{R}$ are continuous functions denoting partial derivatives of the function $Q : \mathbb{R}+^2 \to \mathbb{R}$, which belongs to the class of C^1 and is p-homogeneous.

Systems of type (1.1) are related to various applications in hydrodynamics, Heidelberg ferromagnetism, Magnus theory, condensed matter theory, and dissipative quantum mechanics.

2020 *Mathematics Subject Classification.* 35Q55, 58E05, 58E30.
Key words and phrases. Quasilinear Schrödinger system; Cerami sequence; dual approach; positive solution.

Submitted April 17, 2024. Published March 3, 2025.

By a simple change of variables, system (1.1) is equivalent to the system

$$
\begin{gathered}
-\Delta u+W(\varepsilon x)u-\kappa\Delta(u^2)u=Q_u(u,v)\quad \text{in } \mathbb{R}^N,\\
-\Delta v+V(\varepsilon x)v-\kappa\Delta(v^2)v=Q_v(u,v)\quad \text{in } \mathbb{R}^N,\\
u,v>0\quad \text{in } \mathbb{R}^N
\end{gathered}
\tag{1.2}
$$

Considering the potential values of κ, assumptions regarding potentials, and various nonlinearity types, numerous studies have explored the existence of solutions for system (1.1), particularly when $\kappa\neq 0$, as seen in [5, 6, 8, 10, 11, 19].

Recently, numerous articles have examined the scalar equation:

$$
-\varepsilon^2\Delta u+V(x)u-\kappa\varepsilon^2\Delta(u^2)u=g(u),\quad \text{in } \mathbb{R}^N, \tag{1.3}
$$

where $N\geq 3$, $\kappa\in\mathbb{R}$, $\varepsilon>0$ are real parameters, and $V:\mathbb{R}^N\to\mathbb{R}$ satisfies certain geometries. The function $g:\mathbb{R}\to\mathbb{R}$ is continuous. This kind of equation frequently appears in various models, notably in connection with standing wave phenomena within the quasilinear Schrödinger equation.

$$
-i\varepsilon\frac{\partial z}{\partial t}=-\varepsilon^2\Delta z+F(x)z-\kappa\varepsilon^2\Delta\rho(|z|^2)\rho'(|z|^2)z-f(|z|^2)z,\quad \text{for all } x\in\mathbb{R}^N. \tag{1.4}
$$

where $z:\mathbb{R}\times\mathbb{R}^N\to\mathbb{C}$, F represents the potential, κ is a real constant, and f,ρ are real-valued functions. When $\rho(s)=s$, this equation arises in fields such as fluid mechanics, plasma physics, dissipative mechanics, and condensed matter theory. The stationary solutions of (1.4) take the form

$$
z(t,x)=\exp\Big(-\frac{iEt}{\varepsilon}\Big)u(x),\quad E\in\mathbb{R}, \tag{1.5}
$$

where u represents the solution to equation (1.3) with $V(x)=F(x)-E$ and $g(u)=f(u^2)u$. For the physical motivation of equation (1.4), readers are referred to [13, 12, 14] and references therein.

The semilinear scenario, identified by $\kappa=0$, has undergone thorough examination in recent years. For instance, del Pino and Felmer [17] studied the problem:

$$
-\varepsilon^2\Delta u+V(x)u=q(u),\quad \text{in } \mathbb{R}^N,\quad u>0\quad \text{in } \mathbb{R}^N,\quad u\in H^1(\mathbb{R}^N). \tag{1.6}
$$

where $\varepsilon>0$, $N\geq 3$, and $q:\mathbb{R}\to\mathbb{R}$ is a subcritical nonlinearity. The function V is a locally Hölder continuous potential satisfying

$$
0<\alpha=\inf_{x\in\mathbb{R}^N}V(x)\leq V_0=\inf_{\Omega}V(x)<\min_{\partial\Omega}V(x). \tag{1.7}
$$

In [17], the authors introduced the penalization method and proved that if V satisfies (1.7), then (1.6) has a solution u_ε that concentrates at a minimum of V. Alves, do Ó, and Souto [3] also studied (1.6) and proved the same result as in [17] for V satisfying (1.7), with the subcritical nonlinearity perturbed by a critical term.

Alves [1] studied problem (1.6) with the nonlinearity $q:\mathbb{R}\to\mathbb{R}$ being continuous and having subcritical or critical growth. Alves [1] introduced for the first time two interesting classes of potential V, namely:

Class 1: The potential V satisfies the Palais-Smale (PS) condition, with the following conditions:

- (A1) There exists $V_0>0$ such that $V(x)\geq V_0$, for all $x\in\mathbb{R}^N$, where $V_0=\inf_{\mathbb{R}^N}V(x)$.
- (A2) $V\in C^2(\mathbb{R}^N)$ and $V,\frac{\partial V}{\partial x_i},\frac{\partial^2 V}{\partial x_i\partial x_j}$ are bounded across $\mathbb{R}^N$, for all $i,j\in\{1,2,3,\dots,N\}$.

(A3) V adheres to the Palais-Smale (PS) condition, that is, if $(x_n) \subset \mathbb{R}^N$, with $(V(x_n))$ being bounded and $\nabla V(x_n) \to 0$, then (x_n) possesses a convergent subsequence.

Class 2: The potential V lacks critical points along the boundary of some bounded domain. Within this class of potentials, V satisfies (A1), (A2) and

(A4) there exists a domain $\Lambda \subset \mathbb{R}^N$ where $\nabla V(x) \neq 0$ for all $x \in \partial\Lambda$.

Given that V falls into either Class 1 or Class 2 and taking into account certain conditions met by the nonlinearity, the author demonstrated the existence of a positive solution for $\varepsilon > 0$ sufficiently small.

Alves [2] explored the presence and concentration of solutions for the system derived from (1.1) with $\kappa = 0$:

$$\begin{gathered}
-\varepsilon^2 \Delta u + W(x)u = Q_u(u,v) \quad \text{in } \mathbb{R}^N, \\
-\varepsilon^2 \Delta v + V(x)v = Q_v(u,v) \quad \text{in } \mathbb{R}^N, \\
u, v > 0 \quad \text{in } \mathbb{R}^N, \quad u, v \in H^1(\mathbb{R}^N).
\end{gathered} \tag{1.8}$$

where the functions $W, V : \mathbb{R}^N \to \mathbb{R}$ are Hölder continuous satisfying $W(x), V(x) \geq \alpha > 0$ in $\mathbb{R}^N$ and the condition:

(5) There exists an open and bounded set $\Lambda \subset \mathbb{R}^N$, with $x_0 \in \Lambda$ and $\rho > 0$, such that $W(x), V(x) \geq \rho$, for all $x \in \partial\Lambda$ and $W(x_0), V(x_0) < \rho$.

Severo and Silva [19] employed the variational approach within an appropriate Orlicz space to examine a system of type (1.1) with $\kappa = 1$. Recently, Arruda-Figueiredo and Nascimento [4] considered the two classes of potentials introduced by Alves in [1] and showed the existence of solutions for the system (1.8).

Motivated by these works, and mainly by [1, 2, 4, 17, 19], we study system (1.1) for $\kappa = 1$. We shall refer to the potential V as belonging to Class 1 when it satisfies (A6)–(A8), and as belonging to Class 2 when it satisfies (A6), (A7), (A9) where

(A6) There exist $\mathcal{V}_\infty, \mathcal{V}_0 > 0$ such that $\mathcal{V}_0 \leq \mathcal{V}(x) \leq \mathcal{V}_\infty$, for all $x \in \mathbb{R}^N$, where $\mathcal{V}_0 = \inf_{\mathbb{R}^N} \mathcal{V}(x)$.

(A7) $\mathcal{V} \in C^2(\mathbb{R}^N)$ and $\mathcal{V}$, $\frac{\partial \mathcal{V}}{\partial x_i}$, $\frac{\partial^2 \mathcal{V}}{\partial x_i \partial x_j}$ are bounded in $\mathbb{R}^N$, for $i, j \in \{1, 2, \dots, N\}$.

(A8) $\mathcal{V}$ satisfies the PS-condition, that is, if $(x_n) \subset \mathbb{R}^N$, such that $\mathcal{V}(x_n)$ is bounded and $\nabla \mathcal{V}(x_n) \to 0$, then (x_n) possesses a convergent subsequence.

(A9) There exists a domain $\Lambda \subset \mathbb{R}^N$ exists where $\nabla \mathcal{V}(x) \neq 0$ for every $x \in \partial\Lambda$.

Notation Let $H^1(\mathbb{R}^N)$ be the Sobolev space with norm

$$\|u\|_{H^1(\mathbb{R}^N)} := \Big(\|u\|^2_{L^2(\mathbb{R}^N)} + \|\nabla u\|^2_{L^2(\mathbb{R}^N)} \Big)^{1/2}.$$

Let 2^* be the Sobolev critical exponent

$$2^* = \frac{2N}{N-2} \quad \text{for } N > 2.$$

Now, we present the assumptions on the function Q. Let $\mathbb{R}^2_+ := [0, +\infty) \times [0, +\infty)$, we assume that the nonlinearity $Q \in C^1(\mathbb{R}^2_+, \mathbb{R})$ is p-homogeneous with subcritical growth. More precisely, our hypotheses on Q are:

(A10) There exists $p \in (4, 2.2^*)$, such that $Q(tu, tv) = t^p Q(u, v)$ for all $t > 0$, $(u, v) \in \mathbb{R}^2_+$, where $2^* = \frac{2N}{N-2}$ and $N \geq 3$.

(A11) There exists $C > 0$ such that

$$|Q_u(u,v)| + |Q_v(u,v)| \le C(|u|^{p-1} + |v|^{p-1})$$

for all $(u,v) \in \mathbb{R}^2_+$.

(A12) $Q_u(0,1) = 0, Q_v(1,0) = 0$.

(A13) $Q_u(1,0) = 0, Q_v(0,1) = 0$.

(A14) $Q(u,v) > 0$ for each $u, v > 0$.

(A15) $Q_u(u,v), Q_v(u,v) \ge 0$ for each $(u,v) \in \mathbb{R}^2_+$.

Since Q is a homogeneous function of degree $p > 4$, it follows that

$$pQ(u,v) = uQ_u(u,v) + vQ_v(u,v).$$

Moreover, ∇Q is a homogeneous function of degree $p - 1$.

A prototype of function Q that satisfies (A11)–(A15) is

$$H(u,v) := a|u|^p + \sum_{\alpha_i+\beta_i=p} b_i|u|^{\alpha_i}|v|^{\beta_i} + c|v|^p,$$

where $a, b_i, c \in \mathbb{R}$, $\alpha_i + \beta_i = p$, $\alpha_i, \beta_i \ge 1$, $i \in I$, with I denoting a finite subset of $\mathbb{N}$.

Definition 1.1. We say that the pair $(u,v) \in H^1(\mathbb{R}^N) \cap L^\infty_{\rm loc}(\mathbb{R}^N)$ is a solution to (1.1) if $u, v > 0$ almost everywhere in $\mathbb{R}^N$ and satisfies

$$\varepsilon^2 \int_{\mathbb{R}^N} (1 + 2u^2)\nabla u \nabla \varphi + 2\int_{\mathbb{R}^N} |\nabla u|^2 u\varphi + \int_{\mathbb{R}^N} W(\varepsilon x)u\varphi = \int_{\mathbb{R}^N} Q_u(u,v)\varphi,$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, and

$$\varepsilon^2 \int_{\mathbb{R}^N} (1 + 2v^2)\nabla v \nabla \phi + 2\int_{\mathbb{R}^N} |\nabla v|^2 v\phi + \int_{\mathbb{R}^N} V(\varepsilon x)v\phi = \int_{\mathbb{R}^N} Q_v(u,v)\phi,$$

for all $\phi \in C_0^\infty(\mathbb{R}^N)$.

Theorem 1.2. *Assume that W and V satisfy* (A6) *and that either W or V falls into Class 1 or class 2. Furthermore, suppose that Q satisfies* (A10)–(A15)*. Then system* (1.1) *has a solution for each $\varepsilon \in (0, \varepsilon_0)$. Moreover, $u_\varepsilon, v_\varepsilon \in C^{1,\alpha}_{\rm loc}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, and there exist constants $C_1, C_2, C_3, C_4 > 0$ satisfying*

$$u_\varepsilon(x) \le C_1 \exp(-C_2|x/\varepsilon|), \quad v_\varepsilon(x) \le C_3 \exp(-C_4|x/\varepsilon|), \quad \forall x \in \mathbb{R}^N.$$

Remark 1.3. Theorem 1.2 extends the findings of Arruda-Figueiredo and Nascimento [4, Theorem 1.1] in at least two ways:

(1) The first is that we consider ($\kappa \neq 0$), which leads to entirely different estimates (regarding the functional and the solution to the auxiliary problem) when compared to the case of ($\kappa = 0$).

(2) The second difference is that, unlike [4, Theorem 1.1], we do not require both potentials V and W to belong to the same Class 1 or Class2. We only require that one of the potentials belongs to one of these classes and that the other satisfies condition (A6).

We recall that $J \in C^1(E, \mathbb{R})$ satisfies the Cerami condition on level b, denoted by the $(\mathrm{Ce})_\mathrm{b}$ condition, if any sequence $(u_n) \subset E$ for which

(i) $J(u_n) \to b$,

(ii) $\|J'(u_n)\|_{E'}(\|u_n\| + 1) \to 0$ as $n \to \infty$, possesses a convergent subsequence.

J satisfies the Cerami condition, denoted by (Ce), if it satisfies $(\mathrm{Ce})_{\mathrm{b}}$ for every $b \in \mathbb{R}$. We say that $(u_n) \subset E$ is a $(\mathrm{Ce})_{\mathrm{b}}$ sequence if it satisfies (i) and (ii). We also say that $(u_n) \subset E$ is a (Ce) sequence if it is a $(\mathrm{Ce})_{\mathrm{b}}$ sequence for some $b \in \mathbb{R}$.

To demonstrate our primary finding, we will use the following variant of the Mountain Pass Theorem.

Theorem 1.4 ([20])**.** *Let E be a real Banach space and $J : E \to \mathbb{R}$ be a functional of class C^1. Let S be a closed subset of E, which disconnects (arcwise) E into distinct connected components E_1 and E_2. Suppose further that $J(0) = 0$ and*

1. *$0 \in E_1$ and there exists $\alpha > 0$ such that $J(v) \geq \alpha$ for all $v \in S$.*
2. *There exists $e \in E_2$ such that $J(e) < 0$.*

Then, J possesses a $(Ce)_c$ sequence with $c \geq \alpha$ given by

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)) \geq \alpha,$$

where

$$\Gamma := \{\gamma \in C([0, 1], E) : \gamma(0) = 0, \gamma(1) \in J^{-1}((-\infty, 0]) \cap E_2\}.$$

This article is structured as follows: In Section 2, we reframe the system and introduce an auxiliary system. Initially, we propose an equivalent system through an appropriate variable transformation as discussed in [7, 15]. To address the issue of compactness, we then define the auxiliary system (2.10), following the methodology presented in [8]. Section 3 focuses on the analysis of the positive solution of the auxiliary system (2.10), employing a variant of the Mountain Pass Theorem (Theorem 1.4) that does not require the (PS) condition. This is used to generate a Cerami sequence at the mountain-pass level. Subsequently, we adapt Del Pino's strategies to identify a solution for the auxiliary problem (2.10) and examine certain solution properties of the auxiliary system. Finally, Section 4 is dedicated to the proof of Theorem 1.2.

2. Reformulation of the system and the auxiliary system

Assuming (A6), we consider the closed subspace of $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$,

$$X = \Big\{(w, z) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} [W(\varepsilon x)w^2 + V(\varepsilon x)z^2] < \infty\Big\}$$

which is a Hilbert space when endowed with the norm

$$\|(w, z)\|^2 = \int_{\mathbb{R}^N} [|\nabla w|^2 + |\nabla z|^2 + W(\varepsilon x)w^2 + V(\varepsilon x)z^2].$$

The natural functional associated with (1.2) is

$$\begin{aligned} J_\varepsilon(u, v) = \frac{1}{2} \int_{\mathbb{R}^N} &[(1 + 2u^2)|\nabla u|^2 + (1 + 2v^2)|\nabla v|^2 + W(\varepsilon x)u^2 + V(\varepsilon x)v^2] \\ &- \int_{\mathbb{R}^N} Q(\varepsilon x, u, v), \end{aligned}$$

which is not well-defined in X. To address this challenge, we adopt the variable transformation proposed by Colin and Jeanjean [7], and by Liu, Wang, and Wang [15].

For this we consider $w = f^{-1}(u)$ and $z = f^{-1}(v)$, where $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f'(t) = \begin{cases} \frac{1}{\sqrt{1+2f^2(t)}}, & \text{in } [0, +\infty), \\ -f'(-t), & \text{in } (-\infty, 0]. \end{cases} \tag{2.1}$$

Lemma 2.1. *The function f satisfies the following properties:*

(1) *f is uniquely defined, C^∞, and invertible.*
(2) *$|f'(t)| \le 1$ and $|f(t)| \le |t|$, for $t \in \mathbb{R}$.*
(3) *$\frac{f(t)}{t} \to 1$ as $t \to 0$.*
(4) *$\frac{f(t)}{\sqrt{t}} \to \sqrt[4]{2}$ as $t \to \infty$.*
(5) *$\frac{|f(t)|}{2} \le |t| f'(t) \le |f(t)|$.*
(6) *$|f(t)| \le 2^{1/4} |t|^{1/2}$, for all $t \in \mathbb{R}$.*
(7) *$\frac{f^2(t)}{2} \le t f(t) f'(t) \le f^2(t)$, for all $t \in \mathbb{R}$.*
(8) *There exist constants $C_1, C_2 > 0$, such that:*

$$|f(t)| \ge C_1 |t|, \quad \text{if } |t| \le 1;$$

$$|f(t)| \ge C_2 |t|^{1/2}, \quad \text{if } |t| \ge 1.$$

(9) *$|f(t) f'(t)| \le \frac{1}{\sqrt{2}}$, for all $t \in \mathbb{R}$.*
(10) *The function $t \to f^q(s) f'(s)$ is increasing on $(0, \infty)$ for each $q > 1$.*

With the exception of property (10), all other properties are derived from [9, Lemma 2.1] (see also [7, 16, 15]). For property (10), refer to Reference [8, Remark 3.1].

Following the variable transformation

$$I_\varepsilon(w, z) := J_\varepsilon(f(w), f(z)),$$

we obtain the functional

$$\begin{aligned} I_\varepsilon(w, z) = \frac{1}{2} \int_{\mathbb{R}^N} &[|\nabla w|^2 + |\nabla z|^2 + W(\varepsilon x) f(w)^2 + V(\varepsilon x) f(z)^2] \\ &- \int_{\mathbb{R}^N} Q(\varepsilon x, f(w), f(z)), \end{aligned}$$

which is well-defined in X. More precisely, I_ε is of class $C^1(X, \mathbb{R})$ (because of (A6), (A11), and the properties of f). The Gateaux derivative is

$$\begin{aligned} &I'_\varepsilon(w, z)(\phi, \varphi) \\ &= \int_{\mathbb{R}^N} [\nabla w \nabla \phi + \nabla z \nabla \varphi] + \int_{\mathbb{R}^N} [W(\varepsilon x) f(w) f'(w) \phi + V(\varepsilon x) f(z) f'(z) \varphi] \\ &\quad - \int_{\mathbb{R}^N} [Q_w(\varepsilon x, f(w), f(z)) f'(w) \phi + Q_z(\varepsilon x, f(w), f(z)) f'(z) \varphi]. \end{aligned} \tag{2.2}$$

for all $(w, z), (\phi, \varphi) \in X$.

Let $(w, z) \in X$ be a critical point of the functional I_ε. Then (w, z) constitutes a weak solution to the reformulated system below:

$$\begin{gathered} -\Delta w + W(\varepsilon x) f(w) f'(w) = Q_w(\varepsilon x, f(w), f(z)) f'(w), \quad \text{in } \mathbb{R}^N, \\ -\Delta z + V(\varepsilon x) f(z) f'(z) = Q_z(\varepsilon x, f(w), f(z)) f'(z), \quad \text{in } \mathbb{R}^N, \\ w, z > 0, \quad w, z \in H^1(\mathbb{R}^N). \end{gathered} \tag{2.3}$$

Proposition 2.2. *If* $(w,z) \in X \cap [L^\infty_{\rm loc}(\mathbb{R}^N)]^2$ *is a critical point of* I_ε, *then* $(u,v) = (f(w), f(z))$ *is a solution for* (1.2).

For a poof of the above propostion, see [19, Proposition 2.5].

To apply the variational method and find a solution to (1.2), we will use the *Penalization Method* developed by del Pino and Felmer [17], following the ideas of Alves [2]. Given our interest in securing a positive solution for (1.2), we assume that

$$Q(u,v) = 0 \quad \text{if } u \le 0 \text{ or } v \le 0. \tag{2.4}$$

Let us fix $a > 0$ and let $\eta : \mathbb{R} \to \mathbb{R}$ be a non-increasing C^1 function satisfying

$$\eta \equiv 1 \quad \text{in } (-\infty, a], \quad \eta \equiv 0 \quad \text{in } [5a, +\infty), \quad \eta' \le 0, \quad \text{and} \quad |\eta'| \le \frac{C}{a}, \tag{2.5}$$

where the constant C is independent of a.

Using the function η, we define $\widehat{Q} : \mathbb{R}^2 \to \mathbb{R}$ by

$$\widehat{Q}(s,t) = \eta(|(s,t)|)Q(s,t) + [1 - \eta(|(s,t)|)]A(s^2 + t^2), \tag{2.6}$$

where

$$A := \max\Big\{\frac{Q(s,t)}{s^2 + t^2} : (s,t) \in \mathbb{R}^2, a \le |(s,t)| \le 5a\Big\}. \tag{2.7}$$

Note that $A > 0$ and $A \to 0$ as $a \to 0^+$. Thus, we can assume that:

$$A < \frac{1}{4}\min\{W_0, V_0\}, \quad W(x) \ge W_0 > 0, \quad V(x) \ge V_0 > 0, \tag{2.8}$$

where W_0, V_0 are obtained from (A6).

Now, fixing a bounded domain $\Omega \subset \mathbb{R}^N$, we define the function $H : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ by:

$$H(x,s,t) = \chi_\Omega(x)Q(s,t) + [1 - \chi_\Omega(x)]\widehat{Q}(s,t), \tag{2.9}$$

where χ_Ω denotes the characteristic function of Ω.

Lemma 2.3. *The function* H *and its derivatives* H_s *and* H_t *satisfy the following properties:*

- (A16) $pH(x,s,t) = sH_s(x,s,t) + tH_t(x,s,t)$ *for each* $x \in \Omega$.
- (A17) $2H(x,s,t) \le sH_s(x,s,t) + tH_t(x,s,t)$ *for each* $x \in \mathbb{R}^N \setminus \Omega$.
- (A18) *Fixing* $k = \frac{4p}{p-2}$, *we can choose* $a > 0$ *sufficiently small such that*

$$sH_s(x,s,t) + tH_t(x,s,t) \le \frac{1}{k}[W(x)s^2 + V(x)t^2], \quad \textit{in } \mathbb{R}^N \setminus \Omega,$$
$$\frac{|H_s(x,s,t)|}{a}, \quad \frac{|H_t(x,s,t)|}{a} \le \frac{1}{4}\min\{W_0, V_0\}, \quad \textit{in } \mathbb{R}^N \setminus \Omega.$$

For a proof of the above lemma see [2, Lemma 2.2]. Now, our objective is to study the existence of solutions for the auxiliary system

$$\begin{gathered} -\Delta w + W(\varepsilon x)f(w)f'(w) = H_w(\varepsilon x, f(w), f(z))f'(w), \quad \text{in } \mathbb{R}^N, \\ -\Delta z + V(\varepsilon x)f(z)f'(z) = H_z(\varepsilon x, f(w), f(z))f'(z), \quad \text{in } \mathbb{R}^N, \\ w, z > 0, \quad w, z \in H^1(\mathbb{R}^N). \end{gathered} \tag{2.10}$$

Remark 2.4. If (w,z) is a solution of (2.10) satisfying

$$|(f(w(x)), f(z(x)))| \le a, \quad \forall x \in \mathbb{R}^N \setminus \Omega_\varepsilon,$$

then (w,z) will be a solution of (2.3), where $\Omega_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in \Omega\}$. Thus, our goal is to obtain solutions of (2.10) with this property.

Associated with the system (2.10), we define on X the functional

$$\begin{aligned}\Phi_\varepsilon(w,z) = \frac{1}{2}\int_{\mathbb{R}^N}[|\nabla w|^2 + |\nabla z|^2 + W(\varepsilon x)f^2(w) + V(\varepsilon x)f^2(z)]\\ - \int_{\mathbb{R}^N} H(\varepsilon x, f(w), f(z)).\end{aligned} \tag{2.11}$$

Under conditions (A7) and (A11), it is possible to show that the functional Φ_ε is of class C^1 with Gateaux derivative

$$\begin{aligned}&\Phi'_\varepsilon(w,z)(\phi,\varphi)\\ &= \int_{\mathbb{R}^N}[\nabla w\nabla\phi + \nabla z\nabla\varphi] + \int_{\mathbb{R}^N}[W(\varepsilon x)f(w)f'(w)\phi + V(\varepsilon x)f(z)f'(z)\varphi]\\ &\quad - \int_{\mathbb{R}^N}[H_w(\varepsilon x, f(w), f(z))f'(w)\phi + H_z(\varepsilon x, f(w), f(z))f'(z)\varphi],\end{aligned} \tag{2.12}$$

for any $(w,z),(\phi,\varphi) \in X$. Therefore, the critical points of Φ_ε are precisely the weak solutions of (2.10).

For each $\rho > 0$, consider the set

$$\Sigma_\rho = \left\{(w,z) \in X : \Psi(w,z) = \rho^2\right\}, \tag{2.13}$$

where

$$\Psi(w,z) = \int_{\mathbb{R}^N}[|\nabla w|^2 + |\nabla z|^2 + W(\varepsilon x)f^2(w) + V(\varepsilon x)f^2(z)]. \tag{2.14}$$

Since Ψ is continuous, Σ_ρ is a closed subset in X that disconnects X into

$$X_1 := \{(w,z) \in X : \Psi(w,z) > \rho^2\}, \quad X_2 := \{(w,z) \in X : \Psi(w,z) < \rho^2\}.$$

The following lemma guarantees that the functional Φ_ε meets the geometric conditions required by the Mountain Pass Theorem.

Lemma 2.5. *The functional Φ_ε satisfies the following conditions:*

- (i) *There exist constants $\rho, \alpha > 0$ such that $\Phi_\varepsilon(w,z) \ge \alpha$, for all $(w,z) \in \Sigma_\rho$.*
- (ii) *For every $\varepsilon \in (0,1]$, there exist $(e_1,e_2) \in X_2$ such that $\Phi_\varepsilon(e_1,e_2) \le 0$.*

Proof. Using (A16)–(A18), we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} H(\varepsilon x, f(w), f(z)) \le \int_{\Omega_\varepsilon} H(\varepsilon x, f(w), f(z))\\ + \frac{1}{2k}\int_{\mathbb{R}^N/s\Omega_\varepsilon}[W(\varepsilon x)f(w)^2 + V(\varepsilon x)f(z)^2].\end{aligned}$$

By (2.9) and (A12), we have

$$\begin{aligned}\int_{\mathbb{R}^N} H(\varepsilon x, f(w), f(z)) \le C\int_{\mathbb{R}^N}(|f^2(w)|^{p/2} + |f^2(z)|^{p/2})\\ + \frac{1}{2k}\int_{\mathbb{R}^N\setminus\Omega_\varepsilon}[W(\varepsilon x)f(w)^2 + V(\varepsilon x)f(z)^2].\end{aligned} \tag{2.15}$$

Applying Hölder's inequality and the embedding of $D^{1,2}(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} H(\varepsilon x, f(w), f(z))\,dx \\ &\le C\Big(\int_{\mathbb{R}^N} |f^2(w)|^2\,dx\Big)^{\sigma p/2}\Big(\int_{\mathbb{R}^N} |f^2(w)|^{2^*}\,dx\Big)^{1-\frac{\sigma p}{2}} \\ &\quad + C\Big(\int_{\mathbb{R}^N} |f^2(z)|^2\,dx\Big)^{\sigma p/2}\Big(\int_{\mathbb{R}^N} |f^2(z)|^{2^*}\,dx\Big)^{1-\frac{\sigma p}{2}} \\ &\quad + \frac{1}{2k}\int_{\mathbb{R}^N\setminus\Omega_\varepsilon} \big[W(\varepsilon x)f(w)^2 + V(\varepsilon x)f(z)^2\big]\,dx \\ &\le C\rho^{\sigma p}\Big(\int_{\mathbb{R}^N} |\nabla(f^2(w))|^2\,dx\Big)^{\frac{(1-\frac{\sigma p}{2})2^*}{2}} \\ &\quad + C\rho^{\sigma p}\Big(\int_{\mathbb{R}^N} |\nabla(f^2(z))|^2\,dx\Big)^{\frac{(1-\frac{\sigma p}{2})2^*}{2}} \\ &\quad + \frac{1}{2k}\int_{\mathbb{R}^N\setminus\Omega_\varepsilon} \big[W(\varepsilon x)f(w)^2 + V(\varepsilon x)f(z)^2\big]\,dx \end{aligned} \tag{2.16}$$

where $\sigma := \frac{2.2^*-p}{(2^*-1)p}$ and $C > 0$.

Noting that for each $(w, z) \in \Sigma_\rho$,

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla(f^2(w))|^2 \le 2\int_{\mathbb{R}^N} |\nabla w|^2 \le 2\rho^2, \\ &\int_{\mathbb{R}^N} |\nabla(f^2(z))|^2 \le 2\int_{\mathbb{R}^N} |\nabla z|^2 \le 2\rho^2. \end{aligned} \tag{2.17}$$

From (2.11), (2.17), and (2.16), we obtain

$$\begin{aligned} \Phi_\varepsilon(w, z) \ge \frac{k-1}{2k}\int_{\mathbb{R}^N} \big[|\nabla w|^2 + |\nabla z|^2 + W(\varepsilon x)f(w)^2 + V(\varepsilon x)f(z)^2\big]\,dx \\ - \frac{C}{V_0^{\sigma p/2}}\rho^{\sigma p}C\rho^{(1-\frac{\sigma p}{2})\frac{2^*}{2}} - \frac{C}{W_0^{\sigma p/2}}\rho^{\sigma p}C\rho^{(1-\frac{\sigma p}{2})\frac{2^*}{2}}. \end{aligned}$$

Thus,

$$\Phi_\varepsilon(w, z) \ge \frac{k-1}{2k}\rho^2 C\rho^{(2N+2p)/(N+2)}, \quad \forall (w, z) \in \Sigma_\rho.$$

Since $\frac{2N+2p}{N+2} > 2$ because $p > 4$, we can choose $\alpha > 0$ sufficiently small, such that

$$\Phi_\varepsilon(w, z) \ge \alpha > 0, \quad \forall (w, z) \in \Sigma_\rho,$$

which proves (i). Now, note that, by condition $\mathbf{(H_1)}$, there exist constants $C_3, C_4 > 0$ such that

$$H(\varepsilon x, s, t) \ge C_3|(s, t)|^p - C_4, \quad \forall (x, s, t) \in \Omega \times \mathbb{R}^2. \tag{2.18}$$

Assume without loss of generality that $0 \in \Omega$. Let $\phi \in C_0^\infty(\mathbb{R}^N, [0, 1])$ be such that $\operatorname{supp}\phi \subset B_r(0) \subset \Omega$, for some $r > 0$. Since $B_r(0) \subset \Omega_\varepsilon$ for all $\varepsilon \in (0, 1]$, $f(t\phi) \ge 0$ for all $t \ge 0$, and $W(\varepsilon x) \le W_\infty$, $V(\varepsilon x) \le V_\infty$ in $\mathbb{R}^N$, by (2.18), for all $t \ge 0$, we have

$$\Phi_\varepsilon((t\phi, t\phi)) \le \frac{t^2}{2}\int_{B_r(0)} \Big[2|\nabla\phi|^2 + \frac{1}{2}\int_{B_r(0)} [W_\infty + V_\infty]f^2(t\phi)$$

$$- C_3 \int_{B_r(0)} |(f(t\phi), f(t\phi))|^p + C_4|B_r(0)|\Big].$$

Thus,

$$\begin{aligned}\Phi_\varepsilon((t\phi, t\phi)) \le t^2\Big[\frac{1}{2}\int_{B_r(0)} \big(2|\nabla\phi|^2 + [W_\infty + V_\infty]\phi^2\big) - C_3 \int_{B_r(0)} |f(t\phi)|^p\Big] \\ + C_4|B_r(0)|.\end{aligned}$$

From property (6) of Lemma 2.1, it follows that the function $\frac{f(t)}{t}$ is decreasing for $t > 0$. Since $0 \le t\varphi \le t$, for all $x \in \Omega_\varepsilon$ and $t > 0$, we obtain $f(t)\varphi(x) \le f(t\varphi(x))$. Hence, for all $\varepsilon \in (0, 1]$ and $t \ge 0$, we have

$$\begin{aligned}&\Phi_\varepsilon((t\phi, t\phi)) \\ &\le t^2\Big[\frac{1}{2}\int_{B_r(0)} \big(2|\nabla\phi|^2 + [W_\infty + V_\infty]\phi^2\big) - C_3 \frac{f^p(t)}{t^2}\int_{B_r(0)} |\phi|^p\Big] + C_4|B_r(0)|.\end{aligned} \tag{2.19}$$

From property (5) of Lemma 2.1 and since $p > 4$, we conclude that

$$\lim_{t\to+\infty} \frac{f^p(t)}{t^2} = \lim_{t\to+\infty} \Big(\frac{f(t)}{\sqrt{t}}\Big)^p t^{\frac{p}{2}-2} = +\infty. \tag{2.20}$$

Hence, by (2.19) and (2.20), the proof of (ii) is complete. □

3. Existence of positive solutions for the auxiliary system

By Theorem 1.4 and Lemma 2.5, there exists a Cerami sequence for Φ_ε at the level

$$c_\varepsilon = \inf_{\gamma\in\Gamma_\varepsilon} \max_{t\in[0,1]} \Phi_\varepsilon(\gamma(t)), \tag{3.1}$$

where

$$\begin{gathered}\Gamma_\varepsilon = \big\{\gamma \in C([0, 1], X) : \gamma(0) = 0, \quad \gamma(1) \in \Phi_\varepsilon^{-1}((-\infty, 0]) \cap X_2\big\}, \\ X_2 = \big\{(w, z) \in X : \Psi(w, z) < \rho^2\big\}.\end{gathered}$$

That is, there exists $(w_n, z_n) \subset X$, such that

$$\Phi_\varepsilon(w_n, z_n) = c_\varepsilon + o_n(1), \quad \text{and} \quad (1 + \|(w_n, z_n)\|)\|\Phi'_\varepsilon(w_n, z_n)\|_* = o_n(1). \tag{3.2}$$

Lemma 3.1. *Every Cerami sequence (w_n, z_n) for Φ_ε is bounded in X.*

Proof. Indeed, let

$$(\phi_n, \varphi_n) := \Big(\frac{f(w_n)}{f'(w_n)}, \frac{f(z_n)}{f'(z_n)}\Big).$$

As

$$\nabla\phi_n = \Big(1 + \frac{2f^2(w_n)}{1 + 2f^2(w_n)}\Big)\nabla w_n, \quad \nabla\varphi_n = \Big(1 + \frac{2f^2(z_n)}{1 + 2f^2(z_n)}\Big)\nabla z_n,$$

we have

$$\|\phi_n\| \le 2\|w_n\|, \quad \|\varphi_n\| \le 2\|z_n\|.$$

Hence, by (3.2), $\Phi'_\varepsilon(w_n, z_n)(\phi_n, \varphi_n) = o_n(1)$; thus,

$$\begin{aligned}c_\varepsilon + o_n(1) &= \Phi_\varepsilon(w_n, z_n) - \frac{1}{p}\Phi'_\varepsilon(w_n, z_n)(\phi_n, \varphi_n) \\ &= \int_{\mathbb{R}^N} \Big[\frac{1}{2} - \frac{1}{p}\Big(1 + \frac{2f^2(w_n)}{1 + 2f^2(w_n)}\Big)\Big]|\nabla w_n|^2\end{aligned}$$

$$\begin{aligned}
&+\int_{\mathbb{R}^N}\Big[\frac{1}{2}-\frac{1}{p}\Big(1+\frac{2f^2(z_n)}{1+2f^2(z_n)}\Big)\Big]|\nabla z_n|^2\\
&+\big(\frac{1}{2}-\frac{1}{p}\big)\int_{\mathbb{R}^N}\big[W(\varepsilon x)f^2(w_n)+V(\varepsilon x)f^2(z_n)\big]\\
&+\frac{1}{p}\int_{\mathbb{R}^N}[f(w_n)H_w(\varepsilon x,f(w_n),f(z_n))+f(z_n)H_z(\varepsilon x,f(w_n),f(z_n))]\\
&-\frac{1}{p}\int_{\mathbb{R}^N}pH(\varepsilon x,f(w_n),f(z_n)).
\end{aligned}$$

Since

$$1+\frac{2f^2(t)}{1+2f^2(t)}\le 2,$$

by (A16), we have

$$\begin{aligned}
c_\varepsilon+o_n(1)\ge{}&\big(\frac{1}{2}-\frac{2}{p}\big)\int_{\mathbb{R}^N}(|\nabla w_n|^2+|\nabla z_n|^2)\\
&+\big(\frac{1}{2}-\frac{1}{p}\big)\int_{\mathbb{R}^N}\big[W(\varepsilon x)f^2(w_n)+V(\varepsilon x)f^2(z_n)\big]\\
&+\frac{1}{p}\int_{\mathbb{R}^N\setminus\Omega_\varepsilon}[f(w_n)H_w(\varepsilon x,f(w_n),f(z_n))+f(z_n)H_z(\varepsilon x,f(w_n),f(z_n))]\\
&-\frac{1}{p}\int_{\mathbb{R}^N}pH(\varepsilon x,f(w_n),f(z_n)).
\end{aligned}$$

By (A17) and (A18), it follows that:

$$\begin{aligned}
c_\varepsilon+o_n(1)&\ge\frac{p-4}{2p}\int_{\mathbb{R}^N}\big(|\nabla w_n|^2+|\nabla z_n|^2\big)\\
&\quad+\big(\frac{1}{2}-\frac{1}{p}\big)\int_{\mathbb{R}^N}\big[W(\varepsilon x)f^2(w_n)+V(\varepsilon x)f^2(z_n)\big]\\
&\quad+\int_{\mathbb{R}^N\setminus\Omega_\varepsilon}H(\varepsilon x,f(w_n),f(z_n))\\
&\ge\frac{p-4}{2p}\int_{\mathbb{R}^N}\big(|\nabla w_n|^2+|\nabla z_n|^2\big)\\
&\quad+\Big(\frac{1}{2}-\frac{1}{p}-\frac{1}{2k}\Big)\int_{\mathbb{R}^N}\big[W(\varepsilon x)f^2(w_n)+V(\varepsilon x)f^2(z_n)\big].
\end{aligned}$$

Using $k=\frac{4p}{p-2}$, we obtain

$$c_\varepsilon+o_n(1)\ge c_p\int_{\mathbb{R}^N}\big[(|\nabla w_n|^2+|\nabla z_n|^2)+W(\varepsilon x)f^2(w_n)+V(\varepsilon x)f^2(z_n)\big],\quad (3.3)$$

where

$$c_p:=\min\big\{\frac{p-4}{2p},\frac{p-2}{4p}\big\}>0,\quad\text{since } 4<p<2.2^*.$$

Thus, to show that (w_n,z_n) is bounded, it suffices to demonstrate the existence of a constant $C>0$ such that

$$\int_{\mathbb{R}^N}(W(\varepsilon x)w_n^2+V(\varepsilon x)z_n^2)\le C,\quad\forall n\in\mathbb{N}.$$

Using Lemma 2.1 and (3.3), we obtain

$$\int_{|w_n|\le 1} W(\varepsilon x)w_n^2 \le \frac{1}{C}\int_{|w_n|\le 1} W(\varepsilon x)f(w_n)^2 \le c+o_n(1).$$

Then

$$\begin{aligned}\int_{\{|w_n|>1\}} W(\varepsilon x)w_n^2 &\le W_\infty \int_{\mathbb{R}^N} w_n^{2^*}\\ &\le W_\infty S\Big(\int_{\mathbb{R}^N}|\nabla w_n|^2\Big)^{2^*/2}\\ &\le W_\infty S(c_\varepsilon+o_n(1))^{2^*/2}.\end{aligned}$$

Similarly, we conclude that $(V(\varepsilon x)z_n^2)$ is bounded in $L^1(\mathbb{R}^N)$. Therefore, (w_n,z_n) is bounded in X. □

Lemma 3.2. *Suppose that $(w_n,z_n)\rightharpoonup(w,z)$ in X. Then:*

(i) *Given $\xi>0$, there exists $R>0$ such that*

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N\setminus B_R(0)}\big[|\nabla w_n|^2+|\nabla z_n|^2+W(\varepsilon x)f^2(w_n)+V(\varepsilon x)f^2(z_n)\big]<\xi.$$

(ii)

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathbb{R}^N}\left[W(\varepsilon x)f(w_n)f'(w_n)w_n+V(\varepsilon x)f(z_n)f'(z_n)z_n\right]\\ &=\int_{\mathbb{R}^N}\left[W(\varepsilon x)f(w)f'(w)w+V(\varepsilon x)f(z)f'(z)z\right].\end{aligned}$$

(iii)

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathbb{R}^N}\left[H_w(\varepsilon x,f(w_n),f(z_n))f'(w_n)w_n+H_z(\varepsilon x,f(w_n),f(z_n))f'(z_n)z_n\right]\\ &=\int_{\mathbb{R}^N}\left[H_w(\varepsilon x,f(w),f(z))f'(w)w+H_z(\varepsilon x,f(w),f(z))f'(z)z\right].\end{aligned}$$

(iv) *If $(w_n(x),z_n(x))\to(w(x),z(x))$ a.e. in $\mathbb{R}^N$ and*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\big[W(\varepsilon x)f^2(w_n)+V(\varepsilon x)f^2(z_n)\big]=\int_{\mathbb{R}^N}\big[W(\varepsilon x)f^2(w)+V(\varepsilon x)f^2(z)\big],$$

then

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\big[W(\varepsilon x)f^2(w_n-w)+V(\varepsilon x)f^2(z_n-z)\big]=0.$$

Proof. Consider the cutoff function $\phi_R\in C^\infty(\mathbb{R}^N)$, such that $\phi_R=0$ in $B_{R/2}(0)$, $\phi_R=1$ in $\mathbb{R}^N\setminus B_R(0)$, $0\le\phi_R\le 1$, and $|\nabla\phi_R|\le C/R$, where the constant $C>0$ is independent of R.

Since (w_n,z_n) is bounded, we have

$$\Phi'_\varepsilon(w_n,z_n)\Big(\phi_R\frac{f(w_n)}{f'(w_n)},\phi_R\frac{f(z_n)}{f'(z_n)}\Big)=o_n(1).$$

Thus,

$$\int_{\mathbb{R}^N}\Big(1+\frac{2f^2(w_n)}{1+2f^2(w_n)}\Big)|\nabla w_n|^2\phi_R+\int_{\mathbb{R}^N}\Big(1+\frac{2f^2(z_n)}{1+2f^2(z_n)}\Big)|\nabla z_n|^2\phi_R$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^N}\Big(\frac{f(w_n)}{f'(w_n)}\nabla w_n+\frac{f(z_n)}{f'(z_n)}\nabla z_n\Big)\nabla\phi_R\\
&+\int_{\mathbb{R}^N}W(\varepsilon x)f^2(w_n)\phi_R+\int_{\mathbb{R}^N}V(\varepsilon x)f^2(z_n)\phi_R\\
&=\int_{\mathbb{R}^N}\left[H_w(\varepsilon x,f(w_n),f(z_n))f(w_n)\phi_R+H_z(\varepsilon x,f(w_n),f(z_n))f(z_n)\phi_R\right]+o_n(1).
\end{aligned}
$$

Choosing $R>0$ such that $\Omega_\varepsilon\subset B_{R/2}(0)$, by (A17) and (A18), we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N}(|\nabla w_n|^2+|\nabla z_n|^2)\phi_R+(1-\frac{1}{k})\int_{\mathbb{R}^N}\left[W(\varepsilon x)f^2(w_n)+V(\varepsilon x)f^2(z_n)\right]\phi_R\\
&\le -\int_{\mathbb{R}^N}\Big(\frac{f(w_n)}{f'(w_n)}\nabla w_n+\frac{f(z_n)}{f'(z_n)}\nabla z_n\Big)\nabla\phi_R+o_n(1).
\end{aligned}
$$

Using Lemma 2.1, the Cauchy-Schwarz inequality, the boundedness of (w_n,z_n), and $|\nabla\phi_R|\le C/R$, we obtain:

$$
\begin{aligned}
&(1-\frac{1}{k})\int_{\mathbb{R}^N\setminus B_R(0)}\left[|\nabla w_n|^2+|\nabla z_n|^2+W(\varepsilon x)f^2(w_n)+V(\varepsilon x)f^2(z_n)\right]\\
&\le 2\left(\|w_n\|_{L^2(\mathbb{R}^N)}\|\nabla w_n\nabla\phi_R\|+\|z_n\|_{L^2(\mathbb{R}^N)}\|\nabla z_n\nabla\phi_R\|\right)\\
&\le \frac{Mc}{R}+o_n(1),
\end{aligned}
$$

which completes the proof of (i).

(ii) The proof of (ii) follows from (i) and the property $f(t)f'(t)\le f^2(t)$, we have

$$
\limsup_{n\to\infty}\int_{\mathbb{R}^N\setminus B_R(0)}W(\varepsilon x)f(w_n)f'(w_n)w_n+V(\varepsilon x)f(z_n)f'(z_n)z_n=o_R(1). \tag{3.4}
$$

Since $w_n\to w$ and $z_n\to z$ in $L^s_{\rm loc}(\mathbb{R}^N)$, we have

$$
\begin{gathered}
w_n(x)\to w(x),\quad z_n(x)\to z(x)\quad\text{a.e. in }\mathbb{R}^N,\\
|w_n(x)|\le g_1(x),\quad |z_n(x)|\le g_2(x),\quad g_1,g_2\in L^s(B_R(0)),\quad s\in[1,2^*).
\end{gathered}
$$

By the Lebesgue Dominated Convergence Theorem, we have

$$
\begin{aligned}
&\lim_{n\to\infty}\int_{B_R(0)}\left[W(\varepsilon x)f(w_n)f'(w_n)w_n+V(\varepsilon x)f(z_n)f'(z_n)z_n\right]\\
&=\int_{B_R(0)}\left[W(\varepsilon x)f(w)f'(w)w+V(\varepsilon x)f(z)f'(z)z\right].
\end{aligned}\tag{3.5}
$$

From (3.4) and (3.5), we obtain item (ii).

(iii) Now, using item (i) and that $\Phi'_\varepsilon(w_n,z_n)(w_n,z_n)=o_n(1)$, we have

$$
\begin{aligned}
&\limsup_{n\to\infty}\int_{\mathbb{R}^N\setminus B_R(0)}\big[H_w(\varepsilon x,f(w_n),f(z_n))f'(w_n)w_n\\
&+H_z(\varepsilon x,f(w_n),f(z_n))f'(z_n)z_n\big]=o_R(1).
\end{aligned}\tag{3.6}
$$

Using that $\Omega_\varepsilon \subset B_R(0)$, hypothesis (A18), the convergence $w_n \to w$, $z_n \to z$ in $L^s_{\rm loc}(\mathbb{R}^N)$, and applying the Lebesgue Dominated Convergence Theorem, we obtain

$$\begin{aligned} &\lim_{n\to\infty} \int_{B_R(0)} [H_w(\varepsilon x, f(w_n), f(z_n)) f'(w_n) w_n + H_z(\varepsilon x, f(w_n), f(z_n)) f'(z_n) z_n] \\ &= \int_{B_R(0)} [H_w(\varepsilon x, f(w), f(z)) f'(w) w + H_z(\varepsilon x, f(w), f(z)) f'(z) z]\,. \end{aligned} \tag{3.7}$$

Thus, from (3.6) and (3.7), we obtain item (iii). □

Proposition 3.3. *The functional Φ_ε satisfies the Cerami condition for each level c_ε.*

Proof. Let $(w_n, z_n) \subset X$ be such that

$$\Phi_\varepsilon(w_n, z_n) = c_\varepsilon + o_n(1) \quad \text{and} \quad (1 + \|(w_n, z_n)\|)\|\Phi'_\varepsilon(w_n, z_n)\|_* = o_n(1).$$

By Lemma 3.1, (w_n, z_n) is bounded in X. Thus, up to a subsequence, $(w_n, z_n) \rightharpoonup (w, z)$ in X. Furthermore, since

$$\Phi'_\varepsilon(w_n, z_n)(w_n, z_n) = o_n(1),$$

using (iii) from Lemma 3.2, we obtain

$$\begin{aligned} &\lim_{n\to\infty} \int_{\mathbb{R}^N} \left[|\nabla w_n|^2 + |\nabla z_n|^2 + W(\varepsilon x) f(w_n) f'(w_n) w_n + V(\varepsilon x) f(z_n) f'(z_n) z_n\right] \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^N} [H_w(\varepsilon x, f(w), f(z)) f'(w) w + H_z(\varepsilon x, f(w), f(z)) f'(z) z]\,. \end{aligned} \tag{3.8}$$

Now, using that $\Phi'_\varepsilon(w_n, z_n)(\phi, \varphi) = o_n(1)$ for every $\phi, \varphi \in C_0^\infty(\mathbb{R}^N)$, by passing to the limit combined with the Lebesgue Dominated Convergence Theorem, we obtain $\Phi'_\varepsilon(w, z)(\phi, \varphi) = 0$. It follows that $\Phi'_\varepsilon(w, z)(\phi, \varphi) = 0$ for every $\phi, \varphi \in X$. In particular, $\Phi'_\varepsilon(w, z)(w, z) = 0$, i.e.,

$$\begin{aligned} &\int_{\mathbb{R}^N} \left[|\nabla w|^2 + |\nabla z|^2 + W(\varepsilon x) f(w) f'(w) w + V(\varepsilon x) f(z) f'(z) z\right] \\ &= \int_{\mathbb{R}^N} [H_w(\varepsilon x, f(w), f(z)) f'(w) w + H_z(\varepsilon x, f(w), f(z)) f'(z) z]\,. \end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9), we have

$$\begin{aligned} &\lim_{n\to\infty} \int_{\mathbb{R}^N} \left[|\nabla w_n|^2 + |\nabla z_n|^2 + W(\varepsilon x) f(w_n) f'(w_n) w_n + V(\varepsilon x) f(z_n) f'(z_n) z_n\right] \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^N} \left[|\nabla w|^2 + |\nabla z|^2 + W(\varepsilon x) f(w) f'(w) w + V(\varepsilon x) f(z) f'(z) z\right]. \end{aligned} \tag{3.10}$$

Using (ii) from Lemma 3.2 and (3.10), it follows that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \left[|\nabla w_n|^2 + |\nabla z_n|^2\right] = \int_{\mathbb{R}^N} \left[|\nabla w|^2 + |\nabla z|^2\right].$$

Since $(w_n, z_n) \rightharpoonup (w, z)$ in X, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} [\nabla w_n \cdot \nabla w + \nabla z_n \cdot \nabla z] = \int_{\mathbb{R}^N} \left[|\nabla w|^2 + |\nabla z|^2\right].$$

Thus,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\big[|\nabla(w_n-w)|^2+|\nabla(z_n-z)|^2\big]=0. \tag{3.11}$$

Observing that

$$\|w_n-w\|\le C\Big[\Psi_W(w_n-w)+\Psi_W(w_n-w)^{2^*/2}\Big],$$
$$\|z_n-z\|\le C\Big[\Psi_V(z_n-z)+\Psi_V(z_n-z)^{2^*/2}\Big],$$

where

$$\Psi_{\mathcal{V}}(w_n-w):=\int_{\mathbb{R}^N}|\nabla(w_n-w)|^2+\int_{\mathbb{R}^N}\mathcal{V}(\varepsilon x)f^2(w_n-w),$$

we conclude that

$$\begin{aligned}
&\|(w_n,z_n)-(w,z)\|^2\\
&\le C\big[\Psi_W(w_n-w)+\Psi_W(w_n-w)^{2^*/2}+\Psi_V(z_n-z)+\Psi_V(z_n-z)^{2^*/2}\big]\\
&\le C\big[\Psi_W(w_n-w)+\Psi_V(z_n-z)+(\Psi_W(w_n-w)+\Psi_V(z_n-z))^{2^*/2}\big].
\end{aligned}$$

Using (iv) from Lemma 3.2 and (3.11), up to a subsequence, we have $(w_n,z_n)\to(w,z)$ in X. This completes the proof of the proposition. □

Theorem 3.4. *Suppose* (A6), (A10)-(A15) *are satisfied. Then, for every* $\varepsilon\in(0,1]$, *the auxiliary system* (2.10) *has a weak solution* $(w_\varepsilon,z_\varepsilon)\in X$, *such that*

$$\Phi_\varepsilon(w_\varepsilon,z_\varepsilon)=c_\varepsilon \quad\textit{and}\quad \|(w_\varepsilon,z_\varepsilon)\|^2\le C(c_\varepsilon+c_\varepsilon^{2^*/2}), \tag{3.12}$$

where $C>0$ *is a constant independent of* ε, *and* c_ε *is defined by* (3.1).

Proof. Using Lemma 2.5, Proposition 3.3, and Theorem 1.4, we conclude that the functional Φ_ε has a critical point at

$$c_\varepsilon:=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\Phi_\varepsilon(\gamma(t))\ge\alpha,$$

where

$$\Gamma:=\left\{\gamma\in C([0,1],X):\gamma(0)=0,\gamma(1)\in\Phi^{-1}((-\infty,0])\cap X_2\right\},$$

and α is given in Lemma 2.5. Thus, there exists $(w_\varepsilon,z_\varepsilon)\in X$, such that

$$\Phi_\varepsilon(w_\varepsilon,z_\varepsilon)=c_\varepsilon,\quad\text{and}\quad \Phi'_\varepsilon(w_\varepsilon,z_\varepsilon)=0.$$

Therefore, $(w_\varepsilon,z_\varepsilon)$ is a solution of (2.10).

Consider $(\bar w_\varepsilon,\bar z_\varepsilon)$ as a test function and note that $H(\varepsilon x,s,t)=0$ for all $s,t\le0$ and

$$W(\varepsilon x)f(w_\varepsilon)f'(w_\varepsilon)\bar w_\varepsilon,\quad V(\varepsilon x)f(z_\varepsilon)f'(z_\varepsilon)\bar z_\varepsilon\ge0.$$

Then we obtain

$$\begin{aligned}
\|\bar w_\varepsilon,\bar z_\varepsilon\|^2_{D^{1,2}(\mathbb{R}^N)}&=\int_{\mathbb{R}^N}\big[|\nabla\bar w_\varepsilon|^2+|\nabla\bar z_\varepsilon|^2\big]\\
&\le\int_{\mathbb{R}^N}\nabla w_\varepsilon\nabla\bar w_\varepsilon+\int_{\mathbb{R}^N}W(\varepsilon x)f(w_\varepsilon)f'(w_\varepsilon)\bar w_\varepsilon\\
&\quad+\int_{\mathbb{R}^N}\nabla z_\varepsilon\nabla\bar z_\varepsilon+\int_{\mathbb{R}^N}V(\varepsilon x)f(z_\varepsilon)f'(z_\varepsilon)\bar z_\varepsilon\\
&=\int_{\mathbb{R}^N}H(\varepsilon x,f(w_\varepsilon))f'(w_\varepsilon)\bar w_\varepsilon+\int_{\mathbb{R}^N}H(\varepsilon x,f(z_\varepsilon))f'(z_\varepsilon)\bar z_\varepsilon\le0.
\end{aligned}$$

Therefore, $\|\bar{w}_\varepsilon, \bar{z}_\varepsilon\|^2_{D^{1,2}(\mathbb{R}^N)} = 0$. Hence, $(\bar{w}_\varepsilon, \bar{z}_\varepsilon) = 0$, and consequently, $(w_\varepsilon, z_\varepsilon) = (w_\varepsilon^+, z_\varepsilon^+) \geq 0$ a.e. in $\mathbb{R}^N$. By elliptic regularity, we have $(w_\varepsilon, z_\varepsilon) \in C^{1,\alpha}(\mathbb{R}^N)$ (see the proof of Lemma **??**). Thus, $w_\varepsilon > 0$ in $\mathbb{R}^N$.

I did not find Lemma 3.4

Proof of (3.12). Let

$$(\tilde{w}_\varepsilon, \tilde{z}_\varepsilon) = \Big(\frac{f(w_\varepsilon)}{f'(w_\varepsilon)}, \frac{f(z_\varepsilon)}{f'(z_\varepsilon)}\Big).$$

Since

$$\Phi_\varepsilon(w_\varepsilon, z_\varepsilon) - \frac{1}{p}\Phi'_\varepsilon(w_\varepsilon, z_\varepsilon)(\tilde{w}_\varepsilon, \tilde{z}_\varepsilon) = c_\varepsilon,$$

Using (A16), we obtain

$$\begin{aligned} c_\varepsilon \geq & \int_{\mathbb{R}^N} \Big[\frac{1}{2} - \frac{1}{p}\Big(1 + \frac{2f^2(w_\varepsilon)}{1+2f^2(w_\varepsilon)}\Big)\Big]|\nabla w_\varepsilon|^2 \\ & + \int_{\mathbb{R}^N} \Big[\frac{1}{2} - \frac{1}{p}\Big(1 + \frac{2f^2(z_\varepsilon)}{1+2f^2(z_\varepsilon)}\Big)\Big]|\nabla z_\varepsilon|^2 \\ & + \big(\frac{1}{2} - \frac{1}{p}\big)\int_{\mathbb{R}^N} \big[W(\varepsilon x)f^2(w_\varepsilon) + V(\varepsilon x)f^2(z_\varepsilon)\big] \\ & + \frac{1}{p}\int_{\mathbb{R}^N\setminus\Omega_\varepsilon} [f(w_\varepsilon)H_w(\varepsilon x, f(w_\varepsilon), f(z_\varepsilon)) + f(z_\varepsilon)H_z(\varepsilon x, f(w_\varepsilon), f(z_\varepsilon))] \\ & - \frac{1}{p}\int_{\mathbb{R}^N} pH(\varepsilon x, f(w_\varepsilon), f(z_\varepsilon)). \end{aligned}$$

Using the inequality $1 + \frac{2f^2(w_\varepsilon)}{1+2f^2(w_\varepsilon)} \leq 2$ and (A18), we obtain:

$$\begin{aligned} c_\varepsilon \geq & \big(\frac{1}{2} - \frac{2}{p}\big)\int_{\mathbb{R}^N} \big[|\nabla w_\varepsilon|^2 + |\nabla z_\varepsilon|^2\big] \\ & + \big(\frac{1}{2} - \frac{1}{p}\big)\int_{\mathbb{R}^N} \big[W(\varepsilon x)f^2(w_\varepsilon) + V(\varepsilon x)f^2(z_\varepsilon)\big] \\ & + \frac{1}{p}\int_{\mathbb{R}^N\setminus\Omega_\varepsilon} [f(w_\varepsilon)H_w(\varepsilon x, f(w_\varepsilon), f(z_\varepsilon)) + f(z_\varepsilon)H_z(\varepsilon x, f(w_\varepsilon), f(z_\varepsilon))] \\ & - \frac{1}{p}\int_{\mathbb{R}^N} pH(\varepsilon x, f(w_\varepsilon), f(z_\varepsilon)). \end{aligned}$$

which gives

$$\begin{aligned} c_\varepsilon \geq & \big(\frac{p-4}{2p}\big)\Big[\int_{\mathbb{R}^N} \big[|\nabla w_\varepsilon|^2 + |\nabla z_\varepsilon|^2 + W(\varepsilon x)f^2(w_\varepsilon) + V(\varepsilon x)f^2(z_\varepsilon)\big]\Big] \\ & - \int_{\mathbb{R}^N\setminus\Omega_\varepsilon} H(\varepsilon x, f(w_\varepsilon), f(z_\varepsilon)). \end{aligned}$$

Since $k = \frac{4p}{p-2}$, by (A18), Lemma 2.1, and the above inequality, we have

$$\begin{aligned} c_\varepsilon \geq & \frac{1}{2k}\Big[\int_{\mathbb{R}^N} |\nabla w_\varepsilon|^2 + |\nabla z_\varepsilon|^2 + C\int_{|w_\varepsilon|,|z_\varepsilon|\leq 1} W(\varepsilon x)w_\varepsilon^2 + V(\varepsilon x)z_\varepsilon^2 \\ & + \int_{|w_\varepsilon|,|z_\varepsilon|>1} W(\varepsilon x)f^2(w_\varepsilon) + V(\varepsilon x)f^2(z_\varepsilon)\Big], \end{aligned} \tag{3.13}$$

for some constant $C > 0$, independent of ε.

Using (3.13) and the embedding $H^1(\mathbb{R}^N) \subset D^{1,2}(\mathbb{R}^N)$, we obtain

$$\int_{|w_\varepsilon|>1} W(\varepsilon x)|w_\varepsilon|^2 \le W_\infty \int_{|w_\varepsilon|>1} |w_\varepsilon|^{2^*} \le W_\infty S\Big(\int_{\mathbb{R}^N} |\nabla w_\varepsilon|^2\Big)^{2^*/2}. \tag{3.14}$$

Similarly, the conclusion holds for $(V(\varepsilon x)z_\varepsilon^2)$.

Combining (3.13) and (3.14), we obtain (3.12), wich completes the proof. □

Lemma 3.5. *Let $(w_\varepsilon, z_\varepsilon)$ be the solution of (2.10) obtained in Theorem 3.4, and let the sequences $\varepsilon_n \in (0,1)$ and $(x_n) \subset \mathbb{R}^N$ be such that $\varepsilon_n \to 0$ as $n \to \infty$. The sequences (θ_n) and (ϑ_n) defined by*

$$\theta_n(x) := w_{\varepsilon_n}(x + x_n), \quad \vartheta_n(x) := z_{\varepsilon_n}(x + x_n)$$

belong to $L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ and have subsequences that converge uniformly over compact sets of $\mathbb{R}^N$ to $\theta, \vartheta \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, respectively. Moreover, there exist constants $C_1, C_2, C_3, C_4 > 0$ such that

$$\theta(x) \le C_1 \exp(-C_2|x|), \quad \vartheta(x) \le C_3 \exp(-C_4|x|), \quad \forall x \in \mathbb{R}^N.$$

The proof of the above lemma follows from adapting the arguments used in the proof of [18, Lemma (4.1)], combined with [4, Corollary 4.3].

Lemma 3.6. *Suppose that W and V satisfy (A6), and that either W or V belongs to Class 1 or class 2. Furthermore, suppose that Q satisfies (A10)–(A15). Then*

$$m_\varepsilon := \max_{x \in \partial\Omega_\varepsilon} |(w_\varepsilon(x), z_\varepsilon(x))| \to 0 \quad as \quad \varepsilon \to 0^+,$$

where we define $\Omega_\varepsilon := B_{R_\varepsilon/\varepsilon}$.

Proof. Assume, by contradiction, that the lemma is not true. Then, there exist $\delta > 0$ and a sequence $\varepsilon_n \to 0^+$, such that

$$m_{\varepsilon_n} \ge \delta, \quad \forall n \in \mathbb{N}.$$

Since $w_{\varepsilon_n}, z_{\varepsilon_n} \in C^{1,\alpha}(\mathbb{R}^N)$, there exist $x_n \in \partial B_{R_{\varepsilon_n}/\varepsilon_n}$ such that

$$w_{\varepsilon_n}^2(x_n) + z_{\varepsilon_n}^2(x_n) \ge \delta^2, \quad \forall n \in \mathbb{N}. \tag{3.15}$$

We define the functions $\theta_n, \vartheta_n : \mathbb{R}^N \to \mathbb{R}$ by

$$\theta_n(x) := w_{\varepsilon_n}(x + x_n), \quad \vartheta_n(x) := z_{\varepsilon_n}(x + x_n).$$

By Lemma 3.1, the sequence $(w_{\varepsilon_n}, z_{\varepsilon_n})$ is bounded in X. Thus, by the invariance of $\mathbb{R}^N$ by translation, (θ_n, ϑ_n) is also bounded in X. Moreover, (θ_n, ϑ_n) is a solution of the following system in $\mathbb{R}^N$:

$$\begin{gathered} -\Delta\theta_n + W(\varepsilon_n x + \varepsilon_n x_n)f(\theta_n)f'(\theta_n) = H_w(\varepsilon_n x + \varepsilon_n x_n, f(\theta_n), f(\vartheta_n))f'(\theta_n), \\ -\Delta\vartheta_n + V(\varepsilon_n x + \varepsilon_n x_n)f(\vartheta_n)f'(\vartheta_n) = H_z(\varepsilon_n x + \varepsilon_n x_n, f(\theta_n), f(\vartheta_n))f'(\vartheta_n), \\ \theta_n, \vartheta_n > 0, \quad \theta_n, \vartheta_n \in H^1(\mathbb{R}^N). \end{gathered} \tag{3.16}$$

Note that, up to a subsequence, $(\theta_n, \vartheta_n) \rightharpoonup (\theta, \vartheta)$ in X, for some $(\theta, \vartheta) \in X$. By Lemma 3.5, (θ_n) and (ϑ_n) converge uniformly over compact sets of $\mathbb{R}^N$ to θ and ϑ, respectively. Moreover, $\theta, \vartheta \in C(\mathbb{R}^N)$. Thus, from this fact and the condition above, it follows that:

$$\theta^2(0) + \vartheta^2(0) \ge \delta^2.$$

Hence,

$$\theta \not\equiv 0 \quad \text{or} \quad \vartheta \not\equiv 0. \tag{3.17}$$

Since $(W(\varepsilon_n x_n))$ and $(V(\varepsilon_n x_n))$ are bounded, there exist $\alpha_W, \alpha_V > 0$ such that

$$W(\varepsilon_n x_n) \to \alpha_W, \quad V(\varepsilon_n x_n) \to \alpha_V. \tag{3.18}$$

It follows from (3.16) that

$$\begin{aligned} &\int_{\mathbb{R}^N} \nabla\theta_n \nabla\phi + \int_{\mathbb{R}^N} W(\varepsilon_n x + \varepsilon_n x_n) f(\theta_n) f'(\theta_n)\phi \\ &= \int_{\mathbb{R}^N} H_w(\varepsilon_n x + \varepsilon_n x_n, f(\theta_n), f(\vartheta_n)) f'(\theta_n)\phi + o_n(1), \end{aligned} \tag{3.19}$$

and similarly,

$$\begin{aligned} &\int_{\mathbb{R}^N} \nabla\vartheta_n \nabla\varphi + \int_{\mathbb{R}^N} V(\varepsilon_n x + \varepsilon_n x_n) f(\vartheta_n) f'(\vartheta_n)\varphi \\ &= \int_{\mathbb{R}^N} H_z(\varepsilon_n x + \varepsilon_n x_n, f(\theta_n), f(\vartheta_n)) f'(\vartheta_n)\varphi + o_n(1). \end{aligned} \tag{3.20}$$

Using (3.19) and (3.20), passing to the limit, and the density of $C_0^\infty(\mathbb{R}^N)$ in $H^1(\mathbb{R}^N)$, we obtain

$$\int_{\mathbb{R}^N} [\nabla\theta\nabla\phi + \alpha_W f(\theta) f'(\theta)\phi] = \int_{\mathbb{R}^N} g_1(x, f(\theta), f(\vartheta)) f'(\theta)\phi, \tag{3.21}$$

$$\int_{\mathbb{R}^N} [\nabla\vartheta\nabla\varphi + \alpha_V f(\vartheta) f'(\vartheta)\varphi] = \int_{\mathbb{R}^N} g_2(x, f(\theta), f(\vartheta)) f'(\vartheta)\varphi, \tag{3.22}$$

for all $(\phi, \varphi) \in X$, where

$$\begin{aligned} g_1(x, f(\theta), f(\vartheta)) &:= \tilde{I}(x) Q_u(f(\theta), f(\vartheta)) + (1 - \tilde{I}(x))\widehat{Q}_w(f(\theta), f(\vartheta)), \\ g_2(x, f(\theta), f(\vartheta)) &:= \tilde{I}(x) Q_v(f(\theta), f(\vartheta)) + (1 - \tilde{I}(x))\widehat{Q}_z(f(\theta), f(\vartheta)), \end{aligned}$$

for some function $\tilde{I} \in L^\infty(\mathbb{R}^N)$.

Noting that $(\theta, \vartheta) \in W^{2,2}_{\rm loc}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and

$$\nabla(ff')(w) = (f')^2(w)\nabla w + f(w) f''(w)\nabla w, \quad f''(w) = -2f(w)[f'(w)]^4,$$

for all $w \in H^1(\mathbb{R}^N)$, by property (2) of Lemma 2.1, we have that $(ff')(w) \in H^1(\mathbb{R}^N)$ for all $w \in H^1(\mathbb{R}^N)$. Thus, there exist $\phi_j, \varphi_j \in C_0^\infty(\mathbb{R}^N)$ such that

$$\|\phi_j - (ff')(\theta)\| \le \frac{1}{j}, \quad \|\varphi_j - (ff')(\vartheta)\| \le \frac{1}{j}, \quad \forall j \in \mathbb{N}. \tag{3.23}$$

We assert that (3.17) and (3.21) imply that $\theta \not\equiv 0$ and $\vartheta \not\equiv 0$. Indeed, suppose, by contradiction, that $\theta \not\equiv 0$ and $\vartheta \equiv 0$. Since $f(0) = 0$, by (2.4), (3.15), and (3.19),

$$\int_{\mathbb{R}^N} |\nabla\theta|^2 + \alpha_W \int_{\mathbb{R}^N} f(\theta) f'(\theta)\theta = \int_{\mathbb{R}^N} g_1(x, f(\theta), 0) f'(\theta)\theta = 0.$$

Using that $\theta \ge 0$, we conclude that $\theta \equiv 0$, and thus, $(\theta, \vartheta) = (0, 0)$, which contradicts (3.17). A similar conclusion is obtained if we consider $\theta \equiv 0$ and $\vartheta \not\equiv 0$. Thus, the assertion is true.

Now, suppose W satisfies the (PS) condition, that is, W belongs to Class 1. Choosing $\frac{\partial\phi_j}{\partial x_i}$ as a test function in (3.19), we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \nabla\theta_n \nabla\left(\frac{\partial\phi_j}{\partial x_i}\right) + \int_{\mathbb{R}^N} W(\varepsilon_n x + \varepsilon_n x_n) f(\theta_n) f'(\theta_n)\frac{\partial\phi_j}{\partial x_i} \\ &- \int_{\mathbb{R}^N} H_u(\varepsilon_n x + \varepsilon_n x_n, f(\theta_n), f(\vartheta_n)) f'(\theta_n)\frac{\partial\phi_j}{\partial x_i} = o_n(1). \end{aligned}$$

Thus, exploring the fact that $\theta \not\equiv 0$ and arguing, we can conclude that

$$\nabla W(\varepsilon_n x_n) \to 0, \quad W(\varepsilon_n x_n) \to \alpha_W.$$

Thus, $(\varepsilon_n x_n)$ is a $(\mathrm{PS})_{\alpha_W}$ sequence for W. Therefore, from (A8), $(\varepsilon_n x_n)$ should have a convergent subsequence, but

$$|\varepsilon_n x_n| = R_{\varepsilon_n} = \frac{1}{\varepsilon_n} \to \infty \quad \text{as } n \to \infty.$$

Therefore, the lemma is true for W in Class 1. The case where V belongs to Class 1 is analogous. □

Lemma 3.7. *Suppose that W and V satisfy (A6) and that either W or V belongs to Class 2. Furthermore, suppose that Q satisfies (A10)-(A15). Then*

$$m_\varepsilon := \max_{x \in \partial\Omega_\varepsilon} |(u_\varepsilon(x), v_\varepsilon(x))| \to 0 \quad \text{as } \varepsilon \to 0^+,$$

where we define $\Omega_\varepsilon := \frac{1}{\varepsilon}\Lambda$ with Λ given in hypothesis (A9).

Proof. Assume, for the sake of contradiction, that the statement does not hold. Then, there exists a constant $\delta > 0$ and a sequence $\varepsilon_n \to 0^+$ such that

$$\max_{x \in \partial\Omega_{\varepsilon_n}} |(u_{\varepsilon_n}(x), v_{\varepsilon_n}(x))| \geq \delta, \quad \forall n \in \mathbb{N}.$$

This implies that for each n, there exists a point $x_n \in \partial\Omega_{\varepsilon_n}$ satisfying

$$|(u_{\varepsilon_n}(x_n), v_{\varepsilon_n}(x_n))| \geq \delta.$$

We define the translated functions

$$\theta_n(x) := u_{\varepsilon_n}(x + x_n), \quad \vartheta_n(x) := v_{\varepsilon_n}(x + x_n).$$

Since $(u_{\varepsilon_n}, v_{\varepsilon_n})$ is a solution of the auxiliary system (2.10), the pair (θ_n, ϑ_n) satisfies the system

$$\begin{aligned}
-\Delta\theta_n + W(\varepsilon_n x + \varepsilon_n x_n) f(\theta_n) f'(\theta_n) &= H_w(\varepsilon_n x + \varepsilon_n x_n, f(\theta_n), f(\vartheta_n)) f'(\theta_n),\\
-\Delta\vartheta_n + V(\varepsilon_n x + \varepsilon_n x_n) f(\vartheta_n) f'(\vartheta_n) &= H_z(\varepsilon_n x + \varepsilon_n x_n, f(\theta_n), f(\vartheta_n)) f'(\vartheta_n).
\end{aligned}$$

From the boundedness of $(u_{\varepsilon_n}, v_{\varepsilon_n})$ in $H^1(\mathbb{R}^N)$, the sequences (θ_n, ϑ_n) are also bounded in $H^1(\mathbb{R}^N)$. Specifically, there exists a constant $C > 0$ such that

$$\|\theta_n\|_{H^1(\mathbb{R}^N)} \leq C, \quad \|\vartheta_n\|_{H^1(\mathbb{R}^N)} \leq C, \quad \forall n \in \mathbb{N}.$$

By the Sobolev embedding theorem, up to a subsequence, (θ_n, ϑ_n) converges weakly in $H^1(\mathbb{R}^N)$ and strongly in $L^p_{\text{loc}}(\mathbb{R}^N)$, for $p \in [2, 2^*]$, to some $(\theta, \vartheta) \in H^1(\mathbb{R}^N)$.

The strong convergence in $L^p_{\text{loc}}(\mathbb{R}^N)$ implies that for each compact set $K \subset \mathbb{R}^N$, we have

$$\lim_{n\to\infty} \int_K |\theta_n(x) - \theta(x)|^p dx = 0, \quad \lim_{n\to\infty} \int_K |\vartheta_n(x) - \vartheta(x)|^p dx = 0.$$

In particular, for $K = B_1(0)$ (the unit ball centered at the origin), we have

$$\lim_{n\to\infty} \int_{B_1(0)} |\theta_n(x) - \theta(x)|^2 dx = 0, \quad \lim_{n\to\infty} \int_{B_1(0)} |\vartheta_n(x) - \vartheta(x)|^2 dx = 0.$$

From the assumption $|(u_{\varepsilon_n}(x_n), v_{\varepsilon_n}(x_n))| \geq \delta$, we have

$$\theta_n(0) = u_{\varepsilon_n}(x_n), \quad \vartheta_n(0) = v_{\varepsilon_n}(x_n).$$

Since (θ_n, ϑ_n) converges strongly in $L^p_{\rm loc}(\mathbb{R}^N)$, it follows that

$$\theta(0)^2 + \vartheta(0)^2 = \lim_{n\to\infty} (\theta_n(0)^2 + \vartheta_n(0)^2) \geq \delta^2.$$

This implies that at least one of θ or ϑ is not identically zero. Without loss of generality, assume $\theta \not\equiv 0$.

Using the convergence of (θ_n, ϑ_n) and the continuity of W, V, and H, we can pass to the limit in the weak formulation of the system. This yields

$$\begin{gathered} -\Delta\theta + \alpha_W f(\theta) f'(\theta) = g_1(x, f(\theta), f(\vartheta)) f'(\theta), \\ -\Delta\vartheta + \alpha_V f(\vartheta) f'(\vartheta) = g_2(x, f(\theta), f(\vartheta)) f'(\vartheta), \end{gathered}$$

where

$$\alpha_W = \lim_{n\to\infty} W(\varepsilon_n x_n), \quad \alpha_V = \lim_{n\to\infty} V(\varepsilon_n x_n),$$

and g_1, g_2 are the limits of H_w and H_z, respectively.

Since W or V belongs to Class 2, we have (A9), which implies that $\nabla V(x) \neq 0$ for all $x \in \partial\Lambda$. Now, observe that the sequence $(\varepsilon_n x_n)$ lies on $\partial\Lambda_{\varepsilon_n}$, and as $\varepsilon_n \to 0$, $\varepsilon_n x_n \to x_0 \in \partial\Lambda$, and by the continuity of ∇V, we have $\nabla V(x_0) = 0$. However, this contradicts (A9), which requires that $\nabla V(x_0) \neq 0$ for all $x_0 \in \partial\Lambda$.

The contradiction implies that our initial assumption must be false. Therefore, we conclude that

$$\max_{x\in\partial\Omega_\varepsilon} |(u_\varepsilon(x), v_\varepsilon(x))| \to 0 \quad \text{as } \varepsilon \to 0^+. \qquad \square$$

4. Proof of Theorem 1.2

Proof. Suppose, by contradiction, that there exists $y_\varepsilon \in \mathbb{R}^N \setminus \Omega_\varepsilon$ such that

$$w_\varepsilon(y_\varepsilon) \geq f^{-1}\left(\frac{a}{2}\right).$$

Combining the previous lemma with the fact that $|(w_\varepsilon(x), z_\varepsilon(x))| \to 0$ as $|x| \to +\infty$, see Lemma 3.6, we conclude that there exists a maximum point $x_\varepsilon \in \mathbb{R}^N \setminus \Omega_\varepsilon$ for w_ε.

Since $(w_\varepsilon, z_\varepsilon) \in C^{2,\alpha}_{\rm loc}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is a solution of (2.10), we have

$$W(\varepsilon x_\varepsilon) f(w_\varepsilon(x_\varepsilon)) f'(w_\varepsilon(x_\varepsilon)) = H_w(\varepsilon x_\varepsilon, f(w_\varepsilon(x_\varepsilon)), f(z_\varepsilon(x_\varepsilon))) f'(w_\varepsilon(x_\varepsilon)).$$

Using that f is an increasing function and $f' > 0$ in $(0, \infty)$, we obtain

$$W_0 \frac{a}{2} \leq H_w(\varepsilon x_\varepsilon, f(w_\varepsilon(x_\varepsilon)), f(z_\varepsilon(x_\varepsilon))),$$

which contradicts hypothesis (A18).

Thus, we conclude that

$$w_\varepsilon(x) < f^{-1}\left(\frac{a}{2}\right) \quad \text{in } \mathbb{R}^N \setminus \Omega_\varepsilon.$$

Similarly, we have

$$z_\varepsilon(x) < f^{-1}\left(\frac{a}{2}\right) \quad \text{in } \mathbb{R}^N \setminus \Omega_\varepsilon.$$

Hence,

$$|(f(w_\varepsilon), f(z_\varepsilon))| < a \quad \text{in } \mathbb{R}^N \setminus \Omega_\varepsilon.$$

Therefore, by the definition of H, $(w_\varepsilon, z_\varepsilon)$ is also a solution of (2.3). $\square$

References

[1] C. O. Alves; *Existence of standing waves solutions for a Nonlinear Schrödinger equation in* $\mathbb{R}^N$, J. Elliptic Parabol. Equations, 01 (2015), 231-241.

[2] C. O. Alves; *Local mountain pass for a class of elliptic system*, J. Math. Anal. Appl., (2007), 135-150.

[3] C. O. Alves, J. M. B. do Ó, M. A. S. Souto; *Local mountain-pass for a class of elliptic problems in* $\mathbb{R}^N$ *involving critical growth*, Nonlinear Anal., 46 (2001), 495-510.

[4] S. C. Q. Arruda, G. M. Figueiredo, R. G. Nascimento; *Existence and asymptotic behavior of solutions for a class of semilinear subcritical elliptic systems*, Asymptotic Analysis (2021), 1-20.

[5] C. Chen, H. Yang; *Multiple solutions for a class of quasilinear Schrödinger systems in* $\mathbb{R}^N$, Bull. Malays. Math. Sci. Soc., 42 (2019), 611-636.

[6] J. Chen, Q. Zhang; *Existence of ground state solution of Nehari-Pohožaev type for a quasilinear Schrödinger system*, Differential and Integral Equations, 34(1/2) (2021), 1-20.

[7] M. Colin, L. Jeanjean; *Solutions for a quasilinear Schrödinger equation: a dual approach*, Nonlinear Analysis 56 (2004), 213-226.

[8] F. J. S. A. Corrêa, G. C. G. dos Santos, L. S. Tavares; *Solution for nonvariational quasilinear elliptic systems via sub-supersolution technique and Galerkin method*, Zeitschrift für angewandte Mathematik und Physik, 72 (2021), 1-15.

[9] X.-D. Fang, A. Szulkin; *Multiple solutions for a quasilinear Schrödinger equation*, J. Differ. Equations 254 (2013), 2015–2032.

[10] G. M. Figueiredo, M. F. Furtado; *Multiple positive solutions for a quasilinear system of Schrödinger equations*, Nonlinear Differ. Equ. Appl. NoDEA, 15, no. 3 (2008), 309-334.

[11] Y. Guo, Z. Tang; *Ground state solutions for quasilinear Schrödinger systems*, J. Math. Anal. Appl., 389, no. 1 (2012), 322-339.

[12] R. W. Hasse; *A general method for the solutions of nonlinear soliton and kink Schrödinger equations*, Z. Physik 37 (1980), 83-87.

[13] E. W. Laedke, K. H. Spatschek, L. Stenflo; *Evolution theorem for a class of perturbed envelope soliton solutions*, J. Math. Phys. 24 (1983), 2764-2763.

[14] H. Lange, B. Toomire, P. F. Zweifel; *Timedependent dissipation in nonlinear Schrödinger systems*, J. Math. Phys. 36 (1995), 1274-1283.

[15] J. Liu, Y. Wang, Z. Wang; *Soliton solutions for quasilinear Schrödinger equations II*, J. Differential Equations, 187 (2003), 473-493.

[16] J. M. do Ó, U. Severo; *Solitary waves for a class of quasilinear Schrödinger equations in dimension two*, Calc. Var. Partial Differential Equations, 38 (2010), 275–315.

[17] M. del Pino, P. L. Felmer; *Local Mountain Pass for semilinear elliptic problems in unbounded domains*, Calc. Var. Partial Differential Equations 4 (1996), 121-137.

[18] G. C. G. dos Santos, L. C. Fontinele, R. G. Nascimento, S. C. Q. Arruda; *Solutions for a quasilinear Schrödinger equation: Subcritical and critical cases*, J. Math. Phys., 64, no. 5 (2023).

[19] U. Severo and E. da Silva; *On the existence of standing wave solutions for a class of quasilinear Schrödinger systems*, J. Math. Anal. Appl., 412 (2014), 763-775.

[20] E. A. B. Silva, G. F. Vieira; *Quasilinear asymptotically periodic Schrödinger equations with critical growth*, Calc. Var. Partial Differ. Equations 39 (2010), 1–33.

Addendum posted by the authors on March 27, 2025

A significant portion of this article overlaps with the doctoral thesis "Existência de solução positiva para um sistema de equações de Schrodinger" by Laila Conceição Fontinele. The thesis is publicly available at
`https://scholar.google.com.br/scholar?hl=pt-BR&as_sdt=0%2C5&q=tese+laila+fontinele&btnG=`
and
`https://pdm.propesp.ufpa.br/ARQUIVOS/teses/2022/Tese%20Laila%20Concei%C3%A7%C3%A3o%20Fontinele.pdf`.

Specifically, the following areas show direct overlap between this article and the thesis:

- Definition 1.1, Theorem 1.2, and Remark 1.3 of this article correspond directly to Definição 3.1, Teorema 3.1, and Observação 3.1 in the thesis (page 107).
- Theorem 1.4 of this article corresponds to Teorema 1.2 in the thesis (page 20).
- Section 2: "Reformulation of the system and the auxiliary system" in the article mirrors "Seção 3.2 - A reformulaçõ do sistema e o sistema auxiliar" in the thesis (page 107).
- Theorem 3.4 of this article corresponds to Teorema 3.2 in the thesis (page 120).

We respectfully acknowledge the original work by Dr. Fontinele and her supervisors published in 2022 while this article was published in 2025.

Ayesha Baig
Department of Mathematics and Statistics, Central South University, Changsha 410083, China
Email address: ayee_sha@qq.com

Zhouxin Li (Corresponding Author)
Department of Mathematics and Statistics, Central South University, Changsha 410083, China
Email address: 1735881486@qq.com